International Electronic Journal of Algebra
Volume 33 (2023) 34-53
DOI: 10.24330/ieja.1229771

ON SOME IDEAL STRUCTURE OF LEAVITT PATH ALGEBRAS WITH COEFFICIENTS IN INTEGRAL DOMAINS

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Received: 30 June 2021; Revised: 30 November 2022; Accepted: 18 December 2022
Communicated by Abdullah Harmancı

Dedicated to the memory of Professor Edmund R. Puczyłowski

Abstract. In this paper, we present results concerning the structure of the ideals in the Leavitt path algebra of a (countable) directed graph with coefficients in an integral domain, such as, describing the set of generators for an ideal; the necessary and sufficient conditions for an ideal to be prime; the necessary and sufficient conditions for a Leavitt path algebra to be simple. Besides, some other interesting properties of ideal structure in a Leavitt path algebra are also mentioned.

Mathematics Subject Classification (2020): 16D25, 16W50, 16D70
Keywords: Leavitt path algebra, prime ideal, basic ideal, maximal ideal

1. Introduction

The Leavitt path algebras $L_R(E)$ of the (directed) graph E with coefficients in a unital commutative ring R was introduced in 2011 by M. Tomforde [10]. In [10] the author defined basic ideals, and characterized graded basic ideals of $L_R(E)$ by the saturated hereditary subsets of vertices in E in the case that E is a row-finite graph. For non-row finite graphs, the set of admissible pairs in E are considered instead of saturated hereditary subsets of E^0. However, in this case, basic ideals of $L_R(E)$ are too complicated. In 2015, H. Larki [6] overcame the above complexity by introducing a new definition of basic ideals. In the case when E is row-finite, this definition is equivalent to that in [10].

In this paper, based in the above results and the definition of basic ideals due to H. Larki in [6] and the description of a set of generators for an ideal, the necessary and sufficient conditions for the primeness of ideals, the existence of maximal ideals in Leavitt path algebra with field coefficients (see [2,3,8]), we describe the set of generators for an ideal, the necessary and sufficient conditions for an ideal to be

The first author is funded by Vietnam National University HoChiMinh City (VNU-HCM) under Grant No. B2020-18-02.

prime; the necessary and sufficient conditions for $L_R(E)$ to be simple. Besides, some other interesting properties of ideal structure on $L_R(E)$ are also mentioned.

The paper is organized as follows. We begin by Section 2 to provide some basic facts about Leavitt path algebras. Most of our definitions in this section are from [6,10]. For a countable directed graph E and a unital commutative ring R, an R-algebra $L_R(E)$ is associated. These algebras are defined as the definition of graph C^*-algebras $C^*(E)$, and they have natural $\mathbb{Z}$-grading. In Section 3, we study the structure of generators for an ideal of $L_R(E)$ by replacing some results of K.M. Rangaswamy in [9] on a field with either a unital commutative ring or an integral domain. In Section 4, based on the results of K.M. Rangaswamy in [8] and H. Larki in [6], we give the necessary and sufficient conditions for the primeness of a (graded/non-graded) basic ideal in $L_R(E)$. In Section 5, based on the results of S. Esin and M. Kanuni Er in [3], the necessary and sufficient conditions of the existence of maximal basic ideal of $L_R(E)$ will be discussed.

2. Preliminaries and notation

Throughout this paper, a ring means a unital commutative ring and a graph means a countable directed graph; all graded rings and modules are understood to be $\mathbb{Z}$-graded.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of a set E^0 of verties, a set E^1 of edges, a source function $s : E^1 \to E^0$, and a range function $r : E^1 \to E^0$. We say that E is *countable* if both E^0 and E^1 are countable. A vertex $v \in E^0$ is called a *sink* if $s^{-1}(v) = \emptyset$, and an *infinite emitter* if $|s^{-1}(v)| = \infty$. A vertex v which is either a sink or an infinite emitter called a *singular vertex*, a vertex v which is not a singular vertex called a *regular vertex*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then E is called *row-finite*.

If $e_1, \ldots, e_n$ are edges in E such that $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$, then $p = e_1 \ldots e_n$ is called a *path* of length $|p| = n$ with source $s(p) = s(e_1)$ and range $r(p) = r(e_n)$. Note that we consider the vertices in E^0 to be paths of length zero. Set of all finite paths in E is denoted by $\mathrm{Path}(E)$.

An edge e is called an *exit* for a path $p = e_1 \ldots e_n$ if there exists $1 \leq i \leq n$ such that $e \neq e_i$ and $s(e) = s(e_i)$. If p is a path such that $p \neq v$ and $s(p) = r(p) = v$, then p is called a *closed path* based at v. If $p = e_1 \ldots e_n$ is a closed path such that $s(e_j) \neq s(e_j)$ for every $i \neq j$, then p is called a *cycle*. A graph without any cycles is called *acylic*.

Two cycles c and c' are called to be *equivalent*, denoted by $c \sim c'$, if c arises from c' by a cyclic permutation of the vertices and edges of c', that means there are paths p, q in $\mathrm{Path}(E)$ such that $c = pq$ and $c' = qp$.

We say that a graph E satisfies *Condition* (L) if every cycle in E has an exit, a graph E satisfies *Condition* (K) if every vertex that is the base of a closed path c is also the base of another closed path c' different from c. A cycle c in a graph E is called a *cycle without* (K) if no vertex on c is the base of another cycle c' different from c. We write $u \geq v$ if there exists a path p from vertex u to a vertex v (i.e., $s(p) = u, r(p) = v$). For any vertex v, we define $T(v) := \{w \in E^0 : v \geq w\}$ and $M(v) := \{w \in E^0 : w \geq v\}$. A subset M of E^0 is called *downward directed* if it satisfies the following (MT-3) property:

(MT-3) for any $u, v \in M$, there exists $w \in M$ such that $u \geq w$ and $v \geq w$.

A subset H of E^0 is called *hereditary* if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$; a subset H of E^0 is called *saturated* if, for any regular vertex $v \in E^0$, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$. The set of all hereditary saturated subsets of E^0 is denoted by $\mathcal{H}_E$.

Let $(E^1)^*$ denote the set of formal symbols $\{e^* : e \in E^1\}$. Then, the elements of E^1 are called *read edges*, and the elements of $(E^1)^*$ are called *ghost edges*. For a path $p = e_1 \dots e_n \in \mathrm{Path}(E)$, we define the ghost path of p by $p^* := e_n^* \dots e_1^*$. Note that $v^* = v$ for all $v \in E^0$.

Let E be a graph and R a ring. A Leavitt E-family is a set $\{v, e, e^* : v \in E^0, e \in E^1\} \subseteq R$ such that the following conditions are satisfied:

(A1) $u\,v = \delta_{uv} u$ for all $u, v \in E^0$;

(A2) $s(e)e = er(e) = e$ and $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$;

(CK1) $e^* f = \delta_{ef} r(e)$ for all $e, f \in E^1$;

(CK2) $v = \sum_{e \in s^{-1}(v)} ee^*$ for every regular vertex $v \in E^0$.

The *Leavitt path algebra of* E *with coefficients in* R, denoted by $L_R(E)$, is defined as the universal R-algebra generated by a Leavitt E-family.

The universal property of $L_R(E)$ means that if A is an R-algebra and $\{a_v, b_e, b_{e^*} : v \in E^0, e \in E^1\}$ is a Leavitt E-family in A, then there exists an R-algebra homomorphism $\phi : L_R(E) \to A$ such that $\phi(v) = a_v, \phi(e) = b_e, \phi(e^*) = b_{e^*}$ for all $v \in E^0$ and $e \in E^1$.

