PONTRYAGIN'S PRINCIPLE FOR OPTIMAL CONTROL PROBLEM GOVERNED BY NAVIER-STOKES-VOIGT EQUATIONS

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ABSTRACT. In this paper we establish the Pontryagin principle for two distributed optimal control problems governed by Navier-Stokes-Voigt equations in two circumstances: pointwise control constraints and two point boundary state constraints.

1. INTRODUCTION

The Navier-Stokes-Voigt (NSV) equations was introduced by Oskolkov in [21] as a model of motion of certain linear viscoelastic incompressible fluids.

$$\begin{cases} y_t - \nu\Delta y - \alpha^2\Delta y_t + (y\cdot\nabla)y + \nabla p & = u,\ x\in\Omega, t>0,\\ \nabla\cdot y & = 0,\ x\in\Omega, t>0,\\ y(x,t) & = 0,\ x\in\partial\Omega, t>0,\\ y(x,0) & = y_0(x),\ x\in\Omega. \end{cases} \tag{1.1}$$

Here, Ω is an open domain in $\mathbb{R}^3$ with boundary $\partial\Omega$; $y=(y_1(x,t),y_2(x,t),y_3(x,t))$ is the velocity; $y_0=y_0(x)$ is the initial velocity; $p=p(x,t)$ is the pressure; $\nu>0$ is the kinematic viscosity coefficient; and $\alpha\neq 0$ is the length-scale parameter characterizing the elasticity of the fluid. In [6], the authors also proposed NSV equations, with small values of α, as a regularization of the 3D Navier-Stokes equations for the purpose of direct numerical simulations. The difference between Navier-Stokes equations and NSV equations is appearance of the regularizing term $-\alpha^2\Delta y_t$, which leads to the global well-posedness of NSV equations both forward and backward in time, even in the case of three dimensions. In fact, the Navier-Stokes-Voigt system is perhaps the newest model in the so-called α-models in fluid mechanics (see e.g. [11]). However, it does not require any additional artificial boundary condition (besides the Dirichlet boundary conditions) to get the global well-posedness, which is an appealing advantage compared to other α-models.

In the past years, the existence and long-time behavior of solutions to the Navier-Stokes-Voigt equations has attracted the attention of many mathematicians. In bounded domains or unbounded domains satisfying the Poincaré inequality, there are many results on the existence and long-time behavior of solutions in terms of existence of attractors, see e.g. [5, 7, 9, 10, 12, 13, 22, 33]. Recently, some optimal control problems for NSV equations have been studied, including quadratic optimal control [1], time optimal control [2], optimal control with pointwise control-state constraints [26], optimal control of time-periodic solutions [4], optimal control of feedback control [34] and a numerical scheme for the distributed optimal control

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problem in both time and space variables [3]. In this paper, we establish Pontryagin's principle, as the first necessary optimality conditions, for the two following optimal control problems governed by 3D NSV equations.

Let Ω be an open domain in $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$. Set $Q = \Omega \times (0, T)$, where $T > 0$ is given. The first problem is as following

$$(P_1) \qquad \begin{cases} \min\ J_1(y, u) \\ u \in \mathcal{U}_1, \end{cases}$$

where

- $J_1(y, u)$ is a quadratic objective functional defined by

$$J_1(y,u) = \frac{\alpha_T}{2}\int_\Omega |y(x,T) - y_T(x)|^2 dx + \frac{\alpha_Q}{2}\int_Q |y(x,t) - y_Q(x,t)|^2 dxdt + \frac{\gamma}{2}\int_Q |u(x,t)|^2 dxdt;$$

- $\mathcal{U}_1 = \{u \in \mathbb{L}^2(Q) : u(x,t) \in \mathcal{M} \text{ for a.e. } (x,t) \in Q\}$, where $\mathcal{M}$ is a given closed set in $\mathbb{R}^3$;
- the state y and control u have to fulfill the 3D NSV equations (1.1).

And the second problem is the following

$$(P_2) \qquad \begin{cases} \min\ J_2(y, u) \\ u \in \mathcal{U}_2, \\ (y(0); y(T)) \in S, \end{cases}$$

where

- $J_2(y, u)$ is also a quadratic objective functional defined by

$$J_2(y,u) = \frac{\alpha_Q}{2}\int_Q |y(x,t) - y_Q(x,t)|^2 dxdt + \frac{\gamma}{2}\int_Q |u(x,t)|^2 dxdt;$$

- $\mathcal{U}_2 = \{u \in L^2(0, T; \mathbb{L}^2(\Omega)) : u(t) \in U \text{ for a.e. } t \in (0,T)\}$, where U is a nonempty, closed, bounded, convex subset of $\mathbb{L}^2(\Omega)$;
- S is a nonempty closed convex subset of $V \times V$;
- the state y and control u have to fulfill the 3D NSV equations (1.1).

Problem (P_2) is a representative for several meaningful problems, such as the problem with fixed endpoints: $y(0) = y_0$, $y(T) = y_1$ (in this case, S consists of only one element (y_0, y_1)); the control problem with a final state constraint: $y(0) = y_0$, $y(T) \in W$ $(S = \{y_0\} \times W)$; or the optimal control problem for periodic solutions $(S = \{(x,x) : x \in V\})$. In this paper, we choose a specific objective functional, however, these arguments can be applied in the case of an abstract objective functional with some appropriate assumptions.

Using some ideas presented in [14], we obtain Pontryagin's principle for problem (P_1) by utilizing some regular properties of solutions to NSV equations and techniques of optimal control theory. The achieved result for problem (P_2) follows from a standard scheme: Firstly, we define a penalty functional which helps to transform (P_2) with endpoints constraint to an approximate problem with no endpoint constraint. Then, we apply the Ekeland variational principle to find an optimal pair for the approximate problem. Next, we use spike variation technique to get the necessary conditions for the approximate problem. Lastly, we pass to the limit to derive Pontryagin's principle. This scheme was presented in [15] where the authors considered an optimal control problem governed by evolutionary integral equations. These ideas were applied in [29, 30] for optimal control of fluid dynamic systems to derive Pontryagin's maximum principles. These techniques are then used for optimal control problems governed by a number of equations, such as Cahn-Hilliard-Navier-Stokes model [20], Boussinesq equations [17], primitive equations of the ocean [18, 19], coupled nonlinear wave equations with memory [35], 2D Navier-Stokes equations [31], fluid dynamic systems [32], for several types of constraints including pure state constraints, mixed control-state constraints, two point boundary state constraints. We also apply these ideas into our problem. Our result is closed to the one in [29], where the author dealt with strong solution, since the linearized equations are required being satisfied almost everywhere to evaluate the errors, however, because of regularity of solutions to NSV equations, we can estimate errors even for weak solutions.

The paper is organized as follows. In Section 2, we recall some auxiliary results on the existence and unique of weak solutions to NSV equations and introduce linearized equations as well as its properties. The main results of the paper are presented in Section 3, where we prove Pontryagin's principle for the two above optimal control problems.

2. Preliminaries and auxiliary results

2.1. **Function spaces and inequalities for the nonlinear terms.** For convenience, we set

$$\mathbb{L}^2(\Omega) := L^2(\Omega)^3,\ \ \mathbb{H}_0^1(\Omega) := H_0^1(\Omega)^3,\ \ L^2(0,T;\mathbb{L}^2(\Omega)) \equiv \mathbb{L}^2(Q) := L^2(Q)^3.$$

Define

$$(u,v) := \int_\Omega \sum_{j=1}^3 u_j v_j\, dx, \quad u=(u_1,u_2,u_3), v=(v_1,v_2,v_3)\in \mathbb{L}^2(\Omega),$$

$$((u,v)) := \int_\Omega \sum_{j=1}^3 \nabla u_j\cdot \nabla v_j\, dx, \quad u=(u_1,u_2,u_3), v=(v_1,v_2,v_3)\in \mathbb{H}_0^1(\Omega),$$

and the associated norms $|u|^2 := (u,u), \|u\|^2 := ((u,u))$.

Set

$$\mathcal{V} = \big\{u\in (C_0^\infty(\Omega))^3 \ :\ \nabla\cdot u = 0\big\},$$

and denote by H and V the closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega$ and $\mathbb{H}_0^1(\Omega)$, respectively. Then H, V are Hilbert spaces with scalar products $(.,.)$, $((.,.))$ respectively.

