TOPOLOGICAL ZETA FUNCTIONS OF COMPLEX PLANE CURVE SINGULARITIES

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Abstract. We study topological zeta functions of complex plane curve singularities using toric modifications and further developments. As applications of the research method, we prove that the topological zeta function is a topological invariant for complex plane curve singularities, we give a short and new proof of the monodromy conjecture for plane curves.

1. Introduction

Let f be a non-constant complex function on a smooth complex algebraic variety X, and let X_0 be its zero locus. In 1992, using an embedded resolution of singularities Denef and Loeser [5] introduced the *topological zeta function* for f. Let $h : Y \to (X, X_0)$ be an embedded resolution of singularities of X_0, i.e, a proper morphism $h : Y \to X$ with Y smooth such that the restriction $Y \setminus h^{-1}(X_0) \to X \setminus X_0$ is an isomorphism and $h^{-1}(X_0)$ is a divisor with normal crossings. The exceptional divisors and irreducible components of the strict transform of h are denoted by E_i, where i is in a finite set S. The multiplicities N_i of h^*f on E_i and the discrepancies $\nu_i - 1$ of the Jacobian of h are determined respectively in the formulas $h^{-1}(X_0) = \sum_{i\in S} N_iE_i$ and $K_Y = h^*K_X + \sum_{i\in S}(\nu_i - 1)E_i$. For $I \subseteq S$ we write E_I for the intersection $\bigcap_{i\in I} E_i$ and write E_I° for the set $E_I \setminus \bigcup_{j\notin I} E_j$. For a closed point x in X_0, we denote $S_x := \{i \in S \mid h(E_i) = x\}$. With the function f and the morphism h as above, the associated topological zeta function is defined as follows

$$Z_f^{\mathrm{top}}(s) = \sum_{I\subseteq S} \chi(E_I^\circ) \prod_{i\in I} \frac{1}{N_is + \nu_i}.$$

It was shown that the function $Z_f^{\mathrm{top}}(s)$ is independent of the choice of h (cf. [5, Théorème 3.2]), and its poles are interesting numerical invariants, which concern the monodromy conjecture. The *local* topological zeta function $Z_{f,x}^{\mathrm{top}}(s)$ associated to (f, x) is also defined in the same way where the sum over $I \subseteq S$ is replaced by the sum over $I \subseteq S$ satisfying $I \cap S_x \neq \emptyset$.

It is a fact that the monodromy conjecture is one of important problems in singularity theory, algebraic geometry and other branches of mathematics. In Igusa's original version, it is expected to be a bridge that connects geometry and arithmetic of a integer-coefficient polynomial. The topological version was first stated in [5] using the topological zeta function.

Conjecture 1.1 (Topological monodromy conjecture). *If θ is a pole of $Z_f^{\mathrm{top}}(s)$, then $\exp(2\pi i\theta)$ is an eigenvalue of the monodromy of (f, x) for some closed point x in X_0.*

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Up to now, the positiveness of the conjecture has been confirmed only in particular cases, and finding a proof for the general case is still a widely open problem. Any proof for this conjecture can motivate the development of several fields of mathematics.

In this article, we study the local topological zeta function for reduced complex plane curve singularities (f, O) which have no smooth irreducible components, as well as some related problems in a practical method using toric modifications. The first result, Theorem 3.10, describes explicitly $Z^{\mathrm{top}}_{f,O}(s)$ in terms of the simplified extended resolution graph $\mathbf{G}_{\mathrm{s}}$ of (f, O) defined in [7]. Namely,

$$Z^{\mathrm{top}}_{f,O}(s) = \sum_{\mathcal{B}} \left[\frac{b_1^{\mathcal{B}}}{(N(P^{\mathcal{B}}_{\mathrm{root}})s + \nu(P^{\mathcal{B}}_{\mathrm{root}}))(N(P^{\mathcal{B}}_1)s + \nu(P^{\mathcal{B}}_1))} + Z_{\mathcal{B}}(s) \right],$$

with the sum running over non-top bamboos $\mathcal{B}$ of $\mathbf{G}_{\mathrm{s}}$. Each vertex of a bamboo $\mathcal{B}$ is attached with a primitive vector $P^{\mathcal{B}}_i = (a^{\mathcal{B}}_i, b^{\mathcal{B}}_i)^t$, and if the vertex $P^{\mathcal{B}}_i$ of $\mathbf{G}_{\mathrm{s}}$ is of degree $r^{\mathcal{B}}_i + 1$, we define

$$Z_{\mathcal{B}}(s) = \sum_{i=1}^{k^{\mathcal{B}}} \left[\frac{\det(P^{\mathcal{B}}_i, P^{\mathcal{B}}_{i+1})}{(N(P^{\mathcal{B}}_i)s + \nu(P^{\mathcal{B}}_i))(N(P^{\mathcal{B}}_{i+1})s + \nu(P^{\mathcal{B}}_{i+1}))} - \frac{r^{\mathcal{B}}_i}{N(P^{\mathcal{B}}_i)s + \nu(P^{\mathcal{B}}_i)} \right].$$

Here, the numbers $N(P^{\mathcal{B}}_{\mathrm{root}})$, $\nu(P^{\mathcal{B}}_{\mathrm{root}})$, $N(P^{\mathcal{B}}_i)$ and $\nu(P^{\mathcal{B}}_i)$ concerning the resolution of singularities of (f, O) are also given in Theorem 3.10. Let $\mathcal{B}_0$ denote the first bamboo of $\mathbf{G}_{\mathrm{s}}$. The hypothesis on (f, O) mentioned above means that $a_i = a^{\mathcal{B}_0}_i \geq 2$, $b_i = b^{\mathcal{B}_0}_i \geq 2$ and $(a_i, b_i) = 1$ for all i. Remark that if $a_i = 1$ or $b_i = 1$ for some i, f becomes non-convenient via an analytic change of coordinates described in [6, Lemma 1.3]. In fact, our method also works well in this case, and the restriction of study to the case of reducedness and $a_i \geq 2$, $b_i \geq 2$ and $(a_i, b_i) = 1$ for all i is simply to simplify the notation. Indeed, if $a_i = 1$ for some i, we meet the so-called *exceptional integral* vector $(1, b_i)$ which corresponds to the lowest right end edge of the Newton boundary. In this situation, we add an additional weight vector $(1, b_i)^t + (0, 1)^t$, which is the new right end vertex. If $a_i = 1$ for some i, we may face to this situation several times in higher bamboos $\mathcal{B}$, i.e. $a^{\mathcal{B}}_j = 1$ for some j, while if $a_i \geq 2$, $b_i \geq 2$, $(a_i, b_i) = 1$ for all i, it then follows from [2] that $a^{\mathcal{B}}_j \geq 2$, $b^{\mathcal{B}}_j \geq 2$ and $(a^{\mathcal{B}}_j, b^{\mathcal{B}}_j) = 1$ for all bamboos $\mathcal{B}$ and all j.

As an application of Theorem 3.10, we prove that the local topological zeta function is a topological invariant for reduced complex plane curve singularities (Theorem 4.1). This is in fact not a trivial result because one finds in [3] an example of surface singularities with the same topological type but different local topological zeta functions.

As another application of Theorem 3.10, we revisit the works by Loeser [9] and Rodrigues [12] on the monodromy conjecture for curves with some new ideas. Namely, with the method computing $Z^{\mathrm{top}}_{f,O}(s)$ we prove Conjecture 1.1 for reduced complex plane curves (Theorem 4.2). This result was already made in [9] and [12], our contribution is just a new short proof in terms of an explicit performance of the poles of $Z^{\mathrm{top}}_{f,O}(s)$. We follow the track A'Campo and Oka in [2] and Lê in [7, 8] to reach the proof.

2. Nondegenerate complex plane curve singularities

2.1. **Toric modifications.** Let N be the 2-latice $\{(a, b)^t \mid a, b \in \mathbb{Z}\}$, and N^+ its positive subgroup $\{(a, b)^t \in N \mid a, b \geq 0\}$. We consider $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $N^+_{\mathbb{R}} = N^+ \otimes \mathbb{R}$. By definition, a *simplicial cone subdivision* Σ^* of $N^+_{\mathbb{R}}$ is a sequence $(T_1, \ldots, T_m)$ of primitive weight vectors in N^+ such that $\det(T_i, T_{i+1}) \geq 1$ for all $0 \leq i \leq m$, with $T_0 = (1, 0)^t$ and $T_{m+1} = (0, 1)^t$. A simplicial cone subdivision Σ^* is said to be *regular* if $\det(T_i, T_{i+1}) = 1$ for all $0 \le i \le m$. It is clear that $N^+_{\mathbb{R}}$ is covered by $m+1$ cones $C(T_i, T_{i+1}) = \{xT_i + yT_{i+1} \mid x, y \ge 0\}$ of Σ^*. These cones are in one-to-one correspondence with the matrices $\sigma_i = (T_i, T_{i+1})$; so we shall identify $C(T_i, T_{i+1})$ with σ_i for all $0 \le i \le m$.

It is a fact that each matrix $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}(2, \mathbb{Z})$ defines a birational map

$$\Phi_\sigma : \mathbb{C}^2 \to \mathbb{C}^2$$

sending (x, y) to $(x^a y^b, x^c y^d)$. In toric geometry, one uses such birational map to define toric modifications. For a regular simplicial cone subdivision Σ^* with vertices $T_1, \dots, T_m$, we consider the cones $\sigma_i = (T_i, T_{i+1})$ and the corresponding toric charts $(\mathbb{C}^2_{\sigma_i}; x_i, y_i)$, $0 \le i \le m$, with $\mathbb{C}^2_{\sigma_i}$ a copy of $\mathbb{C}^2$. On the disjoint union $\bigsqcup_{i=0}^m \left(\mathbb{C}^2_{\sigma_i}; x_i, y_i\right)$, as in [11] we consider the equivalence relation given by $(x_i, y_i) \sim (x_j, y_j)$ if and only if $\Phi_{\sigma_j^{-1}\sigma_i}(x_i, y_i) = (x_j, y_j)$. Let X be the quotient of $\bigsqcup_{i=0}^m \left(\mathbb{C}^2_{\sigma_i}; x_i, y_i\right)$ by the previous equivalence relation, which is endowed with the quotient topology. Then X is a smooth complex manifold of dimension 2, with the toric charts $(\mathbb{C}^2_{\sigma_i}; x_i, y_i)$ as local coordinates systems. In other words, we can present

$$X = \bigcup_{i=0}^m \left(\mathbb{C}^2_{\sigma_i}; x_i, y_i\right),$$

where $\mathbb{C}^2_{\sigma_i}$ are viewed as open subsets of X, and two charts $(\mathbb{C}^2_{\sigma_i}; x_i, y_i)$ and $(\mathbb{C}^2_{\sigma_j}; x_j, y_j)$ with nonempty intersection are compatibly glued in such a way that

$$(x_i, y_i) \equiv (x_j, y_j) \quad \text{if and only if} \quad (x_i, y_i) \sim (x_j, y_j). \tag{2.1}$$

We now define $\pi : X \to \mathbb{C}^2$ with $\pi(x_i, y_i) = \Phi_{\sigma_i}(x_i, y_i)$ for (x_i, y_i) in $\mathbb{C}^2_{\sigma_i}$, $0 \le i \le m$. This map is compatible with the glueing and it is called the *toric modification associated to the regular simplicial cone subdivision* Σ^*.