By [10, Proposition 3.4], we see that

$$L_R(E) = \mathrm{span}_R\{pq^* : p, q \in \mathrm{Path}(E), r(p) = r(q)\}$$

and $\lambda v \neq 0$ for all $v \in E^0$ and $\lambda \in R \setminus \{0\}$. This implies that $\lambda pq^* \neq 0$ for all $\lambda \in R \setminus \{0\}$ and $p, q \in \mathrm{Path}(E)$ with $r(p) = r(q)$.

By [10, Proposition 4.7], every Leavitt path algebra $L_R(E)$ is a $\mathbb{Z}$-graded algebra by setting

$$L_R(E)_n := \Big\{\sum_i \lambda_i p_i q_i^* : \ \lambda_i \in R; p_i, q_i \in \mathrm{Path}(E), \text{ and } |p_i| - |q_i| = n \text{ for all } i\Big\}.$$

An ideal I of $L_R(E)$ is said to be a *graded ideal* if $I = \bigoplus_{n\in\mathbb{Z}} (I \cap L_R(E)_n)$.

3. Generators of ideals of $L_R(E)$

Definition 3.1. [6, Definition 3.5] Let E be a graph, R a unital commutative ring, and I an ideal of $L_R(E)$.

i) The ideal I is called *basic* if $\lambda x \in I$ implies $x \in I$, where $\lambda \in R \setminus \{0\}$ and either $x \in E^0$ or x is of the form $v - \sum_{i=1}^{n} e_i e_i^*$ for $v \in E^0$ and $e_i \in s^{-1}(v)$, $(1 \le i \le n)$.

ii) The basic ideal I is called a *maximal basic ideal* of $L_R(E)$ if there are no other basic ideals contained between I and $L_R(E)$.

Note that when E is a row-finite graph, the above definition of basic ideal is equivalent to [10, Definition 7.2] (This is obtained by comparing [6, Theorem 3.10 (4)] with [10, Theorem 7.9 (1)]). Also, suppose that I is a basic ideal of a Leavitt path algebra $L_R(E)$, and let $H = I \cap E^0$. If $\lambda v^H \in I$ for some $v \in B_H$ and $\lambda \in R \setminus \{0\}$, then we have $v^H \in I$.

The generating set for any ideal I of a Leavitt path algebra with coefficients in a field was described in [2, Theorem 2.1]. In the case of coefficients in a unital commutative ring, we have the following result. The idea of the proof is the same as in the proof of [2, Themrem 2.1]. However, we also restate it in a more logical way.

Theorem 3.2. *Let E be a countable graph, R a unital commutative ring, and I a nonzero basic ideal of $L_R(E)$. Then, there exists a generating set for I consisting of elements of the form*

$$\Big(\lambda_1 v + \sum_{i=2}^{n} \lambda_i c^{r_i}\Big)\Big(v - \sum_{e\in S} ee^*\Big),$$

where $v \in E^0$, $\lambda_1, \ldots, \lambda_n \in R$, $r_1, \ldots, r_n$ are positive integers, S is a finite set (possibly empty), $S \subsetneq s^{-1}(v)$, $\lambda_1 \neq 0$, and, whenever $\lambda_i \neq 0$ for some $2 \le i \le n$, c is the unique cycle based at v.

Proof. Put $v_S := v - \sum_{e\in S} ee^*$ and let J the ideal of $L_R(E)$ generated by all the elements of I which have the form described in the statement of the theorem, we show that $I = J$.

It is clear that $J \subseteq I$. Conversely, for any $u \in I \cap E^0$, by choosing $\lambda_1 = 1$, $\lambda_i = 0$ and $S = \emptyset$, we get $u \in J$. It follows that $I \cap E^0 \subseteq J$.

Case 1: There exists an element x of $I \setminus J$ of the form

$$x = \big(\lambda_1 p_1 + \cdots + \lambda_n p_n\big).v_S,$$

where $p_1, \ldots, p_n \in \mathrm{Path}(E), 0 \le l(p_1) \le \ldots \le l(p_n)$ and n is minimal. Among all such x, select one for which $(l(p_1), \ldots, l(p_n))$ is smallest in the lexicographic order.

Then, $\lambda_i \ne 0$ and $r(p_i) = v$ for all $i, 1 \le i \le n$. Let $s(x) = \{s(p_i) \mid 1 \le i \le n\}$. Then, $wx \in I$ for any $w \in s(x)$. But $x = \sum_{w \in s(x)} wx$ and $x \notin J$, it gives $wx \notin J$ for some $w \in s(x)$. By replacing x by wx if necessary, we may assume that $s(p_i) = w$ for all i.

Subcase 1.1: $l(p_1) > 0$.

Let $p_i = f_i.q_i$, where $f_i \in E^1$ and $q_i \in Path(E)$, then

$$f_i^* x = (\lambda_1 f_i q_1 + \lambda_2 f_i^* p_2 + \ldots + \lambda_n f_i^* p_n) v_S.$$

Note that $f_i^* x \in I$ and $f_i^* p_j$ is either 0 or belongs to $\mathrm{Path}(E)$, so that $f_i^* x \in J$ by the minimality of the lengths of p_i. It follows that $f_i f_i^* x \in J$ for any $1 \le i \le n$. Therefore

$$x = \sum_{f_i \in A} f_i f_i^* x \in J, \text{ where } A = \{f_i \mid f_i^* p_i \ne 0, 1 \le i \le n\},$$

a contradiction.

Subcase 1.2: $l(p_1) = 0$ and there exists $f \in S$ such that $f^* p_i \ne 0$ for some $2 \le i \le n$.

By $l(p_1) = 0$, we get $p_1 = w = v$ and $l(p_i) > 0$ for $2 \le i \le n$. Since $f \in S$, we get $ff^*.v_S = 0$, so we have

$$ff^*x = ff^*.v_S + \big(\lambda_2 ff^* p_2 + \cdots + \lambda_n ff^* p_n\big).v_S = \big(\lambda_2 ff^* p_2 + \cdots + \lambda_n ff^* p_n\big).v_S.$$

Note that $ff^*x \in I$ and $ff^* p_i$ is either 0 or belongs to $\mathrm{Path}(E)$, so that $ff^*x \in J$ by the minimality of n. Furthermore, $f^* p_i \ne 0$ for some $2 \le i \le n$ yields that $ff^* p_i = p_i$. Therefore

$$x - ff^*x = \big(\lambda_1 v + \sum_{i; f^* p_i = 0} \lambda_i p_i\big).v_S \in J,$$

by the minimality of n. So we have $x = ff^*xx + (x - ff^*x) \in J$, a contradiction.

Subcase 1.3: $l(p_1) = 0$, $e^* p_i = 0$ for all $e \in S$ and $2 \le i \le n$.

Note that $w = p_1 = v$, and $x \notin J$, so we have $n \geq 2$ and there are two closed simple path $c \neq c'$ based at v such that $p_i = c^{m_i}.c'.q_i$ for some $q_i \in \text{Path}(E)$ and for some i. Pick an integer m for which $l(c^m) > l(p_n)$ and let $y := (c^m)^*.x.c^m$, we get $y \in I$; and by $e^*c = 0$ for all $e \in S$, this yields that $v_S.c = c$. Therefore

$$y = \lambda_1 v + \lambda_2 (c^m)^*.p_2.c^m + \cdots + (c^m)^*.p_n.c^m \in R[c],$$

where $R[t]$ is the polynomial ring over commutative ring R. By $c \neq c'$ and $y \in I$, we get $\lambda_1 v = (c')^*.y.c' \in I$. Since I is a basic ideal of $L_R(E)$, it follows that $v \in I$. But $I \cap E^0 \subseteq J$, so $v \in J$, so that $x = vx \in J$, a contradiction.