Let X be a real Banach space with the norm $\|.\|_X$. We denote by $L^p(0,T;X)$ the standard Banach space of all functions from $(0,T)$ to X, endowed with the norm

$$\|y\|_{L^p(0,T;X)} := \left(\int_0^T \|y(t)\|_X^p dt\right)^{1/p}, \quad 1\le p<\infty,$$
$$\|y\|_{L^\infty(0,T;X)} := \operatorname*{esssup}_{t\in(0,T)} \|y(t)\|_X.$$

When X is a Banach space with the dual space X', we will use $\|.\|_{X'}$ for the norm in X', $\langle .,.\rangle_{X',X}$ for the duality pairing between X' and X. In this case, $L^p(0,T;X)$ is also a Banach space, with the dual space being $L^{p'}(0,T;X')$, where $1/p+1/p'=1$. The pairing between $u\in L^{p'}(0,T;X')$ and $v\in L^p(0,T;X)$ is

$$\langle u,v\rangle_{L^{p'}(0,T;X'),L^p(0,T;X)} = \int_0^T \langle u(t),v(t)\rangle_{X',X} dt.$$

To deal with the time derivative in the state equation, we introduce the common space of functions y whose time derivatives y_t exist as abstract functions

$$W^{1,2}(0,T;X) := \{y\in L^2(0,T;X) : y_t\in L^2(0,T;X)\},$$

endowed with the norm

$$\|y\|_{W^{1,2}(0,T;X)} := \Big(\|y\|^2_{L^2(0,T;X)} + \|y_t\|^2_{L^2(0,T;X)}\Big)^{1/2}.$$

When X is a Hilbert space, $L^2(0,T;X)$ and $W^{1,2}(0,T;X)$ are also Hilbert spaces. We will use the following embedding results:

$$W^{1,2}(0,T;X) \hookrightarrow C([0,T];X) \text{ is continuous (see [24, p. 190])},$$
$$W^{1,2}(0,T;\mathbb{H}^1(\Omega)) \hookrightarrow \mathbb{L}^2(Q) \text{ is compact (see [25])},$$
$$W^{1,2}(0,T;\mathbb{H}^1(\Omega)) \hookrightarrow C([0,T];\mathbb{L}^2(\Omega)) \text{ is compact (see [25])}.$$

We now define the trilinear form b by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.$$

It is easy to check that if $u \in V, v, w \in \mathbb{H}^1_0(\Omega)$ then $b(u,v,w) = -b(u,w,v)$. Hence

$$b(u,v,v) = 0, \ \forall\, u \in V, v \in \mathbb{H}^1_0(\Omega).$$

Lemma 2.1. [8, 27] *We have*

$$|b(u,v,w)| \le C|u|^{1/4}\|u\|^{3/4}\|v\||w|^{1/4}\|w\|^{3/4}, \quad \forall\, u,\ v,\ w \in \mathbb{H}^1_0(\Omega),$$
$$|b(u,v,w)| \le C\|u\|\|v\|\|w\|, \quad \forall\, u,\ v,\ w \in \mathbb{H}^1_0(\Omega),$$
$$|b(u,v,u)| \le C|u|^{1/2}\|u\|^{3/2}\|v\|, \quad \forall\, u, v \in \mathbb{H}^1_0(\Omega).$$

2.2. **Existence and uniqueness of solutions to the Navier-Stokes-Voigt equations.**

Definition 2.1. *For given $u \in L^2(0,T;V')$ and $y_0 \in V$, a function y is called a weak solution to problem* (1.1) *on the interval* $(0,T)$ *if*

$$\begin{cases} y \in C([0,T];V),\ \dfrac{dy}{dt} \in L^2(0,T;V), \\ (y_t(s),v) + \nu((y(s),v)) + \alpha^2((y_t(s),v)) + b(y(s),y(s),v) = \langle u(s),v\rangle_{V',V} \quad \forall v \in V, \\ \qquad\qquad a.e.\ s \in (0,T), \\ y(0) = y_0. \end{cases}$$

We want to give an equivalent formulation as an equation in function spaces. To this aim, we introduce a linear, continuous operator $A : L^2(0,T;V) \to L^2(0,T;V')$ for $y, v \in L^2(0,T;V)$ by

$$\langle Ay, v\rangle_{L^2(0,T;V'),L^2(0,T;V)} = \int_0^T \langle Ay(t), v(t)\rangle_{V',V} dt := \int_0^T ((y(t),v(t)))dt,$$

and a nonlinear operator $B : W^{1,2}(0,T;V) \to L^2(0,T;V')$ for $y \in W^{1,2}(0,T;V)$, $w \in L^2(0,T;V)$ by

$$\langle B(y), w\rangle_{L^2(0,T;V'),L^2(0,T;V)} = \int_0^T \langle B(y)(t), w(t)\rangle_{V',V} dt := \int_0^T b(y(t),y(t),w(t))dt.$$

Now, we have an equivalent formulation with Definition 2.1.

Definition 2.2. *Let $u \in L^2(0,T;V')$ and $y_0 \in V$ be given. A function $y \in W^{1,2}(0,T;V)$ is called a weak solution to the problem* (1.1) *on the interval* $(0,T)$ *if it fulfills*

$$y_t + \nu Ay + \alpha^2 Ay_t + B(y) = u \ \text{in}\ L^2(0,T;V'),$$
$$y(0) = y_0.$$

Theorem 2.1. [5] *For any $y_0 \in V$ and $u \in L^2(0,T;V')$ given, problem* (1.1) *has a unique weak solution y on* $(0,T)$.

Lemma 2.2. i. *Let $\hat{y} \in W^{1,2}(0,T;V)$, $z_0 \in V$ and $f \in L^2(0,T;V')$ be given. Then the following equations*

$$\begin{cases} z_t + \nu Az + \alpha^2 Az_t + B(z,\hat{y}) + B(\hat{y},z) = f \quad \text{in } L^2(0,T;V'), \\ z(0) = z_0. \end{cases}$$

has a unique solution $z \in W^{1,2}(0,T;V)$. Moreover, there exist constants C_1, C_2 independent of f, z_0 such that

$$\|z\|_{W^{1,2}(0,T;V)} \leq C_1 \|f\|_{L^2(0,T;V')} + C_2 \|z_0\|. \tag{2.1}$$

ii. *Let $\hat{y} \in W^{1,2}(0,T;V)$, $z_0 \in V$ and $f \in L^2(0,T;V')$ be given. Then, for each $\rho \geq 0$ the following equations*

$$\begin{cases} z_t + \nu Az + \alpha^2 Az_t + B(z,\hat{y}) + B(\hat{y},z) + \rho B(z,z) = f \quad \text{in } L^2(0,T;V'), \\ z(0) = z_0, \end{cases} \tag{2.2}$$

has a unique solution $z \in W^{1,2}(0,T;V)$. Moreover, there exists a constant C independent of ρ such that

$$\|z\|_{W^{1,2}(0,T;V)} \leq C(1+\rho). \tag{2.3}$$

iii. *Let $\hat{y}, \hat{z} \in W^{1,2}(0,T;V)$, $z_0 \in V$ and $f \in L^2(0,T;V')$ be given. Then the equations below has a unique solution.*

$$\begin{cases} z_t + \nu Az + \alpha^2 Az_t + B(z,\hat{y}) + B(\hat{y},z) + \rho B(z,z) + \rho B(z,\hat{z}) + \rho B(\hat{z},z) \\ \qquad\qquad + \rho B(\hat{z},\hat{z}) = f \quad \text{in } L^2(0,T;V'), \\ z(0) = z_0, \end{cases}$$

Moreover, if $\|f\|_{L^2(0,T;V')} \leq M$ then there exists $C > 0$ independent of f (but dependent of M) such that

$$\|z\|_{W^{1,2}(0,T;V)} \leq C. \tag{2.4}$$

Proof. The proof of existence and uniqueness of solutions to the above equations is standard by Galerkin method. We only present here a proof of (2.3), for (2.1) and (2.4) are proved similarly. In this proof, we use C to denote several positive constants that may depend on $\|\hat{y}\|_{W^{1,2}(0,T;V)}$, $\|f\|_{L^2(0,T;V')}$, T, ν, α, Ω.

From the first equation of (2.2) and the fact that $b(z,z,z) = 0$, $b(\hat{y},z,z) = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\left(|z(t)|^2 + \alpha^2\|z(t)\|^2\right) + \nu\|z(t)\|^2 = -b(z(t),\hat{y}(t),z(t)) + \langle f(t), z(t)\rangle_{V',V}. \tag{2.5}$$

Since $\hat{y} \in W^{1,2}(0,T;V)$ and $W^{1,2}(0,T;V)$ is continuously embedded in $C([0,T];V)$, we get from Young's inequality that

$$\begin{aligned} |b(z(t),\hat{y}(t),z(t))| &\leq C|z(t)|^{1/2}\|z(t)\|^{3/2}\|\hat{y}(t)\| \\ &\leq C|z(t)|^{1/2}\|z(t)\|^{3/2} \\ &\leq C|z(t)|^2 + \frac{\nu}{2}\|z(t)\|^2, \end{aligned}$$

$$|\langle f(t), z(t)\rangle_{V',V}| \leq \|f(t)\|_{V'}\|z(t)\| \leq C\|f(t)\|_{V'}^2 + \frac{\nu}{2}\|z(t)\|^2.$$

From these estimates and (2.5), it follows that

$$\begin{aligned} \frac{d}{dt}(|z(t)|^2 + \alpha^2\|z(t)\|^2) &\leq C|z(t)|^2 + C\|f(t)\|_{V'}^2 \\ &\leq C\|f(t)\|_{V'}^2 + C(|z(t)|^2 + \alpha^2\|z(t)\|^2). \end{aligned}$$

Applying Gronwall's inequality we obtain

$$|z(t)|^2 + \alpha^2\|z(t)\|^2 \le e^{Ct}(|z_0|^2 + \alpha^2\|z_0\|^2) + \int_0^t e^{C(t-s)}\|f(s)\|_{V'}^2 ds \le C. \tag{2.6}$$

Hence, we get that $\|z\|_{L^2(0,T;V)} \le C$. Next, we prove a similar estimation for $\|z_t\|_{L^2(0,T;V)}$.

