Existence and stability for generalized polynomial vector variational inequalities

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Abstract. In this paper, we investigate the generalized polynomial vector variational inequality (GPVVI), which is a natural generalization of generalized polynomial variational inequality (GPVI) and vector variational inequality (VVI). Due to the scalarization method, which is a powerful technique in vector optimization, we establish a relationship between the Pareto solution sets of the GPVVI and the solution set of the GPVI. By using the concept on exceptional family of elements, recession cone, and positive semi-definiteness of matrices, we present sufficient conditions for the nonemptiness and boundedness of the Pareto solution sets of the GVVI. We present sufficient conditions for the upper/lower semicontinuity of the weak Pareto solution map and the stability for GVVIs. Finally, we obtain some applications to polynomial variational inequality. The presented results develop and complement the previous ones.

Key Words general vector variational inequality, Pareto solution, solution existence, stability, upper/lower semicontinuity

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1 Introduction

Variational inequalities theory, which was introduced by Stampacchia [19], is an important branch of the mathematical sciences. The concept of *vector variational inequality* (VVI), which is an adequate tool for studying vector optimization problems and vector equilibrium problems, was proposed by F. Giannessi [2]. By using a stability theorem of Robinson and a scalarization method, Yen and Yao [24] established sufficient conditions for the upper semicontinuity of the solution maps of parametric monotone affine vector variational inequalities (AVVI). A survey of recent results on the VVI was given in [3]. Yen [23] showed that the importance of VVIs for vector optimization is the same as that of variational inequalities for scalar optimization.

General variational inequality (GVI), which was firstly proposed by Noor [15], has been received considerable attention in recent three decades. Author [16] also showed that the minimum of a differentiable hg-convex function on the hg-convex set K in $\mathbb{R}^n$ can be characterized by the GVI. It is well-known that the problem GVI is equivalent to a class of the fixed point problems. These equivalent formulations allow us to approach studying polynomial optimization and fixed point theory via the GVI. Moreover, the GVI contains the class of inverse variational inequalities (see [12, 13]). Recently, existence and stability for generalized polynomial variational inequalities (GPVI) and general polynomial complementarity problems have been studied (see [11, 21]). The obtained results complement ones in [17].

In this paper, we study the *generalized polynomial vector variational inequality* (GPVVI), which is a natural generalization of the GPVI and the VVI. The major contributions to the GPVVI theory are listed as follows:

(*i*) A relationship between the Pareto solution sets of the GPVVI and the solution set of the GPVI is established by using the scalarization method, which is a powerful technique in vector optimization.

(*ii*) By using the concept on exceptional family of elements, recession cone, and positive semi-definiteness of matrices, we present sufficient conditions for the nonemptiness and boundedness of the Pareto solution sets of the GPVVI;

(*iii*) We propose sufficient conditions for the upper/lower semicontinuity of the weak Pareto solution map and characterize the stability for GPVVIs;

(iv) Finally, we give some applications to polynomial variational inequality.

Our results develop and complement the previous ones for VIs, GPVIs and VVIs in [7, 10, 14, 20, 22, 24].

The outline of the paper is as follows. Section 2 gives some preliminaries. The existence of Pareto solutions is established in Section 3. Due to this result, we present sufficient conditions for upper/lower continuity of the weak Pareto solution map of the GPVVI in Section 4. Some stability results for the GPVVI are also proposed Section 4. Some applications to polynomial variational inequality are given in the last section.

2 Preliminaries

Let $\mathbb{R}^s$ be s-dimensional Euclidean space equipped with the standard scalar product and the Euclidean norm, $\mathbb{R}^{s\times s}_S$ be the space of real symmetric $(s \times s)$–matrices equipped with the matrix norm induced by the vector norm in $\mathbb{R}^s$. The scalar product of vectors x, y and the Euclidean norm of a vector x in a finite-dimensional Euclidean space are denoted, respectively, by $\langle x, y\rangle$ and $\|x\|$. Vectors in Euclidean spaces are interpreted as columns of real numbers. The notation $x \geq y$ (resp., $x > y$) means that every component of x is greater or equal (resp., greater) the corresponding component of y. Denote $[q] := \{1, \dots, q\}$. The norm in the product space $X_1 \times \dots \times X_k$ of the normed spaces $X_1, \dots, X_k$ is set to be

$$\|(x_1, \dots, x_k)\| = (\|x_1\|^2 + \dots + \|x_k\|^2)^{\frac{1}{2}}.$$

Let K be a nonempty closed convex set in $\mathbb{R}^s$, $F_i, G : K \to \mathbb{R}^s$, $i = 1, \dots, m$, be vector-valued functions such that, for each $i = 1, \dots, m$, for each $j = 1, \dots, s$, the j-th component F_{ij} of F_i and G_j of G are polynomial functions in the variables $x_1, \dots, x_n$ with $\deg F_{ij} = \delta^{ij}$ and $\deg G_j = \sigma^j$ for some $\delta^{ij}, \sigma^j \in \mathbb{R}_+$. Then, $\deg F_i = \delta^i := \max\{\delta^{ij} : j = 1, \dots, s\}$ and $\deg G = \sigma := \max\{\sigma^j : j = 1, \dots, s\}$. Let $F = (F_1, \dots, F_m)$. For each $x \in K$ and $h \in \mathbb{R}^s$, denote

$$F(x)(h) := \big(\langle F_1(x), h\rangle, \dots, \langle F_m(x), h\rangle\big).$$

The so-called *generalized polynomial vector variational inequality* (GPVVI) is to find a vector $\bar{x} \in \mathbb{R}^s$ such that

$$G(\bar{x}) \in K \text{ and } F(\bar{x})(x - G(\bar{x})) \notin -\mathbb{R}^m_+ \setminus \{0\} \ \ \forall x \in K. \qquad (\text{GPVVI}(F, G, K))$$

To this problem one associates the following: Find a vector $\bar{x} \in \mathbb{R}^s$ such that

$$G(\bar{x}) \in K \text{ and } F(\bar{x})(x - G(\bar{x})) \notin -int(\mathbb{R}^m_+) \ \ \forall x \in K. \qquad (\text{GPVVI}^w(F, G, K))$$

The solution set of GPVVI(F, G, K) and GPVVI$^w(F, G, K)$ are denoted, respectively, by $Sol(F, G, K)$ and $Sol^w(F, G, K)$. The elements of the first set (resp., the second set) are said to be the *Pareto solutions* (resp., the *weak Pareto solutions*) of GPVVI(F, G, K).