Case 2: There exists an element x of $I \setminus J$ of the form

$$x = \big(\lambda_1 p_1 q_1^* + \cdots + \lambda_n p_n q_n^*\big).v_S,$$

where $p_1, q_1, \ldots, p_n, q_n \in \text{Path}(E), 0 \leq l(q_1) \leq \ldots \leq l(q_n)$ and n is minimal. Among all such x, select one for which $(l(q_1), \ldots, l(q_n))$ is smallest in the lexicographic order.

Then, $\lambda_i \neq 0$ for all $i, 1 \leq i \leq n$; $s(p_1) = \ldots = s(p_n)$; $s(q_1) = \ldots = s(q_n) = v$.

Subcase 2.1: $l(q_i) \geq 1$ for all i, $1 \leq i \leq n$.

Then, we can write $q_i = e_i q_i'$, where $e_i \in E^1, q_i' \in \text{Path}(E)$.

If there exists an element $e_i \in S$, then

$$q_i^*.v_S = {q_i'}^* e_i^*.v_S = {q_i'}^*(e_i^* - e_i^*) = 0,$$

so we can remove $\lambda_i p_i q_i^*$ in the expression of x, we get a contradiction with the minimality of n. Therefore $e_i \notin S$ for all i, $1 \leq i \leq n$.

For any $f \in s^{-1}(v) \setminus S$, we have $v_S.f = f$. Therefore

$$xf = \big(\lambda_1 p_1 q_1^* + \cdots + \lambda_n p_n q_n^*\big)f = \sum_{e_i = f} \lambda_i p_i {q_i'}^*.$$

But $xf \in I$, $f = f.r(f) = f\Big(r(f) - \sum\limits_{e \in \emptyset} ee^*\Big)$, and the above expression of xf is equal to zero or is the form of $l(q_i') < l(q_i)$, so we have $xf \in J$. It follows that $xff^* \in J$ for all $f \in s^{-1}(v) \setminus S$.

On the other hand, for any $f \in S$,

$$v_S.ff^* = ff^* - ff^* = 0,$$

so $xff^* = 0$. Then

$$x = xv = x\Big(\sum_{f \in S} ff^* + \sum_{f \in s^{-1}(v) \setminus S} ff^*\Big) = \sum_{f \in s^{-1}(v) \setminus S} xff^* \in F,$$

a contradiction.

Subcase 2.2: $l(q_i) = 0$ for some i, $1 \le i \le n$.

Suppose that $l(q_1) = \ldots = l(q_k) = 0$ and $l(q_i) > 0$ for all $i > k$. Then,

$$x = \big(\lambda_1 p_1 + \cdots + \lambda_k p_k + \lambda_{k+1} p_{k+1} q_{k+1}^* + \cdots + \lambda_n p_n q_n^*\big).v_S.$$

We write $q_i = e_i q_i'$ for $k+1 \le i \le n$ and let $T = \{e_{k+1}, \ldots, e_n\}$. Then, $T \subseteq s^{-1}(v)$. According to the minimality of n and according to the above case, we can assume that $T \cap S = \emptyset$ and $xf \in J$ for all $f \in T$. We have again that

$$q_i^*\Big(v - \sum_{f\in T} ff^*\Big) = q_i'^*(e_i^* - e_i^*) = 0.$$

Therefore,

$$x\Big(v - \sum_{f\in T} ff^*\Big) = (\lambda_1 p_1 + \ldots + \lambda_k p_k)\Big(v - \sum_{f\in T} ff^*\Big).$$

The right side of the above equation has the same form as Case 1, so

$$x\Big(v - \sum_{f\in T} ff^*\Big) \in J.$$

Therefore,

$$x = x\Big(v - \sum_{f\in T} ff^*\Big) + \sum_{f\in T}(xf)f^* \in J,$$

again a contradiction. □

For R is an integral domain and a graph E, the following proposition is an extension of [9, Proposition 2] shows that nonzero basic ideals of $L_R(E)$ containing no vertices are generated by a set of mutually orthogonal polynomials over cycles.

Proposition 3.3. *Let E be a graph, R an integral domain, and N a nonzero basic ideal of $L_R(E)$ which does not contain any vertices of E. Then, N is a nongraded ideal and possesses a generating set of pair-wise mutually orthogonal generator of the form*

$$y = \lambda u + \sum_{i=2}^{n} \lambda_i c^{r_i},$$

where $\lambda, \lambda_i \in R$, $r_i \in \mathbb{N}$, c is a unique cycle without exits based at a vertex $u \in E^0$, $\lambda \neq 0$ and at least one $\lambda_i \neq 0$.

Proof. Let $H = N \cap E^0$ and $S = \{v \in B_H \mid v^H \in N\}$. Since N does not contain any vertices of E, H and S are both empty sets. If N is a graded ideal of $L_R(E)$, then by [1, Theorem 2.4.8, p.42], $N = I(H; S)$, must then be $\{0\}$, a contradiction. Thus, N is a nongraded ideal.

By Theorem 3.2, N is generated by elements of the form

$$y := \Big(\lambda u + \sum_{i=2}^{n} \lambda_i c^{r_i}\Big)\Big(u - \sum_{e\in S} ee^*\Big),$$

where c is a unique cycle without exits based at a vertex $u \in E^0$, $S \subsetneq s^{-1}(u)$, and $\lambda, \lambda_j \in R$, with $\lambda \neq 0$ and at least one $\lambda_j \neq 0$. Since $S \subsetneq s^{-1}(u)$, there is an $f \in s^{-1}(u) \setminus S$. Let $w = r(f)$.

Suppose that f not be the initial edge of c. Then $f^*c = 0$, so $\lambda w = f^*yf \in N$. By N is a basic ideal of $L_R(E)$, it follows that $w \in N \cap E^0$, a contradiction. Thus, f is the initial edge of c. So, there exists a path $\alpha \in \text{Path}(E)$ such that $c = f\alpha$. Let $c' = \alpha f$, we obtain that c' is a cycle based at w, and

$$f^*yf = \lambda w + \sum_{i=2}^{n} \lambda_i (c')^{r_i} \in N.$$

Therefore,

$$c^*yc = \alpha^*(f^*yf)\alpha = \lambda u + \sum_{i=2}^{n} \lambda_i c^{r_i} \in N.$$