As explained in [6], the toric modification π can be decomposed as a composition of finitely many quadratic blowups. The divisor $\pi^{-1}(O)$ has simple normal crossings with m irreducible components $E(T_i)$, named as exceptional divisors, for $1 \le i \le m$. For every $1 \le i \le m$, the exceptional divisor $E(T_i)$ corresponds uniquely to the vertex T_i of Σ^*, and it is covered by two charts $\mathbb{C}^2_{\sigma_{i-1}}$ and $\mathbb{C}^2_{\sigma_i}$, with the equations $y_{i-1} = 0$ and $x_i = 0$ respectively. Therefore, only $E(T_i)$ and $E(T_{i+1})$ intersect for all $1 \le i \le m-1$, and the the intersections are transversal. The noncompact components $E(T_0) = \{x_0 = 0\}$ and $E(T_{m+1}) = \{y_m = 0\}$ are isomorphic to the coordinate axes $x = 0$ and $y = 0$ respectively.

2.2. **A toric resolution for** $f(x, y)$**.** Let $f(x, y) = \sum_{(a,b)\in\mathbb{N}^2} c_{\alpha\beta} x^\alpha y^\beta$ be in $\mathbb{C}\{x, y\}$ such that $f(O) = 0$. Denote by Γ or Γ_f the Newton polyhedron of $f(x, y)$. Clearly, the boundary of Γ contains finitely many facets each of which is completely defined by a positive primitive weight vector of the form $P = (a, b)^t \in N^+$, where (a, b) is a normal vector of the facet. The singularity $f(x, y)$ at O is said to be *nondegenerate with respect to* Γ if it has the form

$$\begin{aligned} & f(x, y) = c x^r y^s f_1(x, y) \cdots f_k(x, y), \\ & f_i(x, y) = \prod_{\ell=1}^{r_i} (y^{a_i} + \xi_{i\ell} x^{b_i}) + (\text{higher terms}), \end{aligned} \tag{2.2}$$

where $c \neq 0$, and for every $1 \leq i \leq k$,

$$(a_i, b_i) = 1, \tag{2.3}$$
$$\xi_{i\ell} \neq 0,\ \xi_{i\ell} \neq \xi_{i\ell'} \text{ if } \ell \neq \ell'.$$

For simplicity, we shall assume that $c = 1$ and $r = s = 0$ in the formula of $f(x, y)$. Then the Newton polyhedron Γ has k primitive weight vectors $P_1 = (a_1, b_1)^t, \dots, P_k = (a_k, b_k)^t$ as k compact facets. We define an ordering on primitive vectors as follows $P < Q$ if $\det(P, Q) > 0$. We order the P_i in such a way that $P_1 < \cdots < P_k$.

Let Σ^* be a regular simplicial subdivision with vertices $T_j = (c_j, d_j)^t$, $1 \leq j \leq m$, augmented by $(c_0, d_0) = (1, 0)$, $(c_{m+1}, d_{m+1}) = (0, 1)$, with $\det(T_j, T_{j+1}) = 1$ for all $0 \leq j \leq m$. We say that Σ^* is *admissible for* $f(x, y)$ if $\{P_1, \dots, P_k\} \subseteq \{T_1, \dots, T_m\}$. Let $\pi : X \to \mathbb{C}^2$ be the toric modification associated to Σ^*. Then π is said to be *admissible for* $f(x, y)$ if Σ^* is admissible for $f(x, y)$. In this case, π is nothing else than a resolution of singularity of $f(x, y)$ at O, with simple normal crossing divisors. We respectively denote by $N(T_j)$ and $\nu(T_j) - 1$ the multiplicity of π^*f and that of $\pi^*(dx \wedge dy)$ on the exceptional divisor $E(T_j)$, for $1 \leq j \leq m$. Since the expression of π on $\mathbb{C}^2_{\sigma_j}$ is $\pi(x_j, y_j) = (x_j^{c_j} y_j^{c_{j+1}}, x_j^{d_j} y_j^{d_{j+1}})$, we have

$$\pi^*(dx \wedge dy)(x_j, y_j) = x_j^{c_j+d_j-1} y_j^{c_{j+1}+d_{j+1}-1} dx_j \wedge dy_j$$

on $\mathbb{C}^2_{\sigma_j}$, thus

$$\nu(T_j) = c_j + d_j, \tag{2.4}$$

for all $1 \leq j \leq m$. It is clear that if F is an irreducible component of the strict transform of $f(x, y)$, and if $f(x, y)$ is reduced, then $\nu(F) = 1$.

We are in fact using the ordering defined above by $P < Q$ if $\det(P, Q) > 0$. To compute the multiplicity $N(T_j)$ of π^*f on $E(T_j)$ we consider the following three cases. The first one is $P_i \leq T_j < P_{i+1}$, for some $1 \leq i \leq k-1$. Since $P_t \leq T_j$ for all $1 \leq t \leq i$, it follows from [2, Section 4.3] that, on the chart $(\mathbb{C}^2_{\sigma_j}; x_j, y_j)$, and for $1 \leq t \leq i$,

$$\pi^*f_t(x_j, y_j) = x_j^{r_t b_t c_j} y_j^{r_t b_t c_{j+1}} \left(\prod_{\ell=1}^{r_t} (x_j^{a_t d_j - b_t c_j} y_j^{a_t d_{j+1} - b_t c_{j+1}} + \xi_{t\ell}) + x_j R_t(x_j, y_j) \right),$$

for some $R_t(x_j, y_j) \in \mathbb{C}\{x_j, y_j\}$. Since $T_j < P_t$ for all $i+1 \leq t \leq k$, it follows similarly as previous, for $i+1 \leq t \leq k$, that

$$\pi^*f_t(x_j, y_j) = x_j^{r_t a_t d_j} y_j^{r_t a_t d_{j+1}} \left(\prod_{\ell=1}^{r_t} \left(1 + \xi_{t\ell} x_j^{b_t c_j - a_t d_j} y_j^{b_t c_{j+1} - a_t d_{j+1}}\right) + x_j R_t(x_j, y_j) \right),$$

for some $R_t(x_j, y_j) \in \mathbb{C}\{x_j, y_j\}$. Thus, on the chart $(\mathbb{C}^2_{\sigma_j}; x_j, y_j)$,

$$\pi^*f(x_j, y_j) = \prod_{t=1}^{i} \pi^*f_t(x_j, y_j) \cdot \prod_{t=i+1}^{k} \pi^*f_t(x_j, y_j) = x_j^{N(T_j)} y_j^{N(T_{j+1})} u(x_j, y_j),$$

with $u(x_j, y_j)$ a unit in $\mathbb{C}\{x_j, y_j\}$, and $N(T_j) = c_j \sum_{t=1}^{i} r_t b_t + d_j \sum_{t=i+1}^{k} r_t a_t$. In the same way, for the second case $T_j < P_1$, we get $N(T_j) = d_j \sum_{t=1}^{k} r_t a_t$, and for the third case

$P_k \leq T_j$, we get $N(T_j) = c_j \sum_{t=1}^{k} r_t b_t$. Thus, by convention that $P_0 := T_0 = (1,0)^t$ and $P_{k+1} := T_{m+1} = (0,1)^t$, we can summarize the three cases by a common formula as follows

$$N(T_j) = c_j \sum_{t=1}^{i} r_t b_t + d_j \sum_{t=i+1}^{k} r_t a_t, \tag{2.5}$$

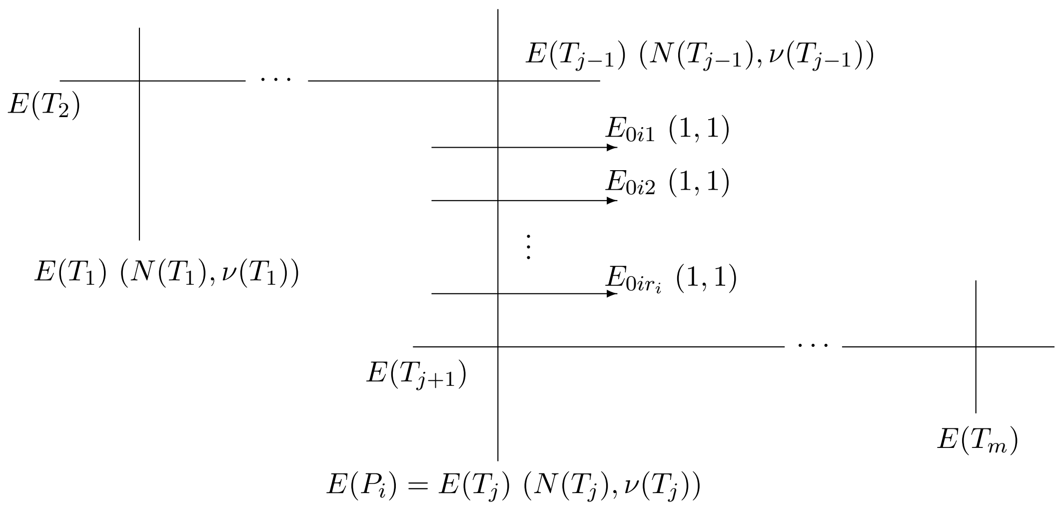
where $P_i \leq T_j < P_{i+1}$, for all $1 \leq j \leq m$.

When $T_j = P_i$ for some i,

$$\pi^* f_i(x_j, y_j) = x_j^{r_i a_i b_i} y_j^{r_i b_i c_{i+1}} \left(\prod_{\ell=1}^{r_i} (y_j + \xi_{i\ell}) + x_j R_i(x_j, y_j) \right),$$

with $R_i(x_j, y_j)$ in $\mathbb{C}\{x_j, y_j\}$. Therefore, there are r_i irreducible components of the strict transform intersecting transversally with $E(P_i)$ at $(0, -\xi_{i\ell})$, $1 \leq \ell \leq r_i$, in the chart $(\mathbb{C}^2_{\sigma_j}; x_j, y_j)$. If $2 \leq j \leq m-1$ and $T_j \neq P_i$ for all $1 \leq i \leq k$, then $E(T_j)$ intersects with exactly two other exceptional divisors and does not intersect with the strict transform. Also, if $T_1 \neq P_1$ (resp. $T_m \neq P_k$), then $E(T_1)$ (resp. $E(T_k)$) intersects with only one divisor.