Multiplying the first equation of (2.2) by $z_t \in L^2(0,T;V)$ pointwise with respect to time and then integrating from 0 to T, we have

$$\begin{aligned}\|z_t\|_{\mathbb{L}^2(Q)^3}^2 + \nu\int_0^T ((z(t), z_t(t)))dt + \alpha^2\|z_t\|_{L^2(0,T;V)}^2 = -\int_0^T b(\hat{y}(t), z(t), z_t(t))dt \\ - \int_0^T b(z(t), \hat{y}(t), z_t(t))dt - \rho\int_0^T b(z(t), z(t), z_t(t))dt + \int_0^T \langle f(t), z_t(t)\rangle_{V',V} dt.\end{aligned} \tag{2.7}$$

After integrating by parts, the left-hand side of (2.7) is

$$\|z_t\|_{L^2(Q)^3}^2 + \frac{\nu}{2}\|z(T)\|^2 - \frac{\nu}{2}\|z_0\|^2 + \alpha^2\|z_t\|_{L^2(0,T;V)}^2. \tag{2.8}$$

From (2.6), by Lemma 2.1 we have following estimates

$$\begin{aligned}\left|\int_0^T b(\hat{y}(t), z(t), z_t(t))dt\right| &\le \frac{\alpha^2}{5}\|z_t\|_{L^2(0,T;V)}^2 + C\|z\|_{L^2(0,T;V)}^2, \\ \left|\int_0^T b(z(t), \hat{y}(t), z_t(t))dt\right| &\le \frac{\alpha^2}{5}\|z_t\|_{L^2(0,T;V)}^2 + C\|z\|_{L^2(0,T;V)}^2, \\ \left|\rho\int_0^T b(z(t), z(t), z_t(t))dt\right| &\le \frac{\alpha^2}{5}\|z_t\|_{L^2(0,T;V)}^2 + C\rho^2, \\ \left|\int_0^T \langle f(t), z_t(t)\rangle_{V',V} dt\right| &\le \frac{\alpha^2}{5}\|z_t\|_{L^2(0,T;V)}^2 + C\|f\|_{L^2(0,T;V')}^2.\end{aligned}$$

Summarizing these estimates and using (2.7), (2.8), we have

$$\begin{aligned}\frac{\alpha^2}{5}\|z_t\|_{L^2(0,T;V)}^2 \le C\|z\|_{L^2(0,T;V)}^2 + C\rho^2 + C\|f\|_{L^2(0,T;V')}^2 + \frac{\nu}{2}\|z_0\|^2 \\ \le C + C\rho^2 \le C(1+\rho^2).\end{aligned}$$

The proof is complete. □

Lemma 2.3. *Let $\hat{y}$ be a given function in $W^{1,2}(0,T;V)$. If y_n converges weakly to y in $W^{1,2}(0,T;V)$ then*

$$\begin{aligned}B(y_n, y_n) &\to B(y,y), \\ B(y_n, \hat{y}) &\to B(y,\hat{y}), \\ B(\hat{y}, y_n) &\to B(\hat{y}, y)\end{aligned}$$

in $L^2(0,T;V')$ as $n \to \infty$.

Proof. The first statement is proved in Lemma 3.1 in [1]. By a similar argument used in that proof, we can easily prove the last two statements. □

3. Pontryagin's Principle

3.1. **Pontryagin's principle of optimal control with pointwise control constraints.** We are going to establish Pontryagin's principle for problem (P_1):

$$\min_u J_1(y,u) := \frac{\alpha_T}{2}\int_\Omega |y(x,T)-y_T(x)|^2 dx + \frac{\alpha_Q}{2}\int_Q |y(x,t)-y_Q(x,t)|^2 dxdt + \frac{\gamma}{2}\int_Q |u(x,t)|^2 dxdt,$$

subject to

$$u \in \mathcal{U}_1 := \{u \in \mathbb{L}^2(Q) : u(x,t) \in \mathcal{M} \text{ for a.e. } (x,t) \in Q\},$$

and

$$\begin{cases} y_t + \nu Ay + \alpha^2 Ay_t + B(y) & = u \text{ in } L^2(0,T;V'), \\ y(0) & = y_0 \text{ in } V. \end{cases} \tag{3.1}$$

Assume that:

- The initial value y_0 is a given function in V. The desired states have to satisfy $y_T \in V$ and $y_Q \in \mathbb{L}^2(Q)$.
- The coefficients α_T, α_Q are non-negative real numbers, where at least one of them is positive to get a non-trivial objective functional. The regularization parameter γ, which measures the cost of the control, is also a positive number.

Set

$$\mathcal{A} = \{(y;u) \in W^{1,2}(0,T;V)\times\mathbb{L}^2(Q) : u \in \mathcal{U}_1 \text{ and } (y;u) \text{ satisfies equation (3.1)}\}.$$

Definition 3.1. *A pair $(\bar{y};\bar{u}) \in \mathcal{A}$ is called an optimal solution to problem (P_1) if there exists $\epsilon > 0$ such that $J_1(y,u) \geq J_1(\bar{y},\bar{u})$ for every $u \in \mathcal{U}_1$ satisfying $\|u-\bar{u}\|_{\mathbb{L}^2(Q)} \leq \epsilon$.*

If we add a asumption that $\mathcal{M}$ is convex then $\mathcal{U}_1$ is a nonempty convex closed set in $\mathbb{L}^2(Q)$. Hence, from Theorem 3.1 in [1] we get the existence of globally optimal solution of optimal problem (P_1). To derive a Pontryagin's principle for the optimal control problem (P_1), we need the two following lemmas.

Lemma 3.1. [23] *Let $v \in \mathcal{U}_1$ and $(\hat{y};\hat{u}) \in \mathcal{A}$. Then, for every $\rho \in (0,1)$ there exists a sequence of Lebesgue measurable sets $E^k_\rho \subset Q$, $H^k_\rho \subset \Omega$ such that*

$$\mu(E^k_\rho) = \rho\mu(Q),$$
$$\frac{1}{\rho}\chi_{E^k_\rho} \rightharpoonup 1 \text{ in } L^\infty(Q) \text{ weakly star as } k \to \infty,$$
$$\int_{E^k_\rho}(|v|^2 - |\hat{u}|^2)dxdt = \rho\int_Q(|v|^2-|\hat{u}|^2)dxdt,$$

where $\mu(.)$ denotes the Lebesgue measure and $\chi_{E^k_\rho}$ is the characteristic function of E^k_ρ.

Lemma 3.2. *Suppose that $(\hat{y};\hat{u}) \in \mathcal{A}$ and $v \in \mathcal{U}_1$. Then there is a sequence $\{\rho\} \subset (0,1)$ and measurable sets $E_\rho \subset Q$ satisfying the following conditions*

- $\rho \to 0^+$,
- $\mu(E_\rho) = \rho\mu(Q)$,
- *If u_ρ is defined by*

$$u_\rho(x,t) := (\hat{u} + \chi_{E_\rho}(v-\hat{u}))(x,t) = \begin{cases} \hat{u}(x,t) & \text{if } (x,t) \in Q\backslash E_\rho, \\ v(x,t) & \text{if } (x,t) \in E_\rho, \end{cases} \tag{3.2}$$

and y_ρ is the state associated to u_ρ then $(y_\rho, u_\rho) \in \mathcal{A}$ and the condition below holds

$$y_\rho = \hat{y} + \rho z + r_\rho, \ \lim_{\rho \to 0^+} \frac{1}{\rho} \|r_\rho\|_{C([0,T];\mathbb{L}^2(\Omega))} = 0, \tag{3.3}$$

$$J_1(u_\rho) = J_1(\hat{u}) + \rho \Delta J_1 + o(\rho), \tag{3.4}$$

where z is the unique solution of the linearized equations

$$\begin{cases} z_t + \nu Az + \alpha^2 Az_t + B(z, \hat{y}) + B(\hat{y}, z) = v - \hat{u} \quad \text{in } L^2(0,T;V'), \\ z(0) = 0, \end{cases}$$

and $\Delta J_1 = \alpha_T(z(T), \hat{y} - y_T) + \alpha_Q(z, \hat{y} - y_Q)_{\mathbb{L}^2(Q)} + \frac{\gamma}{2}(|v|^2_{\mathbb{L}^2(Q)} - |\hat{u}|^2_{\mathbb{L}^2(Q)})$.

Proof. Let v be given in $\mathcal{U}_1$. By Lemma 3.1, for each $\rho \in (0,1)$, there exists a sequence of measurable $E^k_\rho \subset Q$ such that (3.16) and (3.17) are valid. We set

$$u^k_\rho = \hat{u} + \chi_{E^k_\rho}(v - \hat{u}) \tag{3.5}$$

and denote by y^k_ρ the associated state to u^k_ρ. By (3.17), we have

$$\|u^k_\rho - \hat{u}\|^2_{\mathbb{L}^2(Q)} = \rho \int_Q \frac{1}{\rho} \chi_{E^k_\rho} |v(x,t) - \hat{u}(x,t)|^2 dxdt \to \rho \int_Q |v(x,t) - \hat{u}(x,t)|^2 dxdt$$

as $k \to \infty$. Hence, for $\rho > 0$ small enough and k big enough, u^k_ρ belongs to a neighborhood of $\hat{u}$ in $\mathbb{L}^2(Q)$. Since $v - \hat{u} \in \mathbb{L}^2(Q)$, it follows from (3.17) that

$$\frac{1}{\rho} \chi_{E^k_\rho}(v - \hat{u}) \rightharpoonup (v - \hat{u}) \ \text{ in } \mathbb{L}^2(Q) \ \text{ as } k \to \infty. \tag{3.6}$$