If c has an exit at a vertex $v \in E^0$, then there exist $e_1, e_2 \in s^{-1}(v)$ and $\alpha, \beta \in \text{Path}(E)$ such that $e_1 \neq e_2$ and $c = \alpha e_1 \beta$. Then, $e_2^*\alpha^* y \alpha e_2 = \lambda r(e_2) \in N$. By N is a basic ideal of $L_R(E)$, it follows that $r(e_2) \in N$, a contradiction. Then, the cycle c has no exit. In particular, $|s^{-1}(u)| = 1$. Since $S \subsetneq s^{-1}(u)$, this implies that S must be the empty set. Thus,

$$y = \lambda u + \sum_{i=2}^{n} \lambda_i c^{r_i}.$$

If there is another generator of N of the form $y' = \lambda' u + \sum_{i=2}^{n'} \lambda_i' (c')^{s_i}$ with the same vertex u, then, by the uniqueness of c, $c' = c$ and so

$$y' = \lambda' u + \sum_{i=2}^{n'} \lambda_i' c^{s_i}.$$

For a given vertex u and the unique cycle c based at u, let $d(x) \in R[x]$ be the polynomial of the smallest degree with $d(0) \neq 0$ and $d(c) \in N$ (note that $c^0 = u$). By the Polynomial Pseudo-Division Theorem ([7, Theorem 1.3.6, p.19]), any other polynomial $f(x) \in R[x]$ satisfying $f(0) \neq 0$ and $f(c) \in N$, there exist polynomials $g(x), r(x) \in R[x]$ such that

$$\lambda f(x) = g(x).d(x) + r(x),$$

where $\deg(r) < \deg(d)$ and $\lambda \neq 0$ is a power of the leading coefficient of $d(x)$. Then

$$r(c) = \lambda f(c) - g(c)d(c) \in N.$$

By the smallest degree of d, we have $r(x) = 0$. So all those generators $f(c)$ of N involving the same vertex u can be replaced by $d(c)$. Moreover, if $d_1(c')$ is in the other generating set for N such that $c' \sim c$ and $d_1(x) \in R[x]$ be the polynomial of the smallest degree with $d_1(0) \neq 0$ and $d_1(c') \in N$, then there exist some paths p, q such that $c = pq$, $c' = qp$. Therefore, $q^* d_1(c') q = d_1(c)$ and $d_1(c)$ belongs to $\langle d(c) \rangle$ by the minimality of $d(x)$, so we can remove $d_1(c')$ from the generating set for N.

By replacing/removing the generators for N (when necessary), we can get the set of generators for N of the form

$$y_i = \lambda_i u_i + \sum_{j=2}^{n_i} \lambda_{ij} c_i^{r_{ij}},$$

where $\lambda_i, \lambda_{ij} \in R$, c_i is a unique cycle without exits based at a vertex $u_i \in E^0$, $c_i \not\sim c_k$ for $i \neq k$ (and in particular, $u_i \neq u_k$), and for each i, $\lambda_i \neq 0$ and at least one $\lambda_{ij} \neq 0$. Then clearly $y_i y_k = 0 = y_k y_i$ for $i \neq k$. The proof of the theorem is now complete. □

Corollary 3.4. *Let R be an integral domain and E a graph satisfies Condition (L). If I is a nonzero basic ideal of $L_R(E)$, then $I \cap E^0 \neq \emptyset$.*

Proof. Since Condition (L) on a graph E requires that cycles in E have exits, the result follows immediately from Proposition 3.3. □

The following theorem is an extension of [9, Theorem 4] in which R is an integral domain.

Theorem 3.5. *Let E be a graph, R an integral domain. If I is a non-zero basic ideal of $L_R(E)$ with $I \cap E^0 = H$ and $S = \{v \in B_H \mid v^H \in I\}$, then I is generated by $H \cup S^H \cup Y$, where Y is a set of mutually orthogonal elements of the form $\lambda u + \sum_{j=2}^{n} \lambda_i c^{r_i}$ in which c is a unique cycle with no exits in $E^0 \setminus H$ based at a vertex u in $E^0 \setminus H$, $\lambda, \lambda_i \in R$ with $\lambda \neq 0$ and at least one $\lambda_i \neq 0$.*

Proof. Let $J = I(H, S)$ be the ideal of $L_R(E)$ generated by H and $S^H := \{v^H \mid v \in S\}$. Then, $J \subseteq I$. For $J = I$ there is nothing to prove, so we may assume that $J \subsetneq I$. Identifying $L_R(E)/J$ with $L_R(E/(H, S))$ via the isomorphism $L_R(E)/J \cong L_R(E/(H, S))$ (see [6, Theorem 3.10]), we note that the non-zero ideal I/J contains no vertices of $L_R(E/(H, S))$, so by Proposition 3.3, I/J is generated by mutually orthogonal elements of the form $y = \lambda u + \sum_{j=2}^{n} \lambda_j c^{r_j}$, where c is a unique cycle without exits based at a vertex u in $(E/(H, S))^0$, and $\lambda, \lambda_j \in R$ with $\lambda \neq 0$ and at least one $\lambda_j \neq 0$. Observe that

$$(E/(H, S))^0 = E^0 \setminus H \cup \{v' \mid v \in B_H \setminus S\}$$

and that the vertices $v' \in (E/(H,S))^0$ are all sinks, so both u and the vertices on c all belong to $E^0 \setminus H$. Therefore, the ideal I is generated by J and a set Y of mutually orthogonal elements of the form $y = \lambda u + \sum_{j=2}^{n} \lambda_j c^{r_j}$, where c is a unique cycle without exits in $E^0 \setminus H$, based at $u \in E^0 \setminus H$, and $\lambda, \lambda_j \in R$ with $\lambda \neq 0$ and at least one $\lambda_j \neq 0$. □

4. Prime ideals of $L_R(E)$

Prime ideals of Leavitt path algebras with field coefficients were studied in [8] and the characterization of both graded and nongraded prime ideals were given. In the case of coefficients in a unital commutative ring, due to [8, Theorem 3.12], we give the necessary and sufficient conditions for the primeness of a basic ideal of $L_R(E)$ in both graded and nongraded cases.

For the case of a graded basic ideal, we first prove the following lemma.

Lemma 4.1. *Let E be a graph, R a unital commutative ring. If P is an ideal of $L_R(E)$, $H = P \cap E^0$, $S = \{v \in B_H \mid v^H \in P\}$, then the ideal $I(H,S)$ is the graded basic ideal of $L_R(E)$ contains every other graded basic ideal of $L_R(E)$ inside P.*

Proof. Suppose A is a graded basic ideal of $L_R(E)$. By [6, Theorem 3.10 (4)], there is an admissible pair (H_1, S_1) such that $A = I(H_1, S_1)$ and $A \subseteq P$. Then $A \cap E^0 \subseteq P \cap E^0 = H$, so that $H_1 \subseteq H \subseteq I(H,S)$.

For $v \in S_1$, we have v is a breaking vertex for H_1, which means $v \in B_{H_1}$, so we have $0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H_1)| < \infty$. By re-indexing, we may then assume that

$$s^{-1}(v) \cap r^{-1}(E^0 \setminus H_1) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\},$$

where $r(e_i) \notin H$ for $i \leq m$, and $r(e_i) \in H$ for $i > m$. It follows that

$$v^{H_1} = v - \sum_{i=1}^{n} e_i e_i^* = v^H - \sum_{j=m+1}^{n} e_j.r(e_j).e_j^*,\ r(e_j) \in H \text{ for all } j > m.$$

If $m = n$ then clearly $v^{H_1} = v^H \in I(H,S)$; if $m < n$ then by $r(e_j) \in H$ for all $j > m$, we can get $e_j = e_j.r(e_j) \in I(H,S)$ for all $j > m$, we can also get $v^{H_1} \in I(H,S)$. In both cases, we always have $A \subseteq I(H,S)$. □

It is now the necessary and sufficient conditions for the primeness of a basic ideal of $L_R(E)$ in the graded case.

