MEROMORPHIC MAPS OF COMPLEX SUBMANIFOLDS IN $\mathbb{C}^n$

PHAM NGUYEN THU TRANG AND DO DUC THAI AND PHAM NGOC MAI

In memory of Professor Nessim Sibony

Abstract. The purpose of this article is twofold. The first is to establish the Cartan-Nochka second main theorem for algebraically nondegenerate meromorphic mappings from a polydisc in $\mathbb{C}^n$ into a projective algebraic variety sharing hypersurfaces located in N-subgeneral position. The second aim is to establish a sharp non-truncated defect relation for meromorphic mappings from an n-dimensional closed complex submanifold of $\mathbb{C}^l$ into a compact complex manifold X sharing divisors in N-subgeneral position.

1. Introduction

To construct a Nevanlinna theory for meromorphic mappings between complex manifolds of arbitrary dimensions is one of the most important problems of the Value Distribution Theory. Much attention has been given to this problem over the last few decades and several important results have been obtained. For instance, in 1977, W. Stoll [16] introduced to parabolic complex manifolds, i.e manifolds have exhausted functions on the ones with the same role as the radius function in $\mathbb{C}^l$ and constructed a Nevanlinna theory for meromorphic mappings from a parabolic complex manifold into a complex projective space (also see [17], [18]). In the same time, P. Griffiths and J. King [6] constructed a Nevanlinna theory for holomorphic mappings between algebraic varieties by establishing special exhausted functions on affine algebraic varieties. In 1985, H. Fujimoto [5] constructed a Nevanlinna theory for nondegenerate meromorphic mappings from a ball in $\mathbb{C}^n$ into the complex projective space $\mathbb{P}^m(\mathbb{C})$ sharing hyperplanes in general position in $\mathbb{P}^m(\mathbb{C})$. In 2019, Thai-Quang [19] extended the above results of Fujimoto to nondegenerate meromorphic mappings from a ball in $\mathbb{C}^n$ into a complex projective subvariety V of $\mathbb{P}^m(\mathbb{C})$ sharing hypersurfaces of $\mathbb{P}^m(\mathbb{C})$ in N-subgeneral position with respect to V. In 2020, M. Ru and N. Sibony [15] developed Nevanlinna's theory for a class of holomorphic maps when the source is a disc in $\mathbb{C}$.

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There is an interesting problem that is to construct explicitly Nevanlinna's theory for meromorphic mappings from a Stein complex manifold (or a complete Kähler manifold) to a compact complex manifold.

The first main aim of this paper is to deal with the above mentioned problem in a special case when the Stein manifold is the unit polydisc $\mathbb{D}^m$ in $\mathbb{C}^m$ and $f : \mathbb{D}^m \to V$ is an algebraically nondegenerate meromorphic mapping sharing hypersurfaces in N-subgeneral position, where V is a smooth complex projective variety.

Recall the following notion. Let $V \subset \mathbb{P}^n(\mathbb{C})$ be a smooth complex projective variety of dimension $k \geq 1$ and Q_i $(1 \leq i \leq q)$ be q hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. The family of hypersurfaces $\{Q_i\}_{i=1}^q$ is said to be in N-subgeneral position in V if for any $R \subset \{1, \cdots, q\}$ with the cardinality $|R| = N + 1$,

$$\bigcap_{j\in R} Q_j \cap V = \emptyset.$$

If they are in k-subgeneral position, we also say that they are in general position in V.

First of all, we prove the following theorem for algebraically nondegenerate meromorphic mappings from $\mathbb{D}^m$ into the complex projective space $\mathbb{P}^n(\mathbb{C})$.

Theorem 1.1. *Let $f : \mathbb{D}^m \to \mathbb{P}^n(\mathbb{C})$ be an algebraically nondegenerate meromorphic mapping and let Q_i $(1 \leq i \leq q, q > (N - n + 1)(n + 1))$ be hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ of degree d_i, located in N-subgeneral position. Let d be the least common multiple of $d_1, ..., d_q$, i.e., $d = lcm(d_1, ..., d_q)$. Then for every $\epsilon > 0$, the following holds*

$$\| \ (q-(N-n+1)(n+1)-\epsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d_i} N^{[M_0]}(r, \nu_{Q_i(f)}) + O(\log^+(T_f(r))) + O(\log|1-r|^{-1}),$$

for $r \in [r_0, 1]\backslash E$, where $E \subset [0, 1]$ with $\int_E dr/(1 - r) < \infty$ and $M_0 = d^n[(n + 1) + (N - n + 1)(n + 1)^3 I(\epsilon^{-1})]^n - 1$.

Here, by the notation $I(x)$ we denote the smallest integer number which is not smaller than the real number x.

Next, we show the following theorem for algebraically nondegenerate meromorphic mappings from $\mathbb{D}^m$ into a smooth complex projective variety $V \subset \mathbb{P}^n(\mathbb{C})$.

Theorem 1.2. *Let $V \subset \mathbb{P}^n(\mathbb{C})$ be a smooth complex projective variety of dimension $k \geq 1$. Let $Q_1, ..., Q_q$ $(q > (N - k + 1)(k + 1))$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_i, located in $N-$subgeneral position in V. Let d be the least common multiple of $d_1, ..., d_q$, i.e., $d = lcm(d_1, ..., d_q)$. Let $f : \mathbb{D}^m \to V$ be an algebraically nondegenerate meromorphic* *mapping. Then, for every* $\epsilon \in (0, t_k)$,

$$\big|\big|\ (q-(N-k+1)(k+1)-\epsilon)\,T_f(r) \le \sum_{i=1}^{q} \frac{1}{d_i} N^{[M_0]}_{Q_i(f)}(r) + O(\log^+(T_f(r))) + O(\log|1-r|^{-1}),$$

where

$$t_k = \begin{cases} \dfrac{Nq}{2} & \text{if } k=1 \\ k-1 & \text{if } k \ge 2. \end{cases}$$

and

$$M_0 = \begin{cases} [7(\deg V)^2 e d^2 N q \epsilon^{-1}] & \text{if } k=1 \\ \Big[\deg(V)^{k+1} e^k d^{k^2+k} (N-k+1)^k (2k+4)^k q^k \epsilon^{-k}\Big] & \text{if } k \ge 2. \end{cases}$$

Here, by the notation $[x]$ *we denote the biggest integer which does not exceed the real number* x.

The second aim is to establish a sharp non-truncated defect relation for meromorphic mappings from an n-dimensional closed complex submanifold of $\mathbb{C}^l$ into a compact complex manifold X sharing divisors in N-subgeneral position. Namely, we prove the following.

Theorem 1.3. *Let* M *be an* n*-dimensional closed complex submanifold of* $\mathbb{C}^l$ *and* ω *be its Kähler form that is induced from the canonical Kähler form of* $\mathbb{C}^l$*. Let* $\mathcal{L} \to X$ *be a holomorphic line bundle over a compact complex manifold* X*. Fix a positive integer* d *and let* $d_1, d_2, \cdots, d_q$ *be positive divisors of* d*. Let* E *be a* $\mathbb{C}$*-vector subspace of dimension* $m+1$ *of* $H^0(X, \mathcal{L}^d)$*. Let* σ_j $(1 \le j \le q)$ *be in* $H^0(X, \mathcal{L}^{d_j})$ *such that* $\sigma_1^{\frac{d}{d_1}}, \cdots, \sigma_q^{\frac{d}{d_q}} \in E$*. Set* $D_j = (\sigma)_0$ $(1 \le j \le q)$*. Assume that* $D_1, \cdots, D_q$ *are in* N*-subgeneral position with respect to* E*. Let* $f : M \to X$ *be a meromorphic mapping satisfying* $f(M) \not\subset D_j$ *for* $1 \le j \le q$ *and* $\overline{f(M)} \cap B(E) = \varnothing$*. Assume that, there exists a holomorphic section* ν *of* K_M^{-1} *such that for some basis* $\{c_1, c_2, \cdots, c_{m+1}\}$ *of* E *and* l *large enough,*

$$\mathrm{dd^c}\log(|c_1(f)|^2 + \cdots + |c_{m+1}(f)|^2)^{l/d} \ge \mathrm{dd^c}(\nu\bar{\nu}\omega^n).$$

Then,

$$\sum_{i=1}^{q} \bar{\delta}_{f,E}(D_i) \le 2N.$$

2. Nevanlinna theory for meromorphic maps from a polydisc in $\mathbb{C}^m$ into complex projective subvarieties in $\mathbb{P}^n(\mathbb{C})$

In this section we present Nevanlinna theory for meromorphic maps from a polydisc in $\mathbb{C}^m$ into the complex projective space $\mathbb{P}^n(\mathbb{C})$. The general strategy is the same as in the case of meromorphic maps from balls.