Hence, there exists a constant $M_\rho > 0$ such that

$$\left\| \frac{1}{\rho} \chi_{E^k_\rho}(v - \hat{u}) \right\|^2_{\mathbb{L}^2(Q)} \le M^2_\rho, \ \ \forall k \ge 1.$$

Put $z^k_\rho = \frac{1}{\rho}(y^k_\rho - \hat{y})$ and $f_k = \frac{1}{\rho} \chi_{E^k_\rho}(v - \hat{u})$. Then $z^k_\rho \in W^{1,2}(0,T;V)$ and satisfies the following equations

$$\begin{cases} {z^k_\rho}_t + \nu Az^k_\rho + \alpha^2 Az_{\rho t} + B(z^k_\rho, \hat{y}) + B(\hat{y}, z^k_\rho) + \rho B(z^k_\rho, z^k_\rho) = f_k, \\ z^k_\rho(0) = 0. \end{cases}$$

Hence, there exists a constant $C = C(\rho, \hat{y}, v, \hat{u}, T, \nu, \alpha, \Omega)$ such that $\|z^k_\rho\|_{W^{1,2}(0,T;V)} \le C$, thanks to Lemma 2.2. From the boundedness, we can extract a subsequence, denoted again by z^k_ρ, converging weakly to some z_ρ in $W^{1,2}(0,T;V)$ as $k \to \infty$. Then, it follows from Lemma 2.3 and (3.6) that z_ρ is a solution of the following equations

$$\begin{cases} z_{\rho t} + \nu Az_\rho + \alpha^2 Az_{\rho t} + B(z_\rho, \hat{y}) + B(\hat{y}, z_\rho) + \rho B(z_\rho, z_\rho) = v - \hat{u}, \\ z_\rho(0) = 0. \end{cases}$$

By Lemma 2.2, there exists a positive constant C not depending on ρ such that $\|z_\rho\|_{W^{1,2}(0,T;V)} \le C$. Hence, we can assume that $z_\rho \rightharpoonup z$ in $W^{1,2}(0,T;V)$ as $\rho \to 0^+$. We can easily show that z is the unique solution of the following equations

$$\begin{cases} z_t + \nu Az + \alpha^2 Az_t + B(z, \hat{y}) + B(\hat{y}, z) = v - \hat{u}, \\ z(0) = 0, \end{cases}$$

thanks to Lemma 2.3.

Since the embedding $W^{1,2}(0,T;V) \hookrightarrow C([0,T];\mathbb{L}^2(\Omega))$ is compact, we imply that z_ρ^k converges strongly to z_ρ in $C([0,T];\mathbb{L}^2(\Omega))$ as $k \to \infty$. Then, there exists $k(\rho) > 0$ such that

$$\|z_\rho^{k(\rho)} - z_\rho\|_{C([0,T];\mathbb{L}^2(\Omega))} \le \rho. \tag{3.7}$$

We now define

$$E_\rho = E_\rho^{k(\rho)},\ u_\rho = u_\rho^{k(\rho)} = \hat{u} + \chi_{E_\rho}(v - \hat{u}),\ y_\rho = y_\rho^{k(\rho)}.$$

From (3.2) we have $u_\rho \in \mathcal{U}_1$, and then $(y_\rho, u_\rho) \in \mathcal{A}$. It follows from (3.7) that

$$\begin{aligned}\|z_\rho^{k(\rho)} - z\|_{C([0,T];\mathbb{L}^2(\Omega))} &\le \|z_\rho^{k(\rho)} - z_\rho\|_{C([0,T];\mathbb{L}^2(\Omega))} + \|z_\rho - z\|_{C([0,T];\mathbb{L}^2(\Omega))}\\ &\le \rho + \|z_\rho - z\|_{C([0,T];\mathbb{L}^2(\Omega))} \to 0\end{aligned}$$

as $\rho \to 0$. Putting $r_\rho = y_\rho - \hat{y} - \rho z$, we obtain (3.3) from (3.7). It remains to check (3.4). We have

$$\begin{aligned}J_1(u_\rho) &= \frac{\alpha_T}{2}|y_\rho(T) - y_T|^2 + \frac{\alpha_Q}{2}\|y_\rho - y_Q\|^2_{\mathbb{L}^2(Q)} + \frac{\gamma}{2}\|u_\rho\|^2_{\mathbb{L}^2(Q)},\\ J_1(\hat{u}) &= \frac{\alpha_T}{2}|\hat{y}(T) - y_T|^2 + \frac{\alpha_Q}{2}\|\hat{y} - y_Q\|^2_{\mathbb{L}^2(Q)} + \frac{\gamma}{2}\|\hat{u}\|^2_{\mathbb{L}^2(Q)}.\end{aligned}$$

Then, it follows from (3.18) and (3.5) that

$$\begin{aligned}&J_1(u_\rho) - J_1(\hat{u})\\ &\quad= \frac{\alpha_T}{2}(\rho z(T) + r_\rho(T), 2\hat{y} + \rho z(T) - 2y_T + r_\rho(T))\\ &\qquad+ \frac{\alpha_Q}{2}(\rho z + r_\rho, 2\hat{y} + \rho z - 2y_Q + r_\rho)_{\mathbb{L}^2(Q)} + \frac{\gamma}{2}\int_{E_\rho}(|v|^2 - |\hat{u}|^2)dxdt\\ &\quad= \rho\Delta J_1 + \rho S(\rho),\end{aligned}$$

where

$$\Delta J_1 = \alpha_T(z(T), \hat{y} - y_T) + \alpha_Q(z, \hat{y} - y_Q)_{\mathbb{L}^2(Q)} + \frac{\gamma}{2}(|v|^2_{\mathbb{L}^2(Q)} - |\hat{u}|^2_{\mathbb{L}^2(Q)}),$$

$$\begin{aligned}S(\rho) = &\frac{\alpha_T}{2}\rho|z(T)|^2 + \alpha_T(z(T), r_\rho(T)) + \frac{\alpha_T}{\rho}(r_\rho(T), \hat{y} - y_T) + \frac{\alpha_T}{2\rho}|r_\rho(T)|^2\\ &+ \frac{\alpha_Q}{2}\rho\|z\|^2_{\mathbb{L}^2(Q)} + \alpha_Q(z, r_\rho)_{\mathbb{L}^2(Q)} + \frac{\alpha_Q}{2\rho}\|r_\rho\|^2_{\mathbb{L}^2(Q)}.\end{aligned}$$

From (3.3) we deduce that all of the terms in $S(\rho)$ converge to 0 as $\rho \to 0^+$. The proof is complete. □

Theorem 3.1. *If $(\hat{y}, \hat{u})$ is an optimal solution of the problem (P_1) then we have*

$$\bar{\lambda}(x,t)\cdot\hat{u}(x,t) + \frac{\gamma}{2}|\hat{u}(x,t)|^2 = \min_{w\in\mathcal{M}}\big(\bar{\lambda}(x,t)\cdot w + \frac{\gamma}{2}|w|^2\big) \ \text{for a.e. } (x,t)\in Q. \tag{3.8}$$

Here, $\bar{\lambda}$ is the adjoint state, i.e the unique solution of the following adjoint equations

$$\begin{cases}-\bar{\lambda}_t - \nu\Delta\bar{\lambda} + \alpha^2\Delta\bar{\lambda}_t - (\hat{y}\cdot\nabla)\bar{\lambda} + (\nabla\hat{y})^T\bar{\lambda} &= \alpha_Q(\hat{y} - y_Q),\ x\in\Omega, t>0,\\ \nabla\cdot\bar{\lambda} &= 0,\ x\in\Omega, t>0,\\ \bar{\lambda}(x,t) &= 0,\ x\in\partial\Omega, t>0,\\ \bar{\lambda}(T) - \alpha^2\Delta\bar{\lambda}(T) &= \alpha_T(\hat{y}(T) - y_T),\ x\in\Omega.\end{cases}$$

Proof. Let v be some fixed element of $\mathcal{U}_1$. Then for each $\rho \in (0,1)$ there exists a measurable set $E_\rho \in Q$ with $\mu(E_\rho) = \rho\mu(Q)$ that has the following property: If

$$u_\rho(x,t) = \hat{u} + \chi_{E_\rho}(v - \hat{u})(x,t)$$

and y_ρ is the associated state to u_ρ then (3.3), (3.4) hold. Since $(y_\rho, u_\rho) \in \mathcal{A}$, $(y_\rho, u_\rho) \to (\hat{y}, \hat{u})$ as $\rho \to 0^+$ and $(\hat{y}, \hat{u})$ is an optimal solution, we have

$$J_1(y_\rho, u_\rho) \ge J_1(\hat{y}, \hat{u})$$

when ρ is small enough.