Theorem 4.2. *Let E be a graph, R a unital commutative ring, and P a basic ideal of $L_R(E)$ with $P \cap E^0 = H$. Then, P is a graded prime ideal of $L_R(E)$ if and only if R is an integral domain and P satisfies one of the following conditions:*

i) *$P = I(H, B_H)$, and $E^0 \setminus H$ is downward directed;*

ii) $P = I(H, B_H \setminus \{u\})$ *for some* $u \in B_H$ *and* $M(u) = E^0 \setminus H$.

Proof. Let $H = P \cap E^0$, $S = \{w \in B_H : w^H \in P\}$ and $F = E/(H, S)$.

($\Rightarrow$) Suppose P is a graded prime ideal of $L_R(E)$. Then by Lemma 4.1, $P = I(H, S)$. Therefore

$$L_R(E)/P \cong L_R(E)/I(H, S) \cong L_R(F)$$

is a prime ring (By [6, Theorem 3.10 (3)]). Proposition 4.5 in [6] implies that R is an integral domain and F^0 is downward directed.

Let $\text{Sink}(F)$ be the set of sinks in F. If $|\text{Sink}(F)| \geq 2$, then there are $u, v \in \text{Sink}(F), u \neq v$. Since F^0 is downward directed, there exists $y \in F^0$ such that $u \geq y$ and $v \geq y$. Now, both u and v are sinks, so we have $u = y$ and $v = y$. It implies that $u = v$, a contradiction. Thus, $|\text{Sink}(F)| \leq 1$.

If there exists $v' \in F^0$ such that $v' \in B_H \setminus S$, then for all $\alpha \in F^1$ satisfies $s_F(\alpha) \in E^0$, we have $s_F(\alpha) \neq v'$. Hence $v' \in \text{Sink}(F)$. Therefore $B_H \setminus S \subseteq \text{Sink}(F)$. Thus $|B_H \setminus S| \leq 1$.

i) If $B_H \setminus S = \emptyset$, then $S = B_H$, so $E^0 \setminus H = F^0$. Therefore $E^0 \setminus H$ is downward directed;

ii) If $B_H \setminus S = \{u\}$, then $B_H = S \cup \{u\}$, so $u \in B_H$ and $F^0 = (E^0 \setminus H) \cup \{u'\}$.

Clearly, if $v \in M(u)$ then $v \geq u$ and $u \in B_H$, so $v \notin H$, that is $v \in E^0 \setminus H$. Conversely, for all $v \in E^0 \setminus H$, there exists $y \in F^0$ such that $v \geq y$ and $u' \geq y$. By $u' \in \text{Sink}(F)$, it implies $y = u'$, so there is a path $p = p_1 \dots p_n \in \text{Path}(F)$ such that $s_F(p) = v, r_F(p) = u'$. Since $r_F(p_n) = u'$, it follows that $p_n = e'_n$, where $r_E(e_n) \in B_H \setminus (B_H \setminus \{u\}) = \{u\}$, that is $r_E(e_n) = u$. By $r_F(p_{n-1}) = s_F(p_n) \in E^0 \setminus H$, we get $p_{n-1} \in E^0$. By induction, we have $p_1 \dots p_{n-1} \in \text{Path}(E)$, so $(p_1 \dots p_{n-1})e_n \in \text{Path}(E)$, that is $v \geq_E u$. Therefore $v \in M(u)$. Thus, $E^0 \setminus H = M(u)$.

($\Leftarrow$) Since Lemma 4.1, in the both cases we always get P is a graded ideal.

i) If $P = I(H, B_H)$ then $(E/(H, B_H))^0 = E^0 \setminus H$, so $(E/(H, B_H))^0$ is downward directed. Therefore

$$L_R(E/(H; B_H)) \cong L_R(E)/I(H; B_H) = L_R(E)/P$$

is a prime ring. It follows that P is a prime ideal.

ii) If $P = I(H, B_H \setminus \{u\})$ then

$$(E/(H, B_H \setminus \{u\}))^0 = (E^0 \setminus H) \cup \{u'\}.$$

For all $v \in (E/(H, B_H \setminus \{u'\}))^0$, we have $v = u'$ or $v \in E^0 \setminus H = M(u)$, that is $v \geq_E u$, so there is a path $p = p_1 \dots p_n \in \mathrm{Path}(E)$ such that $s(p) = v, r(p) = u$. By replacing the edge p_n in E by the edge p'_n in $E/(H; S)$, we obtain

$$q := p_1 \dots p_{n-1}p'_n \in \mathrm{Path}(E/(H; S)) \text{ and } r_{E/(H;S)}(q) = u'.$$

It implies that $v \geq u'$. Similarly, $w \geq u'$. Therefore $(E/(H; B_H \setminus \{u\}))^0$ is downward directed. Thus

$$L_R(E/(H; B_H \setminus \{u\})) \cong L_R(E)/I(H; B_H \setminus \{u\}) = L_R(E)/P$$

is a prime ring. Therefore P is a prime ideal. □

Corollary 4.3. *Let E be a graph and R a unital commutative ring. If $P = I(H, B_H)$ is a maximal ideal of $L_R(E)$, then $E^0 \setminus H$ is downward directed.*

Proof. If $P = I(H, B_H)$ is a maximal ideal of $L_R(E)$, then P is a graded prime ideal of $L_R(E)$. The result now follows from Theorem 4.2. □

Lemma 4.4. *Let E be a graph, R a unital commutative ring, and P a prime basic ideal of $L_R(E)$, $H = P \cap E^0$, $S = \{v \in B_H : v^H \in P\}$. Then the ideal $I(H, S)$ is also a prime basic ideal of $L_R(E)$.*

Proof. Suppose that A, B are two graded basic ideals of $L_R(E)$ with $AB \subseteq I(H, S)$. Since $AB \subseteq P$ and P is prime, it follows that either $A \subseteq P$ or $B \subseteq P$. By Lemma 4.1, we obtain $A \subseteq I(H, S)$ or $B \subseteq I(H, S)$. Therefore $I(H, S)$ is a prime basic ideal of $L_R(E)$. □

Lemma 4.5. *Let R be an integral domain, E a graph such that E^0 is downward directed. If N is a nonzero basic ideal of $L_R(E)$ which does not contain any vertices of E, then there is a unique cycle c without exits in E and N is a non-graded principal ideal generated by $p(c)$, where $p(x) \in R[x]$.*

Proof. By Proposition 3.3, there is a cycle c without exits (based at a vertex $u \in E^0$) and a polynomial $p(x) \in R[x]$ of the smallest degree such that

$$p(c) := \lambda u + \sum_{i=2}^{n} \lambda_i c^{r_i} \in N.$$

Let $y \in N$, then there exists a cycle c' without exits (based at $w \in E^0$) such that

$$y = \lambda' w + \sum_{i=2}^{m} \lambda'_i (c')^{r_i}.$$

Since E^0 is downward directed, there is a vertex $v \in E^0$ such that $u \geq v$ and $w \geq v$. By c and c' have no exit, we obtain $v \in c^0 \cap (c')^0$, $c = \alpha.w.\beta$, and $c' = \beta.u.\alpha$ for some $\alpha, \beta \in \text{Path}(E)$. Then,

$$\beta^* y \beta = \lambda' u + \sum_{i=2}^{m} \lambda'_i c^{r_i} \in N.$$

Let $f(x) = \lambda' + \sum_{i=2}^{m} \lambda'_i x^{r_i}$. By the Polynomial Pseudo-Division Theorem ([7, Theorem 1.3.6, p.19]), there exist polynomials $q(x), r(x) \in R[x]$ such that

$$\delta.f(x) = p(x).q(x) + r(x),$$

where $0 \leq \deg r(x) < \deg d(x)$ and $\delta \neq 0$ is a power of the leading coefficient of $p(x)$.