Firstly we fix some notations. For a positive real number $r \in (0,1]$, define $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. Let $\partial'\mathbb{D}_r^m := \{|z_j| = r : 1 \leq j \leq m\}$. We will identify $\partial'\mathbb{D}_r^m$ with $[0, 2\pi]^m$ via the isomorphism

$$r(e^{it_1}, \cdots, e^{it_m}) \longmapsto t = (t_1, \cdots, t_m).$$

When $r = 1$, we write $\mathbb{D}$ instead of $\mathbb{D}_1$.

Let $f = (f_1, \cdots, f_{n+1})$ be a meromorphic map from the polydisc $\mathbb{D}^m$ into $\mathbb{P}^n(\mathbb{C})$, where f_j is holomorphic function on $\mathbb{D}^m$ for $1 \leq j \leq n+1$ and $\cap_{j=1}^{n+1}\{f_j = 0\}$ is of codimension ≥ 2. Define

$$g := \max_{1 \leq j \leq m} \log |z_j|$$

which is the pluricomplex Green function on $\mathbb{D}^m$ with pole at the origin, see [10],[2]. Recall that $\mathrm{d}^{\mathrm{c}} = i(\bar{\partial} - \partial)/(4\pi)$, hence $\mathrm{dd}^{\mathrm{c}} = i\partial\bar{\partial}/(2\pi)$. Let

$$\omega := \mathrm{dd}^{\mathrm{c}} \log(|w_1|^2 + \cdots + |w_{n+1}|^2)$$

be the Fubini-Study form of $\mathbb{P}^n(\mathbb{C})$, where $[w_1 : \cdots : w_{n+1}]$ are the homogeneous coordinates on $\mathbb{P}^n(\mathbb{C})$.

Fix a constant $r_0 \in (0,1)$. *The characteristic function of f* is defined by

$$T_f(r) := \int_{\log r_0}^{\log r} ds \int_{\{g<s\}} f^*\omega \wedge (\mathrm{dd}^{\mathrm{c}} g^2)^{m-1}.$$

Let Q be a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$. Set $\mathcal{T}_d := \{(i_1, ..., i_{n+1}) \in \mathbb{N}^{n+1} \; | i_1 + i_2 + \cdots + i_{n+1} = d\}$ and denote by $\|\cdot\|$ the canonical Hermitian metric on the hypersurface line bundle of $\mathbb{P}^n(\mathbb{C})$. If $Q(x) = \sum_{I \in \mathcal{T}_d} a_I x^I$, where the constants a_I are not all zeros, then we set $Q(f) = \sum_{I \in \mathcal{T}_d} a_I f^I$ with $f^I = f_1^{i_1} ... f_{n+1}^{i_{n+1}}$ for $I = (i_1, ..., i_{n+1}) \in \mathcal{T}_d$.

For $1 \leq k \leq \infty$, *the truncated counting function of f to level k with respect to Q* is defined by

$$N_f^{[k]}(r, Q) := \int_{\log r_0}^{\log r} ds \int_{\{g<s\}} \min\{[f^*Q], k\} \wedge (\mathrm{dd}^{\mathrm{c}} g^2)^{m-1}$$

and *the proximity function* is

$$m_f(r, Q) = \frac{1}{(2\pi)^m} \int_{\partial'\mathbb{D}_r^m} \log \frac{1}{||Q \circ f||^2} \, dt.$$

For simplicity, we omit the superscript $^{[k]}$ when $k = \infty$. Applying the Lelong-Jensen formula to g (see [2]), we have

$$T_f(r) = N_f(r, Q) + m_f(r, Q) - m_f(r_0, Q) \tag{1}$$

and

(2)

$$T_f(r) = \frac{1}{(2\pi)^m}\int_{\partial'\mathbb{D}_r^m} \log\|f\|^2 dt + O(1), \quad N_f(r,Q) = \frac{1}{(2\pi)^m}\int_{\partial'\mathbb{D}_r^m} \log\|Q\circ f\|^2 dt + O(1)$$

where $\|f\|^2 := |f_1|^2 + \cdots + |f_{n+1}|^2$.

Recall that $\log^+ x := \max\{\log x, 0\}$ for $x \in \mathbb{R}^+$. For an n-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of non-negative integers, put $|\alpha| := \sum_{j=1}^n \alpha_j$. In what follows, the notation $\lesssim$ means $\leq$ up to a multiplicative constant independent of r.

Proposition 2.1. *Let g be a meromorphic function on $\mathbb{D}^m$. Let $\alpha \in (\mathbb{Z}^+)^m$ and p, p' positive real numbers. Assume that $p|\alpha| < p' < 1$. Then, there exist a constant C independent of r and a subset $E \subset [0,1]$ satisfying $\int_E \frac{dr}{1-r} < \infty$ such that for all $r \in [r_0, 1)\backslash E$ we have*

$$\int_{\partial'\mathbb{D}_r^m} \Big|\frac{D^\alpha g}{g}\Big|^p dt \leq C|1-r|^{p'} T_g(r)^{p'} \tag{3}$$

and

$$m_{\frac{D^\alpha g}{g}}(r,\infty) \leq C\big(\log^+ T_g(r) + \log|1-r|^{-1}\big). \tag{4}$$

Proof. We have

$$m_{\frac{D^\alpha g}{g}}(r,\infty) \leq \int_{\partial'\mathbb{D}_r^m} \log\Big(\Big|\frac{D^\alpha g}{g}\Big| + 1\Big) dt \leq \log \int_{\partial'\mathbb{D}_r^m} \Big|\frac{D^\alpha g}{g}\Big| dt + O(1)$$

by concavity of log function. Using the last inequality, we observe that (4) is deduced directly from (3) by choosing $p = 1/(3|\alpha|), p' = 3p/2$. Hence it remains to prove (3). We will prove it by induction on $|\alpha|$.

Let $r' \in (r_0, 1)$. Consider the case where $|\alpha| = 1$. Without loss of generality, we can suppose $\alpha = (1, 0, \cdots, 0)$. Fix $z' := (z_2, \cdots, z_n)$ such that $\tilde{g}_{z'} := g(\cdot, z')$ is a meromorphic function on $\mathbb{D}$. Note that the last condition holds for almost everywhere $z' \in \partial'\mathbb{D}^{m-1}$. Let $0 < p_1 < p_1' < 1$ be real numbers. Either using [5, Th. 3.1] in dimension one or using directly Riesz's representation formula for $\tilde{g}_{z'}$, one obtains

$$\int_{\partial\mathbb{D}_r} \Big|\frac{Dg}{g}\Big|^{p_1} dt \lesssim |r'-r|^{-p_1'} T_{\tilde{g}_{z'}}^{p_1'}(r'), \tag{5}$$

for every $r \in [r_0, r')$. Integrating (5) on $z' \in \partial'\mathbb{D}_r^{n-1}$ gives

(6)

$$\int_{\partial'\mathbb{D}_r^m} \Big|\frac{Dg}{g}\Big|^{p_1} dt \lesssim C|1-r|^{-p_1'} \int_{z'\in\partial'\mathbb{D}_r^{m-1}} T_{\tilde{g}_{z'}}^{p_1'}(r') \lesssim C|1-r|^{-p_1'} \Big\{\int_{z'\in\partial'\mathbb{D}_r^{m-1}} T_{\tilde{g}_{z'}}(r')\Big\}^{p_1'}.$$

Write $g = g_1/g_2$, where g_1, g_2 are holomorphic on $\mathbb{D}^m$ and have no common divisor. Put $\|g\|^2 := |g_1|^2 + |g_2|^2$. By (2), we have

$$T_{\tilde{g}_{z'}}(r') = \frac{1}{2\pi}\int_{\partial\mathbb{D}_{r'}} \log\|g\|^2 dt + O(1).$$

Integrating the last equality on $z' \in \partial'\mathbb{D}_r^{n-1}$ yields

$$\int_{z'\in\partial'\mathbb{D}_r^{m-1}} T_{\tilde{g}_{z'}}(r') = \frac{1}{2\pi}\int_{\partial\mathbb{D}\times\partial'\mathbb{D}_{r'}^{m-1}} \log\|g\|^2 dt + O(1) \lesssim \int_{\partial'\mathbb{D}_{r'}^m} \log\|g\|^2 dt = T_g(r') \tag{7}$$

because $\log\|g\|^2$ is psh. Combining (7) and (6) gives

$$\int_{\partial'\mathbb{D}_r^m} \Big|\frac{Dg}{g}\Big|^{p_1} dt \lesssim |1-r|^{-p_1'} T_g^{p_1'}(r'). \tag{8}$$

Choose $r' = r + (1-r)/(eT_{\tilde{g}_{z'}}(r'))$. Using (8) and [8, Le. 2.4], we obtain (3) for $\alpha = (1, 0, \cdots, 0)$.