It follows from (3.4) that

$$0 \leq \frac{1}{\rho}\left(J_1(y_\rho, u_\rho) - J_1(\hat{y}, \hat{u})\right) \to \Delta J_1 \quad (\text{as } \rho \to 0^+).$$

Hence, we have

$$0 \leq \Delta J_1 = \alpha_T(z(T), \hat{y} - y_T) + \alpha_Q(z, \hat{y} - y_Q)_{\mathbb{L}^2(Q)} + \frac{\gamma}{2}(|v|^2_{\mathbb{L}^2(Q)} - |\hat{u}|^2_{\mathbb{L}^2(Q)}) \tag{3.9}$$

for every $v \in \mathcal{U}_1$. Here, z is the unique solution of the linearized equations

$$\begin{cases} z_t + \nu Az + \alpha^2 Az_t + B(z, \hat{y}) + B(\hat{y}, z) = v - \hat{u}, \\ z(0) = 0. \end{cases} \tag{3.10}$$

From Theorem 4.1 in [1] we get that the adjoint equations

$$\begin{cases} -\bar{\lambda}_t - \nu\Delta\bar{\lambda} + \alpha^2\Delta\bar{\lambda}_t - (\bar{y}\cdot\nabla)\bar{\lambda} + (\nabla\bar{y})^T\bar{\lambda} & = \alpha_Q(\bar{y} - y_Q),\ x \in \Omega, t > 0, \\ \nabla\cdot\bar{\lambda} & = 0,\ x \in \Omega, t > 0, \\ \bar{\lambda}(x,t) & = 0,\ x \in \partial\Omega, t > 0, \\ \bar{\lambda}(T) - \alpha^2\Delta\bar{\lambda}(T) & = \alpha_T(\bar{y}(T) - y_T),\ x \in \Omega, \end{cases} \tag{3.11}$$

has a unique weak solution $\bar{\lambda} \in W^{1,2}(0,T;V)$.

Multiplying the first equation in (3.11) by z, multiplying the first equation in (3.10) by $\bar{\lambda}$, then integrating over Q and using integration by parts we have

$$\alpha_T\int_\Omega z(x,T)\cdot(\bar{y}(x,T) - y_T(x))dx + \alpha_Q\iint_Q z(x,t)\cdot(\bar{y}(x,t) - y_Q(x,t))dxdt$$
$$= \iint_Q \bar{\lambda}(x,t)\cdot(v(x,t) - \hat{u}(x,t))dxdt.$$

This together with (3.9) give the following inequality

$$\int_Q [\bar{\lambda}(x,t)\cdot(v(x,t) - \hat{u}(x,t)) + \frac{\gamma}{2}(|v(x,t)|^2 - |\hat{u}(x,t)|^2)]dxdt \geq 0. \tag{3.12}$$

This inequality holds for every $v \in \mathcal{U}_1$. Since $\mathcal{M}$ is a separable for it is closed in $\mathbb{R}^3$, there exists a countable dense subset $\mathcal{M}_0 = \{v_i, i \geq 1\}$ of $\mathcal{M}$. For each $v_i \in \mathcal{M}_0$, we set

$$H_{v_i}(x,t) = \bar{\lambda}(x,t)\cdot(v_i - \hat{u}(x,t)) + \frac{\gamma}{2}(v_i^2 - |\hat{u}(x,t)|^2).$$

Thanks to the Lebesgue differential theorem, for every i, there exists a subset $\tilde{Q}_i \subset Q$ such that $\mu(\tilde{Q}_i) = \mu(Q)$ and

$$\lim_{r\to 0}\frac{1}{\mu(B(z_0,r))}\int_{B(z_0,r)} H_{v_i}(x,t)\mathrm{d}x\ \mathrm{d}t = H_{v_i}(z_0)\ \forall z_0 = (x_0,t_0) \in \tilde{Q}_i. \tag{3.13}$$

Define

$$v_r(x,t) = \begin{cases} v_i & \text{if } (x,t) \in B(z_0,r), \\ \hat{u}(x,t) & \text{otherwise.} \end{cases}$$

Since $v_i \in \mathcal{M}$ and $\hat{u} \in \mathcal{U}_1$, we imply that $v_r \in \mathcal{U}_1$. Replacing v in (3.12) by v_r, we get

$$0 \leq \int_{B(z_0,r)} H_{v_i}(x,t)dxdt.$$

Therefore,

$$0 \leq \frac{1}{|B(z_0,r)|}\int_{B(z_0,r)} H_{v_i}(x,t)dxdt.$$

Passing to the limit and using (3.13) we obtain

$$H_{v_i}(x_0,t_0) \geq 0\ \ \forall (x_0,t_0) \in \tilde{Q}_i. \tag{3.14}$$

By setting $\tilde{Q} = \cap_{i\geq 1}\tilde{Q}_i$ we get a subset $\tilde{Q} \subset Q$ such that $\mu(\tilde{Q}) = \mu(Q)$. Since $\mathcal{M}_0 = \{v_i,\ i \geq 1\}$ is dense in $\mathcal{M}$, for any $w \in \mathcal{M}$, there exists a subsequence v_{i_n} that converges to w in $\mathbb{R}^3$. Let (x_0, t_0) be a fixed point in $\tilde{Q}$, then from (3.14) we have

$$H_{v_{i_n}}(x_0, t_0) \geq 0 \ \ \forall\, n.$$

Letting $n \to \infty$, we obtain $0 \leq H_\omega(x_0, t_0)$. Thus, we get that

$$\bar{\lambda}(x_0, t_0) \cdot (w - \hat{u}(x_0, t_0)) + \frac{\gamma}{2}(|w|^2 - |\hat{u}(x_0, t_0)|^2) \geq 0$$

for every $w \in \mathcal{M}$ and for a.e. $(x_0, t_0) \in Q$. This completes the proof. □

Remark 3.1. If we change equality (3.18) in Lemma 3.1 by

$$\int_{E^k_\rho} (v - \hat{u})\hat{u} dxdt = \rho \int_Q (v - \hat{u})\hat{u} dxdt,$$

then by similar arguments as in the proof of Theorem 3.1 we get the following inequality

$$\int_Q (\bar{\lambda} + \gamma\hat{u}) \cdot (v - \hat{u}) dxdt \geq 0.$$

This is exactly the variational inequality stated in [1] for the problem (P_1). Starting with this inequality and following to the arguments used in [28] lead to the pointwise variational inequality below

$$(\bar{\lambda}(x, t) + \gamma\hat{u}(x, t))(w - \hat{u}) \geq 0. \tag{3.15}$$

This variational inequality holds for every $w \in \mathcal{M}$ and for a.e. $(x, t) \in Q$. For a fixed $(x, t) \in Q$, (3.15) is actually the necessary optimality condition for $\hat{u}(x, t)$ to be the optimal solution of the following pointwise optimal problem

$$\min_{w\in\mathcal{M}} \big(\bar{\lambda}(x, t) \cdot w + \frac{\gamma}{2}|w|^2\big),$$

which is stated in Pontryagin's principle (3.8).

3.2. **Pontryagin's principle of optimal control with two points boundary state constraint.** In this section, we are going to derive Pontryagin's principle for problem (P_2):

$$\min_u J_2(y, u) := \frac{\alpha_Q}{2} \int_Q |y(x, t) - y_Q(x, t)|^2 dxdt + \frac{\gamma}{2} \int_Q |u(x, t)|^2 dxdt,$$

subject to

$$u \in \mathcal{U}_2 := \{u \in L^2(0, T; \mathbb{L}^2(\Omega)) \,|\, u(t) \in U \text{ for a.e. } t \in (0, T)\},$$

$$(y(0); y(T)) \in S,$$

and

$$y_t + \nu Ay + \alpha^2 Ay_t + B(y) = u \text{ in } L^2(0, T; V').$$

Assume that:

- The desired state have to satisfy $y_Q \in \mathbb{L}^2(Q)$.
- The coefficient α_Q is a positive real number. The regularization parameter γ, which measures the cost of the control, is also a positive number.

We define the mapping $G : V \times \mathbb{L}^2(Q) \to W^{1,2}(0, T; V)$, $G(y_0, u) = y$, where y is the unique solution of the following equations

$$\begin{cases} y_t + \nu Ay + \alpha^2 Ay_t + B(y, y) = u \ \text{ in } L^2(0, T; V'),\\ y(0) = y_0. \end{cases}$$

The following lemma shows that G is continuous.

Lemma 3.3. [5] *The mapping G defined above is continuous.*

Lemma 3.4. [23, Lemma 4.2] *Let $f \in L^1(0,T)$. Then, for every $\rho \in (0,1)$ there exists a sequence of Lebesgue measurable sets $E_\rho^n \subset [0,T]$ such that*

$$\mu(E_\rho^n) = \rho T, \tag{3.16}$$

$$\frac{1}{\rho}\chi_{E_\rho^n} \rightharpoonup 1 \text{ in } L^\infty(0,T) \text{ weakly star as } n \to \infty, \tag{3.17}$$

$$\int_{E_\rho^n} f(t)dt = \rho \int_0^T f(t)dt, \tag{3.18}$$

where $\mu(.)$ denotes the Lebesgue measure and $\chi_{E_\rho^n}$ is the characteristic function of E_ρ^n.

Let $(\bar{y};\bar{u})$ be an optimal solution of problem (P_2). We define operators $E : \mathbb{L}^2(Q) \to V$ by $E(u) = \xi(T)$ where ξ is the unique solution of the system

$$\begin{cases} \xi_t + \nu A\xi + \alpha^2 A\xi_t + B(z,\bar{y}) + B(\bar{y},z) = u(t) - \bar{u}(t) & \text{a.e. } t \in (0,T), \\ \xi(0) = 0, \end{cases}$$

and $K : V \to V$ by $K(x) = \eta(T)$, where η is the unique solution of the system

$$\begin{cases} \eta_t + \nu A\eta + \alpha^2 A\eta_t + B(z,\bar{y}) + B(\bar{y},z) = 0 & \text{a.e. } t \in (0,T), \\ \eta(0) = x. \end{cases}$$

It is clear that K is linear and bounded, thanks to Lemma 2.2. Set $\mathcal{R} = E(\mathcal{U}_2)$ and

$$Q = \{x_1 - Kx_0 \,|\, (x_0;x_1) \in S\}.$$

To establish Pontryagin's principle for $(\bar{y};\bar{u})$, we need one more assumption that

(H) The set $\mathcal{R} - Q \equiv \{r - q : r \in \mathcal{R}, q \in Q\}$ is finite codimensional in V.