The below is the configuration of the toric resolution for the nondegenerate singularity $f(x,y)$ at O:



2.3. **The topological zeta function of a nondegenerate singularity.** Let $f(x,y)$ be a singularity at O nondegenerate with respect to its Newton polyhedron Γ. Assume that $f(x,y)$ has the form as in (2.2) and (2.3) with $c = 1$ and $r = s = 0$. Recall that $P_i = (a_i, b_i)^t$ for $0 \leq i \leq k+1$, with $(a_0, b_0) = (1,0)$ and $(a_{k+1}, b_{k+1}) = (0,1)$.

Theorem 2.1. *With $f(x,y)$ nondegenerate as previous, $Z^{\mathrm{top}}_{f,O}(s)$ equals*

$$\sum_{i=0}^{k} \frac{\det(P_i, P_{i+1})}{(N(P_i)s + \nu(P_i))(N(P_{i+1})s + \nu(P_{i+1}))} - \frac{s}{s+1} \sum_{i=1}^{k} \frac{r_i}{N(P_i)s + \nu(P_i)},$$

where, for every $0 \leq i \leq k+1$, $\nu(P_i) = a_i + b_i$ and $N(P_i) = a_i \sum_{t=1}^{i} r_t b_t + b_i \sum_{t=i+1}^{k} r_t a_t$.

Proof. We use the toric resolution described in Section 2.2 to compute the topological zeta function. Here is the table with the strata E_I° of $\pi^{-1}(f^{-1}(O))$ and their Euler characteristic:

Strata	Euler char.	Conditions
$E(T_1)^\circ$, $E(T_m)^\circ$	1	
$E(T_j)^\circ$	0	$1<j<m,\ T_j\neq P_i\ (\forall\ 1\le i\le k)$
$E(P_i)^\circ$	$-r_i$	$1\le i\le k$
$E^\circ_{0i\ell}$	0	$1\le i\le k,\ 1\le \ell\le r_i$
$E_{0i\ell}\cap E_{0i\ell'}=\emptyset$	0	$1\le i\le k,\ \ell\neq\ell'$
$E(T_j)\cap E(T_{j+1})=1\mathrm{pt}$	1	$1\le j<m$
$E(T_j)\cap E(T_{j'})=\emptyset$	0	$\lvert j-j'\rvert\ge 2$
$E(T_j)\cap E_{0i\ell}=\emptyset$	0	$1\le i\le k,\ 1\le\ell\le r_i,\ T_j\neq P_i\ (\forall\ i)$
$E(P_i)\cap E_{0i\ell}=1\mathrm{pt}$	1	$1\le i\le k,\ 1\le \ell\le r_i$
$E(P_i)\cap E_{0i'\ell}=\emptyset$	0	$1\le i\neq i'\le k$

By definition, the topological zeta function $Z^{\mathrm{top}}_{f,O}(s)$ is the sum of the following functions

$$Z_1=\frac{1}{N(T_1)s+\nu(T_1)},\ Z_2=\frac{1}{N(T_m)s+\nu(T_m)},\ Z_3=\sum_{i=1}^{k}\frac{-r_i}{N(P_i)s+\nu(P_i)},$$
$$Z_4=\sum_{j=1}^{m-1}\frac{1}{(N(T_j)s+\nu(T_j))(N(T_{j+1})s+\nu(T_{j+1}))},\ Z_5=\sum_{i=1}^{k}\frac{r_i}{(s+1)(N(P_i)s+\nu(P_i))}.$$

For all $0\le i\le k+1$, let j_i be the index with $0\le j_i\le m+1$ and $T_{j_i}=P_i$. Then Z_4 equals

$$\sum_{j=1}^{j_1-1}\frac{1}{(N(T_j)s+\nu(T_j))(N(T_{j+1})s+\nu(T_{j+1}))}+\sum_{j=j_k}^{m-1}\frac{1}{(N(T_j)s+\nu(T_j))(N(T_{j+1})s+\nu(T_{j+1}))}$$

plus

$$\sum_{i=1}^{k-1}\sum_{j=j_i}^{j_{i+1}-1}\frac{1}{(N(T_j)s+\nu(T_j))(N(T_{j+1})s+\nu(T_{j+1}))}.$$

Claim 2.2. *For $0\le i\le k$ and $j_i\le j\le j_{i+1}-1$,*

$$\begin{vmatrix} N(T_{j+1}) & N(T_j)\\ \nu(T_{j+1}) & \nu(T_j)\end{vmatrix}=D_i:=\sum_{t=i+1}^{k}r_ta_t-\sum_{t=1}^{i}r_tb_t.$$

The proof of this claim is trivial, thanks to (2.4), (2.5). If $D_i\neq 0$, then for $j_i\le j\le j_{i+1}-1$,

$$\frac{1}{(N(T_j)s+\nu(T_j))(N(T_{j+1})s+\nu(T_{j+1}))}=\frac{N(T_{j+1})/D_i}{N(T_{j+1})s+\nu(T_{j+1})}-\frac{N(T_j)/D_i}{N(T_j)s+\nu(T_j)}.$$

In particular, D_0 and D_k are automatically nonzero, since $D_0=N(T_1)$ and $D_k=-N(T_m)$. Moreover, $N(P_1)/D_0=b_1$ and $N(P_k)/D_k=-a_k$, hence we have

$$\sum_{j=1}^{j_1-1}\frac{1}{(N(T_j)s+\nu(T_j))(N(T_{j+1})s+\nu(T_{j+1}))}=\frac{b_1}{N(P_1)s+\nu(P_1)}-Z_1,$$
$$\sum_{j=j_k}^{m-1}\frac{1}{(N(T_j)s+\nu(T_j))(N(T_{j+1})s+\nu(T_{j+1}))}=\frac{a_k}{N(P_k)s+\nu(P_k)}-Z_2.$$

For $1 \le i \le k-1$, if $D_i \neq 0$, then

$$I_i := \sum_{j=j_i}^{j_{i+1}-1} \frac{1}{(N(T_j)s+\nu(T_j))(N(T_{j+1})s+\nu(T_{j+1}))} = \frac{N(P_{i+1})/D_i}{N(P_{i+1})s+\nu(P_{i+1})} - \frac{N(P_i)/D_i}{N(P_i)s+\nu(P_i)}$$
$$= \frac{\det(P_i,P_{i+1})}{(N(P_i)s+\nu(P_i))(N(P_{i+1})s+\nu(P_{i+1}))}.$$

Also, if $D_i = 0$, then for $j_i \le j \le j_{i+1}-1$ we have

$$\frac{1}{\lambda_j\lambda_{j+1}} = (a_i+b_i)\left(\frac{c_j}{\lambda_j} - \frac{c_{j+1}}{\lambda_{j+1}}\right) \text{ for } \lambda_j := \frac{N(T_j)}{N(P_i)} = \frac{\nu(T_j)}{\nu(P_i)} = \frac{c_j+d_j}{a_i+b_i};$$

hence

$$I_i = \frac{\det(P_i,P_{i+1})\nu(P_i)/\nu(P_{i+1})}{(N(P_i)s+\nu(P_i))^2} = \frac{\det(P_i,P_{i+1})}{(N(P_i)s+\nu(P_i))(N(P_{i+1})s+\nu(P_{i+1}))}.$$

In conclusion, by the above computation, $Z_{f,O}^{\mathrm{top}}(s)$ equals

$$\sum_{i=0}^{k} \frac{\det(P_i,P_{i+1})}{(N(P_i)s+\nu(P_i))(N(P_{i+1})s+\nu(P_{i+1}))} - \sum_{i=1}^{k} \frac{r_i s}{(s+1)(N(P_i)s+\nu(P_i))},$$

and the theorem is proved. □

We can deduce from the proof of Theorem 2.1 that $-\frac{\nu(P_i)}{N(P_i)}$ is a pole of order 2 of the topological zeta function $Z_{f,O}^{top}(s)$ if and only if $D_i = 0$. Further, also due to Theorem 2.1, we can prove the following proposition. We leave the detailed proof to the reader.

Proposition 2.3. *With $f(x,y)$ nondegenerate as previous, for any $1 \le i \le k$, the rational number $-\frac{\nu(P_i)}{N(P_i)}$ is a pole of $Z_{f,O}^{top}(s)$.*

3. General complex plane curve singularities

3.1. **Toric resolution tree.** Let f be a reduced complex plane curve singularity at O which has no smooth irreducible components, and let $C = f^{-1}(0)$. Using toric modifications with centers determined canonically in terms of Tschirnhausen polynomials (see [2]), Q.T. Lê [7] constructs a resolution of singularity of f at O and a resolution graph $\mathbf{G}_s$ for (f,O). His method allows to arrange the vertices of $\mathbf{G}_s$ into an ordering so that we can consider $\mathbf{G}_s$ as a tree. With the help of [7], $\mathbf{G}_s$ is quite simple but still sufficiently strong to describe combinatorially the monodromy zeta function of (f,O). Further, $\mathbf{G}_s$ is also used in [8] to formulate a recurrence formula for the motivic Milnor fiber of (f,O). It is shown explicitly in this article that we can also compute the topological zeta function and give a new proof of the monodromy conjecture for plane curves in terms of $\mathbf{G}_s$. However, to reach to this goal, we have to construct a more complicated graph $\mathbf{G}$, which is useful for the computation.

