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A simple regularized forward-backward-forward dynamical system for structured monotone inclusions

Pham Ky Anh · Trinh Ngoc Hai

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Abstract In this paper, a regularized dynamical system of forward-backward-forward type method for solving structured monotone inclusions is studied. The novelty of the proposed dynamics consists of the fact that only the algebraic part of the dynamical system needs to be regularized, while the differential part remains unchanged. We obtain strong convergence of the generated trajectories to a solution of the original monotone inclusion and under strong monotonicity conditions we obtain a convergence estimate. A time discretization of the dynamical system by explicit Euler scheme provides an iterative regularization forward-backward-forward splitting method with relaxation parameters. Numerical experiments illustrate the effectiveness of the proposed dynamical system approach with regularization.

Keywords Structured monotone inclusions · Dynamical system · Forward-backward-forward method · Iterative regularization method

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1 Introduction

A dynamical system approach to a monotone inclusion consists of constructing a Cauchy problem, which has a unique global solution, whose limit at infinity exists and solves the original problem.
Having enjoyed many interesting features as well as provided several efficient numerical algorithms, the dynamical system approach has already found various applications and attracted much attention of researchers. For a wide literature, please refer to [1, 2, 6–8, 11–17, 19, 22] and references therein.
In this paper, we are interested in the following structured variational inclusion (VI),

$$\text{Find } u^* \in \mathcal{H} \text{ such that } 0 \in (\mathcal{A}+\mathcal{B})u^*, \tag{VI}$$

where $\mathcal{H}$ is a real Hilbert space, $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximally monotone operator and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is a monotone and Lipschitz continuous operator.
Throughout this paper, we assume that the solution set $\Omega = \mathrm{Zer}(\mathcal{A}+\mathcal{B})$ of the (VI) is nonempty.
When $\mathcal{A} = N_C$, the normal cone of a nonempty closed convex subset C of $\mathcal{H}$, we come to the following variational inequality problem (VIP):

$$\text{Find } u^* \in C \text{ such that } \langle \mathcal{B}u^*, u-u^* \rangle \geq 0,\ \forall u \in C. \tag{VIP}$$

Pham Ky Anh
Department of Mathematics, Vietnam National University, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam
E-mail: anhpk@vnu.edu.vn

Trinh Ngoc Hai
School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, 1 Dai Co Viet, Hai Ba Trung, Hanoi, Vietnam
E-mail: hai.trinhngoc@hust.edu.vn

Barnet and Boţ addressed in [9] a dynamical systems of forward-backward-forward type for finding the zeros of the structured variational inclusion (VI):

$$\begin{cases} z(t) = J_{\gamma(t)\mathcal{A}}\left(x(t) - \gamma(t)\mathcal{B}x(t)\right) \\ 0 = \dot{x}(t) + x(t) - z(t) + \gamma(t)\left(\mathcal{B}x(t) - \mathcal{B}z(t)\right) \\ x(0) = x_0, \end{cases} \tag{1}$$

where $\gamma : [0, +\infty) \to (0, \frac{1}{L})$ is a measurable function, L is a Lipschitz constant of $\mathcal{B}$, $x_0 \in \mathcal{H}$ and $J_{\gamma(t)\mathcal{A}}$ denotes the resolvent of the operator $\gamma(t)\mathcal{A}$ for every $t \in [0, +\infty)$.
Dynamical system (1) is a continuous counterpart of the forward-backward-forward algorithm [25]:

$$\begin{cases} z_n = J_{\gamma_n\mathcal{A}}\left(x_n - \gamma_n\mathcal{B}x_n\right) \\ x_{n+1} = z_n + \gamma_n\left(\mathcal{B}x_n - \mathcal{B}z_n\right) \\ x(0) = x_0. \end{cases} \tag{2}$$

Recently, Boţ, Csetnek, and Vuong have attached to pseudo-monotone variational inequality problem (VIP), a dynamical system, which is a continuous analogue of Tseng's forward-backward-forward algorithm.
Weak convergence of the generated trajectories to a solution of the original problem was established in both works [9, 15].
On the other hand, solutions to monotone inclusions in general are not unique and do not depend continuously on the input data. Besides, approximate methods can in general provide only weak convergence to a solution. In this case the so-called regularization technique is needed to provide strongly convergent algorithms, see, [3–5].
Thus, instead of the monotone inclusion problem (VI), we study the so-called regularized variational inclusion (RVI for short):

$$\text{Find } u \in \mathcal{H} \text{ such that } 0 \in (\mathcal{A} + \mathcal{B})u + \alpha\mathcal{F}u. \tag{RVI}$$

where $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is $\gamma-$strongly monotone and $K-$Lipschitz continuous and $\alpha > 0$ is a regularization parameter. It is well known that for each $\alpha > 0$, the RVI has a unique solution u_α.

To find a special solution to (VI) in a stable manner, we consider the variational inequality problem on the solution set Ω of the (VI):

$$\text{Find } u^\dagger \in \Omega \text{ such that } \left\langle \mathcal{F}u^\dagger, u^* - u^\dagger \right\rangle \geq 0, \ \forall u^* \in \Omega. \tag{3}$$

Since the operator $\mathcal{F}$ is strongly monotone and the solution set $\Omega = \mathrm{Zer}(\mathcal{A}+\mathcal{B})$ of maximally monotone operators is closed and convex, see, [10], problem (3) is uniquely solvable.
In a particular case, when $\mathcal{F} = I - g$, where I is an identity operator and g is a suggested point in $\mathcal{H}$, then the unique solution of the problem (3) is $u^\dagger = P_\Omega(g)$- a projection of g onto Ω. If $g = 0$ then $u^\dagger$ is the minimum-norm solution of the VI.
Very recently, Boţ et all [17] have considered the following regularized forward-backward-forward dynamics:

$$\begin{cases} z(t) = J_{\lambda(t)\mathcal{A}}\left(x(t) - \lambda(t)(\mathcal{B}x(t) + \alpha(t)x(t))\right) \\ 0 = \dot{x}(t) + x(t) - z(t) - \lambda(t)\left(\mathcal{B}x(t) - \mathcal{B}z(t) + \alpha(t)(x(t) - z(t))\right) \\ x(0) = x_0 \in \mathcal{H}, \end{cases} \tag{4}$$

where $\lambda(t)$ and $\alpha(t)$ are Lebesgue measurable functions, and x_0 is a given initial condition. Under certain conditions on the parameters $\lambda(t)$ and $\alpha(t)$, the authors proved the existence and uniqueness of the strong global solution as well as strong convergence of trajectories to the minimum-norm solution of the original monotone inclusion.
We assign to each (RVI) a dynamical system:

$$\begin{cases} z(t) = J_{\lambda(t)\mathcal{A}}\left(u(t) - \lambda(t)(\mathcal{B}u(t) + \alpha(t)\mathcal{F}u(t))\right) \\ \dot{u}(t) = -u(t) + z(t) + \lambda(t)\left(\mathcal{B}u(t) - \mathcal{B}z(t)\right) \\ u(0) = u_0 \in \mathcal{H}, \end{cases} \tag{5}$$

where, $\lambda : [0, +\infty) \to [a, b] \subset (0, \frac{1}{L})$ and $\alpha : [0, +\infty) \to (0, +\infty)$ are continuous functions. Observe that in (5) the differential part needs not to be regularized as in (4). This is no mean trivial in regularizing differential-algebraic equations (1). The new dynamics, when $\mathcal{F} = I$, looks simpler and its trajectories may converge faster to the minimum-norm solution than those of (4).
Motivated by the iterative regularization forward-backward-forward splitting method [20],

$$\begin{cases} v_n = J_{\lambda_n \mathcal{A}}(u_n - \lambda_n(\mathcal{B}u_n + \alpha_n \mathcal{F}u_n)) \\ u_{n+1} = v_n + \lambda_n(\mathcal{B}u_n - \mathcal{B}v_n). \end{cases} \tag{6}$$

we will establish the strong convergence of trajectories to the specially chosen solution $u^\dagger$.
The paper is structured as follows. In Section 2, we recall some notions and concepts which will be frequently used in this paper. In Section 3, we establish the existence and uniqueness of the global solution to (5). Moreover, by proving that $u(t) - u_{\alpha(t)} \to 0$ as $t \to +\infty$, where $u_{\alpha(t)}$ is the unique solution to (RVI) with $\alpha = \alpha(t)$, we establish strong convergence of generated trajectories by (5) to $u^\dagger$. Further, we obtain a convergence rate under the strong monotonicity assumption on $\mathcal{A} + \mathcal{B}$. In Sections 4, we consider some discrete dynamical systems obtained via the explicit time discretization of the corresponding continuous ones. Finally, in Section 5, we perform several numerical experiments to illustrate the effectiveness of the proposed method.