Now we suppose that (3) holds for $\alpha' \in (\mathbb{Z}^+)^m$ in place of α, where $|\alpha'| < |\alpha|$. As already observed, (4) also holds for α' in place of α. By (1) and (2), we get

$$\begin{aligned} T_{D^{\alpha-1}g}(r) &\le T_{D^{\alpha-1}g/g}(r) + T_g(r) + O(1) \\ &\le m_{D^{\alpha-1}g/g}(r,\infty) + N_{D^{\alpha-1}g/g}(r,\infty) + T_g(r) + O(1) \\ &\le C\big(T_g(r) + \log^+ T_g(r) + \log|1-r|^{-1}\big) \end{aligned} \tag{9}$$

by the induction hypothesis and the fact that $N_{D^{\alpha-1}g/g}(r,\infty) \le N_g(r,\infty) \le T_g(r) + O(1)$. Let $\{\alpha_k\}_{1\le k\le|\alpha|}$ be an increasing sequence of m-tuples satisfying

$$\alpha_1 = 0, \quad |\alpha_k| = |\alpha_{k-1}| + 1,$$

for all $k \ge 2$. Let $p|\alpha| < p'' < p'$. By applying (3) to $(D^{\alpha_{k-1}}, \alpha_k - \alpha_{k-1}, p|\alpha|)$, there is a subset $E \subset [r_0, 1)$ with $\int_E dr/(1-r) < \infty$ such that for $r \in [r_0,1)\setminus E$ and $1 \le k \le |\alpha|$, we have

$$\int_{\partial'\mathbb{D}_r^m} \Big|\frac{D^{\alpha_k}g}{D^{\alpha_{k-1}}g}\Big|^{p|\alpha|} \lesssim |1-r|^{p''} T_{D^{\alpha_{k-1}}g}(r)^{2p''} \lesssim |1-r|^{p''} T_g(r)^{2p''} + C|1-r|^{p''}. \tag{10}$$

On the other hand, observe that

$$\int_{\partial'\mathbb{D}_r^m} \Big|\frac{D^{\alpha}g}{g}\Big|^{p} dt = \int_{\partial'\mathbb{D}_r^m} \prod_{k=2}^{|\alpha|} \Big|\frac{D^{\alpha_k}g}{D^{\alpha_{k-1}}g}\Big|^{p} dt \le \frac{1}{|\alpha|}\sum_{k=2}^{|\alpha|}\int_{\partial'\mathbb{D}_r^m} \Big|\frac{D^{\alpha_k}g}{D^{\alpha_{k-1}}g}\Big|^{p|\alpha|} dt$$

which is

$$\lesssim |1-r|^{p''}\sum_{k=1}^{|\alpha|} T_{D^{\alpha_{k-1}}g}(r)^{p''} + |1-r|^{p''} \lesssim |1-r|^{p'} T_g(r)^{p'}$$

by (10) and (9). The proof is finished. □

In order to prove Theorem 1.1, we need a following lemma which is infered from the Lemma 3.1 in [13].

Lemma 2.2. *Let V be a smooth projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension k. Let $Q_1, ..., Q_{N+1}$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of the same degree $d \geq 1$ located in N-subgeneral position. Then there exist k hypersurfaces $P_2, ..., P_{k+1}$ of the forms*

$$P_t = \sum_{j=2}^{N-k+t} c_{tj} Q_j, \ c_{tj} \in \mathbb{C}, \ t = 2, ...k+1,$$

such that $\cap_{t=1}^{k+1} P_t = \emptyset$, where $P_1 = Q_1$.

Let $f : \mathbb{D}^m \to \mathbb{P}^k(\mathbb{C})$ be an linearly nondegenerate meromorphic mapping. Let $\mathcal{M}$ be the field of meromorphic functions on $\mathbb{D}^m$. For $z \in \mathbb{D}^m$, let $\mathcal{M}_z$ be the field of germs of meromorphic functions at z. For $i \in \mathbb{N}$, denote by $\mathcal{F}_z^i$ the subspace of $\mathcal{M}_z$-vector space $\mathcal{M}_z^{k+1}$ generated by $D^\alpha f$ for any m-tuple α with $|\alpha| \leq i$. Set $l_i = \dim \mathcal{F}_z^i$ which is dependent of z.

Repeating the arguments in Proposition 4.5 in [5], we have the following lemma.

Lemma 2.3. *Let $f : \mathbb{D}^m \to \mathbb{P}^k(\mathbb{C})$ be an linearly nondegenerate meromorphic mapping. Then there exist $\alpha_1, ..., \alpha_{k+1} \in \mathbb{N}^m$ such that the following conditions hold:*

(i) $|\alpha_1| + ... + |\alpha_{k+1}| \leq \dfrac{k(k+1)}{2}$, $|\alpha_i| \leq k$ $(1 \leq i \leq k+1)$.

(ii) The germs of $\mathcal{D}^{\alpha_1} f, ..., \mathcal{D}^{\alpha_{l_i}} f$ at any point $z \in \mathbb{D}^m$ is a basic of $\mathcal{F}_z^i$ for $0 \leq i \leq k$. In particular, that $\mathcal{D}^{\alpha_1} f, ..., \mathcal{D}^{\alpha_{k+1}} f$ is a basic of $\mathcal{M}^{k+1}$. Moreover, we have that the generalized Wronskian of f

$$W_{\alpha_1,...,\alpha_{k+1}}(f) := det(\mathcal{D}^{\alpha_i} f_j : 1 \leq i, j \leq k+1) \not\equiv 0.$$

The second property implies, in particular, that $D^{\alpha_1} f, \cdots, D^{\alpha_{k+1}} f$ is a basis of $\mathcal{M}^{k+1}$ because the last space has dimension $k+1$ over $\mathcal{M}$ and $D^{\alpha_1} f, \cdots, D^{\alpha_{k+1}} f$ are linearly independent over $\mathcal{M}$. Moreover, we also have that the *generalized Wronskian of* f

$$W_{\alpha_1,\cdots,\alpha_{k+1}}(f) := \det(D^{\alpha_i} f_j : 1 \leq i, j \leq k+1) \not\equiv 0. \tag{11}$$

By [5, Pro. 4.10], we have

$$\left(\frac{W_{\alpha_1,\cdots,\alpha_{k+1}}(f)}{f_1 \cdots f_{k+1}} \right)_\infty \leq \sum_{j=1}^{k+1} \min\{(f_j)_0, k\} \tag{12}$$

Remark 2.4. By [5], such $\alpha_1, \cdots, \alpha_{k+1}$ also exist for every meromorphic map $f = (f_1, \cdots, f_{k+1})$ from a connected open subset M of $\mathbb{C}^m$ to $\mathbb{P}^k$, where $f_1, \cdots, f_{k+1}$ are holomorphic functions on M with $\cap_{j=1}^{k+1}\{f_j = 0\}$ is of codimension ≥ 2 in M.

The following result will be important for us later.