Write f as follows

$$f = f_1 \cdots f_k, \quad f_i = f_{i1} \cdots f_{ir_i}, \quad f_{i\ell} = f_{i\ell 1} \cdots f_{i\ell r_{i\ell}}, \tag{3.1}$$

where for each (i,ℓ,τ), $f_{i\ell\tau}$ is irreducible in $\mathbb{C}\{x\}[y]$ and of the form

$$f_{i\ell\tau}(x,y) = (y^{a_i} + \xi_{i\ell} x^{b_i})^{A_{i\ell\tau}} + (\text{higher terms}), \tag{3.2}$$

with $\xi_{i\ell}$ being nonzero and distinct. It is clear that (a_i,b_i) is coprime. In this factorization, the (Newton) principal parts of f_i and f_j are weighted homogeneous of different weights for

$i \neq j$, the principal parts of $f_{i\ell}$ and $f_{i\ell'}$ are weighted homogeneous of the same weight (this weight corresponds to (a_i, b_i)). We assume that

$$a_i \geq 2,\ b_i \geq 2 \text{ and } (a_i, b_i) = 1 \quad \text{for all } 1 \leq i \leq k,$$

the assumption guarantees that f has no smooth branches. In fact, if $a_i = 1$ or $b_i = 1$, one may use an analytic change of coordinates (cf. [6, Lemma 1.3]) to make f non-convenient, which we do not want to consider. Put

$$A_i = A_{i1} + \cdots + A_{ir_i}, \quad A_{i\ell} = A_{i\ell 1} + \cdots + A_{i\ell r_{i\ell}}.$$

Then by [2, Section 4.3], the $A_{i\ell\tau}$-th Tschirnhausen approximate polynomial of $f_{i\ell\tau}(x, y)$ has the form

$$h_{i\ell}(x, y) = y^{a_i} + \xi_{i\ell} x^{b_i} + (\text{higher terms}).$$

Put $P_i = (a_i, b_i)^t$ for $1 \leq i \leq k$. These weight vectors correspond to the compact facets of the Newton polyhedron Γ of $f(x, y)$. Suppose that $P_1 < \cdots < P_k$. Let Σ^* be a regular simplicial cone subdivision with vertices $T_j = (c_j, d_j)^t \in N^+$, for $1 \leq j \leq m$, such that $T_1 < \cdots < T_m$ and $\{P_1, \ldots, P_k\} \subseteq \{T_1, \ldots, T_m\}$. We can assume that $T_1 \neq P_1$ and $T_m \neq P_k$ (see [7]). Let π_O be the toric modification associated to Σ^*. Then we construct the first floor of $\mathbf{G}$ as follows: The vertices correspond to the exceptional divisors $E(T_1), \ldots, E(T_m)$ of π_O, the edges are edges joining $E(T_j)$ with $E(T_{j+1})$, for all $1 \leq j \leq m-1$. These vertices and edges form a subgraph $\mathcal{B}_0$ of $\mathbf{G}$, which is named as *the first bamboo of* $\mathbf{G}$. By convention, the coordinates (x, y) will be rewritten as $(x_{\mathcal{B}_0}, y_{\mathcal{B}_0})$.

We construct $\mathbf{G}$ by induction. Assume that $\mathcal{B}_\mathrm{p}$ is a bamboo of $\mathbf{G}$, which consists of vertices $E(T_1^{\mathcal{B}_\mathrm{p}}), \ldots, E(T_{m^{\mathcal{B}_\mathrm{p}}}^{\mathcal{B}_\mathrm{p}})$ with $T_1^{\mathcal{B}_\mathrm{p}} < \cdots < T_{m^{\mathcal{B}_\mathrm{p}}}^{\mathcal{B}_\mathrm{p}}$. Let $\pi_{\mathcal{B}_\mathrm{p}} : X_{\mathcal{B}_\mathrm{p}} \to \mathbb{C}^2$ be the toric modification constructing $\mathcal{B}_\mathrm{p}$, and let $f_{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}})$ be in $\mathbb{C}\{x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}}\}$ for which $\pi_{\mathcal{B}_\mathrm{p}}$ is admissible. Note that $X_{\mathcal{B}_\mathrm{p}}$ is covered by the toric charts $(\mathbb{C}^2_{\mathcal{B}_\mathrm{p},\sigma_j}; x_{\mathcal{B}_\mathrm{p},j}, y_{\mathcal{B}_\mathrm{p},j})$, for $1 \leq j \leq m^{\mathcal{B}_\mathrm{p}}$, and that, for simplicity, we sometimes write their coordinates by (x_j, y_j) instead of $x_{\mathcal{B}_\mathrm{p},j}, y_{\mathcal{B}_\mathrm{p},j}$. Assume that $f_{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}})$ has the form

$$f_{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}}) = U_{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}}) x_{\mathcal{B}_\mathrm{p}}^{N^{\mathcal{B}_\mathrm{p}}} \prod_{i=1}^{k^{\mathcal{B}_\mathrm{p}}} \prod_{\ell=1}^{r_i^{\mathcal{B}_\mathrm{p}}} \prod_{\tau=1}^{r_{i\ell}^{\mathcal{B}_\mathrm{p}}} f_{i\ell\tau}^{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}}),$$

where $N^{\mathcal{B}_\mathrm{p}}$ is in $\mathbb{N}$, $U_{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}})$ is a unit in the ring $\mathbb{C}\{x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}}\}$, and

$$f_{i\ell\tau}^{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}}) = (y_{\mathcal{B}_\mathrm{p}}^{a_i^{\mathcal{B}_\mathrm{p}}} + \xi_{i\ell}^{\mathcal{B}_\mathrm{p}} x_{\mathcal{B}_\mathrm{p}}^{b_i^{\mathcal{B}_\mathrm{p}}})^{A_{i\ell\tau}^{\mathcal{B}_\mathrm{p}}} + (\text{higher terms})$$

are irreducible in $\mathbb{C}\{x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}}\}$, with $\xi_{i\ell}^{\mathcal{B}_\mathrm{p}} \neq 0$ distinct. It follows from [2, Section 4.3] that

$$a_i^{\mathcal{B}_\mathrm{p}} \geq 2,\ b_i^{\mathcal{B}_\mathrm{p}} \geq 2 \text{ and } (a_i^{\mathcal{B}_\mathrm{p}} \geq 2, b_i^{\mathcal{B}_\mathrm{p}} \geq 2) = 1 \quad \text{for all } 1 \leq i \leq k^{\mathcal{B}_\mathrm{p}},$$

because all a_i (for $1 \leq i \leq k$) corresponding to $\mathcal{B}_0$ are greater than or equal to 2. Notice that when $\mathcal{B}_\mathrm{p} = \mathcal{B}_0$, we have $U_{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}}) = 1$, $N^{\mathcal{B}_\mathrm{p}} = 0$, and $f_{\mathcal{B}_\mathrm{p}}$ is nothing but f. Put $P_i^{\mathcal{B}_\mathrm{p}} = (a_i^{\mathcal{B}_\mathrm{p}}, b_i^{\mathcal{B}_\mathrm{p}})^t$ for all $1 \leq i \leq k^{\mathcal{B}_\mathrm{p}}$, and assume that $P_1^{\mathcal{B}_\mathrm{p}} < \cdots < P_{k^{\mathcal{B}_\mathrm{p}}}^{\mathcal{B}_\mathrm{p}}$. By the admissibility for $f_{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}})$ of $\pi_{\mathcal{B}_\mathrm{p}}$, we have $\{P_1^{\mathcal{B}_\mathrm{p}}, \ldots, P_{k^{\mathcal{B}_\mathrm{p}}}^{\mathcal{B}_\mathrm{p}}\} \subseteq \{T_1^{\mathcal{B}_\mathrm{p}}, \ldots, T_{m^{\mathcal{B}_\mathrm{p}}}^{\mathcal{B}_\mathrm{p}}\}$. The vertices $E(P_1^{\mathcal{B}_\mathrm{p}}), \ldots, E(P_{k^{\mathcal{B}_\mathrm{p}}}^{\mathcal{B}_\mathrm{p}})$ are called *the principal vertices of* $\mathcal{B}_\mathrm{p}$. By [2, Section 4.3], the $A_{i\ell\tau}^{\mathcal{B}_\mathrm{p}}$-th Tschirnhausen approximate polynomial of $f_{i\ell\tau}^{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}})$ has the form

$$h_{i\ell}^{\mathcal{B}_\mathrm{p}}(x_{\mathcal{B}_\mathrm{p}}, y_{\mathcal{B}_\mathrm{p}}) = y_{\mathcal{B}_\mathrm{p}}^{a_i^{\mathcal{B}_\mathrm{p}}} + \xi_{i\ell}^{\mathcal{B}_\mathrm{p}} x_{\mathcal{B}_\mathrm{p}}^{b_i^{\mathcal{B}_\mathrm{p}}} + (\text{higher terms}).$$

If $T_j^{\mathcal{B}_{\mathrm{p}}} = P_{i_0}^{\mathcal{B}_{\mathrm{p}}}$, the pullbacks $\pi^*_{\mathcal{B}_{\mathrm{p}}} f_{\mathcal{B}_{\mathrm{p}}}$ and $\pi^*_{\mathcal{B}_{\mathrm{p}}} h_{i_0\ell}$ on the chart $(\mathbb{C}^2_{\mathcal{B}_{\mathrm{p}},\sigma_j}; x_j, y_j)$ are as follows

$$\pi^*_{\mathcal{B}_{\mathrm{p}}} f_{\mathcal{B}_{\mathrm{p}}}(x_j, y_j) = \xi x_j^{N(P_{i_0}^{\mathcal{B}_{\mathrm{p}}})} y_j^{N(T_{j+1}^{\mathcal{B}_{\mathrm{p}}})} \left((y_j + \xi_{i_0\ell}^{\mathcal{B}_{\mathrm{p}}})^{A_{i_0\ell}^{\mathcal{B}_{\mathrm{p}}}} + x_j R(x_j, y_j) \right)$$

and

$$\pi^*_{\mathcal{B}_{\mathrm{p}}} h_{i_0\ell}^{\mathcal{B}_{\mathrm{p}}}(x_j, y_j) = x_j^{a_{i_0}^{\mathcal{B}_{\mathrm{p}}} b_{i_0}^{\mathcal{B}_{\mathrm{p}}}} y_j^{c_{j+1}^{\mathcal{B}_{\mathrm{p}}} b_{i_0}^{\mathcal{B}_{\mathrm{p}}}} (y_j + \xi_{i_0\ell}^{\mathcal{B}_{\mathrm{p}}} + x_j R'(x_j, y_j)),$$

for some ξ in $\mathbb{C}^*$, $R(x_j, y_j)$ and $R'(x_j, y_j)$ in $\mathbb{C}\{x_j, y_j\}$. Without loss of generality we can (and will) assume that $\xi = 1$. By [2], in this step, there is a canonical way to change of variables which uses the Tschirnhausen approximate polynomial $h_{i_0\ell}^{\mathcal{B}_{\mathrm{p}}}$, namely

$$\begin{cases} u = x_j \\ v = \pi^*_{\mathcal{B}_{\mathrm{p}}} h_{i_0\ell}^{\mathcal{B}_{\mathrm{p}}} / x_j^{a_{i_0}^{\mathcal{B}_{\mathrm{p}}} b_{i_0}^{\mathcal{B}_{\mathrm{p}}}} = y_j^{c_{j+1}^{\mathcal{B}_{\mathrm{p}}} b_i^{\mathcal{B}_{\mathrm{p}}}} (y_j + \xi_{i_0\ell} + x_j R'(x_j, y_j)). \end{cases} \tag{3.3}$$

It is easy to obtain the following lemma.

Lemma 3.1. *The inverse modification of (3.3) is of the form*

$$\begin{cases} x_j = u \\ y_j = -\xi_{i_0\ell} + (-\xi_{i_0\ell})^{1/c_{j+1}^{\mathcal{B}_{\mathrm{p}}} b_{i_0}^{\mathcal{B}_{\mathrm{p}}}} v + R''(u, v), \end{cases}$$

for some $R''(u, v)$ in $\mathbb{C}\{u, v\}$.