For the case where $G = id_{\mathbb{R}^s}$, the problem GPVVI(F, G, K) reduces to the following *polynomial vector variational inequality* (VVI) (see [8])

$$\text{Find } \bar{x} \in K \text{ s.t. } F(\bar{x})(x - \bar{x}) \notin -\mathbb{R}^m_+ \setminus \{0\} \ \ \forall x \in K. \qquad (\text{VVI}(F, K))$$

If $m = 1$ and $F = F_1$ then the problem GPVVI(F, G, K) reduces to the following *generalized polynomial variational inequality* (GPVI)

$$\text{Find } \bar{x} \in K \text{ s.t. } \langle F(\bar{x}), x - G(\bar{x}) \rangle \geq 0 \ \ \forall x \in K. \qquad (\text{GVI}(F, G, K))$$

which was firstly proposed by Noor [15]. Specially, if $G = id_{\mathbb{R}^s}$, $m = 1$, and $F = F_1$ then GPVVI(F, G, K) reduces to the following *polynomial variational inequality* (PVI) (see [4, 9])

$$\text{Find } \bar{x} \in K \text{ s.t. } \langle F(\bar{x}), x - \bar{x} \rangle \geq 0 \ \ \forall x \in K. \qquad (\text{PVI}(F, K))$$

For any nonempty closed convex set K of $\mathbb{R}^s$, the asymptotic (recession) cone of K is denoted by

$$K^\infty = \{v \in \mathbb{R}^s : x + tv \in K \ \forall t \geq 0\}.$$

For any cone $C \subset \mathbb{R}^s$, the dual of C is denoted by

$$C^* := \{y \in \mathbb{R}^s : \langle h, y \rangle \geq 0 \ \ \forall h \in C\}.$$

Denote by $\mathcal{P}^{[d,s]}$ the space of all polynomial maps of degree at most d from $\mathbb{R}^s$ to $\mathbb{R}^s$ with the norm defined by $\|L\| := \big(\|L_1\| + \ldots + \|L_s\|\big)^{\frac{1}{2}}$ for all $L = (L_1, \ldots, L_s) \in \mathcal{P}^{[d,s]}$. Let

$$\Omega := \underbrace{\mathcal{P}^{[\delta,s]} \times \ldots \times \mathcal{P}^{[\delta,s]}}_{m\ times} \times \mathcal{P}^{[\sigma,s]}.$$

A multifunction $\Phi : X \subset \mathbb{R}^q \rightrightarrows \mathbb{R}^s$ is said to be *upper semicontinuous* at $\bar{z} \in \mathbb{R}^q$ if for each open set V containing $\Phi(\bar{z})$ there exists $\delta > 0$ such that $\Phi(z) \subset V$ for every $z \in \mathbb{R}^s$ satisfying $\|z - \bar{z}\| < \delta$. We also say that Φ is *lower semicontinuous* at $\bar{z}$ if $\Phi(\bar{z}) \neq \emptyset$ and, for each open set V satisfying $\Phi(\bar{z}) \cap V \neq \emptyset$, there exists $\epsilon > 0$ such that $\Phi(z) \cap V \neq \emptyset$ for every $z \in \mathbb{R}^s$ satisfying $\|z - \bar{z}\| < \epsilon$. If Φ is lower and upper semicontinuous at $\bar{z}$ then Φ is called *continuous* at $\bar{z}$.

Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be two continuous functions and let $\bar{x} \in \mathbb{R}^n$. A set of points $\{x^k\} \subset \mathbb{R}^n$ is called an *exceptional family of elements for the pair* (f, g) with respect to $\bar{x} \in \mathbb{R}^n$ if $\|x^k\| \to \infty$ as $k \to \infty$; and for each x^k, there exist $\{x^k\} \subset \mathbb{R}^s$ satisfying $\|x^k\| \to \infty$ as $k \to \infty$ and $\alpha^k > 0$ such that $g(x^k) \in K$ and

$$-f(x^k) - \alpha^k g(x^k) \in \mathcal{N}_K(g(x^k)),$$

where $\mathcal{N}_K(g(x^k))$ is the normal cone of K at $g(x^k)$.

For any nonempty closed convex set K of $\mathbb{R}^s$, the projection of x on K is denoted by $\Pi_K(\hat{x})$.

For any continuous mapping $g : \mathbb{R}^s \to \mathbb{R}^s$ and $\Omega_r^{\hat{x}} := \{x \in \mathbb{R}^s : \|g(x)\| < r\}$, where $r > \|\Pi_K(\hat{x})\|$ for any given $\hat{x} \in \mathbb{R}^s$, the *topological degree of* g *at* $\Pi_K(\hat{x})$ *relative to* $\Omega_r^{\hat{x}}$ is denoted by $deg(g(\cdot), \Omega_r^{\hat{x}}, \Pi_K(\hat{x}))$.

Let K be a nonempty closed convex set of $\mathbb{R}^s$ and $g : \mathbb{R}^s \to \mathbb{R}^s$ be a continuous mapping. We shall say that the pair (g, K) has the *property* $\mathcal{A}$ if it satisfies the following conditions

(a_1) the boundedness of $\|g(x)\|$ implies the boundedness of $\|x\|$;
(a_2) $deg(g(\cdot), \Omega_r^{\hat{x}}, \Pi_K(\hat{x}))$ is defined and nonzero.

We have the following lemma.

Lemma 2.1. *Consider the problem $GVI(f,g,K)$ with K being a nonempty closed convex set and $f,g:\mathbb{R}^s\to\mathbb{R}^s$ being continuous mappings.*

If the pair (g,K) has the property $\mathcal{A}$, then there exists either a solution of $GVI(f,g,K)$ or an exceptional family of elements for the pair (f,g) with respects to any given $\hat{x}\in\mathbb{R}^s$.