2 Preliminaries

We begin by recalling some notations and concepts of multi-valued (set-valued) operators. Let $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued operator acting in a Hilbert space $\mathcal{H}$. The graph of $\mathcal{A}$ is defined by

$$\mathrm{Graph}(\mathcal{A}) = \{(x, u) : x \in \mathcal{H}, u \in \mathcal{A}x\}.$$

A multi-valued operator $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is called: (i) *monotone*, if $\langle u - v, x - y\rangle \geq 0$ for all x, $y \in \mathcal{H}$ and $u \in \mathcal{A}x$, $v \in \mathcal{A}y$; (ii) γ - *strongly monotone*, if there exists $\gamma > 0$ such that $\langle u - v, x - y\rangle \geq \gamma ||x - y||^2$ for all x, $y \in \mathcal{H}$ and $u \in \mathcal{A}x$, $v \in \mathcal{A}y$; (iii) *maximally monotone*, if $\mathcal{A}$ is monotone and its graph is not properly contained in the graph of any other monotone operator.
The resolvent $J_{\lambda\mathcal{A}} = (I + \lambda\mathcal{A})^{-1}$ of the maximal operator $\lambda\mathcal{A}$ for $\lambda > 0$ is a single-valued operator, defined on the whole space $\mathcal{H}$ and it is firmly nonespansive, i.e.,

$$\langle J_{\lambda\mathcal{A}}u - J_{\lambda\mathcal{A}}v, u - v\rangle \geq \|J_{\lambda\mathcal{A}}u - J_{\lambda\mathcal{A}}v\|^2, \ \forall u, v \in \mathcal{H}.$$

Moreover, according to [24], for all $r, s > 0$ and for all $x \in \mathcal{H}$ we have

$$\frac{s - r}{s} \langle J_{s\mathcal{A}}x - J_{r\mathcal{A}}x, J_{s\mathcal{A}}x - x\rangle \geq ||J_{s\mathcal{A}}x - J_{r\mathcal{A}}x||^2.$$

In particular, we get

$$||J_{s\mathcal{A}}x - J_{r\mathcal{A}}x|| \leq \frac{|s - r|}{s}\|x - J_{s\mathcal{A}}x\|.$$

Recall that a single-valued operator $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is called: (i) Lipschitz ($L-$Lipschitz) continuous, if there exists $L > 0$, such that $\|\mathcal{B}x - \mathcal{B}y\| \leq L\|x - y\|$ for all $x, y \in \mathcal{H}$; (ii) monotone, if $\langle \mathcal{B}x - \mathcal{B}y, x - y\rangle \geq 0$ for all $x, y \in \mathcal{H}$; (iii) strongly ($\gamma-$ strongly) monotone, if there exists a constant $\gamma > 0$, such that $\langle \mathcal{B}x - \mathcal{B}y, x - y\rangle \geq \gamma\|x - y\|^2$ for all $x, y \in \mathcal{H}$.
By [18, Lemma 2.4], if $\mathcal{A} : \mathcal{A} \to 2^{\mathcal{H}}$ is a maximally monotone operator and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is a Lipschitz continuous and monotone operator, then the sum $\mathcal{A} + \mathcal{B}$ is a maximally monotone operator.
Observe that since $\mathcal{A}$ is maximally monotone, $\mathcal{B}$ is Lipschitz continuous and monotone and for each $\alpha > 0$, $\alpha\mathcal{F}$ is Lipschitz continuous and strongly monotone, the operator $\mathcal{A} + \mathcal{B} + \alpha\mathcal{F}$ is maximally and strongly monotone, hence its solution set $\mathrm{Zer}\,(\mathcal{A} + \mathcal{B} + \alpha\mathcal{F}) = \{u_\alpha\}$ is a singleton (see [10, Corollary 23.37]).
We call u_α the regularized solution to (RVI) and collect some of its properties in the following lemma, see [20]:

Lemma 2.1 *(i) The net $\{u_\alpha\}$ is bounded;*
(ii) There exists a positive constant N such that for all $\alpha > 0$ and $\beta > 0$, $||u_\alpha - u_\beta|| \le \frac{|\alpha-\beta|}{\alpha} N$;
(iii) $\lim\limits_{\alpha\to 0^+} u_\alpha = u^\dagger$.

Under the strong monotonicity condition of $\mathcal{A}+\mathcal{B}$, we obtain an estimate $\|u_\alpha - u^\dagger\| = \mathcal{O}(\alpha)$.

Lemma 2.2 *If $\mathcal{A}+\mathcal{B}$ is ρ-strongly monotone and $\mathcal{F}$ is γ-strongly monotone then*

$$\|u^\dagger - u_\alpha\| \le \frac{\alpha}{\alpha\gamma+\rho}\|\mathcal{F}u^\dagger\|.$$

Proof We have $0 \in (\mathcal{A}+\mathcal{B})u^\dagger$ and $-\alpha\mathcal{F}u_\alpha \in (\mathcal{A}+\mathcal{B})u_\alpha$. Using the ρ-strong monotonicity of $\mathcal{A}+\mathcal{B}$, we obtain

$$\left\langle \alpha\mathcal{F}u_\alpha, u^\dagger - u_\alpha \right\rangle \ge \rho\|u^\dagger - u_\alpha\|^2.$$

On the other hand, since $\mathcal{F}$ is γ-strongly monotone, it holds that

$$\alpha\left\langle \mathcal{F}u^\dagger, u^\dagger - u_\alpha\right\rangle - \alpha\gamma\|u^\dagger - u_\alpha\|^2 \ge \alpha\left\langle \mathcal{F}u_\alpha, u^\dagger - u_\alpha\right\rangle \ge \rho\|u^\dagger - u_\alpha\|^2.$$

Using the Cauchy-Schwarz inequality, we have

$$\|u^\dagger - u_\alpha\| \le \frac{\alpha}{\alpha\gamma+\rho}\|\mathcal{F}u^\dagger\|.$$

■

Lemma 2.3 [26] *Let $\{a_k\}$, $\{\zeta_k\}$, $\{\theta_k\}$ be sequences of non-negative real numbers satisfying $\zeta_k \in (0,1)$ and*

$$\begin{cases} a_{k+1} \le (1-\zeta_k)a_k + \theta_k \ \forall k \ge 0 \\ \lim_{k\to\infty} \zeta_k = 0;\ \sum_{k=0}^{\infty} \zeta_k = \infty;\ \lim_{k\to\infty} \frac{\theta_k}{\zeta_k} = 0. \end{cases}$$

Then, $a_k \to 0$ as $k \to \infty$.

Finally, we will denote by $AC_{\text{loc}}([0,+\infty),\mathcal{H})$, $L^1_{\text{loc}}([0,+\infty),\mathcal{H})$ the spaces of locally absolutely continuous functions and locally integrable functions, respectively. For more details, see, [9].

3 The dynamical approach with regularization

3.1 The regularized forward-backward-forward dynamics

Consider the dynamical system of equations (5), where, $\lambda : [0,+\infty) \to [a,b] \subset (0, \frac{1}{L})$ and $\alpha : [0,+\infty) \to (0,+\infty)$ are continuous functions.
Following [9], we call $u : [0,+\infty) \to \mathcal{H}$ a strong global solution of (5) if the following properties hold:

i) $u \in AC_{\text{loc}}[0,+\infty)$, i.e., $u(t)$ is absolutely continuous on each interval $[0,T]$, for any $0 < T < +\infty$;
ii) For almost everywhere $t \in [0,+\infty)$ the second equation of (5) holds, where $z(t)$ is defined by the first relation of (5);
iii) $u(0) = u_0$.

Define the function $f : (0,+\infty)\times(0,+\infty)\times\mathcal{H} \to \mathcal{H}$ as

$$f(\alpha,\lambda,u) := ((I-\lambda\mathcal{B})\circ J_{\lambda\mathcal{A}}\circ(I-\lambda\mathcal{B}-\alpha\lambda\mathcal{F}) - (I-\lambda\mathcal{B}))\,u.$$

Thus (5) is reduced to an initial-value problem for the non-autonomous differential equation

$$\begin{cases} \dot{u}(t) = f(\alpha(t),\lambda(t),u(t)) \\ u(0) = u_0. \end{cases} \tag{1}$$

Theorem 3.1 *Let $\alpha : [0,+\infty) \to (0,\alpha^*)$ and $\lambda : [0,+\infty) \to [a,b] \subset (0, \frac{1}{L})$ be two continuous functions. Then for each $u_0 \in \mathcal{H}$, there exists a unique global solution $u \in AC_{loc}([0,+\infty),\mathcal{H})$, satisfying equation (1) for almost every $t \in [0,+\infty)$, and $u(0) = u_0$.*

Proof For applying the Cauchy-Lipschitz-Picard theorem on the existence and uniqueness of the global solution, see, [21, Prop. 6.2.1] to (1), we need to verify the following conditions:

(i) $\forall u \in \mathcal{H},\ f(\cdot,\cdot,u) \in L^1_{\text{loc}}([0,+\infty),\mathcal{H})$;
(ii) $\forall t \in [0,+\infty),\ f(\alpha(t),\lambda(t),\cdot) : \mathcal{H} \to \mathcal{H}$ is continuous, moreover, $\forall u,v \in \mathcal{H},\ \|f(\alpha(t),\lambda(t),u) - f(\alpha(t),\lambda(t),v)\| \le \omega(t,\|u\|+\|v\|)\|u-v\|$, where $\forall r > 0,\ \omega(t,r) \in L^1_{\text{loc}}[0,+\infty)$.
(iii) $\forall t \in [0,+\infty),\ \|f(\alpha(t),\lambda(t),u)\| \le \sigma(t)(1+\|u\|)$, where $\sigma \in L^1_{\text{loc}}[0,+\infty)$.