Let ϵ be some given positive number. Set

$$V_\epsilon(0) = \{x \in V : \|x\| \le \epsilon\},$$

and

$$R_\epsilon = \left\{(y^0;z_T) \in V \times V : z_T = Ky^0 + w, \text{ for some } w \in \mathcal{R}, y^0 \in V_\epsilon(0)\right\}.$$

By Lemma 3.5 in [16], we know that

$$\mathcal{R} - Q \text{ is finite codimensional in } V \times V \text{ if and only if } R_\epsilon - S \text{ is so in } V \times V. \tag{3.19}$$

Theorem 3.2. *Assume that $(\bar{y};\bar{u})$ is an optimal solution of problem (P_2) and hypothesis (H) is satisfied. Then there exists a nontrivial pair $(\beta_0;\lambda) \in R \times W^{1,2}(0,T;V)$ such that*

$$-\lambda_t + \nu A\lambda - \alpha^2 A\lambda_t - \tilde{B}(\bar{y},\lambda) = 2\beta_0(y_Q - \bar{y}) \text{ in } L^2(0,T;V'),$$

$$\begin{aligned}(\lambda(0), x_0 - \bar{y}(0)) &+ \alpha^2((\lambda(0), x_0 - \bar{y}(0))) \\ &\le (\lambda(T), x_1 - \bar{y}(T))) + \alpha^2((\lambda(T), x_1 - \bar{y}(T))) \ \forall (x_0;x_1) \in S,\end{aligned}$$

and

$$(\bar{u}(t),\lambda(t)) + \beta_0|\bar{u}(t)|^2 = \min_{u\in U}[(u(t),\lambda(t)) + \beta_0|u(t)|^2] \text{ a.e. } t \in [0,T].$$

Here,

$$\langle \tilde{B}(\bar{y},\lambda), w\rangle_{L^2(0,T;V'),L^2(0,T;V)} = \int_0^T b(\bar{y},\lambda,w)dxdt - \int_0^T b(w,\bar{y},\lambda)dxdt.$$

Proof. By setting $L(y,u) = J_2(y,u) - J_2(\bar{y},\bar{u})$, we now consider the optimal control problem with the objective functional being $L(y,u)$. We notice that $L(y,u) \geq 0$ for every $(y,u) \in W^{1,2}(0,T;V) \times \mathcal{U}_2$ that is close enough to $(\bar{y},\bar{u})$. We follow six steps as in [16] to prove Pontryagin's principle.

Step 1: Metric space

For each $k > 0$, we set

$$\mathcal{U}_{ad}(\bar{u},k) = \{u \in \mathcal{U}_2 \,:\, |u(t) - \bar{u}(t)| \leq k \text{ for a.e. } t \in (0,T)\}. \tag{3.20}$$

We endow this space with Ekeland's metric d defined by

$$d(u,v) = \mu(\{t \in [0,T] :\, u(t) \neq v(t)\}),$$

where μ denotes the Lebesgue measure in $\mathbb{R}$. We can easily see that if $u_n \to u$ in $\mathcal{U}_{ad}(\bar{u},k)$ then $u_n \to u$ in $\mathbb{L}^2(Q)$.

Step 2: Approximate problem

For each $\varepsilon > 0$, we define $L_\varepsilon : V \times \mathcal{U}_{ad}(\bar{u},k) \to \mathbb{R}$ by

$$L_\varepsilon(y_0,u) = \{[L(y,u)+\varepsilon]^2 + d_S^2(y(0),y(T))\}^{1/2},$$

where $y = G(y_0,u)$ and

$$d_S(y_1,y_2) = d((y_1;y_2),S) := \inf_{(x_1;x_2)\in S}\{\|y_1 - x_1\|^2 + \|y_2 - x_2\|^2\}^{1/2}.$$

For fixed $\varepsilon > 0$ and $k \in \mathbb{Z}^+$, we consider the approximate problem

$$(P_{2,k,\varepsilon}) \qquad \begin{cases} \min\ L_\varepsilon(y_0,u) \\ u \in \mathcal{U}_{ad}(\bar{u},k),\ y_0 \in V. \end{cases}$$

For each $k \in \mathbb{Z}^+$, we choose $\varepsilon_k \leq \dfrac{1}{k^8}$ and denote by $(P_{2,k})$ the approximate problem (P_{2,k,ε_k}) and by L_k the penalty function L_{ε_k}.

Since $L(y,u)$ is continuous on $W^{1,2}(0,T;V) \times \mathbb{L}^2(Q)$, $L_\varepsilon(y_0,u)$ is continuous on $V \times \mathcal{U}_{ad}(\bar{u},k)$, by Lemma 3.3. Because $L_\varepsilon(y_0,u) \geq \varepsilon$ for every $(y_0,u) \in V \times \mathcal{U}_{ad}(\bar{u},k)$ we obtain that $\varepsilon \leq \inf_{(y_0,u)\in V\times\mathcal{U}_{ad}(\bar{u},k)} L_\varepsilon(y_0,u) < +\infty$, hence that

$$L_\varepsilon(\bar{y}(0),\bar{u}) = \varepsilon \leq \inf_{(y_0,u)\in V\times\mathcal{U}_{ad}(\bar{u},k)} L_\varepsilon(y_0,u) + \varepsilon.$$

Thanks to Ekeland variational principle (see [16, Corollary 2.2]), for every $k \geq 1$, there exists a pair $(y_{0k};u_k) \in V \times \mathcal{U}_{ad}(\bar{u},k)$ such that

$$\|y_{0k} - \bar{y}(0)\|^2 + d^2(u_k,\bar{u}) \leq \varepsilon_k \leq \frac{1}{k^8}, \tag{3.21}$$

and

$$\begin{aligned} &L_k(y_{0k},u_k) \leq L_k(y_0,u) + \sqrt{\varepsilon_k}[\|y_{0k} - y_0\|^2 + d^2(u_k,u)]^{1/2}] \\ &\quad \forall\ (y_0,u) \in V \times \mathcal{U}_{ad}(\bar{u},k). \end{aligned} \tag{3.22}$$

By the definition of ε_k, we get from (3.20) and (3.21) that

$$\begin{aligned} \int_0^T |u_k(t) - \bar{u}(t)|^2 dt &= \int_0^T \chi_{e_k}|u_k(t) - \bar{u}(t)|^2 dt \\ &\leq k^2 \int_0^T \chi_{e_k} dt = \mu(e_k)k^2 = d(u_k,\bar{u})k^2 \leq \frac{1}{k^2}, \end{aligned} \tag{3.23}$$

where $e_k = \{t \in [0,T] \,:\, u_k(t) \neq \bar{u}(t)\}$.

Step 3: Diffuse perturbations

To exploit the necessary conditions for $(y_{0k};u_k)$, we introduce the following particular perturbation of $(y_{0k};u_k)$.

For given $(y^0; u^0) \in V \times \mathcal{U}_2$, we set

$$u_{0k}(t) = \begin{cases} u^0(t) & \text{if } |u^0(t) - \bar{u}(t)| \le k, \\ \bar{u}(t) & \text{if } |u^0(t) - \bar{u}(t)| > k. \end{cases}$$

It is clear that for each $k \ge 1$, u_{0k} belongs to $\mathcal{U}_{ad}(\bar{u}, k)$. We have

$$\|u_{0k} - u^0\|^2_{\mathbb{L}^2(Q)} = \int_0^T |u_{0k}(t) - u^0(t)|^2 dt = \int_0^T \chi_{A_k}(t)|\bar{u}(t) - u^0(t)|^2 dt,$$

where $A_k = \{t \in (0,T) : |u^0(t) - \bar{u}(t)| > k\}$. Since $\chi_{A_k}(t) \to 0$ for a.e. $t \in (0,T)$, it follows from Lebesgue's dominated convergence theorem that

$$u_{0k} \to u^0 \text{ in } \mathbb{L}^2(Q) \text{ as } k \to +\infty. \tag{3.24}$$

Set $y_k = G(y_{0k}, u_k)$. By Lemma 3.4, for each $\rho \in (0,1)$, there exists e sequence of Lebesgue measurable sets $E^\rho_{k,n} \subset [0,T]$, $n = 1, 2, ...$ such that

$$\mu(E^\rho_{k,n}) = \rho T, \tag{3.25}$$

$$\frac{1}{\rho}\chi_{E^\rho_{k,n}} \rightharpoonup 1 \text{ in } L^\infty(0,T) \text{ weakly star as } n \to \infty, \tag{3.26}$$

$$\int_{E^\rho_{k,n}} \left(|u_{0k}(t)|^2 - |u_k(t)|^2\right) dt = \rho \int_0^T \left(|u_{0k}(t)|^2 - |u_k(t)|^2\right) dt. \tag{3.27}$$

Consider the following equations

$$\begin{cases} r_t + \nu Ar + \alpha^2 Ar_t + B(r, y_k) + B(y_k, r) + \rho B(z_k, z_k) + \rho B(z_k, r) \\ +\rho B(r, z_k) + \rho B(r, r) = (\frac{1}{\rho}\chi_{E^\rho_{k,n}} - 1)(u_{0k} - u_k) \text{ in } L^2(0,T;V'), \\ r(0) = 0. \end{cases} \tag{3.28}$$