Proof. Its proof is similar to that of [6, Theorem 2.2]. □

3 Existence of Pareto solutions

Our goal in this section is twofold. Firstly, we use the scalarization method to establish a relationship between the Pareto solution sets of the GVVI and the solution set of the GVI. Secondly, by the above relation and the exceptional family of elements, we present sufficient conditions for the nonemptiness and boundedness of the Pareto solution sets of the GPVVI. To do this, let

$$\Delta := \left\{\xi=(\xi_1,\dots,\xi_m)\in\mathbb{R}^s_+ : \sum_{i=1}^m \xi_i = 1\right\}$$

and

$$ri\Delta := \big\{\xi=(\xi_1,\dots,\xi_m)\in\Delta : \xi_i>0 \ \ \forall i\in[m]\big\}.$$

For some $\xi=(\xi_1,\dots,\xi_m)\in\Delta$, consider the following variational inequality

$$\text{Find } \bar{x}\in K \text{ s.t. } \left\langle \sum_{i=1}^m \xi_i F_i(\bar{x}), x-G(\bar{x})\right\rangle \geq 0 \ \ \forall x\in K. \qquad (\mathrm{GVI}_\xi(F,G,K))$$

Denote by $Sol(\mathrm{GVI}_\xi(F,G,K))$ the solution set of $(\mathrm{GVI})_\xi(F,G,K)$. Let $F_\xi := \sum_{i=1}^m \xi_i F_i$. Denote by $F^\infty := (F_1^\infty,\dots,F_m^\infty)$ the leading term of F with degree δ, that is, $F^\infty(\lambda h)=\lambda^\delta F^\infty(h)$.

The following lemma shows a relationship between the Pareto solution sets of $\mathrm{GPVVI}(F,G,K)$ and the solution set of $(\mathrm{GVI})_\xi(F,G,K)$.

Lemma 3.1. *Let K be a nonempty closed convex set. It holds that*

$$\bigcup_{\xi\in ri\Delta} Sol(\mathrm{GVI}_\xi(F,G,K)) \subset Sol(F,G,K) \qquad (1)$$

and

$$Sol^w(F,G,K) = \bigcup_{\xi\in\Delta} Sol(\mathrm{GVI}_\xi(F,G,K)). \qquad (2)$$

Proof. To prove the inclusion (1), we let any $\xi \in ri\Delta$ and $\bar{x} \in Sol(\mathrm{GVI}_\xi(F,G,K))$. For every $x \in K$, we have $G(\bar{x}) \in K$ and

$$0 \le \left\langle \sum_{i=1}^m \xi_i F_i(\bar{x}), x - G(\bar{x}) \right\rangle = \sum_{i=1}^m \xi_i \langle F_i(\bar{x}), x - G(\bar{x})\rangle = \xi^T F_\xi(\bar{x})(x - \bar{x}). \qquad (3)$$

Hence, there no exist $x \in K$ such that $G(\bar{x}) \in K$ and

$$F(\bar{x})(x - G(\bar{x})) \in -\mathbb{R}^m_+ \setminus \{0\},$$

that is, $\bar{x} \in Sol(F,G,K)$.

We now show that (2) holds. Indeed, for any $\bar{x} \in Sol(\mathrm{GVI}_\xi(F,G,K))$ with $\xi \in \Delta$, we have (3). Then, there no exist $\bar{x} \in \mathbb{R}^n$ satisfying $G(\bar{x}) \in K$ and $x \in K$ such that $F_\xi(\bar{x})(x - G(\bar{x})) \in -int(\mathbb{R}^m_+) \setminus \{0\}$. It implies $\bar{x} \in Sol^w(F,G,K)$. Conversely, for any $\bar{x} \in Sol^w(F,G,K)$, we have

$$\{F_\xi(\bar{x})(x - G(\bar{x})) : x \in K\} \cap -int(\mathbb{R}^s_+) = \emptyset$$

and $G(\bar{x}) \in K$. According to the separation theorem (see, for instance, [18, Theorem 11.3]), there exists $\hat{\xi} \in \mathbb{R}$ with $\|\hat{\xi}\| = 1$ such that

$$\inf_{x\in K} \langle \hat{\xi}, F_\xi(\bar{x})(x - G(\bar{x}))\rangle \ge \sup_{y\in -int(\mathbb{R}^s_+)} \langle \hat{\xi}, y\rangle. \qquad (4)$$

If there exists $j \in [m]$ such that $\hat{\xi}_j < 0$ then $\sup_{y\in -int(\mathbb{R}^s_+)} \langle \hat{\xi}, y\rangle = +\infty$; then, the left hand of (4) equals to $+\infty$. This follows $K = \emptyset$, contrary to the assumption. Hence, $\hat{\xi} \in \Delta$ and

$$\hat{\xi}^T F_\xi(\bar{x})(x - G(\bar{x})) \ge 0$$

for every $x \in K$. Therefore, $\bar{x} \in Sol(\mathrm{GVI}_\xi(F,G,K))$. The lemma is proved. □ □

The main result in this section is established as follows.

Theorem 3.1. *Consider the problem* GPVVI(F,G,K) *with* $Ker(G^\infty) = \{0\}$. *Suppose that the pair* (G,K) *has the property* $\mathcal{A}$. *Then, the following statements hold:*

(*i*) *If there exists* $\xi \in \Delta$ *such that*

$$Sol(F_\xi^\infty + \rho G^\infty, G^\infty, K^\infty) = \{0\} \tag{5}$$

for every $\rho \geq 0$ *then* $\mathrm{GVI}_\xi(F, G, K)$ *has a solution; hence,* $Sol^w(F, G, K)$ *is nonempty. In addition, if*

$$Sol^w(F^\infty, G^\infty, K^\infty) = \{0\} \tag{6}$$

then $Sol^w(F, G, K)$ *is a nonempty compact set;*

(*ii*) *If there exists* $\xi \in ri\Delta$ *satisfying* (5) *for every* $\rho \geq 0$ *then* $Sol(F, G, K)$ *is nonempty.*

Proof. (*i*) Suppose that there exists $\xi \in \Delta$ satisfying (5) for every $\rho \geq 0$ and the problem $\mathrm{GVI}_\xi(F, G, K)$ has no solution. According to Lemma 2.1, there exists an exceptional family of elements for the pair (F_ξ, G) with respect to $0 \in \mathbb{R}^s$, i.e., there exist $\{x^k\} \subset \mathbb{R}^s$ satisfying $\|x^k\| \to \infty$ as $k \to \infty$ and $\alpha^k > 0$ such that $G(x^k) \in K$ and

$$\langle F_\xi(x^k) + \alpha^k G(x^k), y - G(x^k)\rangle \geq 0 \ \ \forall y \in K.$$

This follows that

$$\left\langle \sum_{i=1}^m \xi_i F_i(x^k), y - G(x^k) \right\rangle + \alpha^k \langle G(x^k), y - G(x^k)\rangle \geq 0 \ \ \forall y \in K. \tag{7}$$