Lemma 2.5. *Let $\alpha_1, \cdots, \alpha_{k+1}$ be as above. Let U be a connected domain of $\mathbb{C}^m$ and $\Phi : U \to \mathbb{D}^m$ a biholomorphism. Denote by $\alpha'_1, \cdots, \alpha'_{k+1}$ be the set of m-tuples satisfying Properties $(i), (ii)$ for $f \circ \Phi$ in place of f. Then there exists a nowhere vanishing holomorphic function ψ on U such that*

$$W_{\alpha'_1,\cdots,\alpha'_{k+1}}(f \circ \Phi) = \psi W_{\alpha_1,\cdots,\alpha_{k+1}}(f) \circ \Phi. \tag{13}$$

Proof. The inequality (12) is deduced from [5, Pro. 4.10]. Now we prove (13). Observe that

$$D^{\alpha}(f \circ \Phi) = \sum_{\beta:\beta\leq\alpha} g_\beta D^\beta f \circ \Phi,$$

for some holomorphic functions g_β on U. Applying the last equality to $\alpha = \alpha_j$ for $1 \leq j \leq k+1$ and using the fact that $\{D^{\alpha_j} f\}_{1\leq j\leq k+1}$ generates $\mathcal{M}^{k+1}$, we see that $D^{\alpha'_j}(f \circ \Phi)$ is a linear combination of $\{D^{\alpha_j} f \circ \Phi\}_{1\leq j\leq k+1}$ with coefficients in $\mathcal{M}$. This yields that there exists a meromorphic function ψ on U such that (13) holds. It remains to show that ψ is holomorphic and nowhere vanishing on U. By applying the above arguments to Φ^{-1}, we can find a meromorphic function ψ' on $\mathbb{D}^m$ for which

$$W_{\alpha_1,\cdots,\alpha_{k+1}}(f) = \psi' W_{\alpha'_1,\cdots,\alpha'_{k+1}}(f \circ \Phi) \circ \Phi^{-1}.$$

Thus we obtain

$$W_{\alpha_1,\cdots,\alpha_{k+1}}(f) \circ \Phi = (\psi' \circ \Phi) W_{\alpha'_1,\cdots,\alpha'_{k+1}}(f \circ \Phi) = (\psi' \circ \Phi)\psi W_{\alpha_1,\cdots,\alpha_{k+1}}(f) \circ \Phi$$

which implies that

$$(\psi' \circ \Phi)\psi \equiv 1$$

on the complement of the divisor of $W_{\alpha_1,\cdots,\alpha_{k+1}}(f) \circ \Phi$ in U. Since $W_{\alpha_1,\cdots,\alpha_{k+1}}(f)$ is not identically zero and Φ is biholomorphic, that complement is an open dense subset of U. This yields that $(\psi' \circ \Phi)\psi \equiv 1$ on U. In other words, ψ is nowhere vanishing. The proof is finished. □

Proof of Theorem 1.1

Fix a homogeneous coordinates $(\omega_0 : \cdots : \omega_n)$ of $\mathbb{P}^n(\mathbb{C})$ and take a reduced representation $(f_0 : \cdots : f_n)$ of f. It suffices to prove that the theorem holds in the case where all of $d_i's$ are equal to d. Indeed, if the theorem holds in that case, then for ϵ and M_0 as in the statement of the theorem, we have

$$\| \ (q-(N-n+1)(n+1)-\epsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d} N^{[M_0]}(r, \nu_{Q_i^{d/d_i}(f)}) + O(\log^+ T_f(r)) + O(\log|1-r|^{-1}).$$

Assume that $z \in \mathbb{D}^m$ is a zero of $Q_i(f)$ with multiplicity u. Then z is a zero of $Q_i^{d/d_i}(f)$ with multiplicity $u\frac{d}{d_i}$. Therefore

$$N^{[M_0]}(r, \nu_{Q_i^{d/d_i}(f)}) \leq \frac{d}{d_i} N^{[M_0]}(r, \nu_{Q_i(f)}).$$

This implies that

$$\begin{aligned}\| \ (q - (N-n+1)(n+1) - \epsilon) T_f(r) &\leq \sum_{i=1}^{q} \frac{1}{d} N^{[M_0]}(r, \nu_{Q_i^{d/d_i}(f)}) + O(\log^+ T_f(r)) \\ &\quad + O(\log |1-r|^{-1}). \\ &\leq \sum_{i=1}^{q} \frac{1}{d_i} N^{[M_0]}(r, \nu_{Q_i(f)}) + O(\log^+ T_f(r)) \\ &\quad + O(\log |1-r|^{-1}).\end{aligned}$$

Hence, we may assume that $Q_1, ..., Q_q$ $(q > (N-n+1)(n+1))$ have the same degree of d.

Denote by $\mathcal{I}$ the set of all permutations of $\{1, ..., q\}$. Then $n_0 := \sharp \mathcal{I} = q!$ and we may assume that $\mathcal{I} = \{\sigma_1, ..., \sigma_{n_0}\}$, where $\sigma_i = (\sigma_i(1), ..., \sigma_i(q)) \in \mathbb{N}^q$ and $\sigma_1 < \sigma_2 < ... < \sigma_{n_0}$ in the lexicographic order.

Since $\{Q_i\}_{i=1}^q$ are hypersurfaces located in N-subgeneral position and by Lemma 2.2, for each $\sigma_i \in \mathcal{I}$, there exist n hypersurfaces, we may assume that $P_{i,2}, ..., P_{i,n+1}$ which have the forms

$$P_{i,t} = \sum_{j=2}^{N-n+t} c_{i,t}^j Q_{\sigma_i(j)}, \ c_{i,t}^j \in \mathbb{C}, \ t \in 2, ..., n+1 \tag{14}$$

such that $\bigcap_{t=1}^{n+1} P_{i,t} = \emptyset$, where $P_{i,1} = Q_{\sigma_i(1)}$. Fix an element $\sigma_{i_0} \in \mathcal{I}$. Denote by $S_{\sigma_{i_0}}$ the set of all points $z \in \mathbb{D}^m \setminus \bigcup_{i=1}^q Q_i(f)^{-1}(\{0\})$ such that

$$|Q_{\sigma_i(1)}(f)(z)| \leq |Q_{\sigma_i(2)}(f)(z)| \leq ... \leq |Q_{\sigma_i(q)}(f)(z)|.$$

For $z \in S_{\sigma_{i_0}}$, by (14), there exists a positive constant $C \geq 1$ which is chosen common for all $\sigma_i \in \mathcal{I}$, such that

$$|P_{i,t}(f)(z)| \leq C \max_{1 \leq j \leq N-n+t} |Q_{\sigma_i(j)}(f)(z)|, \ 1 \leq t \leq n+1.$$

Moreover, by the compactness of $\mathbb{P}^n(\mathbb{C})$, there is a constant B (chosen commonly for all σ_i) such that

$$B = \max_{\omega \in \mathbb{P}^n(\mathbb{C})} \frac{\sqrt{|\omega_0|^2 + \cdots + |\omega_n|^2}^d}{\max_{j=1}^{N+1} |Q_{\sigma_i(j)}(\omega_0, ..., \omega_n)|},$$

where $\omega = (\omega_0 : \cdots : \omega_n)$. Then, we have

$$\|f(z)\|^d \leq B \max\{|Q_{\sigma_i(1)}(f)(z)|, ..., |Q_{\sigma_i(N+1)}(f)(z)|\} \ \forall \ z \in \mathbb{C}. \tag{15}$$

Put $\lambda = d[(n+1)+(N-n+1)(n+1)^3 I(\epsilon^{-1})]$. Denote V_λ the space of homogeneous polynomials in $\mathbb{C}[X_0,...,X_n]$ of degree λ and the zero polynomial. By arranging the lexicographic order, the n-tuples $(j)=(j_1,...,j_n)$ of non-negative integers satisfies that $\|(j)\| := \sum_{k=1}^n j_k \le \frac{\lambda}{d} = (n+1)+(N-n+1)(n+1)^3 I(\epsilon^{-1})$. Define the spaces

$$I^i_{(j)} = \sum_{(e)\ge(j)} P^{e_1}_{i,1}\cdots P^{e_n}_{i,n} V_{\lambda - d\|(e)\|}.$$