By Lemma 2.2, for each $\rho \in (0,1)$, $k \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+$, the system above have a unique solution. We denote this solution by $r^\rho_{k,n}$. From (3.26), we have

$$\left(\frac{1}{\rho}\chi_{E^\rho_{k,n}} - 1\right)(u_{0k} - u_k) \rightharpoonup 0 \text{ in } \mathbb{L}^2(Q) \text{ as } n \to \infty. \tag{3.29}$$

This implies that sequence $\left\{\left(\frac{1}{\rho}\chi_{E^\rho_{k,n}} - 1\right)(u_{0k} - u_k)\right\}_n$ is bounded in $\mathbb{L}^2(Q)$. Hence, $\{r^\rho_{k,n}\}_n$ is bounded in $W^{1,2}(0,T;V)$, thanks to Lemma 2.2. As a consequence, we can extract a subsequence, denoted again by $\{r^\rho_{k,n}\}_n$, weakly convergent in the space $W^{1,2}(0,T;V)$ as $n \to \infty$. From Lemma 2.3 and (3.29), we can pass to the limit in the equations (3.28) and obtain that

$$r^\rho_{k,n} \rightharpoonup 0 \text{ in } W^{1,2}(0,T;V) \text{ as } n \to \infty.$$

Since $W^{1,2}(0,T;V)$ is compactly embedded in the space $C([0,T];\mathbb{L}^2(\Omega))$, we have

$$r^\rho_{k,n} \to 0 \text{ in } C([0,T];\mathbb{L}^2(\Omega)) \text{ as } n \to \infty.$$

Hence, there exists $n(\rho) \in \mathbb{N}^*$ such that

$$\|r^\rho_{k,n}\|_{C([0,T];\mathbb{L}^2(\Omega))} \le \rho \ \ \forall n \ge n(\rho).$$

Set $E^\rho_k = E^\rho_{k,n(\rho)}$. For each $\rho \in (0,1)$, we set $y^\rho_{0k} = y_{0k} + \rho y^0$ and

$$u^\rho_k(t) = \begin{cases} u_k(t), & \text{on } [0,T]\backslash E^\rho_k, \\ u_{0k}(t), & \text{on } E^\rho_k. \end{cases}$$

It is clear that $u^\rho_k \to u_k$ in $\mathcal{U}_{ad}(\bar{u}, k)$ as $\rho \to 0$. Hence, we have

$$u^\rho_k \to u_k \text{ in } \mathbb{L}^2(Q), \ y^\rho_{0k} \to y_{0k} \text{ in } V \text{ as } \rho \to 0. \tag{3.30}$$

Set $y_k^\rho := G(y_{0k}^\rho, u_k^\rho)$. Then, it follows from Lemma 3.3 that

$$y_k^\rho \to y_k \ \text{ in } W^{1,2}(0,T;V) \ \text{ as } \rho \to 0.$$

Hence, $\{y_k^\rho\}_\rho$ is bounded when ρ is close enough to 0. From Lemma 2.2, the equations below has a unique solution $z_k \in W^{1,2}(0,T;V)$.

$$\begin{cases} z_{kt} + \nu A z_k + \alpha^2 A z_{kt} + B(z_k, y_k) + B(y_k, z_k) = u_{0k} - u_k \ \text{ in } L^2(0,T;V'), \\ z_k(0) = y^0. \end{cases} \tag{3.31}$$

Set $r_k^\rho := r_{k,n(\rho)}^\rho$, then we have

$$r_k^\rho = \frac{y_k^\rho - y_k - \rho z_k}{\rho} \ \text{ and } \|r_k^\rho\|_{C([0,T];\mathbb{L}^2(\Omega))} \le \rho. \tag{3.32}$$

Step 4: Necessary conditions for $(y_{0k}; u_k)$
From (3.30) and Lemma 3.3 we imply that

$$L_k(y_{0k}^\rho, u_k^\rho) = L_k(y_{0k}, u_k) + o(1) \ \text{ as } \rho \to 0.$$

Hence, it follows from (3.22) and (3.25) that

$$\begin{aligned} -\frac{1}{k^4}&\left[\|y^0\| + T\right] \\ &\leqslant \frac{1}{\rho}\left[L_k\left(y_{0k}^\rho, u_k^\rho\right) - L_k\left(y_{0k}, u_k\right)\right] \\ &= \{1/\left[2L_k\left(y_{0k}, u_k\right) + o(1)\right]\}\left\{\frac{1}{\rho}\left[\left(L\left(y_{0k}^\rho, u_k^\rho\right) + \varepsilon_k\right)^2\right.\right. \\ &\quad \left.\left. - \left(L(y_{0k}, u_k) + \varepsilon_k\right)^2\right] + \frac{1}{\rho}\left[d_S^2\left(y_{0k}^\rho, y_k^\rho(T)\right) - d_S^2(y_{0k}, y_k(T))\right]\right\}. \end{aligned} \tag{3.33}$$

By using a similar argument as in [16, p. 154], we imply that

$$\begin{aligned} \lim_{\rho\to 0}\frac{1}{\rho}&\left[d_S^2\left(y_{0k}^\rho, y_k^\rho(T)\right) - d_S^2\left(y_{0k}, y_k(T)\right)\right] \\ &= 2d_S\left(y_{0k}, y_k(T)\right)\left[\langle a_k, y^0\rangle_{V',V} + \langle b_k, z_k(T)\rangle_{V',V}\right], \end{aligned}$$

where $(a_k; b_k) \in \partial d_S\left(y_{0k}, y_k(T)\right)$ (the subdifferential of d_S at $(y_{0k}; y_k(T))$) and

$$\|a_k\|_{V'}^2 + \|b_k\|_{V'}^2 = \begin{cases} 1, & \text{if } (y_{0k}; y_k(T)) \notin S, \\ 0, & \text{if } (y_{0k}; y_k(T)) \in S. \end{cases}$$

It follows from (3.27) and (3.32) that

$$\begin{aligned} \lim_{\rho\to 0^+}\frac{1}{\rho}&\left[\left(L\left(y_{0k}^\rho, u_k^\rho\right) + \varepsilon_k\right)^2 - \left(L(y_{0k}, u_k) + \varepsilon_k\right)^2\right] \\ &= 2\big(L(y_{0k}, u_k) + \varepsilon_k\big)\left[2\int_0^T (z_k(t)(y_k(t) - y_Q(t))dt\right. \\ &\quad \left. + \int_0^T (|u_{0k}(t)|^2 - |u_k(t)|^2)dt\right]. \end{aligned}$$

Hence, from (3.33) we obtain that

$$-\frac{1}{k^4}(\|y^0\|+T) \leq \frac{1}{2L_k(y_{0k},u_k)}\Big\{2L(y_{0k},u_k)\Big[2\int_0^T (z_k(t),y_k(t)-y_Q(t))dt + \int_0^T (|u_{0k}(t)|^2-|u_k(t)|^2)dt\Big] + 2d_S(y_{0k},y_k(T))\big[\langle a_k,y^0\rangle_{V',V} + \langle b_k,z_k(T)\rangle_{V',V}\big]\Big\}.$$

Setting

$$\zeta_k = 2\int_0^T (z_k(t),y_k(t)-y_Q(t))dt + \int_0^T (|u_{0k}(t)|^2-|u_k(t)|^2)dt,$$

$$(\varphi_k,\psi_k) = \frac{d_S(y_{0k},y_k(T))}{L_k(y_{0k},u_k)}(a_k,b_k), \quad \beta_k = \frac{L(y_{0k},u_k)}{L_k(y_{0k},u_k)},$$

we get that

$$-\frac{1}{k^4}\big[\|y^0\|+T\big] \leq \langle\varphi_k,y^0\rangle_{V',V} + \langle\psi_k,z_k(T)\rangle_{V',V} + \beta_k\zeta_k, \tag{3.34}$$

and

$$\|\varphi_k\|^2_{V'} + \|\psi_k\|^2_{V'} + \beta_k^2 = 1. \tag{3.35}$$

Since $(\varphi_k;\psi_k) \in \partial d_S(y_{0k},y_k(T))$, we have

$$\langle\varphi_k,x_0-y_{0k}\rangle_{V',V} + \langle\psi_k,x_1-y_k(T)\rangle_{V',V} \leqslant 0 \quad \forall\,(x_0;x_1)\in S. \tag{3.36}$$

Step 5: Passing to the limit

By (3.35), from sequence $(\varphi_k;\psi_k;\beta_k)_k$ we can extract a subsequence, denoted again by $(\varphi_k;\psi_k;\beta_k)_k$, where $(\varphi_k;\psi_k)$ weakly* converging to some $(\varphi_0;\psi_0)$ in $V'\times V'$ and $\beta_k\to\beta_0$ in $\mathbb{R}$.

From (3.23), (3.24) and (3.21), we have

$$u_k\to\bar{u}, \;\; u_{0k}\to u^0 \;\text{ in } \mathbb{L}^2(Q), \tag{3.37}$$

$$y_{0k}\to\bar{y}(0) \quad\text{in } V \text{ as } k\to\infty. \tag{3.38}$$

Hence, we obtain that

$$y_k\to\bar{y} \;\text{ in } W^{1,2}(0,T;V) \;\text{ as } k\to\infty, \tag{3.39}$$

thanks to Lemma 3.3.