By the fact that $\|x^k\| \to \infty$ as $k \to \infty$, we may assume that $\|x^k\| > 0$ for all $k \to \infty$ and $\frac{x^k}{\|x^k\|} \to \bar{h}$ for some $\bar{h} \in \mathbb{R}^s$ with $\|\bar{h}\| = 1$. Applying [18, Theorem 8.2] to $G(x^k) \in K$ and $\frac{1}{\|x^k\|^\sigma} \to 0$, we have

$$\frac{1}{\|x^k\|^\sigma} G(x^k) \to G^\infty(\bar{h}) \in K^\infty.$$

From the assumption that $Ker(G^\infty) = \{0\}$ it follows that $G^\infty(\bar{h}) \neq 0$. For some $y \in K$, dividing both sides of the inequality (7) by $\|x^k\|^{\sigma+\delta}$, we obtain that

$$A(x^k) + \alpha^k \|x^k\|^{\sigma-\delta} B(x^k) \geq 0, \tag{8}$$

where

$$A(x^k) := \left\langle \sum_{i=1}^m \frac{\xi_i F_i(x^k)}{\|x^k\|^\delta}, \frac{y - G(x^k)}{\|x^k\|^\sigma} \right\rangle \tag{9}$$

and

$$B(x^k) := \left\langle \frac{G(x^k)}{\|x^k\|^\sigma}, \frac{y - G(x^k)}{\|x^k\|^\sigma} \right\rangle \tag{10}$$

We have

$$\lim_{k\to\infty} B(x^k) = -\|G^\infty(\bar{h})\| < 0 \tag{11}$$

and

$$\lim_{k\to\infty} A(x^k) = \left\langle \sum_{i=1}^m \frac{\xi_i F_i(x^k)}{\|x^k\|^\delta}, \frac{y - G(x^k)}{\|x^k\|^\sigma} \right\rangle = -\left\langle \sum_{i=1}^m \xi_i F_i^\infty(\bar{h}), G^\infty(\bar{h}) \right\rangle. \tag{12}$$

We now show that $\{\rho^k := \alpha^k \|x^k\|^{\sigma-\delta}\}$ is bounded. Indeed, suppose, on the contrary, that $\rho^k \to +\infty$ as $k \to +\infty$. Then, dividing both sides of the equality (8) by ρ^k and letting $k \to \infty$ yields

$$-\|G^\infty(\bar{h})\| \geq 0,$$

contrary to (11). Hence $\{\rho^k\}$ is bounded. Without loss of generality, we may assume that $\rho^k \to \bar{\rho}$ for some $\bar{\rho} \in \mathbb{R}_+$. Fix $w \in K$. For every $h \in K^\infty$, we have $z := w + h\|x^k\|^\sigma \in K$. From (7) it follows that

$$\left\langle \sum_{i=1}^m \xi_i F_i(x^k) + \rho^k \|x^k\|^{-\sigma+\delta} G(x^k), w + h\|x^k\|^\sigma - G(x^k) \right\rangle \geq 0.$$

Dividing both sides of last inequality by $\|x^k\|^{\delta+\sigma}$ and letting $k \to +\infty$ yields:

$$\left\langle \sum_{i=1}^m \xi_i F_i^\infty(\bar{h}) + \bar{\rho}\, G^\infty(\bar{h}), h - G^\infty(\bar{h}) \right\rangle \geq 0.$$

This leads to $0 \neq \bar{h} \in Sol^w(F_\xi^\infty + \bar{\rho}\, G^\infty, G^\infty, K^\infty) = \{0\}$, a contradiction. Hence, $\mathrm{GVI}_\xi(F, G, K)$ has a solution. From Lemma 3.1 it follows that $Sol^w(F, G, K)$ is nonempty.

Let any a sequence $\{z^k\} \subset Sol^w(F, G, K)$ such that $z^k \to \bar{z}$ as $k \to +\infty$ for some $\bar{z} \in \mathbb{R}^s$. For each k, there exists $\xi^k \in \Delta$ such that $z^k \in Sol^w(F_{\xi^k}, G, K)$; that is, for some $y \in K$,

$$G(z^k) \in K \text{ and } \left\langle \sum_{i=1}^m \xi_i^k F_i(z^k), y - G(z^k) \right\rangle \geq 0. \tag{13}$$

From the boundedness of the sequence $\{\xi^k\}$ it follows that a subsequence $\{k_j\} \subset \{k\}$ such that $\xi^{k_j} \to \hat{\xi}$ as $j \to \infty$ for some $\hat{\xi} \in \Delta$. Passing the inequality (13) to limits as $k \to \infty$, we have

$$G(\bar{z}) \in K \text{ and } \left\langle \sum_{i=1}^{m} \hat{\xi}_i F_i(\bar{z}), y - G(\bar{z}) \right\rangle \geq 0.$$

This follows that $\bar{z} \in Sol^w(F, G, K)$; hence, $Sol^w(F, G, K)$ is closed.

Finally, we show the boundedness of $Sol^w(F, G, K)$. Suppose, on the contrary, that $Sol^w(F, G, K)$ is unbounded. Then, there exists a sequence $\{y^k\} \subset Sol^w(F, G, K)$ such that $\|y^k\| \to +\infty$ as $k \to +\infty$. Without loss of generality we may assume that $\|y^k\| \to \infty$ as $k \to \infty$, $\|y^k\| \neq 0$ for all k, and $\|y^k\|^{-1} y^k \to \bar{v}$ with $\|\bar{v}\| = 1$. For each k, there exists $\xi^k \in \Delta$ such that $y^k \in Sol^w(F_{\xi^k}, G, K)$; that is, $G(y^k) \in K$ and

$$\left\langle \sum_{i=1}^{m} \xi_i^k F_i(y^k), y - G(y^k) \right\rangle \geq 0 \quad \forall y \in K. \tag{14}$$