Then, $r(c) = \delta.f(c) - p(c)q(c) \in N$. By the minimality of $p(x)$, we get $r(x) = 0$. It follows that

$$f(c) = p(c)q(c) \in \langle p(c) \rangle.$$

Therefore $y = \alpha^*.f(c).\alpha \in \langle p(c) \rangle$, we then conclude that $N = \langle p(c) \rangle$. □

Recall that a ring R is prime if the zero ideal $\{0\}$ is a prime ideal in R. It is known that a commutative ring is a prime ring if and only if it is an integral domain. The following is the necessary and sufficient conditions for the primeness of a basic ideal of $L_R(E)$ in the non-graded case.

Theorem 4.6. *Let E be a graph, R a unital commutative ring, and P a basic ideal of $L_R(E)$ with $P \cap E^0 = H$. Then, P is a non-graded prime ideal of $L_R(E)$ if and only if R is an integral domain and $P = I(H, B_H) + \langle f(c) \rangle$, where c is a cycle without (K) in E based at a vertex v, $M(v) = E^0 \setminus H$ and $f(x)$ is an irreducible polynomial in $R[x, x^{-1}]$.*

Proof. Let $H = P \cap E^0$, $S = \{w \in B_H : w^H \in P\}$ and $F = E/(H, S)$.

($\Rightarrow$) Suppose P is a non-graded prime ideal of $L_R(E)$. Then by Lemma 4.1, $I(H, S) \subsetneq P$. Since [6, Theorem 3.10(3)], there is an R-isomophism

$$\phi : L_R(E)/I(H, S) \to L_R(F).$$

Let $N = \phi(P/I(H, S))$. By P is a prime basic ideal of $L_R(E)$, and by Lemma 4.4, $I(H, S)$ is a graded prime basic ideal of $L_R(E)$. Since $L_R(E)/I(H; S) \cong L_R(F)$, it follows that $L_R(F)$ is a prime ring. By [6, Prop. 4.5], R is an integral domain and F^0 is downward directed. By an argument analogous to the proof of Theorem 4.2, we get $S = B_H$ or $S = B_H \setminus \{u\}$ for some $u \in B_H$ such that $E^0 \setminus H = M(u)$.

- If $S = B_H \setminus \{u\}$ and $E^0 \setminus H = M(u)$ then $F^0 = (E^0 \setminus H) \cup \{u'\}$ and $w \geq u'$ for all $w \in F^0$, where $u' = \phi(u^H + I(H,S))$. Note that $u \notin S$, so $u^H \notin P$, imply $u' \notin N$. If there is $v' \in N \cap F^0$ then $v' \neq u'$, so we have $v' \in (N \cap E^0) \cap (E^0 \setminus H)$. Therefore, there exists $v \in (P \cap E^0) \cap (E^0 \setminus H)$ such that $v' = \phi(v + I(H,S))$. But $(P \cap E^0) \cap (E^0 \setminus H) = H \cap (E^0 \setminus H) = \emptyset$. It follows that v doesn't exist, therefore N contains no vertices of F. By Lemma 4.5, there is a cycle c without exits in F, based at a vertex $v \in F^0$ such that N is the principal ideal generated by $p(c)$, where $p(x) \in R[x]$. But the cycle c without exits and $w \geq u'$ for all $w \in F^0$, so we get a contradiction. Therefore this case is impossible.

- If $S = B_H$ then $F^0 = E^0 \setminus H$, so we have

$$N \cap F^0 = (N \cap E^0) \cap (E^0 \setminus H) = H \cap (E^0 \setminus H) = \emptyset.$$

By Lemma 4.5, there is a cycle c without exits in F, based at a vertex $v \in F^0$ such that N is the principal ideal generated by $p(c)$, where $p(x) \in R[x]$. Clearly, c is a cycle without (K) and $P = I(H, B_H) + \langle f(c) \rangle$. For $w \in F^0$, by $v \in F^0$ and F^0 is downward directed, there is a vertex $w_1 \in F^0$ such that $w \geq w_1$ and $v \geq w_1$. Since the cycle c without exits and v is the base of c, we get $v = w_1$, that is $w \geq v$. It follows that $w \in M(v)$. Therefore $M(v) = F^0$. Since N is a prime ideal in $L_R(F)$, Proposition 10.2 in [5] now yields vNv is a prime ideal in $vL_R(F)v$, generated by $vf(c)v = f(c)$. It is easy to see that $vL_R(F)v \cong R[x, x^{-1}]$ with the isomorphism θ maps v to 1, c to x, and c^* to x^{-1}, it follows that θ maps $f(c)$ to $f(x)$. Since $f(x)$ is a generator of a prime ideal in the Euclidean domain $R[x, x^{-1}]$, $f(x)$ must then be an irreducible polynomial in $R[x, x^{-1}]$.

($\Leftarrow$) Suppose R is an integral domain, and $P = I(H, B_H) + \langle f(c) \rangle$, where

(a) c is a cycle without (K) in E based at a vertex v;

(b) $M(v) = E^0 \setminus H$; and

(c) $f(x)$ is an irreducible polynomial in $R[x, x^{-1}]$.

Now hypothesis (b) implies $F^0 = E^0 \setminus H = M(v)$. Therefore F is downward directed and contains the cycle c. As c is a cycle without (K) in E, the downward directed property implies that c has no exit in the graph F. By [6, Theorem 3.10 (3)], there is an R-isomorphism

$$\phi : L_R(E)/I(H,S) \to L_R(F).$$

Let $N = \phi(P/I(H,S))$, then, by the hypothesis (c), the ideal N is generated by $f(c)$. Since $vL_R(F)v \cong R[x, x^{-1}]$ with the isomorphism θ maps v to 1, c to x, and c^* to x^{-1}, it follows that θ maps $f(c)$ to $f(x)$. As the polynomial $f(x)$ is irreducible in $R[x, x^{-1}]$, the ideal vNv, being generated by $vf(c)v = f(c) = \theta^{-1}(f(x))$, is a maximal ideal of $vL_R(F)v$. Now, if A, B are two ideals of $L_R(F)$ such that $AB \subseteq N$, then $vAv.vBv \subseteq vABv \subseteq vNv$, so either vAv or vBv is included in vNv. Without loss of generality, we can assume that $vAv \subseteq vNv$. If there is $w \in vAv \cap F^0$, then $w \geq v$, imply that $v \in A$. But then $vL_R(F)v \subseteq A$, and so $vL_R(F)v \subseteq vAv \subseteq vNv$, this fact contradicts the maximality of vNv in $vL_R(F)v$. Thus vAv does not contain any vertices of F, hence by Lemma 4.5, A will be generated by a polynomial $q(c)$. Since

$$q(c) = vq(c)v \in vAv \subseteq vNv \subseteq N,$$

we conclude that $A \subseteq N$. Thus N is a prime ideal of $L_R(F)$. It follows that P is a prime ideal of $L_R(E)$. Now, if P is a graded ideal, then by $P \cap E^0 = H$, Lemma 4.1 yields $P = I(H, B_H)$, so we have $N = 0$, hence vNv must be 0. But vNv is generated by $f(c) \neq 0$, so we get a contradiction. Hence, P must be a non-graded prime ideal of $L_R(E)$. □

Recall that the Leavitt path algebra $L_R(E)$ is called *basically simple* if the only basic ideal of $L_R(E)$ are $\{0\}$ and $L_R(E)$. In [10, Theorem 7.20], it is shown the necessary and sufficient condition for the basically simplicity of $L_R(E)$ when E is row-finite. For E is a countable graph, we have the following.