Indeed, the function $\alpha \longmapsto f(\alpha,\lambda,u)$ is continuous on $[0,+\infty)$. Further, due to [9, Lemma 1], the function $\lambda \longmapsto f(\alpha,\lambda,u)$ is continuous on $(0,+\infty)$ and $\lim\limits_{\lambda \downarrow 0} f(\alpha,\lambda,u) = 0$, for every $u \in \text{Dom}\mathcal{A}$.
Let $u^\dagger \in \text{Dom}\mathcal{A}$ be a solution to (3), then the function $\lambda \longmapsto f(\alpha,\lambda,u^\dagger)$ can be extended continuously on the interval $[0,+\infty)$, hence the function

$$t \longmapsto \varphi(t) := \|f(\alpha(t),\lambda(t),u^\dagger)\| \tag{2}$$

is continuous on $[0,+\infty)$.
In order to have more compact notations, we set $\mathcal{C} := I - \lambda\mathcal{B},\ \mathcal{C}_\alpha := \mathcal{C} - \alpha\lambda\mathcal{F},\ J := J_{\lambda\mathcal{A}}$ and rewrite the right-hand-side of (1) as $f(\alpha,\lambda,u) = (\mathcal{C} \circ J \circ \mathcal{C}_\alpha - \mathcal{C})u$.
Next we show that the function $f(\alpha,\lambda,u)$ is globally Lipschitz continuous w.r.t. the third variable. For all $u,v \in \mathcal{H}$, we have $\|f(\alpha,\lambda,u) - f(\alpha,\lambda,v)\|^2 \equiv T_1 + T_2$, where $T_1 := \|\mathcal{C} \circ J \circ \mathcal{C}_\alpha u - \mathcal{C} \circ J \circ \mathcal{C}_\alpha v\|^2$, $T_2 := \|\mathcal{C}u - \mathcal{C}v\|^2 - 2\langle \mathcal{C} \circ J \circ \mathcal{C}_\alpha u - \mathcal{C} \circ J \circ \mathcal{C}_\alpha v, \mathcal{C}u - \mathcal{C}v\rangle$.
Further, using the $L-$ Lipschitz continuity of $\mathcal{B}$, we obtain $T_1 = \|J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v\|^2 + \lambda^2\|\mathcal{B} \circ J \circ \mathcal{C}_\alpha u - \mathcal{B} \circ J \circ \mathcal{C}_\alpha v\|^2 - 2\lambda\langle \mathcal{B} \circ J \circ \mathcal{C}_\alpha u - \mathcal{B} \circ J \circ \mathcal{C}_\alpha v, J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v\rangle \le (1+\lambda^2L^2)\|J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v\|^2 - 2\lambda\langle \mathcal{B} \circ J \circ \mathcal{C}_\alpha u - \mathcal{B} \circ J \circ \mathcal{C}_\alpha v, J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v\rangle$. Since J is firmly nonexpansive, one gets $\|J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v\|^2 \le \langle J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v, \mathcal{C}_\alpha u - \mathcal{C}_\alpha v\rangle$, which ensures that

$$T_1 \le (1+\lambda^2L^2)\langle J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v, \mathcal{C}_\alpha u - \mathcal{C}_\alpha v\rangle - 2\lambda\langle \mathcal{B} \circ J \circ \mathcal{C}_\alpha u - \mathcal{B} \circ J \circ \mathcal{C}_\alpha v, J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v\rangle. \tag{3}$$

Rewritting $T_2 = \|\mathcal{C}u - \mathcal{C}v\|^2 - 2\langle J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v, \mathcal{C}u - \mathcal{C}v\rangle + 2\lambda\langle \mathcal{B} \circ J \circ \mathcal{C}_\alpha u - \mathcal{B} \circ J \circ \mathcal{C}_\alpha, \mathcal{C}u - \mathcal{C}v\rangle$ and using the firm nonexpansiveness of J, we find
$T_1 + T_2 \le (1+\lambda^2L^2-2)\langle J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v, \mathcal{C}_\alpha u - \mathcal{C}_\alpha v\rangle + 2\langle J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v, \mathcal{C}_\alpha u - \mathcal{C}_\alpha v\rangle - 2\lambda\langle \mathcal{B} \circ J \circ \mathcal{C}_\alpha u - \mathcal{B} \circ J \circ \mathcal{C}_\alpha v, J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v\rangle + \|\mathcal{C}u - \mathcal{C}v\|^2 - 2\langle J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v, \mathcal{C}u - \mathcal{C}v\rangle + 2\lambda\langle \mathcal{B} \circ J \circ \mathcal{C}_\alpha u - \mathcal{B} \circ J \circ \mathcal{C}_\alpha v, \mathcal{C}u - \mathcal{C}v\rangle$. Taking into account (3), the monotonicity of $\mathcal{B}$ and the relation between $\mathcal{C}_\alpha$ and $\mathcal{C}$, we come to the estimation

$$T_1 + T_2 \le \|\mathcal{C}u - \mathcal{C}v\|^2 + 2\langle J \circ \mathcal{C}_\alpha u - J \circ \mathcal{C}_\alpha v, \alpha\lambda(\mathcal{F}v - \mathcal{F}u)\rangle + 2\lambda\langle \mathcal{B} \circ J \circ \mathcal{C}_\alpha u - \mathcal{B} \circ J \circ \mathcal{C}_\alpha, \mathcal{C}u - \mathcal{C}v\rangle. \tag{4}$$

Using the $L-$ Lipschitz continuity of $\mathcal{B}$ and $K-$ Lipschitz continuity of F we have $\|\mathcal{C}u - \mathcal{C}v\| \le (1+\lambda L)\|u-v\|$ and $\|\mathcal{C}_\alpha u - \mathcal{C}_\alpha v\| \le (1+\lambda L+\alpha\lambda K)\|u-v\|$. Thus, from (4) we get

$$\|f(\alpha,\lambda,u) - f(\alpha,\lambda,v)\| \le M(t)\|u-v\|, \tag{5}$$

where $M(t) = \sqrt{(1+\lambda L)^2 + 2\alpha\lambda L(1+\lambda L+\alpha\lambda K) + 2\lambda L(1+\lambda L)(1+\lambda L+\alpha\lambda K)}$. Due to the assumptions $\alpha(t) \in (0,\alpha^*)$ and $\lambda(t) \in [a,b] \subset (0,\frac{1}{L})$, there exists a number $M^* > 0$, such as $M(t) \le M^*$ for all $t \in [0,+\infty)$. Thus, the global Lipschitz continuity of $f(\alpha,\lambda,u)$ w.r.t. the third variable is proved.
Due to (2), the function $\sigma(t) := \max\{\varphi(t) + M^*\|u^\dagger\|, M^*\}$ is continuous on $[0,+\infty)$. Thus, $\|f(\alpha(t),\lambda(t),u)\| \le \|f(\alpha(t),\lambda(t),u^\dagger)\| + \|f(\alpha(t),\lambda(t),u) - f(\alpha(t),\lambda(t),u^\dagger)\| \le \varphi(t) + M^*\|u - u^\dagger\| \le \varphi(t) + M^*\|u^\dagger\| + M^*\|u\| \le \sigma(t)(1+\|u\|)$. Thus condition (iii) is satisfied.
Finally, recalling that for each $u \in \mathcal{H}$, the function $\alpha \longmapsto f(\alpha,\lambda,u)$ is continuous on $[0,+\infty)$, while the function $\lambda \longmapsto f(\alpha,\lambda,u)$ is continuous on $(0,+\infty)$ we can conclude that the function $t \longmapsto f(\alpha(t),\lambda(t),u)$ is measurable. Condition (iii) ensures the local integrability of $f(\cdot,\cdot,u)$ for each $u \in \mathcal{H}$, which means Condition (i).
The proof of Theorem 3.1 is complete. ■

3.2 Convergence analysis

Let $\alpha : [0,+\infty) \to (0,+\infty)$ be a continuous function. For each $t \in [0,+\infty)$, there exists a unique solution $u_{\alpha(t)}$ to the problem

$$\text{Find } u \in \mathcal{H} \text{ such that } 0 \in (\mathcal{A}+\mathcal{B})u + \alpha(t)\mathcal{F}u. \tag{RVI$_t$}$$

In what follows, we will call $u_{\alpha(t)}$ a regularized solution to (RVI$_t$) for short.
The following result is a direct consequence of Lemma 2.1.

Lemma 3.1 *Suppose $\alpha : [0,+\infty) \to (0,+\infty)$ is a continuous function. Then it holds:*

(i) The set $\{u_{\alpha(t)}\}$ is uniformly bounded on the interval $[0,+\infty)$;
(ii) There exists a constant $N > 0$, such that for all $t, s \in [0,+\infty)$;

$$\|u_{\alpha(t)} - u_{\alpha(s)}\| \le N \frac{|\alpha(t)-\alpha(s)|}{\alpha(s)};$$

(iii) If $\lim_{t\to+\infty} \alpha(t) = 0$ then $u_{\alpha(t)} \to u^\dagger$, as $t \to +\infty$.

Theorem 3.2 *Let $\alpha(t)$ be a positive and strictly decreasing and continuously differentiable function, satisfying the following conditions*

A1) $\lim_{t\to+\infty} \alpha(t) = 0$;
A2) $\int\limits_0^\infty \alpha(t)\mathrm{dt} = +\infty$;
A3) $\lim_{t\to+\infty} \frac{\dot\alpha(t)}{\alpha^2(t)} = 0$.