By Lemma 2.2, it follows from (3.31), (3.37),(3.38) that

$$z_k\to z \text{ in } W^{1,2}(0,T;V) \text{ as } k\to\infty,$$

where z is the unique solution of the equations below.

$$\begin{cases} z_t+\nu Az+\alpha^2Az_t+B(z,y)+B(y,z)=u^0-\bar{u} \;\text{ in } L^2(0,T;V'),\\ z(0)=y^0. \end{cases} \tag{3.40}$$

As a consequence, we obtain that

$$\zeta_k\to\zeta = 2\int_Q (z(t),\bar{y}(t)-y_Q(t))dt + \int_0^T (|u^0(t)|^2-|\bar{u}(t)|^2)dt \;\text{ as } k\to\infty. \tag{3.41}$$

Letting k tend to $+\infty$ in (3.34), (3.36) yields

$$0\leq\langle\varphi_0,y^0\rangle_{V',V}+\langle\psi_0,z(T)\rangle_{V',V}+\beta_0\zeta \;\;\forall\,(y^0;u^0)\in V\times\mathcal{U}_2, \tag{3.42}$$

$$\langle\varphi_0,x_0-\bar{y}(0)\rangle_{V',V}+\langle\psi_0,x_1-\bar{y}(T)\rangle_{V',V}\leq 0 \;\;\forall\,(x_0;x_1)\in S. \tag{3.43}$$

Let $\lambda \in W^{1,2}(0,T;V)$ be the unique weak solution to

$$\begin{cases} -\lambda_t + \nu A\lambda - \alpha^2 A\lambda_t - \tilde{B}(\bar{y},\lambda) = 2\beta_0(y_Q - \bar{y}) \text{ in } L^2(0,T;V'), \\ \lambda(T) + \alpha^2 A\lambda(T) = \psi_0 \text{ in } V'. \end{cases} \tag{3.44}$$

where

$$\langle \tilde{B}(\bar{y},\lambda), w\rangle_{L^2(0,T;V'),L^2(0,T;V)} = \int_0^T b(\bar{y},\lambda,w)dxdt - \int_0^T b(w,\bar{y},\lambda)dxdt,$$

$$\langle \lambda(T) + \alpha^2 A\lambda(T), w\rangle_{V',V} = (\lambda(T),w) + \alpha^2((\lambda(T),w)).$$

Multiplying (3.44) by z and integrating by parts yield

$$\begin{aligned} &-(\lambda(T),z(T)) + (\lambda(0),y^0) + \int_0^T (z_t,\lambda)dt + \nu\int_0^T ((Az,\lambda))dt \\ &\quad - \alpha^2((\lambda(T),z(T))) + \alpha^2((\lambda(0),y^0)) + \alpha^2\int_0^T ((z_t,\lambda))dt + \int_0^T b(\bar{y},z,\lambda)dt \\ &\qquad + \int_0^T b(z,\bar{y},\lambda)dt = 2\beta_0\int_0^T (y_Q - \bar{y}, z)dt. \end{aligned}$$

This together with 3.40 and 3.41 imply that

$$\begin{aligned} &-(\psi_0,z(T)) + (\lambda(0),y^0) + \alpha^2((\lambda(0),y^0)) + \int_0^T (u^0 - \bar{u},\lambda)dt \\ &\qquad = \beta_0\zeta - \beta_0\int_0^T (|u^0(t)|^2 - |\bar{u}(t)|^2)dt. \end{aligned}$$

Hence, from (3.42) we have

$$\begin{aligned} 0 \le \langle \varphi_0, y^0\rangle_{V',V} + (\lambda(0),y^0) + \alpha^2((\lambda(0),y^0)) + \int_0^T (u^0 - \bar{u},\lambda)dt \\ + \beta_0\int_0^T (|u^0(t)|^2 - |\bar{u}(t)|^2)dt \quad \forall (y^0;u^0) \in V\times\mathcal{U}_2. \end{aligned}$$

Taking $u^0 = \bar{u}$ yields that

$$\lambda(0) + \alpha^2 A\lambda(0) = -\varphi_0 \quad \text{ in } V', \tag{3.45}$$

and that

$$\int_0^T (\bar{u},\lambda)dt + \beta_0\int_0^T |\bar{u}(t)|^2dt \le \int_0^T (u^0,\lambda)dt + \beta_0\int_0^T |u^0(t)|^2dt \quad \forall u^0 \in \mathcal{U}_2. \tag{3.46}$$

Here, $\lambda(0) + \alpha^2 A\lambda(0)$ is considered as an element of V' by

$$\langle \lambda(0) + \alpha^2 A\lambda(0), w\rangle_{V',V} = (\lambda(0),w) + \alpha^2((\lambda(0),w)).$$

Since U is separable, there exists a countable dense set $U_0 = \{u_i, i \ge 1\} \subset U$. For each $u_i \in U_0$, we denote

$$g_i(s) = (\bar{u}(s),\lambda) + \beta_0|\bar{u}(s)|^2 - (u_i,\lambda) - \beta_0|u_i|^2.$$

Then $g_i \in L^1(0,T)$. Hence, there exists a measurable set $F_i \subset [0,T]$ with $\mu(F_i) = T$ such that any point in F_i is a Lebesgue point of g_i. Namely,

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{t-\delta}^{t+\delta} |g_i(s) - g_i(t)|\,ds = 0, \quad \forall\, t \in F_i.$$

Now, for any $t \in F_i$, we define

$$v_\delta(s) = \begin{cases} \bar{u}(s), & |s-t| > \delta, \\ u_i, & |s-t| \le \delta. \end{cases}$$

By the definition, $v_\delta \in \mathcal{U}_2$. Hence, by (3.46) we obtain that

$$\int_{t-\delta}^{t+\delta} g_i(s)ds \le 0, \quad \forall \delta > 0.$$

Dividing by $\delta > 0$ and sending $\delta \to 0$, we obtain $g_i(t) \le 0$. That means

$$(\bar{u}(s), \lambda(s)) + \beta_0|\bar{u}(s)|^2 \le (u_i, \lambda(s)) + \beta_0|u_i|^2, \quad \forall\, s \in F \equiv \bigcap_{i\ge 1} F_i, \;\; u_i \in U_0.$$

As U_0 is a countable dense subset of U, we have $|F| = T$ and

$$(\bar{u}(s), \lambda(s)) + \beta_0|\bar{u}(s)|^2 \le (u, \lambda(s)) + \beta_0|u(s)|^2, \;\; \forall\, s \in F, \; \forall u \in U. \tag{3.47}$$

From (3.43), (3.44), (3.45) we obtain that

$$(\lambda(0), x_0 - \bar{y}(0)) + \alpha^2((\lambda(0), x_0 - \bar{y}(0))) - (\lambda(T), x_1 - \bar{y}(T)) \\ - \alpha^2((\lambda(T), x_1 - \bar{y}(T))) \le 0 \;\; \forall\, (x_0; x_1) \in S.$$

Step 6: Nontriviality of $(\beta_0; \lambda)$

We have to show that $(\beta_0; \lambda) \neq 0$. If $\beta_0 \neq 0$, we are done. Otherwise, $\beta_k \to 0$ as $k \to \infty$. Then, from (3.34) and (3.36) we have

$$\begin{aligned} &\langle \varphi_k, y^0 - (x_0 - \bar{y}(0))\rangle_{V',V} + \langle \psi_k, z(T) - (x_1 - \bar{y}(T)))\rangle\rangle_{V',V} \\ &\ge -\frac{1}{k^4}\left[\|y^0\| + T\right] - \beta_k|\zeta_k| - |y_{0k} - \bar{y}(0)| - |y_k(T) - \bar{y}(T)| - |z(T) - z_k(T)| \\ &\equiv -\theta_k, \quad \forall (x_0; x_1) \in S, \; y^0 \in V, \; u^0 \in U. \end{aligned} \tag{3.48}$$

From (3.35) we imply the existence of a positive real number $\delta > 0$ such that

$$\|\varphi_k\|^2_{V'} + \|\psi_k\|^2_{V'} \ge \delta.$$

From the boundedness of U and the definition of u_{0k} we imply that u_{0k} converges to u^0 uniformly in u^0. As a consequence, $\beta_k|\zeta_k| \to 0$ and $|z(T) - z_k(T)| \to 0$ as $k \to \infty$ uniformly in $u^0 \in \mathcal{U}_2$. Hence, $\theta_k \to 0$ uniformly in $x_0, x_1 \in S$, $y^0 \in V_\epsilon(0)$, $u^0 \in \mathcal{U}_2$, thanks to (3.38), (3.39), (3.41).

By the definition of R_ϵ, (3.48) may be rewritten as follows

$$\left\langle \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}, \begin{pmatrix} \zeta_0 \\ \zeta_1 \end{pmatrix} \right\rangle_{V'\times V', V\times V} \ge -\theta_k \quad \forall \begin{pmatrix} \zeta_0 \\ \zeta_1 \end{pmatrix} \in R_\epsilon - S + \begin{pmatrix} \bar{y}(0) \\ \bar{y}(T) \end{pmatrix}.$$

By (3.19), $R_\epsilon - S$ is finite codimensional in $V \times V$, then so does $R_\epsilon - S - \begin{pmatrix} \bar{y}(0) \\ \bar{y}(T) \end{pmatrix}$. Therefore, $(\varphi_0; \psi_0) \neq 0$, thanks to Lemma 3.6 in [16, Chapter 4]. This implies that $\lambda \neq 0$ and the proof is completed. ☐

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