Applying [18, Theorem 8.2] to $G(y^k) \in K$ and $\frac{1}{\|y^k\|^{\sigma}} \to 0$, we have

$$\frac{1}{\|y^k\|^{l-1}} G(y^k) \to G^{\infty}(\bar{v}) \in K^{\infty}$$

as $k \to \infty$. From the boundedness of the sequence $\{\xi^k\}$ it follows that a subsequence $\{k_j\} \subset \{k\}$ such that $\xi^{k_j} \to \bar{\xi}$ as $j \to \infty$ for some $\bar{\xi} \in \Delta$. Fix $w \in K$. For every $v \in K^{\infty}$, we have

$$z := w + v\|y^{k_j}\|^{\sigma} \in K.$$

From (14) it follows that

$$\left\langle \sum_{i=1}^{m} \xi_i^{k_j} F_i(y^{k_j}), w + v\|y^{k_j}\|^{\sigma} - G(y^{k_j}) \right\rangle \geq 0.$$

Dividing both sides of the last equality by $\|y^k\|^{\delta+\sigma}$ and letting $k \to \infty$ yields

$$\left\langle \sum_{i=1}^{m} \bar{\xi}_i F_i^{\infty}(\bar{v}), v - G^{\infty}(\bar{v}) \right\rangle \geq 0.$$

Then, $0 \neq \bar{v} \in Sol(F^{\infty}_{\bar{\xi}}, G^{\infty}, K^{\infty}) \subset Sol^w(F^{\infty}, G^{\infty}, K^{\infty})$, contrary to the assumption that $Sol^w(F^{\infty}, G^{\infty}, K^{\infty}) = \{0\}$. Therefore, $Sol^w(F, G, K)$ is bounded. Therefore, $Sol^w(F, G, K)$ is a compact set.

(ii) By the similar arguments as in the part (ii), the nonemptiness of $Sol(F, G, K)$ are obvious. The proof is complete. □

4 Some stability results

In this section, we study the upper/lower semicontinuity of the set-valued maps $Sol^w((\cdot, K))$ of GPVVI^w and $Sol((\cdot, K))$ of GPVVI with respect to the change in the problem parameters. Namely, we study the upper/lower semicontinuity of the set-valued maps

$$\Omega \ni \omega = (F', G') \longmapsto Sol^w((F', G', K)),$$

and

$$\Omega \ni \omega = (F', G') \longmapsto Sol((F', G', K)),$$

where

$$\Omega := \underbrace{\mathcal{P}^{[\delta,n]} \times \ldots \times \mathcal{P}^{[\delta,n]}}_{m\ times} \times \mathcal{P}^{[\sigma,n]}.$$

The following lemma is useful for proving the main theorem.

Lemma 4.1. [1, p. 114] *Let I be an index set and let X and Y be topological spaces. Then, the union $\Phi = \cup_{i \in I} \Phi_i$ of a family of lower semicontinuous set-valued mappings Φ_i from X into Y is also a lower semicontinuous set-valued mapping from X into Y.*

A sufficient condition for the continuity of $Sol^w(\text{GPVVI}(\cdot, K))$ on Ω is proposed below.

Theorem 4.1. *Consider the problems* GPVVI(ω, K) *depending on the parameter $\omega = (F, G) \in \Omega$. Suppose that*

(a) *the pair* (G, K) *has the property* $\mathcal{A}$ *and* $Ker(G^{\infty}) = \{0\}$ *for every* $\omega \in \Omega$,

(b) *the condition* (5) *is satisfied for some* $\xi \in \Delta$ *and*

(c) F_i *is strictly monotone respect to* G *on* K *for every* $i \in [m]$ *for every* $\omega \in \Omega$.

Then, the following statements hold:

(*i*) $Sol(\mathrm{GPVI}_{\xi}(\cdot, K))$ *is single-valued on* Ω;

(*ii*) $Sol(\mathrm{GPVI}_{\xi}(\cdot, K))$ *is continuous on* Ω;

(*iii*) $Sol^w(\mathrm{GPVVI}(\cdot, K))$ *is continuous on* Ω.

Proof. (*i*) By Theorem 3.1, for some $\bar{\omega} \in \Omega$, we obtain that $Sol(\mathrm{GPVI}_{\xi}(\bar{\omega}, K))$ is nonempty. On the contrary, suppose that there exist $\bar{z}$ and $\bar{x}$ such that $\bar{z} \neq \bar{x}$ and $\{\bar{z}, \bar{x}\} \subset Sol(\mathrm{GPVI}_{\xi}(\bar{\omega}, K))$. Then, we have $G(\bar{z}) \in K$, $G(\bar{x}) \in K$, and

$$\langle F_{\xi}(\bar{z}), G(\bar{x}) - G(\bar{z})\rangle \geq 0 \quad \text{and} \quad \langle F_{\xi}(\bar{x}), G(\bar{z}) - G(\bar{x})\rangle \geq 0.$$

It implies that $\langle F_{\xi}(\bar{x}) - F_{\xi}(\bar{z}), G(\bar{x}) - G(\bar{z})\rangle \leq 0$. This contradicts the assumption that F_i is strictly monotone respect to G on K for every $i \in [m]$. Therefore, $Sol(\mathrm{GPVI}_{\xi}(\cdot, K))$ is single-valued on Ω.

(*ii*) We next show that the solution map $Sol(\mathrm{GPVI}_{\xi}(\cdot, K))$ is lower semicontinuous on Ω. Indeed, on the contrary, suppose that there exists $\bar{\omega} \in \Omega$ such that $Sol(\mathrm{GPVI}_{\xi}(\cdot, K))$ is not lower semicontinuous at $\bar{\omega}$. That is, there exist $\bar{x} \in Sol(\mathrm{GPVI}_{\xi}(\bar{\omega}, K))$ and a sequence $\{\omega^s\} \subset \Omega$ satisfying $\omega^s \to \bar{\omega}$ such that, for any $z^s \in Sol(\mathrm{GPVI}_{\xi}(\omega^s, K))$ satisfying $z^s \to \bar{z}$, one has $\bar{z} \neq \bar{x}$. By the part (*i*), we have $\{\bar{x}\} = Sol(\mathrm{GPVI}_{\xi}(\bar{\omega}, K))$. Since $z^s \in Sol(\mathrm{GPVI}_{\xi}(\omega^s, K))$, we obtain that

$$G(z^s) \in K \text{ and } \langle F_{\xi}(z^s), z - G(z^s)\rangle \geq 0 \ \ \forall z \in K. \tag{15}$$

For each $z \in K$, passing (15) to limits as $s \to \infty$ gives

$$G(\bar{z}) \in K \text{ and } \langle F_{\xi}(\bar{z}), z - G(\bar{z})\rangle \geq 0.$$

This implies that $\bar{z} \in Sol(\mathrm{GPVI}_{\xi}(\bar{\omega}, K))$. Hence, $\bar{z} = \bar{x}$, a contradiction. It follows that $Sol(\mathrm{GPVI}_{\xi}(\cdot, K))$ is lower semicontinuous on Ω. Since the map $Sol(\mathrm{GPVI}_{\xi}(\cdot, K))$ is single-valued on Ω, we conclude that $Sol(\mathrm{GPVI}_{\xi}(\cdot, K))$ is continuous on Ω.