Set $M^i_{(j)} := \dim \frac{I^i_{(j)}}{I^i_{(j')}}$, where (j') follows (j) in the ordering. Then

$$M^i_{(j)} = d^n, \quad \text{and } d\|(j)\| < \lambda - nd.$$

Put $M := \dim V_\lambda$. Then

$$M = \binom{n+\lambda}{n} = \frac{(\lambda+1)\cdots(\lambda+n)}{n!}.$$

Therefore

$$M - 1 \le \begin{pmatrix} \lambda+n \\ n \end{pmatrix} - 1 \le \lambda^n - 1 = d^n[(n+1)+(N-n+1)(n+1)^3 I(\epsilon^{-1})]^n - 1 = M_0.$$

For $(j') = (j'_1,...,j'_n) \in \mathbb{N}^n$ with $\|(j')\| \le \frac{\lambda}{d}$, we may choose a basis of $I^i_{(j')}$ such that every element $\psi^i_{(j'),l}$ of its has a form

$$\psi^i_{(j'),l} = P^{j'_1}_{i,1}...P^{j'_n}_{i,n} h_l, \text{ where } (j') = (j'_1,...,j'_n),\ h_l \in V_{\lambda - d\|(j')\|}. \tag{16}$$

After that, we give $M^i_{(j)}$ supplementary elements which have form as (16) in $I^i_{(j)}$, where (j') follows (j) in the ordering, such that they are a basis of $I^i_{(j)}$. We continue this process until $I^i_{(j)} = V_\lambda$ and we obtain a basis of V_λ which includes M elements as follows.

$$\psi^i_l = P^{j_1}_{i,1}...P^{j_n}_{i,n} h_l, \text{ where } (j) = (j_1,...,j_n),\ h_l \in V_{\lambda - d\|(j)\|},\ 1 \le l \le M.$$

Then we have

$$\begin{aligned}|\psi^i_l(f)(z)| &\le |P_{i,1}(f)(z)|^{j_1}...|P_{i,n}(f)(z)|^{j_n}|h_l(f)(z)| \\ &\le C_1 |P_{i,1}(f)(z)|^{j_1}...|P_{i,n}(f)(z)|^{j_n}\|f(z)\|^{\lambda - d\|(j)\|} \\ &= C_1\left(\frac{|P_{i,1}(f)(z)|}{\|f(z)\|^d}\right)^{j_1}....\left(\frac{|P_{i,n}(f)(z)|}{\|f(z)\|^d}\right)^{j_n}\|f(z)\|^\lambda,\end{aligned}$$

where C_1 is a positive constant independent from l, i, f and z. Therefore

$$\begin{aligned}\log\prod_{l=1}^M |\psi^i_l(f)(z)| \le \sum_{(j)} M^i_{(j)}\left(j_1\log\frac{|P_{i,1}(f)(z)|}{\|f(z)\|^d} + ... + j_n \log\frac{|P_{i,n}(f)(z)}{\|f(z)\|^d}\right) \\ + M\lambda\log\|f(z)\| + M\log C_1,\end{aligned}$$

If we put

$$c_j^i = \sum_{(j)} M^i_{(j)} j_k,\ 1 \le k \le n,$$
$$c = \min_{j,i} c_j^i,$$

then from above inequality, we get

$$\log \prod_{s=1}^{M} |\psi_l^i(f)(z)| \le \log \left(\prod_{j=1}^{n} \left(\frac{|P_{i,j}(f)(z)|}{\|f(z)\|^d} \right)^{c_j^i} \right) + M\lambda \log \|f(z)\| + M \log C_1.$$

It implies that

$$\begin{aligned} \log \frac{\|f(z)\|^{d\sum_{j=1}^n c_j^i}}{\prod_{j=1}^n |P_{i,j}(f)(z)|^{c_j^i}} &\le - \log \prod_{s=1}^{M} |\psi_l^i(f)(z)| + M\lambda \log \|f(z)\| + M \log C_1 \\ &= \log \frac{\|f(z)\|^{M\lambda}}{\prod_{s=1}^{M} |\psi_l^i(f)(z)|} + M \log C_1. \end{aligned} \tag{17}$$

Take a basis $\{\phi_1, ..., \phi_M\}$ of V_λ, $\psi_s^i(f) = L_s^i(\phi(f))$ $(1 \le s \le M)$, where L_s^i are linear forms.

Consider the meromorphic mapping ϕ with a reduced representation

$$\phi = (\phi_1(f) : \cdots : \phi_M(f)).$$

Since f is algebraically nondegenerate over $\mathbb{D}^m$, it implies that $\phi = (\phi_1(f) : \cdots : \phi_M(f))$ is linearly nondegenerate over $\mathbb{D}^m$. Then, from Lemma 2.3, there exist $\alpha_1, ..., \alpha_M \in \mathbb{N}^m$ such that

$$W(\phi) := W_{\alpha_1,...,\alpha_M}(\phi_1(f), ..., \phi_M(f)) = \det\left(\mathcal{D}^{\alpha_k}\phi_j : 1 \le k, j \le M\right) \not\equiv 0.$$

Hence from Lemma 2.5, we have

$$\begin{aligned} W_{\alpha_1,...,\alpha_M}(\psi^i(f)) &= W_{\alpha_1,...,\alpha_M}(\psi_1^i(f), ..., \psi_M^i(f)) \\ &= \det\left(\mathcal{D}^{\alpha_k}\psi_j^i : 1 \le k, j \le M\right) = C_2 W(\phi) \not\equiv 0, \end{aligned}$$

where C_2 is a nonzero constant.

From (15), for $z \in S_{\sigma_{i_0}}$ we have

$$\begin{aligned}
\log \prod_{i=1}^{q} \frac{\|f(z)\|^d}{|Q_i(f)(z)|} &\leq \log \left(B^{q-N-1} \prod_{j=1}^{N+1} \frac{\|f(z)\|^d}{|Q_{\sigma_{i_0}(j)}(f)(z)|} \right) \\
&\leq \log \left(B^{q-N-1} C^n \frac{\|f(z)\|^{(N+1)d}}{\prod_{j=1}^{N-n+1} |Q_{\sigma_{i_0}(j)}(f)(z)| \prod_{j=2}^{n+1} |P_{i_0,j}(f)(z)|} \right) \\
&\leq \log \left(B^{q-N-1} C^n \frac{\|f(z)\|^{(N+1)d}}{|P_{i_0,1}(f)(z)|^{(N-n+1)} \prod_{j=2}^{n+1} |P_{i_0,j}(f)(z)|} \right) \\
&\leq \log \left(B^{q-N-1} C^n D^{N-n} \frac{\|f(z)\|^{[(N+1)d+(N-n)nd]}}{\prod_{j=1}^{n+1} |P_{i_0,j}(f)(z)|^{(N-n+1)}} \right) \\
&\leq \log \frac{\|f(z)\|^{(N-n+1)(n+1)d}}{\prod_{j=1}^{n+1} |P_{i_0,j}(f)(z)|^{(N-n+1)}} + O(1) \\
&\leq \log \frac{\|f(z)\|^{(N-n+1)nd}}{\prod_{j=1}^{n} |P_{i_0,j}(f)(z)|^{(N-n+1)}} + O(1),
\end{aligned} \tag{18}$$

where the term $O(1)$ does not depend on z and D is chosen common for all $\sigma_i \in \mathcal{I}$, such that

$$|P_{i,j}(f)(z)| \leq D\|f(z)\|^d, \ \forall \ z \in S_{\sigma_{i_0}}.$$

Therefore, by using (17) and (18) we have

$$\begin{aligned}
&\log \frac{\|f(z)\|^{qdc} |W(\phi)(z)|^{(N-n+1)}}{\prod_{j=1}^{q} |Q_j(f)(z)|^c} \\
&= \log \left(\prod_{i=1}^{q} \frac{\|f(z)\|^{dc}}{|Q_i(f)(z)|^c} \times |W(\phi)(z)|^{(N-n+1)} \right) \\
&\leq \log \frac{\|f(z)\|^{(N-n+1)ndc} |W(\phi)(z)|^{(N-n+1)}}{\prod_{j=1}^{n} |P_{i_0,j}(f)(z)|^{(N-n+1)c}} + O(1) \\
&\leq \log \frac{\|f(z)\|^{(N-n+1)d\sum_{j=1}^{n} c_j^i} |W(\phi)(z)|^{(N-n+1)}}{\prod_{j=1}^{n} |P_{i_0,j}(f)(z)|^{(N-n+1)c_j^i}} + O(1) \\
&\leq \log \frac{\|f(z)\|^{(N-n+1)M\lambda} |W(\psi^{i_0})(z)|^{(N-n+1)}}{\prod_{j=1}^{M} |\psi_j^{i_0}(f)(z)|^{(N-n+1)}} + O(1),
\end{aligned} \tag{19}$$