Theorem 4.7. *Let E be a graph and R be a unital commutative ring. Then the Leavitt path algebra $L_R(E)$ is basically simple if and only if E satisfies the following conditions:*

i) $\mathcal{H}_E = \{\emptyset, E^0\}$;
ii) *The graph E satisfies Condition (L);*
iii) *R is a field.*

Proof. Suppose that $L_R(E)$ is basically simple. Then the only basic ideal of $L_R(E)$ are $\{0\}$ and $L_R(E)$, both of which are graded. By [6, Theorem 3.18] we have that E satisfies Condition (K). It then follows from [6, Theorem 3.10 (4)] and the fact that $L_R(E)$ is basically simple, that the only saturated hereditary subsets of E are $\emptyset$ and E^0. Hence $\mathcal{H}_E = \{\emptyset, E^0\}$ and the graph E satisfies Condition (L). Therefore, it is suffices to show that R is a field.

Suppose J is a nonzero ideal of R. Then $J.L_R(E)$ is an ideal of $L_R(E)$, so either

$$L_R(E).J = 0 \text{ or } L_R(E).J = L_R(E).$$

If $L_R(E).J = 0$, then for $0 \neq \lambda \in J$ and $v \in E^0$, $\lambda v \in L_R(E).J = 0$, imply $\lambda v = 0$, which contradicts Proposition 3.4 in [10]. Therefore $L_R(E).J = L_R(E)$.

For $\lambda \in R$ and $v \in E^0$, we have $\lambda v \in L_R(E) = J.L_R(E)$, hence there are $x \in L_R(E)$ and $\lambda' \in J$ such that $\lambda v = \lambda' x$. By [10, Proposition 4.7], $x \in L_R(E)_0$, that is $x = \lambda_1 v$, where $\lambda_1 \in R$. It implies that $(\lambda - \lambda'\lambda_1)v = 0$, so $\lambda = \lambda'\lambda_1 \in JR \subseteq J$. Thus $R = J$, hence R is a field.

The converse follows from ([1, Theorem 2.9.1, p.68]). □

5. Maximal basic ideals of $L_R(E)$

Recall that any ideal in a unital ring is contained in a maximal ideal, hence maximal ideals always exist. In [3], the author studied when maximal ideals exist and also the conditions on the graph E and the field K for which every ideal of $L_K(E)$ is contained in a maximal ideal. In this section, we discuss the necessary and sufficient conditions of the existence of maximal basic ideal of $L_R(E)$.

We begin with the two following lemmas.

Lemma 5.1. *Let R be an integral domain and E a graph. If $H \in \mathcal{H}_E$, then $I(H, B_H)$ is a maximal ideal in $L_R(E)$ if and only if H is a maximal element in $\mathcal{H}_E$ and the quotient graph $E/(H, B_H)$ satisfies Condition (L).*

Proof. Let $F = E/(H, B_H)$ and assume that $I(H, B_H)$ is a maximal ideal in $L_R(E)$, then $L_R(E)/I(H, B_H) \cong L_R(F)$ is a simple ring. By Theorem 4.7, R is a field, F satisfies Condition (L), and $\mathcal{H}_F = \{\emptyset, F^0\}$. If H is not a maximal element in $\mathcal{H}_E$, then there exists H' in $\mathcal{H}_E$ such that $H \subsetneq H'$. Then $I(H', B_{H'})/I(H, B_H)$ is a proper ideal of $L_R(E)/I(H, B_H)$, which contradicts the simplicity of $L_R(F)$.

Conversely, if H is a maximal element in $\mathcal{H}_E$ such that $F = E/(H, B_H)$ satisfies Condition (L), and there exists a proper ideal N of $L_R(E)$ containing $I(H, B_H)$, then by [6, Theorem 3.10 (4)] there exists an admissible pair (H_1, S_1) such that $\mathrm{gr}(N) = I(H_1, S_1)$, where $H_1 \in \mathcal{H}_E$ and $S_1 \subseteq B_{H_1}$. Hence,

$$I(H, B_H) \subseteq \mathrm{gr}(N) = I(H_1, S_1) \subseteq N.$$

By [1, Proposition 2.5.4, p.46], $H \subseteq H_1$ and $B_H \subseteq S_1 \cup H_1 \subseteq B_{H_1} \cup H_1$. Since H is maximal in $\mathcal{H}_E$, it follows that $H = H_1$, and so $B_H \subseteq S_1 \cup H \subseteq B_H \cup H$, implies $S_1 = B_H$. Hence, $I(H, B_H) = \mathrm{gr}(N)$. On the other hand, by Theorem 3.5, N is generated by $H \cup S^H \cup Y$, where Y is a set of mutually orthogonal elements of the form $\lambda u + \sum_{i=2}^{n} \lambda_i c^{r_i}$ in which c is a unique cycle without exits in $E^0 \setminus H$, based at a vertex u in $E^0 \setminus H$, $\lambda, \lambda_i \in R$ with $\lambda \neq 0$ and at least one $\lambda_i \neq 0$. As $F = E/(H, B_H)$ satisfies Condition (L), $Y = \emptyset$, that is $N = I(H, B_H)$. Hence, $I(H, B_H)$ is a maximal ideal in $L_R(E)$. □

Lemma 5.2. *Let R be an integral domain, E a graph, and H a maximal element in $\mathcal{H}_E$. Then $E/(H, B_H)$ not satisfying Condition (L) if and only if there is a maximal non-graded basic ideal M containing $I(H, B_H)$ with $H = M \cap E^0$.*

Proof. Suppose H is a maximal element in $\mathcal{H}_E$ and $F := E/(H, B_H)$ not satisfying Condition (L). Then, there exists a cycle c without exits in F, based at $u \in E^0 \setminus H$. Let N be an ideal of $L_R(E)$ generated by $H \cup S^H \cup Y$, where Y is a set of mutually orthogonal elements of the form $\lambda u + \sum_{i=2}^{n} \lambda_i c^{r_i}$ in which $\lambda, \lambda_i \in R$ with $\lambda \neq 0$ and at least one $\lambda_i \neq 0$. Then clearly N is a non-graded basic ideal of $L_R(E)$.

If M is a graded maximal basic ideal of $L_R(E)$ such that $N \subseteq M$, then there exists an admissible pair (H', S') such that $M = I(H', S')$. Since every maximal ideal in a Leavitt path algebra is prime, M is a graded prime ideal. By Theorem 4.2, $M = I(H', S')$ with either $S' = B_{H'}$ and $E^0 \setminus H'$ is downward directed or $S' = B_{H'} \setminus \{u\}$ with $u \in B_{H'}$ such that $M(u) = E^0 \setminus H'$. However, if the second case happens, then M can not be a maximal ideal, as $M \subsetneq I(H', B_{H'})$. Thus $M = I(H', B_{H'})$. Now, by Lemma 5.1, H' is a maximal element in $\mathcal{H}_E$ and $E \setminus (H', B_{H'})$ satisfies Condition (L). This contradicts the fact that H is a maximal element in $\mathcal{H}_E$ and $I(H, B_H) = \mathrm{gr}(N) \subseteq N \subsetneq I(H', B_{H'}) = M$. Thus, N is not contained in a maximal graded basic ideal of $L_R(E)$.