(*iii*) By Lemma 3.1, we have

$$Sol^w(\mathrm{GPVVI}(\omega, K)) = \cup_{\xi \in \Delta} Sol(\mathrm{GPVI}_{\xi}(\omega, K)).$$

Therefore, we deduce that $Sol^w(\text{GPVVI}(\cdot, K))$ is lower semicontinuous on Ω by using Lemma 4.1.

Finally, we show that $Sol^w(\text{GPVVI}(\cdot, K))$ is upper semicontinuous on Ω. Indeed, suppose that $Sol^w(\text{GPVVI}(\cdot, K))$ is not upper semicontinuous at $\bar{\omega}$ for some $\bar{\omega} \in \Omega$. Then, there exist an open U containing $Sol^w(\text{GPVVI}(\bar{\omega}, K))$, a sequence $\omega^s \to \bar{\omega}$ and $x^s \in Sol^w(\text{GPVVI}(\bar{\omega}, K))$ such that $x^s \notin U$ for every s. By the fact that $x^s \in Sol^w(\text{GPVVI}(\bar{\omega}, K))$ and Lemma 3.1, there exist $\bar{\xi} \in \Delta$ such that $x^s \in Sol(\text{GPVI}_{\bar{\xi}}(\omega, K)$. From the assumption that F_i is strictly monotone respect to G on K for every $i \in [m]$ and the above arguments in part (i) it follows that $Sol(\text{GPVI}_{\xi}(\cdot, K))$ is single-valued. Suppose that $Sol(\text{GPVI}_{\xi}(\bar{\omega}, K)) = \{\bar{x}\}$. Then, $Sol(\text{GPVI}_{\xi}(\cdot, K))$ is continuous at $\bar{\xi}$. Hence, $x^s \to \bar{x} \in Sol(\text{GPVI}_{\xi}(\bar{\omega}, K)) \subset U$. This contradicts the openness of the set U. Thus $Sol^w(\text{GPVVI}(\cdot, K))$ is upper semicontinuous on Ω and (iii) is shown. The proof is complete. □

The following lemma is a Hartman-Stampacchia type theorem for the GPVVI.

Lemma 4.2. *Consider the problem* GPVVI(F, G, K) *with* $Ker(G^\infty) = \{0\}$. *Suppose that* G *and* K *satisfy the assumptions* (a_1) *and* (a_2). *Then, for each* $\xi \in \Delta$, GPVI$_\xi(F, G, K)$ *has a solution if* K *is a nonempty compact convex set.*

Proof. On the contrary, suppose that the problem GVI$_\xi(F, G, K)$ has no solution for some $\xi \in \Delta$. According to Lemma 2.1, there exists an exceptional family of elements for the pair (F_ξ, G) with respect to $0 \in \mathbb{R}^s$, i.e., there exist $\{x^k\} \subset \mathbb{R}^s$ satisfying $\|x^k\| \to \infty$ as $k \to \infty$ and $\alpha^k > 0$ such that $G(x^k) \in K$ and

$$-F_\xi(x^k) - \alpha^k G(x^k) \in \mathcal{N}_K(G(z^k)).$$

By the fact that $\|x^k\| \to \infty$ as $k \to \infty$, we may assume that $\|x^k\| > 0$ for all $k \to \infty$. Without loss of generality, we may assume that $\frac{x^k}{\|x^k\|} \to \bar{h}$ for some $\bar{h} \in \mathbb{R}^s$ with $\|\bar{h}\| = 1$. Applying [18, Theorem 8.2] to $G(x^k) \in K$ and $\frac{1}{\|x^k\|^\sigma} \to 0$, we have

$$\frac{1}{\|x^k\|^\sigma} G(x^k) \to G^\infty(\bar{h}) \in K^\infty = \{0\}.$$

From the assumption that $Ker(G^\infty) = \{0\}$ it follows that $\bar{h} = 0$, a contradiction. Therefore, GPVI$_\xi(F, G, K)$ has a solution. □

Remark 4.1. *Applying Lemma 4.2 to the case where $G = id_{\mathbb{R}^n}$, we get well-known Hartman-Stampacchia's theorem for the PVI (see [9]).*

We obtain the following stability result.

Theorem 4.2. *Suppose that G and K satisfy the assumptions (a_1) and (a_2), and $Ker(G^\infty) = \{0\}$. The following statements are valid:*

(i) If F_i^∞ is strictly copositive with respect to G^∞ on K^∞ for every $i \in [m]$ then $Sol^w(F+\tilde{F}, G+\tilde{G}, K)$ is a nonempty compact set for every $(\tilde{F}, \tilde{G}) \in \mathcal{P}^{[m-1,n]} \times \mathcal{P}^{[l-1,n]}$;

(ii) If F_i^∞ is copositive with respect to G^∞ on K^∞ for every $i \in [m]$ and if there exists $\xi \in \Delta$ such that

$$Sol(F_\xi^\infty, G^\infty, K^\infty) = \{0\} \tag{16}$$

then $Sol^w(F+\tilde{F}, G+\tilde{G}, K)$ is nonempty for every $(\tilde{F}, \tilde{G}) \in \mathcal{P}^{[m-1,n]} \times \mathcal{P}^{[l-1,n]}$. In addition, if (16) holds for every $\xi \in \Delta$ then $Sol^w(F+\tilde{F}, G+\tilde{G}, K)$ is a compact set;

(iii) If F_i is copositive with respect to G on K for every $i \in [m]$ then $Sol(F + c, G, K)$ is nonempty for every $c \in \mathbb{R}^{m\times n}$ satisfying

$$\sum_{i=1}^m \xi_i c_i \in int\{G^\infty(Sol^w(F_\xi^\infty, G^\infty, K^\infty))\}^* \tag{17}$$

for some $\xi \in \Delta$. In addition, if (17) holds for every $\xi \in \Delta$ then $Sol^w(F + \tilde{F}, G + \tilde{G}, K)$ is a compact set.