where $\psi^{i_0} = (\psi_1^{i_0}(f), ..., \psi_M^{i_0}(f))$ and the term $O(1)$ depends only on λ and $\{Q_j\}_{j=1}^q$. It implies that

$$\log \frac{\|f(z)\|^{[qdc-(N-n+1)M\lambda]} |W(\phi)(z)|^{(N-n+1)}}{\prod_{j=1}^{q} |Q_j(f)(z)|^c} \leq \log \frac{|W(\psi^{i_0})(z)|^{(N-n+1)}}{\prod_{j=1}^{M} |\psi_j^{i_0}(f)(z)|^{(N-n+1)}} + O(1)$$

for all z in $S_{\sigma_{i_0}}$ which are not zero of any functions $Q_j(f), P_{i_0,j}(f)$ $(1 \le i \le q)$. Therefore, we have

$$\log \frac{\|f(z)\|^{qdc-(N-n+1)M\lambda}|W(\phi)(z)|^{(N-n+1)}}{\prod_{j=1}^{q}|Q_j(f)(z)|^c} \le \sum_{i=1}^{n_0} \log^+ \frac{|W(\psi^i)(z)|^{(N-n+1)}}{\prod_{j=1}^{M}|\psi_j^i(f)(z)|^{(N-n+1)}} + O(1).$$

Integrating both sides of the above inequality over $\partial\mathbb{D}_r^m$ and using the Lemma 2.4, we obtain

$$\begin{aligned}\| \ (qdc-(N-n+1)M\lambda)\, T_f(r) + (N-n+1)N(r,\nu_{W(\phi)}) - c\sum_{j=1}^{q} N(r,\nu_{Q_j(f)}) \\ \le O(\log^+(T_f(r))) + O(\log|1-r|^{-1}).\end{aligned}$$

This implies that

$$\begin{aligned}\| \left(qd - \frac{(N-n+1)M\lambda}{c}\right) T_f(r) \le \sum_{j=1}^{q} N(r,\nu_{Q_j(f)}) - \frac{N-n+1}{c}N(r,\nu_{W(\phi)}) \\ + O(\log^+(T_f(r))) + O(\log|1-r|^{-1}),\end{aligned} \tag{20}$$

On the other hand, since the number of nonnegative integer n-tuples $(j_1,...,j_n)$ with $\sum_{s=1}^{n} j_s \le \frac{\lambda}{d}$ is equal to the number of nonnegative integer $(n+1)$-tuples $(j_1,...,j_{n+1})$ with $\sum_{s=1}^{n+1} j_s = \frac{\lambda}{d}$, which is $\binom{\lambda/d+n}{n}$ and the sum below is independent of j, we get

$$\begin{aligned} c_j^i &= \sum_{\|(j)\|\le\lambda/d} M^i_{(j)} j_s \ge \sum_{\|(j)\|\le\lambda/d-n} M^i_{(j)} j_s \\ &= \sum_{\|(j)\|\le\lambda/d-n} d^n j_s = \frac{d^n}{n+1} \sum_{\|(j)\|\le\lambda/d-n} \sum_{s=1}^{n+1} j_s \\ &= \frac{d^n}{n+1} \sum_{\|(j)\|\le\lambda/d-n} \frac{\lambda}{d} = \frac{d^n\lambda}{(n+1)d}\binom{\lambda/d}{n} = \frac{d^n\lambda(\lambda/d-1)\cdots(\lambda/d-n)}{(n+1)!d}. \end{aligned} \tag{21}$$

We note that for each positive number $x \in (0, \frac{1}{(n+1)^2}]$, we always have

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k \le 1 + \sum_{k=1}^{n}\frac{n^k}{k!(n+1)^{2k-2}}x \le 1+(n+1)x.$$

Applying this inequality for $0 < \dfrac{(n+1)d}{\lambda-(n+1)d} = \dfrac{(n+1)d}{(N-n+1)(n+1)^3 I(\epsilon^{-1})d} \le \dfrac{1}{(n+1)^2}$, we get

$$\left(1+\frac{(n+1)d}{\lambda-(n+1)d}\right)^n \le 1 + (n+1)\frac{(n+1)d}{\lambda-(n+1)d}.$$

So from (21), we obtain

$$\begin{aligned}
\frac{(N-n+1)M\lambda}{c} &\le d(N-n+1)(n+1)\frac{(\lambda+1)\cdots(\lambda+n)}{(\lambda-d)\cdots(\lambda-nd)}\\
&\le d(N-n+1)(n+1)\prod_{j=1}^{n}\frac{\lambda+j}{\lambda-(n+1)d+jd}\\
&\le d(N-n+1)(n+1)\left(\frac{\lambda}{\lambda-(n+1)d}\right)^n\\
&= d(N-n+1)(n+1)\left(1+\frac{(n+1)d}{\lambda-(n+1)d}\right)^n\\
&\le d(N-n+1)(n+1)\left(1+(n+1)\frac{(n+1)d}{\lambda-(n+1)d}\right)\\
&\le d(N-n+1)(n+1)\left(1+(n+1)\frac{(n+1)d}{(N-n+1)(n+1)^3I(\epsilon^{-1})d}\right)\\
&\le d(N-n+1)(n+1)\left(1+\frac{1}{(N-n+1)(n+1)I(\epsilon^{-1})}\right)\\
&= d(N-n+1)(n+1)+\frac{d}{I(\epsilon^{-1})}.
\end{aligned}$$

Therefore, we get

$$qd-\frac{(N-n+1)M\lambda}{c}\ge qd-d(N-n+1)(n+1)-\frac{d}{I(\epsilon^{-1})}=d[q-(N-n+1)(n+1)-\frac{1}{I(\epsilon^{-1})}].$$

Thus, from (20) we obtain

$$\begin{aligned}
\| \ d[q-(N-n+1)(n+1)-\frac{1}{I(\epsilon^{-1})}]T_f(r) &\le \sum_{j=1}^{q}N(r,\nu_{Q_j(f)})-\frac{N-n+1}{c}N(r,\nu_{W(\phi)})\\
&\quad +O(\log^+(T_f(r)))+O(\log|1-r|^{-1}).
\end{aligned}\tag{22}$$

Claim.

$$\sum_{j=1}^{q}N(r,\nu_{Q_j(f)})-\frac{N-n+1}{c}N(r,\nu_{W(\phi)})\le\sum_{j=1}^{q}N^{[M-1]}(r,\nu_{Q_j(f)}).\tag{23}$$

Indeed, fix $z\in\mathbb{D}^m$, since $\{Q_i\}_{i=1}^q$ is in $N-$subgeneral position, there are at most N indices $j\in\{1,...,q\}$ such that $\nu_{Q_j(f)}(z)>0$. Without loss of generality, we may assume that

$$\nu_{Q_1(f)}(z)\ge\cdots\ge\nu_{Q_N(f)}(z)>0=\nu_{Q_{N+1}(f)}(z)=\cdots=\nu_{Q_q(f)}(z),$$

We note that $\sigma_1=(1,2,...,q)$ and

$$\begin{aligned}
&\nu_{P_{1,1}(f)}(z)=\nu_{Q_1(f)}(z)\\
&\nu_{P_{1,j}(f)}(z)\ge\nu_{Q_{N-n+j}(f)}(z)\ (2\le j\le n).
\end{aligned}$$