Fix i_0 in $\{1, \dots, k^{\mathcal{B}_{\mathrm{p}}}\}$ and ℓ_0 in $\{1, \dots, r_i^{\mathcal{B}_{\mathrm{p}}}\}$. Since $\xi_{i_0\ell_0} \neq 0$, it follows from Lemma 3.1 that the pullback $\pi^*_{\mathcal{B}_{\mathrm{p}}} f_{\mathcal{B}_{\mathrm{p}}}$ is of the following form, in the Tschirnhausen coordinates (u, v),

$$\pi^*_{\mathcal{B}_{\mathrm{p}}} f_{\mathcal{B}_{\mathrm{p}}}(u, v) = U'(u, v) u^{N(P_{i_0}^{\mathcal{B}_{\mathrm{p}}})} \prod_{i=1}^{k'} \prod_{\ell=1}^{r'_i} \prod_{\tau=1}^{r'_{i\ell}} f'_{i\ell\tau}(u, v),$$

where $U'(u, v)$ is a unit in $\mathbb{C}\{u, v\}$, and

$$f'_{i\ell\tau}(u, v) = (v^{a'_i} + \xi'_{i\ell} u^{b'_i})^{A'_{i\ell\tau}} + (\text{higher terms})$$

are irreducible in $\mathbb{C}\{u\}[v]$, with $\xi'_{i\ell} \in \mathbb{C}^*$ distinct. The Newton polyhedron of $\pi^*_{\mathcal{B}_{\mathrm{p}}} f_{\mathcal{B}_{\mathrm{p}}}(u, v)$ again gives rise to an admissible toric modification, which constructs a bamboo $\mathcal{B}$ whose vertices are denoted by $E(T_1^{\mathcal{B}}), \dots, E(T_{m^{\mathcal{B}}}^{\mathcal{B}})$ with $T_1^{\mathcal{B}} < \dots < T_{m^{\mathcal{B}}}^{\mathcal{B}}$. In $\mathbf{G}$, we connect $E(T_1^{\mathcal{B}})$ to $E(P_{i_0}^{\mathcal{B}_{\mathrm{p}}})$ by a single edge, and this edge is taken into account of $\mathcal{B}$.

Definition 3.2. The graph $\mathbf{G}$ is called the *toric resolution tree* $\mathbf{G}$ of (f, O). The bamboo $\mathcal{B}$ constructed as above is called *the successor* (in $\mathbf{G}$) *of* $\mathcal{B}_{\mathrm{p}}$ *at* $E(P_{i_0}^{\mathcal{B}_{\mathrm{p}}})$ *associated to* ℓ_0. The bamboo $\mathcal{B}_{\mathrm{p}}$ is called *the predecessor* (in $\mathbf{G}$) *of* $\mathcal{B}$. A bamboo of $\mathbf{G}$ which has no successor is called a *top bamboo* of $\mathbf{G}$. A bamboo of $\mathbf{G}$ which is not a top bamboo is called a *non-top bamboo* of $\mathbf{G}$. Let $\mathbf{B}^{\mathrm{nt}}$ denote the set of all the non-top bamboos of $\mathbf{G}$.

Notation 3.3. Since each bamboo $\mathcal{B} \neq \mathcal{B}_0$ determines uniquely $P_{i_0}^{\mathcal{B}_{\mathrm{p}}}$, hence from now on, we denote $P_{\mathrm{root}}^{\mathcal{B}} := P_{i_0}^{\mathcal{B}_{\mathrm{p}}}$. Remark again that $E(P_{\mathrm{root}}^{\mathcal{B}})$ is not a vertex of $\mathcal{B}$, it is a vertex of $\mathcal{B}_{\mathrm{p}}$.

Remark that every top bamboo has a unique vertex and a unique edge. The number of top bamboos of $\mathbf{G}$ is nothing else than the number of irreducible components of the singularity (f, O). The below illustrates a toric resolution tree of a plane curve singularity (where the bamboos containing a unique white vertex are top bamboos):

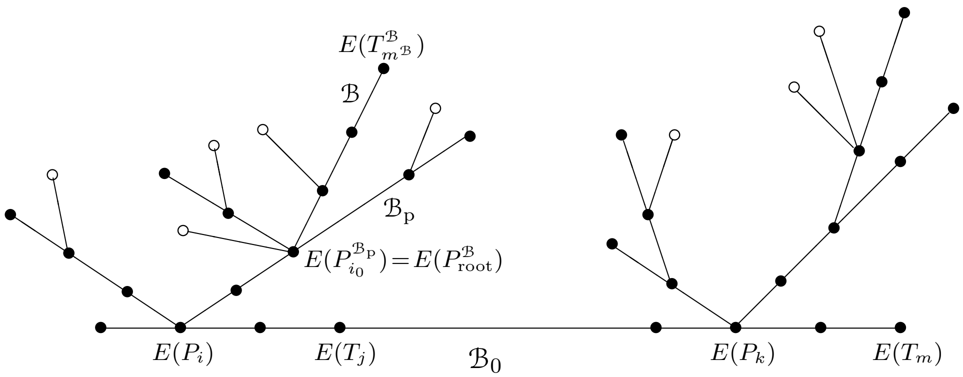


Figure 1. A toric resolution tree of a plane curve singularity

Notation 3.4. It is convenient to denote

$$(x_{\mathcal{B}}, y_{\mathcal{B}}) := (u, v), \quad f_{\mathcal{B}} := \pi^*_{\mathcal{B}_{\rm p}} f_{\mathcal{B}_{\rm p}}, \quad U_{\mathcal{B}} := U',$$
$$(a^{\mathcal{B}}_i, b^{\mathcal{B}}_i) := (a'_i, b'_i),\ A^{\mathcal{B}}_{i\ell\tau} := A'_{i\ell\tau},\ \xi^{\mathcal{B}}_{i\ell} := \xi'_{i\ell},\ k^{\mathcal{B}} := k',\ r^{\mathcal{B}}_i := r'_i,\ r^{\mathcal{B}}_{i\ell} := r'_{i\ell}.$$

Then we rewrite the initial expansion of $f_{\mathcal{B}}(x_{\mathcal{B}}, y_{\mathcal{B}})$ as follows

$$\begin{gathered} f_{\mathcal{B}} = U_{\mathcal{B}} x_{\mathcal{B}}^{N(P^{\mathcal{B}}_{\rm root})} \cdot f^{\mathcal{B}}_1 \cdots f^{\mathcal{B}}_{k^{\mathcal{B}}}, \quad f^{\mathcal{B}}_i = f^{\mathcal{B}}_{i1} \cdots f^{\mathcal{B}}_{ir^{\mathcal{B}}_i}, \quad f^{\mathcal{B}}_{i\ell} = f^{\mathcal{B}}_{i\ell 1} \cdots f^{\mathcal{B}}_{i\ell r^{\mathcal{B}}_{i\ell}}, \\ f^{\mathcal{B}}_{i\ell\tau}(x_{\mathcal{B}}, y_{\mathcal{B}}) = (y_{\mathcal{B}}^{a^{\mathcal{B}}_i} + \xi^{\mathcal{B}}_{i\ell} x_{\mathcal{B}}^{b^{\mathcal{B}}_i})^{A^{\mathcal{B}}_{i\ell\tau}} + (\text{higher terms}), \end{gathered} \tag{3.4}$$

where $a^{\mathcal{B}}_i \geq 2$, $a^{\mathcal{B}}_i \geq 2$, $(a^{\mathcal{B}}_i, b^{\mathcal{B}}_i) = 1$, and $f^{\mathcal{B}}_{i\ell\tau}(x_{\mathcal{B}}, y_{\mathcal{B}})$ are irreducible in $\mathbb{C}\{x_{\mathcal{B}}, y_{\mathcal{B}}\}$, and the complex numbers $\xi^{\mathcal{B}_{\rm p}}_{i\ell}$ are nonzero and distinct.

Notation 3.5. We denote $P^{\mathcal{B}}_0 := (1,0)^t$, $P^{\mathcal{B}}_{k^{\mathcal{B}}+1} := (0,1)^t$; also, if $\mathcal{B} = \mathcal{B}_0$, we write simply k for $k^{\mathcal{B}_0}$, and P_i for $P^{\mathcal{B}_0}_i$, for $0 \leq i \leq k+1$.

Remark 3.6. To a bamboo $\mathcal{B}$ of $\mathbf{G}$ we associate a unique bamboo $\mathcal{B}_{\rm s}$ whose vertices are the principal vertices of $\mathcal{B}$ together with $E(T^{\mathcal{B}}_1)$ and $E(T^{\mathcal{B}}_{m^{\mathcal{B}}})$. All the edges of $\mathcal{B}_{\rm s}$ consist of the one connecting $E(T^{\mathcal{B}}_1)$ with $E(P^{\mathcal{B}}_1)$, the ones connecting $E(P^{\mathcal{B}}_i)$ with $E(P^{\mathcal{B}}_{i+1})$ for all $1 \leq i \leq k^{\mathcal{B}} - 1$, and the one connecting $E(P^{\mathcal{B}}_{k^{\mathcal{B}}})$ with $E(T^{\mathcal{B}}_{m^{\mathcal{B}}})$. Working with the bamboos $\mathcal{B}_{\rm s}$ and using the method in constructing $\mathbf{G}$ we obtain a tree, which recovers the simplified extended resolution graph $\mathbf{G}_{\rm s}$ in [8].

3.2. **Multiplicities and discrepancies.** Let $\mathcal{B}$ be a bamboo of $\mathbf{G}$ and $\mathcal{B}_{\rm p}$ be the predecessor of $\mathcal{B}$ in $\mathbf{G}$. First, using the notation in Section 3.1 (in particular, Notation 3.3) and the same method of computation as in Section 2.2 we obtain the following lemmas.

Lemma 3.7. *For $\mathcal{B} = \mathcal{B}_0$, and $1 \leq j \leq m$ with $P_i \leq T_j < P_{i+1}$, we have*

$$N(T_j) = c_j \sum_{t=1}^{i} b_t A_t + d_j \sum_{t=i+1}^{k} a_t A_t,$$

where $A_i = \sum_{\ell=1}^{r_i} \sum_{\tau=1}^{r_{i\ell}} A_{i\ell\tau}$.

As above, suppose that $\mathcal{B}$ has all vertices $E(T^{\mathcal{B}}_j)$, with $T^{\mathcal{B}}_j = (c^{\mathcal{B}}_j, d^{\mathcal{B}}_j)^t$ and $T^{\mathcal{B}}_1 < \cdots < T^{\mathcal{B}}_{m^{\mathcal{B}}}$, and it has $E(P^{\mathcal{B}}_1), \ldots, E(P^{\mathcal{B}}_{k^{\mathcal{B}}})$ as the principal vertices.