Proof. (i) For some $z^0 \in K$, for each $s = 1, 2, \ldots$, denote

$$K_s = \{z \in \mathbb{R}^n : z \in K, \|z - z^0\| \leq s\}.$$

Then, we may assume that K_s is nonempty compact set. For some $(\tilde{F}, \tilde{G}) \in \mathcal{P}^{[m-1,n]} \times \mathcal{P}^{[l-1,n]}$, by using Lemma 4.2, we obtain that GPVI$(F_\xi + \tilde{F}_\xi, G, K_s)$ has a solution for every s. Let any $x^s \in Sol(F_\xi + \tilde{F}_\xi, G + \tilde{G}, K_s)$. We prove that $\{x^s\}$ is bounded. Indeed, suppose, on the contrary, that $\{x^s\}$ is unbounded. Then, we may assume that $x^s > 0$ for every s and $x^s/\|x^s\| \to \bar{v}$ for some $\bar{v} \in \mathbb{R}^n$. By the fact that $x^s \in Sol(F_\xi + \tilde{F}_\xi, G + \tilde{G}, K_s)$, we have $G(x^s) + \tilde{G}(x^s) \in K_s$ and

$$\left\langle \sum_{i=1}^{m} \xi_i(F_i(x^s) + \tilde{F}_i(x^s)), z - G(x^s) - \tilde{G}(x^s) \right\rangle \geq 0 \tag{18}$$

for every $z \in K_s$. Since $G(x^s) + \tilde{G}(x^s) \in K$ and $\frac{1}{\|x^s\|^\sigma} \to 0$, applying [18, Theorem 8.2], we have

$$\frac{1}{\|x^s\|^\sigma}\Big(G(x^s) + \tilde{G}(x^s)\Big) \to G^\infty(\bar{v}) \in K^\infty$$

as $s \to \infty$. Multiplying both sides of the inequality (18) by $\|x^s\|^{-(\delta+\sigma)}$ and taking $s \to \infty$ yields

$$\left\langle \sum_{i=1}^{m} \xi_i F_i^\infty(\bar{v}), G^\infty(\bar{v}) \right\rangle \leq 0. \tag{19}$$

This contradicts the assumption that F_i^∞ is strictly copositive with respect to G^∞ on K^∞ for every $i \in [m]$. Therefore, $\{x^s\}$ is bounded. We may assume, without loss of generality, that $x^s \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$. Since $G(x^s) + \tilde{G}(x^s) \in K$ and since K is closed, we have $G(\bar{x}) + \tilde{G}(\bar{x}) \in K$. Passing (18) to limits as $s \to \infty$, we obtain

$$\left\langle \sum_{i=1}^{m} \xi_i(F_i(\bar{x}) + \tilde{F}_i(\bar{x})), z - G(\bar{x}) - \tilde{G}(\bar{x}) \right\rangle \geq 0. \tag{20}$$

Hence, $\bar{x} \in Sol(F_\xi + \tilde{F}_\xi, G + \tilde{G}, K) \subset Sol^w(F + \tilde{F}, G + \tilde{G}, K)$.

Suppose that $Sol^w(F + \tilde{F}, G + \tilde{G}, K)$ is unbounded. Then, there exists $\{y^s\} \subset Sol^w(F + \tilde{F}, G + \tilde{G}, K)$ such that $\|y^s\| \to \infty$ and $y^s/\|y^s\| \to \bar{y}$ for some $\bar{y} \in \mathbb{R}^n$. By Lemma 2.1, there exists $\xi^s \in \Delta$ such that $y^s \in Sol(F_{\xi^s} + \tilde{F}_{\xi^s}, G + \tilde{G}, K)$ for each s. By the fact that $\|\xi^s\| = 1$ for every s, we may assume that $\xi^s \to \hat{\xi}$ for some $\hat{\xi} \in \Delta$. Repeating the above arguments, we obtain that $\left\langle \sum_{i=1}^{m} \xi_i F_i^\infty(\bar{y}), G^\infty(\bar{y}) \right\rangle \leq 0$. This is contradicts the assumption that F_i^∞ is strictly copositive with respect to G^∞ on K^∞ for every $i \in [m]$. Thus, $Sol^w(F + \tilde{F}, G + \tilde{G}, K)$ is bounded. The closedness of $Sol^w(F + \tilde{F}, G + \tilde{G}, K)$ follows from a similar argument as in the proof of Theorem 3.1 (the part (i)). Therefore, $Sol^w(F + \tilde{F}, G + \tilde{G}, K)$ is compact set.

(ii) By the similar arguments in Part (i), we obtain (18) and (19). Since F_i^∞ is copositive with respect to G^∞ on K^∞, we have

$$\left\langle \sum_{i=1}^m \xi_i F_i^\infty(\bar v), G^\infty(\bar v)\right\rangle \ge 0.$$

By this and (19), we have

$$\left\langle \sum_{i=1}^m \xi_i F_i^\infty(\bar v), G^\infty(\bar v)\right\rangle = 0. \tag{21}$$

For any $v \in K^\infty \setminus \{0\}$, one has $y^s := z^0 + \frac{\|G(x^s)+\tilde G(x^s)\|}{\|v\|} v \in K$ and

$$\|y^s - z^0\| = \|G(x^s) + \tilde G(x^s)\| \le s.$$

Hence, $y^s \in K_s$. By (18), one has

$$\left\langle \sum_{i=1}^m \xi_i (F_i(x^s) + \tilde F_i(x^s)), \frac{\|G(x^s)+\tilde G(x^s)\|}{\|v\|} v - G(x^s) - \tilde G(x^s)\right\rangle \ge 0.$$

Multiplying both sides of the last inequality by $\|x^s\|^{-(\delta+\sigma)}$ and letting $s \to \infty$ yields

$$\left\langle \sum_{i=1}^m \xi_i F_i^\infty(\bar v), \frac{\|G^\infty(\bar v)\|}{\|v\|} v\right\rangle \ge \left\langle \sum_{i=1}^m \xi_i F_i^\infty(\bar v), G^\infty(\bar v)\right\rangle = 0.$$

Hence, $\left\langle \sum_{i=1}^m \xi_i F_i^\infty(\bar v), v\right\rangle \ge 0$, that is, $\sum_{i=1}^m \xi_i F_i^\infty(\bar v) \in (K^\infty)^*$. By this and (21), we deduce that

$$G^\infty(\bar v) \in K^\infty, \ \ \sum_{i=1}^m \xi_i F_i^\infty(\bar v) \in (K^\infty)^* \ \text{ and } \left\langle \sum_{i=1}^m \xi_i F_i^\infty(\bar v), G^\infty(\bar v)\right\rangle = 0.,$$

that is,

$$0 \ne \bar v \in Sol(F_\xi^\infty, G^\infty, K^\infty), \tag{22}$$

contrary to the assumption that $Sol(F_\xi^\infty, G^\infty, K^\infty) = \{0\}$. Therefore, $\{x^s\}$ is bounded. The nonemptiness of $Sol^w(F + \tilde F, G + \tilde G, K)$ follows from a similar argument as in Part (i).