For $\psi^1 = P_{1,1}^{j_1}...P_{1,n}^{j_n}h \in \{\psi_s^1\}_{s=1}^M$, we have

$$\psi^1(f)(z) = P_{1,1}^{j_1}(f)(z)...P_{1,n}^{j_n}(f)(z).h(f)(z).$$

Hence

$$\begin{aligned}
\sum_{j=1}^{q} \nu_{Q_j(f)}(z) - &\frac{N-n+1}{c}\nu_{W(\phi)}(z) \\
&\le \sum_{j=1}^{N} \nu_{Q_j(f)}(z) - \frac{N-n+1}{c}\nu_{W(\phi)}(z) \\
&= \sum_{j=1}^{N} \nu_{Q_j(f)}(z) - \frac{N-n+1}{c}\nu_{W(\psi^1(f))}(z) \\
&\le \sum_{j=1}^{N} \nu_{Q_j(f)}(z) - \frac{N-n+1}{c}\sum_{s=1}^{M}\max\{\nu_{\psi_s^1(f)}(z) - M + 1, 0\} \\
&\le \sum_{j=1}^{N} \nu_{Q_j(f)}(z) - \frac{N-n+1}{c}\sum_{(j)} M_{(j)}^1 \sum_{k=1}^{n} j_k \max\{\nu_{P_{1,k}(f)}(z) - M + 1, 0\} \\
&= \sum_{j=1}^{N} \nu_{Q_j(f)}(z) - \frac{N-n+1}{c}\sum_{j=1}^{n} c_j^1 \max\{\nu_{P_{1,j}(f)}(z) - M + 1, 0\} \\
&\le \sum_{j=1}^{N} \nu_{Q_j(f)}(z) - (N-n+1)\sum_{j=1}^{n} \max\{\nu_{P_{1,j}(f)}(z) - M + 1, 0\} \\
&\le \sum_{j=1}^{N} \nu_{Q_j(f)}(z) - \sum_{j=1}^{N} \max\{\nu_{Q_j(f)}(z) - M + 1, 0\} \\
&\le \sum_{j=1}^{q} \min\{\nu_{Q_j(f)}(z), M-1\}) = \sum_{j=1}^{q} \nu_{Q_j(f)}^{[M-1]}(z).
\end{aligned}$$

Integrating both sides of the above inequality, we get the conclusion of the claim.

Now, combining (22) and (23), we obtain

$$\begin{aligned}
\| \left(q - (N-n+1)(n+1) - \frac{1}{I(\epsilon^{-1})}\right) T_f(r) &\le \sum_{j=1}^{q} \frac{1}{d} N^{[M-1]}(r, \nu_{Q_j(f)}) + O(\log^+(T_f(r))) \\
&\le \sum_{j=1}^{q} \frac{1}{d} N^{[M_0]}(r, \nu_{Q_j(f)}) + O(\log^+(T_f(r))) \\
&+ O(\log|1-r|^{-1}).
\end{aligned}$$

This completes the proof of the theorem. ■

Remark. By a little modification of the above proof, we get the following.

Theorem 1.1'. *Let $f : \mathbb{D}^m \to \mathbb{P}^n(\mathbb{C})$ be an algebraically nondegenerate meromorphic mapping and let Q_i $(1 \le i \le q, q > (N-n+1)(n+1))$ be hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ of degree d_i, located in N-subgeneral position. Let d be the least common multiple of $d_1, ..., d_q$, i.e., $d = lcm(d_1, ..., d_q)$. Then for every $\epsilon \in (0,1)$, the following holds*

$$\| \left(q - (N-n+1)(n+1) - \frac{1}{[\epsilon^{-1}]}\right) T_f(r) \le \sum_{i=1}^{q} \frac{1}{d_i} N^{[M_0]}(r, \nu_{Q_i(f)}) + O(\log^+(T_f(r))) + O(\log|1-r|^{-1}),$$

for $r \in [r_0, 1] \backslash E$, where $E \subset [0,1]$ with $\int_E dr/(1-r) < \infty$ and

$$M_0 = d^n \left((n+1) + (N-n+1)(n+1)^3[\epsilon^{-1}]\right)^n - 1.$$

Here, by the notation $[x]$ we denote the biggest integer which does not exceed the real number x.

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we need some estimate on the Chow weight of J. Evertse and R. Ferretti [4] and S. D. Quang [14].

First of all, we recall the notion of Chow weights and Hilbert weights from [4].

Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a projective variety of dimension k and degree Δ. To X we associate, up to a constant scalar, a unique polynomial

$$F_X(\mathbf{u}_0, \dots, \mathbf{u}_k) = F_X(u_{00}, \dots, u_{0n}; \dots; u_{k0}, \dots, u_{kn})$$

in $k+1$ blocks of variables $\mathbf{u}_i = (u_{i0}, \dots, u_{in}), i = 0, \dots, k$, which is called the Chow form of X, with the following properties: F_X is irreducible in $\mathbb{C}[u_{00}, \dots, u_{kn}]$, F_X is homogeneous of degree Δ in each block $\mathbf{u}_i, i = 0, \dots, k$, and $F_X(\mathbf{u}_0, \dots, \mathbf{u}_k) = 0$ if and only if $X \cap H_{\mathbf{u}_0} \cap H_{\mathbf{u}_k} \neq \emptyset$, where $H_{\mathbf{u}_i}, i = 0, \dots, k$, are the hyperplanes given by

$$u_{i0}x_0 + \cdots + u_{in}x_n = 0.$$

Let F_X be the Chow form associated to X. Let $\mathbf{c} = (c_0, \dots, c_n)$ be a tuple of real numbers. Let t be an auxiliary variable. We consider the decomposition

$$\begin{aligned} &F_X(t^{c_0}u_{00}, \dots, t^{c_n}u_{0n}; \dots; t^{c_0}u_{k0}, \dots, t^{c_n}u_{kn}) \\ &\qquad = t^{e_0}G_0(\mathbf{u}_0, \dots, \mathbf{u}_n) + \cdots + t^{e_r}G_r(\mathbf{u}_0, \dots, \mathbf{u}_n). \end{aligned} \tag{24}$$

with $G_0, \dots, G_r \in \mathbb{C}[u_{00}, \dots, u_{0n}; \dots; u_{k0}, \dots, u_{kn}]$ and $e_0 > e_1 > \cdots > e_r$. The Chow weight of X with respect to c is defined by

$$e_X(\mathbf{c}) := e_0. \tag{25}$$

For each subset $J = \{j_0, ..., j_k\}$ of $\{0, ..., n\}$ with $j_0 < j_1 < \cdots < j_k$, we define the bracket

$$[J] = [J](\mathbf{u}_0, \ldots, \mathbf{u}_n) := \det(u_{ij_t}), i, t = 0, \ldots, k, \tag{26}$$

where $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$ denotes the blocks of $n+1$ variables. Let $J_1, \ldots, J_\beta$ with $\beta = \binom{n+1}{k+1}$ be all subsets of $\{0, ..., n\}$ of cardinality $k+1$.

Then the Chow form F_X of X can be written as a homogeneous polynomial of degree Δ in $[J_1], \ldots, [J_\beta]$. We may see that for $\mathbf{c} = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ and for any J among $J_1, \ldots, J_\beta$,

$$\begin{aligned}&[J](t^{c_0}u_{00}, \ldots, t^{c_n}u_{0n}, \ldots, t^{c_0}u_{k0}, \ldots, t^{c_n}u_{kn})\\ &\qquad = t\sum_{j\in J} c_j [J](u_{00}, \ldots, u_{0n}, \ldots, u_{k0}, \ldots, u_{kn}).\end{aligned} \tag{27}$$

For $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$, we write $\mathbf{x}^{\mathbf{a}}$ for the monomial $x_0^{a_0} \cdots x_n^{a_n}$. Let $I = I_X$ be the prime ideal in $\mathbb{C}[x_0, \ldots, x_n]$ defining X. Denote $\mathbb{C}[x_0, \ldots, x_n]_m$ the vector space of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ of degree m (including 0). Put $I_m := C[x_0, \ldots, x_n]_m \cap I$ and define the Hilbert function H_X of X by, for $m = 1, 2, ...,$

$$H_X(m) := \dim(\mathbb{C}[x_0, ..., x_n]_m / I_m). \tag{28}$$

By the usual theory of Hilbert polynomials,

$$H_X(m) = \Delta \cdot \frac{m^n}{n!} + O(m^{n-1}). \tag{29}$$

The m-th Hilbert weight $S_X(m, \mathbf{c})$ of X with respect to the tuple $\mathbf{c} = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ is defined by

$$S_X(m, \mathbf{c}) := \max\left(\sum_{i=1}^{H_X(m)} \mathbf{a}_i \cdot \mathbf{c}\right), \tag{30}$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{H_X(m)}}$ whose residue classes modulo I form a basis of $\mathbb{C}[x_0, \ldots, x_n]_m / I_m$.