Lemma 3.8. *For $\mathcal{B} \neq \mathcal{B}_0$ and $1 \leq j \leq m^{\mathcal{B}}$ with $P_i^{\mathcal{B}} \leq T_j^{\mathcal{B}} < P_{i+1}^{\mathcal{B}}$, we have*

$$N(T_j^{\mathcal{B}}) = c_j^{\mathcal{B}} N(P_{\text{root}}^{\mathcal{B}}) + c_j^{\mathcal{B}} \sum_{t=1}^{i} b_t^{\mathcal{B}} A_t^{\mathcal{B}} + d_j^{\mathcal{B}} \sum_{t=i+1}^{k^{\mathcal{B}}} a_t^{\mathcal{B}} A_t^{\mathcal{B}},$$

where $A_i^{\mathcal{B}} = \sum_{\ell=1}^{r_i^{\mathcal{B}}} \sum_{\tau=1}^{r_{i\ell}^{\mathcal{B}}} A_{i\ell\tau}^{\mathcal{B}}$.

Consider the Tschirnhausen coordinates $(x_{\mathcal{B}_{\mathrm{p}}}, y_{\mathcal{B}_{\mathrm{p}}})$, which is used to construct $\mathcal{B}_{\mathrm{p}}$, and consider the 2-form $\omega_{\mathcal{B}_{\mathrm{p}}} = dx_{\mathcal{B}_{\mathrm{p}}} \wedge dy_{\mathcal{B}_{\mathrm{p}}}$ on $(\mathbb{C}^2; x_{\mathcal{B}_{\mathrm{p}}}, y_{\mathcal{B}_{\mathrm{p}}})$ (note that $(x_{\mathcal{B}_0}, y_{\mathcal{B}_0}) = (x, y)$ and $\omega := dx \wedge dy$). Let $\pi_{\mathcal{B}_{\mathrm{p}}} : X_{\mathcal{B}_{\mathrm{p}}} \to (\mathbb{C}^2; x_{\mathcal{B}_{\mathrm{p}}}, y_{\mathcal{B}_{\mathrm{p}}})$ be the toric modification constructing $\mathcal{B}_{\mathrm{p}}$. Suppose that j' is the index such that $T_{j'}^{\mathcal{B}_{\mathrm{p}}} = P_{\text{root}}^{\mathcal{B}}$. Then, in the chart $(\mathbb{C}^2_{\mathcal{B}_{\mathrm{p}},j'}; x_{\mathcal{B}_{\mathrm{p}},j'}, y_{\mathcal{B}_{\mathrm{p}},j'})$ of $X_{\mathcal{B}_{\mathrm{p}}}$, we have

$$\Phi_{\mathcal{B}_{\mathrm{p}}}^* \omega = x_{\mathcal{B}_{\mathrm{p}},j'}^{\nu(P_{\text{root}}^{\mathcal{B}})-1} y_{\mathcal{B}_{\mathrm{p}},j'}^{\nu-1} dx_{\mathcal{B}_{\mathrm{p}},j'} \wedge dy_{\mathcal{B}_{\mathrm{p}},j'}$$

for some ν in $\mathbb{N}^*$, where $\Phi_{\mathcal{B}_{\mathrm{p}}}$ is the composition of the toric modifications along the series of consecutive bamboos from $\mathcal{B}_0$ to $\mathcal{B}_{\mathrm{p}}$ in $\mathbf{G}$. Via the change of variables in Lemma 3.1, this form $\Phi_{\mathcal{B}_{\mathrm{p}}}^* \omega$ becomes

$$\widetilde{U}(x_{\mathcal{B}}, y_{\mathcal{B}}) x_{\mathcal{B}}^{\nu(P_{\text{root}}^{\mathcal{B}})-1} \omega_{\mathcal{B}},$$

where $\widetilde{U}(x_{\mathcal{B}}, y_{\mathcal{B}})$ is a unit in $\mathbb{C}\{x_{\mathcal{B}}, y_{\mathcal{B}}\}$. Here, due to Notation 3.4, we replace (u, v) by $(x_{\mathcal{B}}, y_{\mathcal{B}})$ when applying Lemma 3.1.

Lemma 3.9. *With the previous notation and hypothesis, for $\mathcal{B} = \mathcal{B}_0$ and $1 \leq j \leq m$, we have $\nu(T_j) = c_j + d_j$; otherwise, for $1 \leq j \leq m^{\mathcal{B}}$,*

$$\nu(T_j^{\mathcal{B}}) = c_j^{\mathcal{B}} \nu(P_{\text{root}}^{\mathcal{B}}) + d_j^{\mathcal{B}}.$$

Proof. The case $\mathcal{B} = \mathcal{B}_0$ is similar as in the nondegenerate case. Now we consider the case $\mathcal{B} \neq \mathcal{B}_0$. In the chart $(\mathbb{C}^2_{\mathcal{B},j}; x_{\mathcal{B},j}, y_{\mathcal{B},j})$ of $X_{\mathcal{B}}$, we have

$$\pi_{\mathcal{B}}^* \left(x_{\mathcal{B}}^{\nu(P_{\text{root}}^{\mathcal{B}})-1} \omega_{\mathcal{B}} \right) = x_{\mathcal{B},j}^{c_j^{\mathcal{B}} \nu(P_{\text{root}}^{\mathcal{B}}) + d_j^{\mathcal{B}} - 1} y_{\mathcal{B},j}^{c_{j+1}^{\mathcal{B}} \nu(P_{\text{root}}^{\mathcal{B}}) + d_{j+1}^{\mathcal{B}} - 1} dx_{\mathcal{B},j} \wedge dy_{\mathcal{B},j}.$$

Hence $\nu(T_j^{\mathcal{B}}) = c_j^{\mathcal{B}} \nu(P_{\text{root}}^{\mathcal{B}}) + d_j^{\mathcal{B}}$ and the lemma is proved. □

3.3. **The topological zeta function.** Let $f(x, y)$ be a reduced complex plane curve singularity at $O = (0, 0)$, in which its initial expansion is given in (3.1) and (3.2) (with respect to $\mathcal{B}_0$) and the initial expansion of $f_{\mathcal{B}}$ in the Tschirnhausen coordinates $(x_{\mathcal{B}}, y_{\mathcal{B}})$ with respect to $\mathcal{B}$ is given in (3.4). The main result can be stated using $\mathbf{G}_{\mathrm{s}}$ (i.e., only principal vertices of $\mathbf{G}$) and proved using $\mathbf{G}$. We use all the notation in Section 3.2. Let $\mathbf{B}$ be the set of the bamboos of $\mathbf{G}$. Note that we can identify $\mathbf{B}$ with the set of the bamboos of $\mathbf{G}_{\mathrm{s}}$.

Theorem 3.10. *With the previous notation, put $Z_{\mathcal{B}}(s) = 0$ for $\mathcal{B}$ being a top bamboo, and*

$$Z_{\mathcal{B}}(s) = \sum_{i=1}^{k^{\mathcal{B}}} \left[\frac{\det(P_i^{\mathcal{B}}, P_{i+1}^{\mathcal{B}})}{(N(P_i^{\mathcal{B}})s + \nu(P_i^{\mathcal{B}}))(N(P_{i+1}^{\mathcal{B}})s + \nu(P_{i+1}^{\mathcal{B}}))} - \frac{r_i^{\mathcal{B}}}{N(P_i^{\mathcal{B}})s + \nu(P_i^{\mathcal{B}})} \right]$$

otherwise, where $\nu(P_i^{\mathcal{B}}) = a_i^{\mathcal{B}} \nu(P_{\text{root}}^{\mathcal{B}}) + b_i^{\mathcal{B}}$ and

$$N(P_i^{\mathcal{B}}) = a_i^{\mathcal{B}} N(P_{\text{root}}^{\mathcal{B}}) + a_i^{\mathcal{B}} \sum_{t=1}^{i} b_t^{\mathcal{B}} A_t^{\mathcal{B}} + b_i^{\mathcal{B}} \sum_{t=i+1}^{k^{\mathcal{B}}} a_t^{\mathcal{B}} A_t^{\mathcal{B}}.$$

Then, the topological zeta function of (f,O) *is given by*

$$Z^{\mathrm{top}}_{f,O}(s)=\sum_{\mathcal{B}\in\mathbf{B}}\left[\frac{b_1^{\mathcal{B}}}{(N(P^{\mathcal{B}}_{\mathrm{root}})s+\nu(P^{\mathcal{B}}_{\mathrm{root}}))(N(P^{\mathcal{B}}_1)s+\nu(P^{\mathcal{B}}_1))}+Z_{\mathcal{B}}(s)\right],$$

with $N(P^{\mathcal{B}_0}_{\mathrm{root}})=0$, $\nu(P^{\mathcal{B}_0}_{\mathrm{root}})=1$, *and* $N(P^{\mathcal{B}}_1)=\nu(P^{\mathcal{B}}_1)=b^{\mathcal{B}}_1=1$ *for any top bamboo* $\mathcal{B}$.

Proof. Let us regard each bamboo $\mathcal{B}$ of $\mathbf{G}$ as a subgraph of $\mathbf{G}$ with the edge connecting $E(T^{\mathcal{B}}_1)$ to $E(P^{\mathcal{B}}_{\mathrm{root}})$ included. Remark that the vertex $E(P^{\mathcal{B}}_{\mathrm{root}})$ belongs to the predecessor bamboo $\mathcal{B}_{\mathrm{p}}$ of $\mathcal{B}$ in $\mathbf{G}$, and that each top bamboo consists of a unique vertex and a unique edge.