Further, assume that $\lambda(t)$ is a continuous function, mapping the interval $[0,+\infty)$ into a finite interval $[a,b] \subset (0,\frac{1}{L})$. Then the trajectory $u(t)$ defined by (5) strongly converges to $u^\dagger$ as $t \to +\infty$.

Proof First observe that by Condition (A1), there exists a positive number α^* such that $\alpha(t) \le \alpha^*$ for all $t \in [0,+\infty)$. Thus, all the conditions of Theorem 3.1 are satisfied, hence there exists a unique global solution $u \in AC_{loc}([0,+\infty),\mathcal{H})$ to (5). From Condition (A3), it follows that the function $\alpha(t)$ is absolutely continuous.
Further, we show that the regularized solution $u_{\alpha(t)}$ is locally absolutely continuous on $[0,+\infty)$, hence it is differentiable almost everywhere. Indeed, for any $T > 0$ and for any $t, s \in [0,T]$, one has $\alpha(t), \alpha(s) \ge \alpha(T)$, hence, according to Lemma 3.1 (ii) $\|u_{\alpha(t)} - u_{\alpha(s)}\| \le N\frac{|\alpha(t)-\alpha(s)|}{\alpha(T)}$, which implies the absolute continuity of $u_{\alpha(t)}$ on the interval $[0,T]$.
Moreover, from the relation $\|u_{\alpha(t+\Delta)} - u_{\alpha(t)}\| \le N\frac{|\alpha(t+\Delta)-\alpha(t)|}{\alpha(t+\Delta)}$, we have

$$\left\|\frac{\mathrm{d}}{\mathrm{dt}} u_{\alpha(t)}\right\| \le N\frac{|\dot\alpha(t)|}{\alpha(t)}. \tag{6}$$

Now let us consider the Lyapunov function $V(t) := \frac{1}{2}\|u(t) - u_{\alpha(t)}\|^2$. From (6), we have

$$\begin{aligned}\dot V(t) &= \langle \dot u(t) - \dot u_{\alpha(t)}, u(t) - u_{\alpha(t)}\rangle \\ &\le \left\langle -u(t) + z(t) + \lambda(t)\left(\mathcal{B}u(t) - \mathcal{B}z(t)\right), z(t) - u_{\alpha(t)}\right\rangle + N\frac{|\dot\alpha(t)|}{\alpha(t)}\|u(t) - u_{\alpha(t)}\| + \\ &\quad + \langle -u(t) + z(t) + \lambda(t)\left(\mathcal{B}u(t) - \mathcal{B}z(t)\right), u(t) - z(t)\rangle. \end{aligned} \tag{7}$$

Using the L-Lipschitz continuity of $\mathcal{B}$, we have

$$\langle -u(t) + z(t) + \lambda(t)\left(\mathcal{B}u(t) - \mathcal{B}z(t)\right), u(t) - z(t)\rangle \le -(1-\lambda(t)L)\|u(t) - z(t)\|^2. \tag{8}$$

On the other hand, from the definition of $z(t)$ in (5), we get

$$u(t) - \lambda(t)\left[\mathcal{B}u(t) + \alpha(t)\mathcal{F}u(t)\right] \in z(t) + \lambda(t)\mathcal{A}z(t),$$

or equivalently,

$$u(t) - z(t) - \lambda(t)\left[\mathcal{B}u(t) - \mathcal{B}z(t) + \alpha(t)\mathcal{F}u(t)\right] \in \lambda(t)(\mathcal{A}+\mathcal{B})z(t). \tag{9}$$

Since $u_{\alpha(t)}$ is a solution of ($\mathrm{RVI_t}$), it follows that

$$-\lambda(t)\alpha(t)\mathcal{F}u_{\alpha(t)} \in \lambda(t)(\mathcal{A}+\mathcal{B})u_{\alpha(t)}. \tag{10}$$

Using (9), (10) and the monotonicity of $(\mathcal{A}+\mathcal{B})$, we find

$$\left\langle u(t)-z(t)-\lambda(t)\left[\mathcal{B}u(t)-\mathcal{B}z(t)\right]-\lambda(t)\alpha(t)\left[\mathcal{F}u(t)-\mathcal{F}u_{\alpha(t)}\right], z(t)-u_{\alpha(t)}\right\rangle \geq 0. \tag{11}$$

Combining (11), the γ-strong monotonicity and the K-Lipschitz continuity of $\mathcal{F}$, we obtain

$$\begin{aligned}&\left\langle u(t)-z(t)-\lambda(t)\left[\mathcal{B}u(t)-\mathcal{B}z(t)\right], z(t)-u_{\alpha(t)}\right\rangle \geq \lambda(t)\alpha(t)\left\langle \mathcal{F}u(t)-\mathcal{F}u_{\alpha(t)}, z(t)-u_{\alpha(t)}\right\rangle\\ &\geq -K\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|\|z(t)-u(t)\| + \gamma\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|^2.\end{aligned} \tag{12}$$

From (7), (8) and (12), we have

$$\begin{aligned}\dot{V}(t) \leq\ & K\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|\|z(t)-u(t)\| - \gamma\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|^2+\\ &+N\frac{|\dot{\alpha}(t)|}{\alpha(t)}\|u(t)-u_{\alpha(t)}\| - (1-\lambda(t)L)\|u(t)-z(t)\|^2.\end{aligned} \tag{13}$$

Note that

$$K\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|\|z(t)-u(t)\| - \frac{\gamma}{4}\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|^2 - (1-\lambda(t)L)\|u(t)-z(t)\|^2 \leq 0 \tag{14}$$

if

$$K^2 - \frac{\gamma(1-\lambda(t)L)}{\alpha(t)\lambda(t)} \leq 0$$

or equivalently

$$\alpha(t) \leq \frac{\gamma(1-\lambda(t)L)}{K^2\lambda(t)}. \tag{15}$$

Since $\lambda(t) \in [a,b] \subset (0;\frac{1}{L})$ and $\alpha(t) \to 0$, we can assume that for all $t \in [0,+\infty)$, $\alpha(t) \leq \frac{\gamma(1-La)}{K^2 b}$, which ensures (15). On the other hand, it holds that

$$-\frac{\gamma}{4}\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|^2 + N\frac{|\dot{\alpha}(t)|}{\alpha(t)}\|u(t)-u_{\alpha(t)}\| \leq \frac{N^2\dot{\alpha}^2(t)}{\alpha^3(t)\gamma\lambda(t)}. \tag{16}$$

Combining (13), (14) and (16), we have

$$\dot{V}(t) + \gamma\lambda(t)\alpha(t)V(t) \leq \frac{N^2\dot{\alpha}^2(t)}{\alpha^3(t)\gamma\lambda(t)}.$$

The last inequality can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(V(t)e^{\int_0^t \gamma\lambda(u)\alpha(u)du}\right) \leq \frac{\mathrm{d}}{\mathrm{dt}}\left(\int_0^t e^{\int_0^u \gamma\lambda(s)\alpha(s)ds}\frac{N^2\dot{\alpha}^2(u)}{\alpha^3(u)\gamma\lambda(u)}du\right)$$

or

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(V(t)e^{\int_0^t \gamma\lambda(u)\alpha(u)du} - \int_0^t e^{\int_0^u \gamma\lambda(s)\alpha(s)ds}\frac{M^2\dot{\alpha}^2(u)}{\alpha^3(u)\gamma\lambda(u)}du\right) \leq 0.$$

It implies that the function

$$h(t) := V(t)e^{\int_0^t \gamma\lambda(u)\alpha(u)du} - \int_0^t e^{\int_0^u \gamma\lambda(s)\alpha(s)ds}\frac{M^2\dot{\alpha}^2(u)}{\alpha^3(u)\gamma\lambda(u)}du$$

is decreasing and hence, $h(t) \leq h(0) = V(0)$ for all $t \geq 0$. We obtain

$$V(t) \leq e^{-\int_0^t \gamma\lambda(u)\alpha(u)du}\left(\int_0^t e^{\int_0^u \gamma\lambda(s)\alpha(s)ds}\frac{M^2\dot{\alpha}^2(u)}{\alpha^3(u)\gamma\lambda(u)}du + V(0)\right) \tag{17}$$

Under the assumptions that $\int_0^\infty \alpha(t)dt = \infty$, $\lambda(t) \in [a,b] \subset (0;\frac{1}{L})$, we have

$$\lim_{t\to\infty} e^{\int_0^t \gamma\lambda(u)\alpha(u)du} = \infty. \tag{18}$$

If

$$\int_0^\infty e^{\int_0^u \gamma\lambda(s)\alpha(s)du} \frac{N^2\dot{\alpha}^2(u)}{\alpha^3(u)\gamma\lambda(u)} du < \infty,$$

then from (17) it implies that $V(t) \to 0$. In the opposite case, applying l'Hospital's rule, we have

$$\begin{aligned}\lim_{t\to\infty} V(t) &= \lim_{t\to\infty} \frac{e^{\int_0^t \gamma\lambda(s)\alpha(s)ds} \frac{M^2\dot{\alpha}^2(t)}{\alpha^3(t)\gamma\lambda(t)}}{e^{\int_0^t \gamma\lambda(u)\alpha(u)du}\gamma\lambda(t)\alpha(t)} \\ &= \lim_{t\to\infty} \frac{N^2\dot{\alpha}^2(t)}{\alpha^4(t)\gamma^2\lambda^2(t)} \\ &= 0. \end{aligned} \tag{19}$$

The last equality comes from the conditions $\frac{\dot{\alpha}(t)}{\alpha^2(t)} \to 0$, $\lambda(t) \in [a, b] \subset (0; \frac{1}{L})$ for all $t \geq 0$. Taking into account Lemma 3.1-(iii), we obtain the desired result. ∎

Remark 3.1 It is worthy to note that if $\mathcal{F} = I$- an identity operator, then the dynamical systems (1) and (5) are of the same complexity, however the trajectories of the first system converge weakly, while those of the second one converge strongly to a solution of the original variational inclusion.