Repeating the above arguments as in Part (i), we obtain that $0 \ne \bar y \in Sol(F_{\hat\xi} + \tilde F_{\hat\xi}, G + \tilde G, K)$, contrary to the assumption that (16) holds for every $\xi \in \Delta$. Thus, $Sol^w(F + \tilde F, G + \tilde G, K)$ is bounded.

(iii) Let $c \in \mathbb{R}^{m\times n}$ satisfying $\sum_{i=1}^m \xi_i c_i \in int\{G^\infty(Sol(F_\xi^\infty, G^\infty, K^\infty))\}^*$. By the similar arguments in Part (i) with $\tilde{F} \equiv c \in \mathbb{R}^m$ and $\tilde{G} \equiv 0$, we obtain (18)–(21). Repeating the arguments in Part (i), one gets $\bar{v} \in Sol(F_\xi^\infty, G^\infty, K^\infty)$. From (18), taking $z = 0$, we have

$$\left\langle \sum_{i=1}^m \xi_i (F_i(x^s) + c_i), -G(x^s) \right\rangle \geq 0.$$

This implis that

$$\left\langle \sum_{i=1}^m \xi_i c_i, G(x^s) \right\rangle \leq -\left\langle \sum_{i=1}^m \xi_i F_i(x^s), G(x^s) \right\rangle \leq 0,$$

since F_i is copositive with respect to G on K for every $i \in [m]$. Dividing both sides of the last inequality by $\|x^s\|^\sigma$ and letting $s \to \infty$ yields

$$\left\langle \sum_{i=1}^m \xi_i c_i, G^\infty(\bar{v}) \right\rangle \leq 0,$$

contrary to the assumption that $\sum_{i=1}^m \xi_i c_i \in int\{G^\infty(Sol(F_\xi^\infty, G^\infty, K^\infty))\}^*$. Therefore, $Sol^w(F + c, G, K)$ is nonempty. The boundedness and closedness of $Sol^w(F + c, G, K)$ follows from a similar analysis as in Part (i). □

By Theorem 4.2, we get the following corollary.

Corollary 4.1. *Suppose that the assumptions in Theorem 4.2 are satisfied. The following statements are valid:*

(i') If F_i^∞ is strictly copositive with respect to G^∞ on K^∞ for every $i \in [m]$ then $Sol(F + \tilde{F}, G + \tilde{G}, K)$ is nonempty for every $(\tilde{F}, \tilde{G}) \in \mathcal{P}^{[m-1,n]} \times \mathcal{P}^{[l-1,n]}$;

(ii') If F_i^∞ is copositive with respect to G^∞ on K^∞ for every $i \in [m]$ and (16) *holds for some $\xi \in ri\Delta$ then $Sol(F + \tilde{F}, G + \tilde{G}, K)$ is nonempty for every $(\tilde{F}, \tilde{G}) \in \mathcal{P}^{[m-1,n]} \times \mathcal{P}^{[l-1,n]}$.*

(iii') If F_i is copositive with respect to G on K for every $i \in [m]$ then, for every $c \in \mathbb{R}^{m\times n}$ satisfying (17) *for some $\xi \in ri\Delta$, the set $Sol(F + c, G, K)$ is nonempty.*

Proof. The desired conclusion follows from a similar argument as in the proof of Theorem 4.2. □

5 Applications for polynomial variational inequality

Recall that if $G = id_{\mathbb{R}^s}$, $m = 1$, and $F = F_1$ then GPVVI(F, G, K) reduces to the following *polynomial variational inequality* (PVI) (see [4, 9])

$$\text{Find } \bar{x} \in K \text{ s.t. } \langle F(\bar{x}), x - \bar{x} \rangle \geq 0 \ \ \forall x \in K. \qquad \text{(PVI}(F, K))$$

Applying Theorem 4.2 (ii) to the problem $(PVI(F, K))$, we obtain the following corollary.

Corollary 5.1. *Consider the problem $(PVI(F, K))$ with K being nonempty. Then, the following statements are valid:*

(a) If F^∞ is strictly copositive on K^∞ then $Sol(F + \tilde{F}, K)$ is a nonempty compact set for every $\tilde{F} \in \mathcal{P}^{[m-1,n]}$;

(b) If F^∞ is copositive on K^∞ and $Sol(F^\infty, K^\infty) = \{0\}$ then $Sol(F + \tilde{F}, K)$ is a nonempty compact set for every $\tilde{F} \in \mathcal{P}^{[m-1,n]}$;

(c) If F^∞ is copositive on K^∞ then $Sol(F + c, K)$ is a nonempty compact set for every $c \in \mathbb{R}^{m\times n}$ satisfying $c \in int\{Sol(F^\infty, K^\infty)\}^*$.

Remark 5.1. *Corollary 5.1* (*c*) *is an extension of [7, Theorem 4.1] because the assumption "$0 \in K$" is omitted. The obtained results in Corollary 5.1 are also generalizations of Theorems 6.3–6.5 in [10]. Corollary 5.1 gives a partial answer to Question 5.2 in [22]. Theorem 6.2 in [5] gives a result on the solution existence for weakly homogeneous variational inequality under the assumption $int(K^*) \neq \emptyset$. However, this condition is omitted in Corollary 5.1.*

6 Conclusions

In the presented paper, the sufficient conditions for the existence of Pareto solutions of the GPVVI are established (Theorem 3.1). Due to this result, we present a sufficient condition for upper/lower continuity of the weak Pareto solution map of the GVVI (Theorem 4.1). Some stability results for the GPVVI are also proposed (Theorem 4.2). Our results develop and complement the previous ones for VIs, GPVIs and VVIs in [7, 10, 14, 20, 22, 24].

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