From the definition of $Z^{\mathrm{top}}_{f,O}(s)$, if for each bamboo $\mathcal{B}$ of $\mathbf{G}$ which is not a top bamboo, we define $Z'_{\mathcal{B}}(s)$ as the sum of

$$\frac{\delta(\mathcal{B})}{N(T^{\mathcal{B}}_1)s+\nu(T^{\mathcal{B}}_1)},\ \frac{1-\delta(\mathcal{B})}{(N(P^{\mathcal{B}}_{\mathrm{root}})s+\nu(P^{\mathcal{B}}_{\mathrm{root}}))(N(T^{\mathcal{B}}_1)s+\nu(T^{\mathcal{B}}_1))},\ \frac{1}{N(T^{\mathcal{B}}_{m^{\mathcal{B}}})s+\nu(T^{\mathcal{B}}_{m^{\mathcal{B}}})},$$
$$\sum_{i=1}^{k^{\mathcal{B}}}\frac{-r^{\mathcal{B}}_i}{N(P^{\mathcal{B}}_i)s+\nu(P^{\mathcal{B}}_i)},\ \text{and } Z:=\sum_{j=1}^{m^{\mathcal{B}}-1}\frac{1}{(N(T^{\mathcal{B}}_j)s+\nu(T^{\mathcal{B}}_j))(N(T^{\mathcal{B}}_{j+1})s+\nu(T^{\mathcal{B}}_{j+1}))},$$

with $\delta(\mathcal{B}_0)=1$ and $\delta(\mathcal{B})=0$ whenever $\mathcal{B}\neq\mathcal{B}_0$, and if for each top bamboo $\mathcal{B}$, we define

$$Z'_{\mathcal{B}}(s)=\frac{1}{(N(P^{\mathcal{B}}_{\mathrm{root}})s+\nu(P^{\mathcal{B}}_{\mathrm{root}}))(s+1)},$$

then $Z^{\mathrm{top}}_{f,O}(s)=\sum_{\mathcal{B}\in\mathbf{B}}Z'_{\mathcal{B}}(s)$. Similarly as in the nondegenarate case (Theorem 2.1), we have

$$Z'_{\mathcal{B}_0}(s)=\sum_{i=0}^{k}\frac{\det(P_i,P_{i+1})}{(N(P_i)s+\nu(P_i))(N(P_{i+1})s+\nu(P_{i+1}))}-\sum_{i=1}^{k}\frac{r_i}{N(P_i)s+\nu(P_i)}.$$

Now we consider a bamboo $\mathcal{B}$ of $\mathbf{G}$ which is neither the first bamboo $\mathcal{B}_0$ nor a top bamboo. By the same method of computation as in the proof of Theorem 2.1 we get

$$\begin{aligned}Z=&\sum_{i=1}^{k^{\mathcal{B}}-1}\frac{\det(P^{\mathcal{B}}_i,P^{\mathcal{B}}_{i+1})}{(N(P^{\mathcal{B}}_i)s+\nu(P^{\mathcal{B}}_i))(N(P^{\mathcal{B}}_{i+1})s+\nu(P^{\mathcal{B}}_{i+1}))}\\&+\frac{\det(T^{\mathcal{B}}_1,P^{\mathcal{B}}_1)}{(N(T^{\mathcal{B}}_1)s+\nu(T^{\mathcal{B}}_1))(N(P^{\mathcal{B}}_1)s+\nu(P^{\mathcal{B}}_1))}+\frac{\det(P^{\mathcal{B}}_{k^{\mathcal{B}}},T^{\mathcal{B}}_{m^{\mathcal{B}}})}{(N(P^{\mathcal{B}}_{k^{\mathcal{B}}})s+\nu(P^{\mathcal{B}}_{k^{\mathcal{B}}}))(N(T^{\mathcal{B}}_{m^{\mathcal{B}}})s+\nu(T^{\mathcal{B}}_{m^{\mathcal{B}}}))}.\end{aligned}$$

It follows that

$$\begin{aligned}Z'_{\mathcal{B}}(s)=&\frac{b^{\mathcal{B}}_1}{(N(P^{\mathcal{B}}_{\mathrm{root}})s+\nu(P^{\mathcal{B}}_{\mathrm{root}}))(N(P^{\mathcal{B}}_1)s+\nu(P^{\mathcal{B}}_1))}+\frac{a^{\mathcal{B}}_{k^{\mathcal{B}}}}{N(P^{\mathcal{B}}_{k^{\mathcal{B}}})s+\nu(P^{\mathcal{B}}_{k^{\mathcal{B}}})}\\&+\sum_{i=1}^{k^{\mathcal{B}}-1}\frac{\det(P^{\mathcal{B}}_i,P^{\mathcal{B}}_{i+1})}{(N(P^{\mathcal{B}}_i)s+\nu(P^{\mathcal{B}}_i))(N(P^{\mathcal{B}}_{i+1})s+\nu(P^{\mathcal{B}}_{i+1}))}-\sum_{i=1}^{k^{\mathcal{B}}}\frac{r^{\mathcal{B}}_i}{N(P^{\mathcal{B}}_i)s+\nu(P^{\mathcal{B}}_i)}.\end{aligned}$$

Since $a^{\mathcal{B}}_{k^{\mathcal{B}}}=\det(P^{\mathcal{B}}_{k^{\mathcal{B}}},P^{\mathcal{B}}_{k^{\mathcal{B}}+1})$, $N(P^{\mathcal{B}}_{k^{\mathcal{B}}+1})=0$, $\nu(P^{\mathcal{B}}_{k^{\mathcal{B}}+1})=1$, the theorem is now proved. □

This theorem gives immediately the following corollary.

Corollary 3.11. *Every pole of* $Z^{\mathrm{top}}_{f,O}(s)$ *has the form* $-\frac{\nu(P^{\mathcal{B}}_i)}{N(P^{\mathcal{B}}_i)}$ *for some* $\mathcal{B}$ *in* $\mathbf{B}$ *and some* i *with* $1\leq i\leq k^{\mathcal{B}}$.

In fact, we can go further to state that every number $-\frac{\nu(P_i^{\mathcal{B}})}{N(P_i^{\mathcal{B}})}$ is a pole of $Z_{f,O}^{\mathrm{top}}(s)$. However, its proof is rather long while all we need for the proof of the main theorem (Theorem 4.2) is only Corollary 3.11. So we skip proving this stronger statement.

4. Applications of Theorem 3.10

4.1. **The topological invariance of the zeta function.** Recall that two analytic function germs (f,x) and (g,y) on $\mathbb{C}^n$ are topologically equivalent if there are neighborhoods U of x and V of y in $\mathbb{C}^n$, and a homeomorphism $\varphi : U \to V$ such that $g \circ \varphi = f$. In [3], Artal Bartolo, Cassou-Noguès, Luengo and Melle Hernández introduce an example which shows that the topological zeta function of a germ of a complex hypersurface singularity is *not* a topological invariant of the singularity. However, in this section we shall prove that when $n=2$ the topological zeta function of a complex singularity is exactly a topological invariant.

Theorem 4.1. *For reduced complex plane curve singularities, the local topological zeta function is a topological invariant.*

Proof. In the toric resolution tree $\mathbf{G}$ of the reduced singularity (f,O), consider a sequence of consecutive bamboos from the first one $\mathcal{B}_0$ to a top one, say $(\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_{g+1})$ with $\mathcal{B}_i$ is the predecessor of $\mathcal{B}_{i+1}$. Then the sequence of vertices

$$(P_{\mathrm{root}}^{\mathcal{B}_1},\ldots,P_{\mathrm{root}}^{\mathcal{B}_{g+1}})$$

corresponds one-to-one to an irreducible component D of (f,O), hence by [2, Remark 4.5.4], to the sequence of Puiseux pairs of the irreducible component of (f,O). Let D' be another irreducible component of (f,O), which corresponds to a sequence of consecutive bamboos $(\mathcal{B}'_0=\mathcal{B}_0,\mathcal{B}'_1,\ldots,\mathcal{B}'_{g'+1})$. Let θ be the index such that

$$P_{\mathrm{root}}^{\mathcal{B}_t}=P_{\mathrm{root}}^{\mathcal{B}'_t},\ 0\le t\le\theta,\quad \text{and } P_{\mathrm{root}}^{\mathcal{B}_{\theta+1}}\neq P_{\mathrm{root}}^{\mathcal{B}'_{\theta+1}}.$$

Via Notation 3.3, fixing a bamboo $\mathcal{B}$ of $\mathbf{G}$ we introduce new notations as follows: If $P_i^{\mathcal{B}}=(a_i^{\mathcal{B}},b_i^{\mathcal{B}})$ is the weight vector in the initial expansion of $\Phi_{\mathcal{B}}^*D=f_{i\ell\tau}^{\mathcal{B}}$ (for some ℓ and τ), with $\Phi_{\mathcal{B}}$ defined in the paragraph right before Lemma 3.9, then we put

$$a(P_i^{\mathcal{B}}):=a_i^{\mathcal{B}},\quad b(P_i^{\mathcal{B}}):=b_i^{\mathcal{B}},\quad A_D(P_i^{\mathcal{B}})=A_{i\ell\tau}^{\mathcal{B}}.$$

By [2, Lemma 3.4.2], the intersection number $I(D,D';O)$ is computed as follows

$$I(D,D';O)=\sum_{t=0}^{\theta}a(P_{\mathrm{root}}^{\mathcal{B}_t})b(P_{\mathrm{root}}^{\mathcal{B}_t})A_D(P_{\mathrm{root}}^{\mathcal{B}_t})A_{D'}(P_{\mathrm{root}}^{\mathcal{B}'_t})+I_{\theta+1},$$

where $I_{\theta+1}$ is equal to

$$\min\Big\{a(P_{\mathrm{root}}^{\mathcal{B}_{\theta+1}})b(P_{\mathrm{root}}^{\mathcal{B}'_{\theta+1}})A_D(P_{\mathrm{root}}^{\mathcal{B}_{\theta+1}})A_{D'}(P_{\mathrm{root}}^{\mathcal{B}'_{\theta+1}}),\ a(P_{\mathrm{root}}^{\mathcal{B}'_{\theta+1}})b(P_{\mathrm{root}}^{\mathcal{B}'_{\theta+1}})A_D(P_{\mathrm{root}}^{\mathcal{B}_{\theta+1}})A_{D'}(P_{\mathrm{root}}^{\mathcal{B}'_{\theta+1}})\Big\}$$

if $\theta<\min\{g,g'\}$, and

$$I_{\theta+1}=b(P_{\mathrm{root}}^{\mathcal{B}_{\theta+1}})A_D(P_{\mathrm{root}}^{\mathcal{B}_{\theta+1}})A_{D'}(P_{\mathrm{root}}^{\mathcal{B}'_{\theta+1}})$$

if $\theta=g'=\min\{g,g'\}$. This means that the simplified extended resolution graph $\mathbf{G}_{\mathrm{s}}$ of (f,O) defined in [8] (see Remark 3.6) completely determines the Puiseux pairs of all the irreducible components and the intersection numbers of any couple of them. Thus, by Brieskorn [4], $\mathbf{G}_{\mathrm{s}}$ is a topological invariant of the singularity (f,O).

Clearly, the statement in Theorem 3.10 can be stated using $\mathbf{G}_{\mathrm{s}}$ (i.e., using data from the principal vertices of the bamboos $\mathcal{B}$ of $\mathbf{G}$). Then the topological zeta function of (f,O) is completely determined by $\mathbf{G}_{\mathrm{s}}$ of (f,O). Since $\mathbf{G}_{\mathrm{s}}$ is a topological invariant of (f,O), so is the topological zeta function of (f,O). $\square$

4.2. **A new proof of the monodromy conjecture for complex plane curves.** In 1975, A'Campo introduced in [1, Theorem 3] a celebrated formula computing the monodromy zeta function of an isolated singularity in terms of its embedded resolution. For complex plane curve singularities, a reduced one is always isolated, so we can apply the formula of A'Campo.