Remark 3.2 An example of $\alpha(t)$ satisfying the conditions in Theorem 3.2 is $\alpha(t) = \frac{1}{(t+1)^p}$, where $0 < p < 1$.

The condition $\lambda(t) \in (0, \frac{1}{L})$ makes the algorithm not applicable when the constant L is unknown or difficult to estimate. In the following corollary, we propose a different way of choosing $\lambda(t)$, without knowing the constant L.

Corollary 3.1 *Theorem 3.2 remains true if we replace the condition* $\lambda(t) \in [a, b] \subset (0, \frac{1}{L})$ *with the following conditions:* $\lambda(t) \to 0$, $\frac{\dot{\alpha}(t)}{\alpha(t)^2\lambda(t)} \to 0$ *as* $t \to \infty$ *and* $\int_0^\infty \lambda(t)\alpha(t)dt = \infty$.

Proof In the proof of Theorem 3.2, the condition $\lambda(t) \in [a, b] \subset (0, \frac{1}{L})$ is used to obtain (15), (18) and (19). However, these results are still true under the new conditions of $\lambda(t)$, listed in Corollary 3.1. ∎

Remark 3.3 An example of $\lambda(t)$ and $\alpha(t)$ satisfying the conditions in Corollary 3.1 is: $\lambda(t) = \frac{1}{(t+1)^\alpha}$ and $\alpha(t) = \frac{1}{(t+1)^\beta}$, where $\alpha, \beta > 0$, $\alpha + \beta < 1$.

Lemma 3.2 *Suppose that* $\alpha(t)$ *satisfies all the conditions in Theorem 3.2. Then for an arbitrary* $\epsilon > 0$ *we have*

$$\alpha(t)e^{\epsilon t} \to \infty \text{ as } t \to \infty.$$

Proof From Conditions (A3), we have $\frac{\dot{\alpha}(t)}{\alpha(t)} \to 0$ as $t \to \infty$. Hence, for all $\epsilon > 0$, there exists $t_0 > 0$ such that

$$\frac{\dot{\alpha}(t)}{\alpha(t)} \geq -\frac{\epsilon}{2} \ \forall t \geq t_0.$$

It implies that

$$\alpha(t) \geq e^{-\frac{\epsilon}{2}(t-t_0)}\alpha(t_0) \ \forall t \geq t_0.$$

We obtain the desired result. ∎

Corollary 3.2 *In Theorem 3.2, suppose that* $\mathcal{A}+\mathcal{B}$ *is* ρ*-strongly monotone instead of monotone. Then, we have*

$$\|u(t) - u^\dagger\|^2 = \mathcal{O}(\alpha(t)) \text{ as } t \to \infty.$$

Proof Since $\mathcal{A}+\mathcal{B}$ is ρ-strongly monotone, instead of (11), we have

$$\left\langle u(t)-z(t)-\lambda(t)\left[\mathcal{B}u(t)-\mathcal{B}z(t)\right]-\lambda(t)\alpha(t)\left[\mathcal{F}u(t)-\mathcal{F}u_{\alpha(t)}\right], z(t)-u_{\alpha(t)}\right\rangle \geq \rho\|z(t)-u_{\alpha(t)}\|^2. \tag{20}$$

Combining (20), the γ-strong monotonicity and the K-Lipschitz continuity of $\mathcal{F}$, we obtain

$$\begin{aligned}
&\left\langle u(t)-z(t)-\lambda(t)\left[Bu(t)-Bz(t)\right], z(t)-u_{\alpha(t)}\right\rangle \geq \\
&\quad\geq \lambda(t)\alpha(t)\left\langle \mathcal{F}u(t)-\mathcal{F}u_{\alpha(t)}, z(t)-u_{\alpha(t)}\right\rangle + \rho\|z(t)-u_{\alpha(t)}\|^2 \\
&\quad\geq -K\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|\|z(t)-u(t)\| + \gamma\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|^2 + \rho\|z(t)-u_{\alpha(t)}\|^2.
\end{aligned}$$

Hence, instead of (13), now we have

$$\begin{aligned}
\dot{V}(t) &\leq K\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|\|z(t)-u(t)\| - \gamma\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|^2 + \\
&\quad + M\frac{|\dot{\alpha}(t)|}{\alpha(t)}\|u(t)-u_{\alpha(t)}\| - (1-\lambda(t)L)\|u(t)-z(t)\|^2 - \rho\|z(t)-u_{\alpha(t)}\|^2.
\end{aligned} \tag{21}$$

Since $\lambda(t) \in [a,b] \subset (0, \frac{1}{L})$, there exists $\epsilon > 0$ such that $1-\epsilon-\lambda(t)L > 0$. Due to the condition $\alpha(t) \to 0$, there exists $t_0 > 0$ such that

$$\alpha(t) \leq \frac{\gamma(1-\epsilon-\lambda(t)L)}{K^2\lambda(t)} \ \forall t \geq t_0,$$

and hence for all $t \geq t_0$, we have

$$K\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|\|z(t)-u(t)\| - \frac{\gamma}{4}\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|^2 - (1-\epsilon-\lambda(t)L)\|u(t)-z(t)\|^2 \leq 0. \tag{22}$$

On the other hand, due to Condition (A3), there exists a constant $Q > 0$ satisfying

$$\frac{N^2\dot{\alpha}^2(t)}{\alpha^3(t)\gamma\lambda(t)} \leq Q\alpha(t) \ \forall t \geq 0.$$

Combining this inequality and (16), we get

$$-\frac{\gamma}{4}\lambda(t)\alpha(t)\|u(t)-u_{\alpha(t)}\|^2 + N\frac{|\dot{\alpha}(t)|}{\alpha(t)}\|u(t)-u_{\alpha(t)}\| \leq Q\alpha(t) \ \forall t \geq 0. \tag{23}$$

Moreover, it is easy seen that

$$\epsilon\|u(t)-z(t)\|^2 + \rho\|z(t)-u_{\alpha(t)}\|^2 \geq \frac{\epsilon\rho}{\epsilon+\rho}\|u(t)-u_{\alpha(t)}\|^2. \tag{24}$$

From (21), (22), (23), (24), we have

$$\dot{V}(t) + \frac{2\epsilon\rho}{\epsilon+\rho}V(t) \leq Q\alpha(t) \ \forall t \geq t_0.$$

Hence

$$V(t) \leq e^{-\frac{2\epsilon\rho}{\epsilon+\rho}t}\left(\int_{t_0}^t e^{\frac{2\epsilon\rho}{\epsilon+\rho}u}Q\alpha(u)du + V(t_0)e^{\frac{2\epsilon\rho}{\epsilon+\rho}t_0}\right).$$

Using Lemma 3.2 and the l'Hospital's rule, we have

$$\begin{aligned}
\lim_{t\to\infty}\frac{V(t)}{\alpha(t)} &= \lim_{t\to\infty}\frac{\int_{t_0}^t e^{\frac{2\epsilon\rho}{\epsilon+\rho}u}Q\alpha(u)du}{e^{\frac{2\epsilon\rho}{\epsilon+\rho}t}\alpha(t)} \\
&= \lim_{t\to\infty}\frac{e^{\frac{2\epsilon\rho}{\epsilon+\rho}t}Q\alpha(t)}{e^{\frac{2\epsilon\rho}{\epsilon+\rho}t}\left(\frac{2\epsilon\rho}{\epsilon+\rho}\alpha(t)+\dot{\alpha}(t)\right)} \\
&= Q\frac{2\epsilon\rho}{\epsilon+\rho},
\end{aligned} \tag{25}$$

where the last equality is inferred from Condition (A3). Combining (25), Lemma 2.2 and the inequality

$$\|u(t)-u^\dagger\|^2 \leq 2\|u(t)-u_\alpha(t)\|^2 + 2\|u_\alpha(t)-u^\dagger\|^2,$$

we obtain the desired result. ■

3.3 Discrete dynamical system

Now, we consider a time discretization of the dynamical system (5):

$$\begin{cases} u^0 \in \mathcal{H}, \\ z^k = J_{\lambda_k \mathcal{A}}\left(u^k - \lambda_k(\mathcal{B}u^k + \alpha_k \mathcal{F}u^k)\right), \\ \frac{u^{k+1}-u^k}{h_k} = -u^k + z^k + \lambda_k\left(\mathcal{B}u^k - \mathcal{B}z^k\right), \end{cases} \tag{26}$$

where the parameters satisfy the following conditions:

(B1) $\lambda_k \in [a,b] \subset \left(0, \frac{1}{L}\right)$; $\alpha_k > 0$ for all $k \geq 0$;
(B2) $\lim_{k\to\infty} \alpha_k = 0$;
(B3) $\lim_{k\to\infty} \frac{|\alpha_{k+1}-\alpha_k|}{\alpha_k^2} = 0$;
(B4) $\sum_{k=0}^{\infty} \alpha_k = \infty$;
(B5) $h_k \in [c,d] \subset \left(0, \frac{2(1-bL)}{1+b^2L^2}\right)$.