If N is maximal, then the result will come out. If not, then there exists an ideal N_1 such that $N_0 := N \subsetneq N_1 \subsetneq L_R(E)$. Continuing in this manner, we obtain a chain of proper ideals $\{N_i\}$ with

$$N = N_0 \subsetneq N_1 \subsetneq \ldots \subsetneq N_i \subsetneq N_{i+1} \subsetneq \ldots$$

If the chain is finite, then we are done; otherwise, by $I(H, B_H) = \mathrm{gr}(N) \subseteq \mathrm{gr}(N_i)$ for all i, and by H is maximal in $\mathcal{H}_E$, we conclude that $\mathrm{gr}(N_i) = I(H, B_H)$ for all i and N_i is generated by $I(H, B_I) \cup \langle f_i(c) \rangle$ for some polynomial $f_i(x) \in R[x]$. By the same argument of proving Theorem 4.6, this yields to a sequence of polynomials $f_i(x)$ with $f_0(x) = f(x)$ and $f_{i+1}(x) \mid f_i(x)$. As there are only finitely many factors of $\lambda + \sum_{i=2}^{n} \lambda_i x^{r_i}$, the sequence stabilizes at an irreducible polynomial $f(x)$ that divides $\lambda + \sum_{i=2}^{n} \lambda_i x^{r_i}$. Hence the ideal generated by $I(H, B_H) \cup \langle f(c) \rangle$ is a maximal non-graded basic ideal.

Conversely, if M is a non-graded maximal basic ideal of $L_R(E)$, then by Theorem 4.6, $M = I(H, B_H) + \langle f(c) \rangle$, where c is a cycle without (K), based at a vertex $u \in E^0$, $M(u) = E^0 \setminus H$ and $f(x)$ is an irreducible polynomial in $R[x, x^{-1}]$. If there exists an admissible pair (H', S') such that

$$\mathrm{gr}(M) = I(H, B_H) \subsetneq I(H', S'),$$

then $H \subsetneq H'$, hence there is a vertex v in $H' \setminus H$. Since $v \in M(u), u \geq v$. It implies that c and hence $f(c) \in I(H', S')$. By M is non-graded, $M \subsetneq I(H', S')$. By the maximality of M, we get $I(H', S') = L_R(E)$. Thus, $I(H, B_H)$ is a maximal among the graded basic ideal of $L_R(E)$. Now, if $H \subseteq H_1$ then $I(H, B_H) \subseteq I(H_1, B_{H_1})$, hence by the non-graded maximality of $I(H, B_H)$, we obtain $I(H, B_H) = I(H_1, B_{H_1})$. In particular, $H = H_1$, it yields H is maximal in $\mathcal{H}_E$. Finally, by Lemma 5.1, it is clear that $E \setminus (H, B_H)$ does not satisfy Condition (L). □

From Lemmas 5.1 and 5.2, we deduce that there is a maximal element in $\mathcal{H}_E$ if and only if there exists a maximal basic ideal in $L_R(E)$.

Theorem 5.3. *Let R be an integral domain, E a graph. Then, $L_R(E)$ has a maximal basic ideal if and only if $\mathcal{H}_E$ has a maximal element.*

Proof. Suppose $L_R(E)$ has a maximal basic ideal M. If M is a graded ideal then the result will come from Lemma 5.1; otherwise, the result will come from Lemma 5.2.

Conversely, suppose $\mathcal{H}_E$ has a maximal element H. If $E \setminus (H, B_H)$ satisfies Condition (L) the result will come from Lemma 5.1; otherwise, the result will come from Lemma 5.2. □

The following is the condition when every basic ideal of a Leavitt path algebra with coefficients in a unital commutative ring is contained in a maximal ideal.

Theorem 5.4. *Let R be an integral domain, E be a graph. Then the following are equivalent:*

i) *For every element $X \in \mathcal{H}_E$ there exists a maximal element Z in $\mathcal{H}_E$ such that $X \subseteq Z$.*
ii) *Every basic ideal in $L_R(E)$ is contained in a maximal basic ideal.*

Proof. By Lemmas 5.1, 5.2 and a similar argument as in the proof [3, Theorem 3.5], we obtain the result. □

Let E be a graph and R a unital commutative ring. Recall that for any ideal N of a graded ring $L_R(E)$, $\mathrm{gr}(N)$ denotes the largest graded ideal of $L_R(E)$ contained in N. It was proved in Lemma 4.1 that $\mathrm{gr}(N)$ is the ideal generated by the admissible pair (H, S), where $H = N \cap E^0$ and $S = \{v \in B_H \mid v^H \in N\}$. Note that if N is a maximal basic ideal of $L_R(E)$ which is a graded ideal, then clearly $N = \mathrm{gr}(N)$ is also maximal graded basic ideal of $L_R(E)$. We now discuss for a non-graded maximal basic ideal of $L_R(E)$.

Now we prove that a unique maximal basic ideal in $L_R(E)$ has to be a graded ideal.

Proposition 5.5. *Let R be an integral domain and E a graph. If $L_R(E)$ has a unique maximal basic ideal M, then M must be a graded ideal.*

Proof. Suppose M is a non-graded maximal basic ideal of $L_R(E)$. Then, M is prime. Let $H = M \cap E^0$, then by Theorem 4.6, $M = I(H, B_H) + \langle p(c) \rangle$, where c is a cycle without (K), based at a vertex u, $M(u) = E^0 \setminus H$ and $p(x)$ is an irreducible polynomial in $R[x, x^{-1}]$. If there exists an admissible pair (H', S') such that

$$\text{gr}(M) = I(H, B_H) \subsetneq I(H', S'),$$

then $H \subsetneq H'$, hence there is a vertex v in $H' \setminus H$. By $v \in M(u)$, $u \geq v$. It implies that c and hence $p(c) \in I(H', S')$. By M is non-graded, $M \subsetneq I(H', S')$. By the maximality of M, we get $I(H', S') = L_R(E)$. Thus, $I(H, B_H)$ is a maximal among the graded basic ideal of $L_R(E)$. Let $F = E \setminus (H, B_H)$, then $L_R(E)/I(H, B_H) \cong L_R(F)$ has no proper graded maximal basic ideal. So $F^0 = E^0 \setminus H$ has no proper nonempty hereditary saturated subsets and so no proper ideal of $L_R(F)$ contains any vertices. Moreover, since c is a cycle without exits based at $u \in E^0 \setminus H$ and $M(u) = E^0 \setminus H$ implies that c is the only cycle without exits in $E^0 \setminus H$. By Lemma 4.5, every proper ideal of $L_R(F)$ has the form $\langle f(c) \rangle$, where $f(x) \in R[x]$. Therefore, if $q(x) \in R[x]$ is an irreducible polynomial different from $p(x)$, then $\langle q(c) \rangle$ will be a maximal basic ideal of $L_R(F)$ different from $\langle p(c) \rangle$. Then $N = I(H, B_H) + \langle q(c) \rangle$ is a maximal basic ideal of $L_R(E)$ not equal M, this contradicts the uniqueness of M. Hence M must be a graded basic ideal of $L_R(E)$. □

Acknowledgements. The authors would like to thank the referee for the valuable suggestions and comments. This work was written while the first author was working at the Vietnam Institute for Advanced Study in Mathematics (VIASM), the author would like to thank the institute for providing a fruitful research environment and working condition.

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