Let $f(x,y)$ be a complex plane curve singularity at the origin O of $\mathbb{C}^2$. Its Milnor fiber F_O is the intersection of $f^{-1}(\eta)$ with a small ball around O for $\eta > 0$ very small (see Milnor [10]). The complex vector spaces $H^q(F_O, \mathbb{C})$ (resp. $H^*(F_O,\mathbb{C})$) admit an automorphism $M_O^{(q)}$ (resp. M_O) generated by going once around a loop around O with the starting point η.

Theorem 4.2. *Let (f,O) be a reduced complex plane curve singularity. If θ is a pole of $Z_{f,O}^{\mathrm{top}}(s)$, then $\exp(2\pi\sqrt{-1}\theta)$ is an eigenvalue of M_O.*

Proof. By the Weierstrass preparation theorem, we can assume that $f(x,y)$ is in $\mathbb{C}\{x\}[y]$. Denote by n the degree of the polynomial $f(x,y)$ in the variable y. It is sufficient to consider the poles different from 1 of $Z_{f,O}^{\mathrm{top}}(s)$. By Corollary 3.11, every pole different from 1 is of the form $-\nu(P_i^{\mathcal{B}})/N(P_i^{\mathcal{B}})$ for some $\mathcal{B} \in \mathbf{B}^{\mathrm{nt}}$ and some i with $1 \leq i \leq k^{\mathcal{B}}$, where $\mathbf{B}^{\mathrm{nt}}$ is the set of all the non-top bamboos of $\mathbf{G}$ (see Definition 3.2, in Figure 1 non-top bamboos are bamboos containing black vertices).

The proof is by induction with many steps. The first step is to verify for the case where the number $k = k^{\mathcal{B}_0}$ of compact facets of Γ_f is ≥ 2. The second one is to do for $k=1$ and the number $r_1 = r_1^{\mathcal{B}_0}$ of successors of $\mathcal{B}_0$ in $\mathbf{G}$ is ≥ 2. Finally, for the case $k = r_1 = 1$ we prove by induction on n.

Let $\Delta^{(1)}(t)$ be the characteristic polynomial of $M_O^{(1)}$. By Milnor [10], $\Delta^{(1)}(t)$ is symmetric, hence $\Delta^{(1)}(t) = (1-t)Z_{f,O}^{\mathrm{mon}}(t)$, where $Z_{f,O}^{\mathrm{mon}}(t)$ is the monodromy zeta function of (f,O). We recall the computation of $Z_{f,O}^{\mathrm{mon}}(t)$ in [7, Theorem 3.5], under the light of [1, Theorem 3], as follows

$$Z_{f,O}^{\mathrm{mon}}(t) = \frac{1}{1-t^{N(T_1)}} \prod_{\mathcal{B}\in\mathbf{B}^{\mathrm{nt}}} \frac{\prod_{i=1}^{k^{\mathcal{B}}}(1-t^{N(P_i^{\mathcal{B}})})^{r_i^{\mathcal{B}}}}{1-t^{N(T^{\mathcal{B}}_{m^{\mathcal{B}}})}}. \tag{4.1}$$

Notice that $N(T_1)$ and $N(T^{\mathcal{B}}_{m^{\mathcal{B}}})$ are independent of T_1 and $T^{\mathcal{B}}_{m^{\mathcal{B}}}$ for any $\mathcal{B}$ in $\mathbf{B}^{\mathrm{nt}}$, because

$$N(P_1) = b_1 N(T_1), \quad N(P^{\mathcal{B}}_{k^{\mathcal{B}}}) = a^{\mathcal{B}}_{k^{\mathcal{B}}} N(T^{\mathcal{B}}_{m^{\mathcal{B}}}). \tag{4.2}$$

Hence, from (4.1), if $k = k^{\mathcal{B}_0} \geq 2$, then $Z_{f,O}^{\mathrm{mon}}(t)$ equals

$$\frac{(1-t^{N(P_1)})^{r_1}}{1-t^{N(T_1)}} \cdot \frac{(1-t^{N(P_k)})^{r_k}}{1-t^{N(T_m)}} \prod_{i=2}^{k-1}(1-t^{N(P_i)})^{r_i}$$

times

$$\prod_{\mathcal{B}_0\neq\mathcal{B}\in\mathbf{B}^{\mathrm{nt}}} \frac{(1-t^{N(P^{\mathcal{B}}_{k^{\mathcal{B}}})})^{r^{\mathcal{B}}_{k^{\mathcal{B}}}}}{1-t^{N(T^{\mathcal{B}}_{m^{\mathcal{B}}})}} \prod_{i=1}^{k^{\mathcal{B}}-1}(1-t^{N(P_i^{\mathcal{B}})})^{r_i^{\mathcal{B}}}.$$

In this formula, observe that the complex numbers $\exp\left(-2\pi\sqrt{-1}\nu(P_i^{\mathcal{B}})/N(P_i^{\mathcal{B}})\right)$ are surely eigenvalues of $M_O^{(1)}$ if either $\mathcal{B} = \mathcal{B}_0$ and $2 \leq i \leq k-1$ or $\mathcal{B} \neq \mathcal{B}_0$ and $1 \leq i \leq k^{\mathcal{B}} - 1$.

Also in the case $k \geq 2$, we consider the complex numbers $t_1 = \exp\big(-2\pi\sqrt{-1}\nu(P_1)/N(P_1)\big)$ and $t_{k^{\mathcal{B}}} = \exp\big(-2\pi\sqrt{-1}\nu(P^{\mathcal{B}}_{k^{\mathcal{B}}})/N(P^{\mathcal{B}}_{k^{\mathcal{B}}})\big)$ for every $\mathcal{B}$ in $\mathbf{B}^{\mathrm{nt}}$. By (4.2) and the recurrence formula of $\nu(P^{\mathcal{B}}_i)$ in Theorem 3.10, we get

$$t_1^{N(T_1)} = \exp\big(-2\pi\sqrt{-1}(a_1+b_1)/b_1\big) = \exp\big(-2\pi\sqrt{-1}a_1/b_1\big)$$

and

$$t_{k^{\mathcal{B}}}^{N(T^{\mathcal{B}}_{m^{\mathcal{B}}})} = \exp\Big(-2\pi\sqrt{-1}(a^{\mathcal{B}}_{k^{\mathcal{B}}}\nu(P^{\mathcal{B}}_{\mathrm{root}}) + b^{\mathcal{B}}_{k^{\mathcal{B}}})/a^{\mathcal{B}}_{k^{\mathcal{B}}}\Big) = \exp\Big(-2\pi\sqrt{-1}b^{\mathcal{B}}_{k^{\mathcal{B}}}/a^{\mathcal{B}}_{k^{\mathcal{B}}}\Big).$$

Since $a_1, b_1 \geq 2$ and $a^{\mathcal{B}}_{k^{\mathcal{B}}}, b^{\mathcal{B}}_{k^{\mathcal{B}}} \geq 2$ are coprime pairs for every $\mathcal{B}$ in $\mathbf{B}^{\mathrm{nt}}$, it implies that a_1/b_1 and $b^{\mathcal{B}}_{k^{\mathcal{B}}}/a^{\mathcal{B}}_{k^{\mathcal{B}}}$ is not in $\mathbb{Z}$, hence t_1 (resp. $t_{k^{\mathcal{B}}}$) is a zero of

$$\frac{(1-t^{N(P_1)})^{r_1}}{1-t^{N(T_1)}} \quad (\text{resp. } \frac{(1-t^{N(P^{\mathcal{B}}_{k^{\mathcal{B}}})})^{r^{\mathcal{B}}_{k^{\mathcal{B}}}}}{1-t^{N(T^{\mathcal{B}}_{m^{\mathcal{B}}})}}).$$

So t_1 and $t_{k^{\mathcal{B}}}$, for all $\mathcal{B}$ in $\mathbf{B}^{\mathrm{nt}}$, are eigenvalues of $M^{(1)}_O$, thus the proof for $k \geq 2$ completes.

We now consider the case $k = 1$, that is, the initial expansion of $f(x,y)$ at O has the form $(y^{a_1} + \xi x^{b_1})^A +$ (higher terms), with ξ in $\mathbb{C}^*$ and A in $\mathbb{N}^*$. If $r_1 \geq 2$, then by (4.2), the same arguments as in the case $k \geq 2$ still holds, and we thus have that $\exp\big(-2\pi\sqrt{-1}\nu(P_1)/N(P_1)\big)$ is an eigenvalue of $M^{(1)}_O$, where $P_1 = (a_1, b_1)^t$. Assume that $r_1 = 1$. We are going to prove the theorem by induction of the degree $n = a_1 A$ of the polynomial f in the variable y. Obviously, the theorem holds for $A = 1$. Assume that the theorem already holds for every function germ of degree in y less than n. Let $\mathcal{B}_1$ be the unique successor of $\mathcal{B}_0$. Since $N(T^{\mathcal{B}_1}_1) = A$, the function $Z^{\mathrm{mon}}_{f,O}(t)$ equals

$$\frac{(1-t^{a_1b_1A})(1-t^A)}{(1-t^{b_1A})(1-t^{a_1A})} \cdot \frac{1}{1-t^{N(T^{\mathcal{B}_1}_1)}} \cdot \prod_{\mathcal{B}_0 \neq \mathcal{B} \in \mathbf{B}^{\mathrm{nt}}} \frac{(1-t^{N(P^{\mathcal{B}}_{k^{\mathcal{B}}})})^{r^{\mathcal{B}}_{k^{\mathcal{B}}}}}{1-t^{N(T^{\mathcal{B}}_{m^{\mathcal{B}}})}} \prod_{i=1}^{k^{\mathcal{B}}-1} (1-t^{N(P^{\mathcal{B}}_i)})^{r^{\mathcal{B}}_i}.$$

By (4.1) we get

$$Z^{\mathrm{mon}}_{f,O}(t) = \frac{(1-t^{a_1b_1A})(1-t^A)}{(1-t^{b_1A})(1-t^{a_1A})} Z^{\mathrm{mon}}_{\pi_1^*f,O'}(t),$$

where O' is the origin of the system of Tschirnhausen coordinates after the toric modification π_1 admissible for f. Clearly, t_1 is a root of the polynomial

$$\frac{(1-t^{a_1b_1A})(1-t^A)}{(1-t^{b_1A})(1-t^{a_1A})},$$

and the degree of π_1^*f in y is less than n. This completes the proof. □

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