The last equation in (26) can be rewritten explicitly as

$$u^{k+1} = (1-h_k)u^k + h_k\left(z^k + \lambda_k\left(\mathcal{B}u^k - \mathcal{B}z^k\right)\right).$$

Theorem 3.3 *The sequence $\{u^k\}$ generated by (26) converges strongly to a solution of (VI).*

Proof Denote by u_{α_k} the unique solution of the α_k-regularized variational inclusion:

$$\text{Find } u_{\alpha_k} \in \mathcal{H} \text{ such that } 0 \in (\mathcal{A}+\mathcal{B}+\alpha_k\mathcal{F})u_{\alpha_k}.$$

According to Lemma 3.1, we have $u_{\alpha_k} \to u^\dagger$ as $k \to \infty$. It remains to prove that $\|u^k - u_{\alpha_k}\| \to 0$. From Lemma 3.1, it implies that there exists $P > 0$ satisfying

$$\|u_{\alpha_k} - u_{\alpha_{k+1}}\| \leq P\frac{|\alpha_k - \alpha_{k+1}|}{\alpha_k} \ \forall k \geq 0.$$

Applying the L-Lipschitz continuity and the monotonicity of $\mathcal{B}$, we have

$$\begin{aligned} \left\|u^{k+1} - u_{\alpha_{k+1}}\right\|^2 &= \left\|u^k - u_{\alpha_k} + h_k\left(-u^k + z^k + \lambda_k\left(\mathcal{B}u^k - \mathcal{B}z^k\right)\right) + u_{\alpha_k} - u_{\alpha_{k+1}}\right\|^2 \\ &\leq \|u^k - u_{\alpha_k}\|^2 + h_k^2\left(1+\lambda_k^2L^2\right)\|u^k - z^k\|^2 + P^2\frac{|\alpha_k - \alpha_{k+1}|^2}{\alpha_k^2} + \\ &\quad + 2P\frac{|\alpha_k - \alpha_{k+1}|}{\alpha_k}\|u^k - u_{\alpha_k}\| + 2Ph_k(1+\lambda_kL)\frac{|\alpha_k - \alpha_{k+1}|}{\alpha_k}\|u^k - z^k\| + \\ &\quad + 2h_k\left\langle -u^k + z^k + \lambda_k\left(\mathcal{B}u^k - \mathcal{B}z^k\right), u^k - z^k\right\rangle + \\ &\quad + 2h_k\left\langle -u^k + z^k + \lambda_k\left(\mathcal{B}u^k - \mathcal{B}z^k\right), z^k - u_{\alpha_k}\right\rangle. \end{aligned} \tag{27}$$

It holds that

$$\left\langle -u^k + z^k + \lambda_k\left(\mathcal{B}u^k - \mathcal{B}z^k\right), u^k - z^k\right\rangle \leq (-1+\lambda_kL)\|u^k - z^k\|^2. \tag{28}$$

Similarly to (12), we have

$$\left\langle -u^k + z^k + \lambda_k\left(\mathcal{B}u^k - \mathcal{B}z^k\right), z^k - u_{\alpha_k}\right\rangle \leq K\lambda_k\alpha_k\|u^k - u_{\alpha_k}\|\|z^k - u^k\| - \gamma\lambda_k\alpha_k\|u^k - u_{\alpha_k}\|^2. \tag{29}$$

Combining (27), (28) and (29), we get

$$\begin{aligned} \left\|u^{k+1} - u_{\alpha_{k+1}}\right\|^2 &\leq (1 - 2h_k\gamma\lambda_k\alpha_k)\|u^k - u_{\alpha_k}\|^2 + \left(h_k^2 + \lambda_k^2L^2h_k^2 - 2h_k + 2h_k\lambda_kL\right)\|u^k - z^k\|^2 + \\ &\quad + 2P\frac{|\alpha_k - \alpha_{k+1}|}{\alpha_k}\|u^k - u_{\alpha_k}\| + 2Ph_k(1+\lambda_kL)\frac{|\alpha_k - \alpha_{k+1}|}{\alpha_k}\|u^k - z^k\| + \\ &\quad + 2h_kK\lambda_k\alpha_k\|u^k - u_{\alpha_k}\|\|z^k - u^k\| + P^2\frac{|\alpha_k - \alpha_{k+1}|^2}{\alpha_k^2}. \end{aligned} \tag{30}$$

Under the conditions $\lambda_k \in [a,b] \subset (0, \frac{1}{L})$, $h_k \in [c,d] \subset \left(0, \frac{2(1-bL)}{1+b^2L^2}\right)$, there exists $\xi > 0$ such that

$$h_k^2 + \lambda_k^2 L^2 h_k^2 - 2h_k + 2h_k\lambda_k L \leq -2\xi < 0 \; \forall k \geq 0. \tag{31}$$

We have

$$2Ph_k(1+\lambda_k L)\frac{|\alpha_k - \alpha_{k+1}|}{\alpha_k}\|u^k - z^k\| \leq \xi\|u^k - z^k\|^2 + \frac{P^2 h_k^2(1+bL)^2}{\xi}\left(\frac{|\alpha_k - \alpha_{k+1}|}{\alpha_k}\right)^2, \tag{32}$$

$$2h_k K\lambda_k\alpha_k\|u^k - u_{\alpha_k}\|\|z^k - u^k\| \leq \xi\|u^k - z^k\|^2 + \frac{(h_k Kb\alpha_k)^2}{\xi}\|u^k - u_{\alpha_k}\|^2 \tag{33}$$

and

$$2P\frac{|\alpha_k - \alpha_{k+1}|}{\alpha_k}\|u^k - u_{\alpha_k}\| \leq h_k\gamma a\alpha_k\|u^k - u_{\alpha_k}\|^2 + \frac{P^2|\alpha_k - \alpha_{k+1}|^2}{\alpha_k^3 h_k\gamma a}. \tag{34}$$

Combining (30), (32), (33) and (34), we have

$$\begin{aligned}\left\|u^{k+1} - u_{\alpha_{k+1}}\right\|^2 \leq &\left(1 - h_k\gamma\lambda_k\alpha_k + \frac{(h_k Kb\alpha_k)^2}{\xi}\right)\|u^k - u_{\alpha_k}\|^2 + \frac{P^2|\alpha_k - \alpha_{k+1}|^2}{\alpha_k^3 h_k\gamma a} + \\ &+ \left[\frac{P^2 h_k^2(1+bL)^2}{\xi} + P^2\right]\frac{|\alpha_k - \alpha_{k+1}|^2}{\alpha_k^2}. \end{aligned}\tag{35}$$

Since $\lambda_k \in [a,b] \subset (0, \frac{1}{L})$, $\alpha_k \to 0$, without loss of generality we can suppose that $\zeta_k := h_k\gamma\lambda_k\alpha_k - \frac{(h_k Kb\alpha_k)^2}{\xi} \in (0,1)$ for all $k \geq 0$. Moreover, from the condition $\sum_{k=0}^{\infty}\alpha_k = \infty$, it implies that $\sum_{k=0}^{\infty}\zeta_k = \infty$. On the other hand, using Condition (B2)-(B3), we infer that $\lim_{k\to\infty}\frac{\theta_k}{\zeta_k} = 0$, where

$$\theta_k := \frac{P^2|\alpha_k - \alpha_{k+1}|^2}{\alpha_k^3 h_k\gamma a} + \left[\frac{P^2 h_k^2(1+bL)^2}{\xi} + P^2\right]\frac{|\alpha_k - \alpha_{k+1}|^2}{\alpha_k^2}.$$

Applying Lemma 2.3, from (35) we have $\|u^k - u_{\alpha_k}\| \to 0$ and obtain the desired result. ■

Remark 3.4 Theorem 3.3 recovers Theorem 2 in [20] as a special case, when the stepsize $h_k = 1$ for all $k \geq 0$. Moreover, Algorithm (26) may be considered as an iterative regularization forward-backward-forward splitting method with relaxation parameters. Note that Condition (B5) can be replaced by those of: $h_k > 0$ and $\lim_{k\to\infty} h_k \in \left(0, \frac{2(1-bL)}{1+b^2L^2}\right)$.

The following corollary allows us to apply Algorithm (26) when the constant L is unknown.

Corollary 3.3 *Theorem 3.3 remains true if we replace the conditions (B1)-(B5) by the following ones:* $\alpha_k, \lambda_k > 0$, $\lim_{k\to\infty}\lambda_k = 0$, $\lim_{k\to\infty}\frac{\alpha_k}{\lambda_k} = 0$, $\sum_{k=0}^{\infty}\alpha_k\lambda_k = \infty$, $\lim_{k\to\infty}\frac{|\alpha_{k+1}-\alpha_k|}{\alpha_k^2\sqrt{\lambda_k}} = 0$ *and* $h_k \in [c,d] \subset (0,2)$.

Proof First, we note that under the new conditions of parameters, (31) is still true and we arrive at (35). Since $\lim_{k\to\infty}\frac{\alpha_k}{\lambda_k} = 0$, $\sum_{k=0}^{\infty}\alpha_k\lambda_k = \infty$, without loss of generality, we may assume that $\zeta_k \in (0,1)$ for all $k \geq 0$. We have $\sum_{k=0}^{\infty}\zeta_k = \infty$. On the other hand, since $\lim_{k\to\infty}\frac{|\alpha_{k+1}-\alpha_k|}{\alpha_k^2\sqrt{\lambda_k}} = 0$, it implies that $\lim_{k\to\infty}\frac{\theta_k}{\zeta_k} = 0$. This completes the proof.■

Remark 3.5 We can choose the parameters satisfying the condition in Corollary 3.3 as follows: $\lambda_k = \frac{1}{(k+1)^p}$, $\alpha_k = \frac{1}{(k+1)^q}$, where $q > p > 0$, $p + q \leq 1$.

4 Numerical experiments

In this section, we present some numerical experiments to illustrate the effectiveness of our algorithms. These experiments were conducted using Matlab software, running on a PC with CPU i5 10400 and 16Gb RAM.

Example 4.1 (Application to variational inequalities) Let $\mathcal{A} = N_C$, where

$$C := \{x \in \mathbb{R}^m, -5 \le x_i \le 5 \ \forall i = 1, 2, \dots, m\},$$

$\mathcal{B} : \mathbb{R}^m \to \mathbb{R}^m$, $\mathcal{B}(x) = Bx$ for all $x \in \mathbb{R}^m$, where $B = (b_{ij})$ is an $m \times m$ matrix,

$$b_{ij} = \begin{cases} -1 & \text{if } j = m + 1 - i, \ j > i \\ 1 & \text{if } j = m + 1 - i, \ j < i \\ 0 & \text{otherwise.} \end{cases}$$

It is easy seen that the mapping $\mathcal{A}$ is maximally monotone, $\mathcal{B}$ is monotone and 1-Lipschitz continuous on $\mathbb{R}^m$. The variational inclusion (VI) now becomes the variational inequality

$$\text{Find } u^* \in C \text{ such that } \langle \mathcal{B}u^*, u - u^* \rangle \ge 0, \ \forall u \in C \tag{VIP}$$

and the dynamical system (5) has the following form

$$\begin{cases} z(t) = P_C\left(u(t) - \lambda(t)(\mathcal{B}u(t) + \alpha(t)\mathcal{F}u(t))\right) \\ \dot{u}(t) = -u(t) + z(t) + \lambda(t)\left(\mathcal{B}u(t) - \mathcal{B}z(t)\right) \\ u(0) = u_0 \in \mathbb{R}^m. \end{cases}$$

Let $m = 5$. We implement Algorithm (5) with $\mathcal{F}(u) = u$ for all $u \in \mathbb{R}^m$, $\alpha(t) = \frac{10}{(t+1)^{0.5}}$, $\lambda(t) = \frac{10}{(t+1)^{0.4}}$, the starting point is randomly generated. The trajectories of $u(t)$ are presented in Figure 1. We can see that the solution of the dynamical system (5) converges to a solution $u^\dagger = (0, 0, 0, 0, 0)^T$ of the problem (VIP).

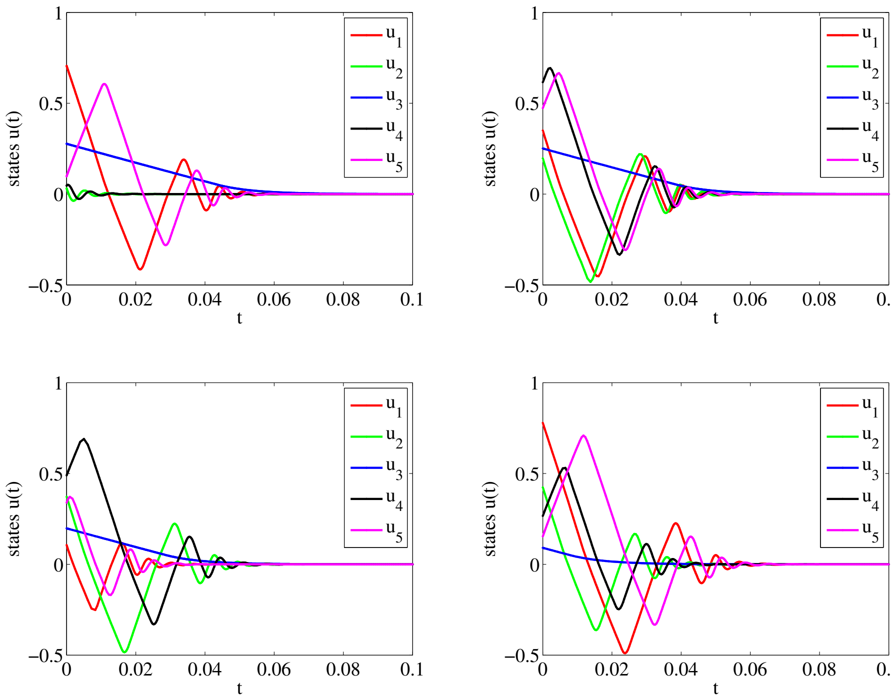


Fig. 1 Performance of Algorithm (5) in Example 4.1 with different starting points

Next, we study effect of the step size on performance of Algorithm (5). Choosing $m = 500$, $u(0) = (1, 1, \dots, 1)^T$, we test Algorithm (5) with different constant functions $\lambda_k(t)$, $k =$

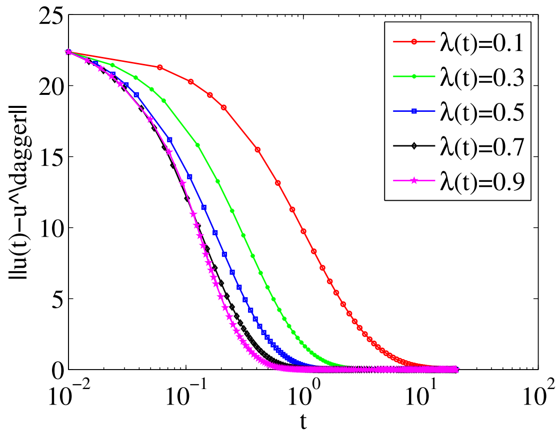


Fig. 2 Performance of Algorithm (5) with different step sizes

$1, \ldots, 5$. The results are presented in Figure 2. We found that the larger $\lambda_k(t)$, the faster the algorithm converges.

To end this example, we compare Algorithm (26) with the Regularization forward-backward-forward splitting method (RFBFSM) [20, Algorithm 2] and the Halpern-type forward-backward-forward splitting method (HFBFSM) [23, Algorithm 3.11]. All these algorithms converge strongly under the same assumptions. In these algorithms, we use the same starting point, which is randomly generated, and choose $\alpha_k = \frac{1}{(k+1)^{0.5}}$, $\lambda_k = 0.5$ for all $k \geq 0$. The mapping $\mathcal{F}$ in the two regularization methods is the identity one. In Algorithm (26), the relaxation parameter is $h_k = 0.7 + \frac{0.9}{\ln(k+2)}$ for all $k \geq 0$. Comparison results are presented in Figure 3. We can see that the two regularization algorithms are clearly superior to the Halpern-type one. Compared with RFBFSM, the new algorithm gives slightly better results. This advantage is more clear when we use overrelaxion parameters in Algorithm (26). To do this, according to Corollary 3.3, in our algorithm, we choose $\lambda_k = \frac{1}{k^{0.1}}$, $h_k = 1.05$ for all $k \geq 0$. The other settings in the both algorithms are unchanged. We test the two algorithms in two cases : $C := \{x \in \mathbb{R}^m : -5 \leq x_i \leq 5\ \forall i = 1, \ldots, m\}$ and $C := \{x \in \mathbb{R}^m : 2x_1^2 + x_2^2 + \ldots + x_m^2 \leq 1\}$. The stopping rule in the both algorithms is $\|u^k - u^\dagger\| \leq 10^{-4}$. We obtain the comparison results as in Table 1. Our algorithm shows better behaviors in term of iterations and computational time.

(a) $C := \{x \in \mathbb{R}^m : -5 \leq x_i \leq 5\ \forall i = 1, \ldots, m\}$

	Algorithm (26)		RFBFSM	
	Times(s)	Iter.	Times(s)	Iter.
m=50	0.0037	27	0.0058	37
m=1000	0.7145	32	1.0173	45
m=5000	18.6659	34	26.8957	49

(b) $C := \{x \in \mathbb{R}^m : 2x_1^2 + x_2^2 + \ldots + x_m^2 \leq 1\}$

	Algorithm (26)		RFBFSM	
	Times(s)	Iter.	Times(s)	Iter.
m=50	0.3497	24	0.3803	34
m=1000	2.9248	24	3.3418	36
m=5000	32.1197	35	45.0149	49

Table 1 Comparison results of Algorithm (26) with RFBFSM in two cases: $C := \{x \in \mathbb{R}^m : -5 \leq x_i \leq 5\ \forall i = 1, \ldots, m\}$ and $C := \{x \in \mathbb{R}^m : 2x_1^2 + x_2^2 + \ldots + x_m^2 \leq 1\}$

Example 4.2 In this example, we compare Algorithm (5) with the one introduced by Boţ et al. [17, Algorithm (5.2)]. Let $\mathcal{H} = l^2$, $\mathcal{A} = N_C$, where $C := \{u \in l^2 : u_1 = 0\}$,

$$\mathcal{B}(u) = (u_1, 0, u_3, \ldots)$$

for all $u = (u_1, u_2, u_3, \ldots) \in l^2$. It is easy seen that $\mathcal{A}$ is maximally monotone, $\mathcal{B}$ is monotone and 1-Lipschitz continuous on l^2. In our algorithm, let $\mathcal{F}$ be the identity mapping. In the both algorithms, we choose $\lambda(t) = \lambda \in (0, 1)$, $\alpha(t) = \frac{1}{(t+1)^{0.5}}$ for all $t \geq 0$. These parameters satisfy all conditions in Theorem 3.2 and [17, Theorem 5.8]. It holds that $J_{\lambda \mathcal{A}}(u) = P_C(u) =$

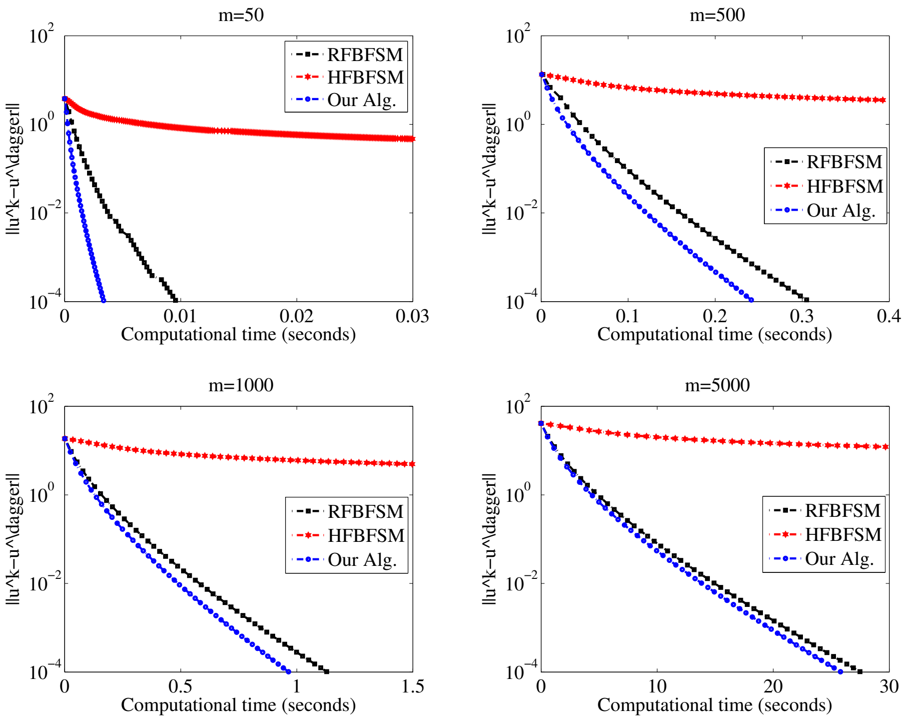


Fig. 3 Comparison of the algorithms in Example 4.1

$(0, u_2, u_3, \ldots)$. From the first equation in (5), we have

$$\begin{cases} z_1(t) = 0, \\ z_2(t) = (1 - \lambda\alpha(t))u_2(t), \\ z_i(t) = (1 - \lambda - \lambda\alpha(t))u_i(t) \ \forall i \geq 3. \end{cases}$$

The dynamical system (5) now becomes

$$\begin{cases} \dot{u}_1(t) = -(1 - \lambda)u_1(t), \\ \dot{u}_2(t) = -\lambda\alpha(t)u_2(t), \\ \dot{u}_i(t) = -\Big(\lambda + \lambda\alpha(t) - \lambda^2 - \lambda^2\alpha(t)\Big)u_i(t), \ \forall i \geq 3, \\ u(0) \in \mathcal{H}. \end{cases}$$

Solving it, we have

$$\begin{cases} u_1(t) = u_1(0)e^{-\int_0^t (1-\lambda)du}, \\ u_2(t) = u_2(0)e^{-\int_0^t \lambda\alpha(u)du}, \\ u_i(t) = u_i(0)e^{-\int_0^t [\lambda+\lambda\alpha(u)-\lambda^2-\lambda^2\alpha(u)]du}, \ \forall i \geq 3. \end{cases}$$

Similarly, the Boţ's algorithm has the following form:

$$\begin{cases} \dot{x}_1(t) = -(1 - \lambda(1 + \alpha(t)))x_1(t), \\ \dot{x}_2(t) = -\lambda\alpha(t)(1 - \lambda\alpha(t))x_2(t), \\ \dot{x}_i(t) = -\Big(\lambda + \lambda\alpha(t) - \lambda^2(1 + \alpha(t))^2\Big)x_i(t), \ \forall i \geq 3, \\ x(0) \in \mathcal{H}. \end{cases}$$

Thus,

$$\begin{cases} x_1(t) = x_1(0)e^{-\int_0^t (1-\lambda)du}e^{\int_0^t \lambda\alpha(u)du}, \\ x_2(t) = x_2(0)e^{-\int_0^t \lambda\alpha(u)du}e^{\int_0^t \lambda^2\alpha^2(u)du}, \\ x_i(t) = x_i(0)e^{-\int_0^t [\lambda+\lambda\alpha(u)-\lambda^2-\lambda^2\alpha(u)]du}e^{\int_0^t \lambda^2\alpha(u)(1+\alpha(u))du}, \ \forall i \geq 3. \end{cases}$$

Assume that the two algorithms start at a same point, i.e. $x(0) = u(0)$. There exists $\mu > 0$ such that

$$x_i(t) \geq \mu.u_i(t)e^{\int_0^t \lambda^2\alpha^2(u)du} = \mu.u_i(t)e^{\int_0^t \frac{\lambda^2}{u+1}du} = \mu(t+1)^{\lambda^2}u_i(t) \;\forall i \geq 1, t \geq 0.$$

Consequently,

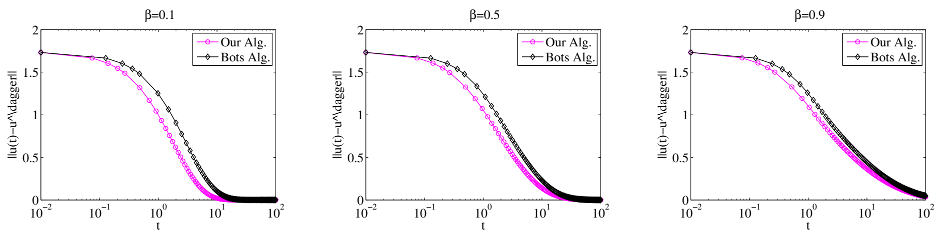
$$\lim_{t\to\infty} \frac{\|u(t) - u^\dagger\|}{\|x(t) - u^\dagger\|} = 0.$$

This means that our algorithm converges to the solution faster than the Boţ's one.

Example 4.3 We compare Algorithm (5) with the Boţ's algorithm [17, Algorithm (5.2)] using the example (6.2) in [17]. Let $C = \{x \in \mathbb{R}^3 : 3x_1 - x_2 + x_3 = 0\}$, $\mathcal{A} = N_C$, $\mathcal{B} = Bx$, where

$$B = \begin{pmatrix} 0 & 0.1 & 0.5 \\ -0.1 & 0 & -0.4 \\ -0.5 & 0.4 & 0 \end{pmatrix}.$$

Following [17], in the both algorithms, we choose $\alpha(t) = \frac{1}{(t+1)^\beta}$, $\lambda(t) = 0.5$ for all $t \geq 0$. Note that, unlike the assertion in [17], if $\beta = 0$, the algorithms remain regularized with the regularization parameter $\alpha(t) = 1$ for all $t \geq 0$. However, this contradicts the condition $\alpha(t) \to 0$. Hence, we test these algorithms with $\beta \in \{0.1,\ 0.5,\ 0.9\}$ and the starting point $x(0) = u(0) = (1, 1, 1)^T$. The comparison results are presented in Figure 4. As we can see, the new algorithm converges faster than the existing one. This happens because in the Boţ's algorithm, the dynamical system needs to be regularized in both the algebraic part and the differential part. Meanwhile, in our algorithm, only the algebraic part of the dynamical system needs to be regularized.



(a) (b) (c)

Fig. 4 Comparison of the two algorithms in Example 4.3 with different regularization parameters

5 Conclusion

We introduce a regularized forward-backward-forward continuous dynamics and proved strong convergence of its trajectories to a specially chosen solution of the original variational inclusion. Under the strong monotonicity assumptions, we obtained a convergence rate. Time discretization of the continuous dynamics provides a iterative regularization forward-backward-forward splitting method with relaxation parameters. Some simple numerical examples were given to illustrate the performance of the proposed algorithm.

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