The following theorem is due to J. Evertse and R. Ferretti [4] for the case of number field, but it automatically holds for the case of complex field.

Theorem 3.1 (Theorem 4.1 [4])**.** *Let $X \subset \mathbb{P}^n(\mathbb{C})$ be an algebraic variety of dimension k and degree Δ. Let $m > \Delta$ be an integer and let $\mathbf{c} = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}_{\geqslant 0}$. Then*

$$\frac{1}{mH_X(m)} S_X(m, \mathbf{c}) \geq \frac{1}{(n+1)\Delta} e_X(\mathbf{c}) - \frac{(2n+1)\Delta}{m} \cdot (\max_{i=0,\ldots,n} c_i).$$

The following below estimate of Chow weight is due to S. D. Quang [14].

Theorem 3.2 (Theorem 4.1 [14]). *Let Y be a projective subvariety of $\mathbb{P}^R(\mathbb{C})$ of dimension $n \ge 1$ and degree D. Let m $(m \ge n)$ be an integer and let $\mathbf{c} = (c_0, \dots, c_R)$ be a tuple of nonnegative reals. Let $\{i_0, \dots, i_m\}$ be a subset of $\{0, \dots, R\}$ such that*

$$Y \cap \{y_{i_0} = 0, \dots, y_{i_m} = 0\} = \emptyset$$

and $Y \not\subset \{y_{i_j} = 0\}$ for all $j = 0, \dots, m$. Then

$$e_Y(\mathbf{c}) \ge \frac{D}{m-n+1}(c_{i_0} + \cdots + c_{i_m}).$$

Proof of Theorem 1.2

Firstly, we will prove the theorem for the case where all hypersurfaces Q_i $(1 \le i \le q)$ are of the same degree d and $\|Q_i\| = 1$. Consider a reduced representation $\tilde{f} = (f_0, \dots, f_n) : \mathbb{C}^m \to \mathbb{C}^{n+1}$ of f. For a point $z \in \mathbb{C} \setminus \bigcup_{i=1}^q Q_i(\tilde{f})^{-1}(\{0\})$, there exits a permutation $(i_1, \dots, i_q)$ of $\{1, \dots, q\}$ such that

$$|Q_{i_1}(\tilde{f})(z)| \le |Q_{\sigma_i(2)}(\tilde{f})(z)| \le \cdots \le |Q_{i_q}(\tilde{f})(z)|.$$

Since $Q_1, \dots, Q_q$ are in $N-$subgeneral position in V, by the compactness of V, there exists a positive constant A, which is chosen common for all z, such that

$$||\tilde{f}(z)||^d \le A\max_{1\le j\le N+1}|Q_{i_j}(\tilde{f})(z)|.$$

Therefore, we have

$$\prod_{i=1}^q \frac{||\tilde{f}(z)||^d}{|Q_i(\tilde{f})(z)|} \le A^{q-N-1}\prod_{j=1}^{N+1}\frac{||\tilde{f}(z)||^d}{|Q_{i_j}(\tilde{f})(z)|}.$$

We consider the mapping Φ from V into $\mathbb{P}^{q-1}(\mathbb{C})$, which maps a point $x \in V$ into the point $\Phi(x) \in \mathbb{P}^{q-1}(\mathbb{C})$ given by

$$\Phi(x) = (Q_1(x) : \cdots : Q_q(x)).$$

Let $Y = \Phi(V)$. Since $V \cap \bigcap_{j=1}^q Q_j = \emptyset$, Φ is a finite morphism on V and Y is a complex projective subvariety of $\mathbb{P}^{q-1}(\mathbb{C})$ with $\dim Y = k$ and $\Delta := \deg Y \le d^k.\deg V$. For every $\mathbf{a} = (a_1, \dots, a_q) \in \mathbb{Z}^q_{\ge 0}$ and $\mathbf{y} = (y_1, \dots, y_q)$, put $\mathbf{y}^{\mathbf{a}} = y_1^{a_1} \dots y_2^{a_2} \dots y_q^{a_q}$. Let u be a positive integer. Set

$$n_u := H_Y(u) - 1,\ l_u := \binom{q+u-1}{u} - 1,$$

and define the space

$$Y_u = \mathbb{C}[y_1, \dots, y_p]_u/(I_Y)_u,$$

which is a vector space of dimension $n_u + 1$. Fix a basis $\{v_0, \dots, v_{n_u}\}$ of Y_u and consider the meromorphic mapping F with a reduced representation

$$\tilde{F} = (v_0(\Phi \circ \tilde{f}), \dots, v_{n_u}(\Phi \circ \tilde{f})) : \mathbb{C}^m \to \mathbb{C}^{n_u+1}.$$

Since f is algebraically nondegenerate, it implies that F is linearly nondegenerate,

Now, fix an index $i \in \{1, ..., n_0\}$ and a point $z \in S(i)$. Define

$$\mathbf{c}_z = (c_{1,z}, \dots, c_{q,z}) \in \mathbb{Z}^q,$$

where

$$c_{j,z} := \log \frac{||\tilde{f}(z)||^d}{|Q_{i_j}(\tilde{f})(z)|} \text{ for } j = 1, ..., q.$$

We see that $c_{j,z} \geq 0$ for all j. By the definition of the Hilbert weight, there are $\mathbf{a}_{1,z}, ..., \mathbf{a}_{H_Y(u),z} \in \mathbb{N}^{l_u}$ with

$$\mathbf{a}_{i,z} = (a_{i,1,z}, \dots, a_{i,q,z}), a_{i,j,z} \in \{1, ..., l_u\},$$

such that the residue classes modulo $(I_Y)_u$ of $\mathbf{y}^{\mathbf{a}_{1,z}}, ..., \mathbf{y}^{\mathbf{a}_{H_Y(u),z}}$ form a basis of $\mathbb{C}[y_1, ..., y_q]_u/(I_Y)_u$ and

$$S_Y(u, \mathbf{c}_z) = \sum_{i=1}^{H_Y(u)} \mathbf{a}_{i,z} \cdot \mathbf{c}_z.$$

We see that $\mathbf{y}^{\mathbf{a}_{i,z}} \in Y_m$ (modulo $(I_Y)_m$). Then we may write

$$\mathbf{y}^{\mathbf{a}_{i,z}} = L_{i,z}(v_0, \dots, v_{H_Y(u)}),$$

where $L_{i,z}$ $(1 \leq i \leq H_Y(u))$ are independent linear forms. We have

$$\begin{aligned}\log \prod_{i=1}^{H_Y(u)} |L_{i,z}(\tilde{F}(z))| &= \log \prod_{i=1}^{H_Y(u)} \prod_{1 \leq j \leq q} |Q_j(\tilde{f}(z))|^{a_{i,j,z}} \\ &= -S_Y(m, \mathbf{c}_z) + duH_Y(u) \log ||\tilde{f}(z)|| + O(uH_Y(u)).\end{aligned}$$

This implies that

$$\begin{aligned}\log \prod_{i=1}^{H_Y(u)} \frac{||\tilde{F}(z)|| \cdot ||L_{i,z}||}{|L_{i,z}(\tilde{F}(z))|} =& S_Y(u, \mathbf{c}_z) - duH_Y(u) \log ||\tilde{f}(z)|| \\ &+ H_Y(u) \log ||\tilde{F}(z)|| + O(uH_Y(u)).\end{aligned}$$

Here we note that $L_{i,z}$ depends on i and z, but the number of these linear forms is finite. Denote by $\mathcal{L}$ the set of all $L_{i,z}$ occuring in the above inequalities. Then we have

$$\begin{aligned}S_Y(u, \mathbf{c}_z) \leq& \max_{\mathcal{J} \subset \mathcal{L}} \log \prod_{L \in \mathcal{J}} \frac{||\tilde{F}(z)|| \cdot ||L||}{|L(\tilde{F}(z))|} + duH_Y(u) \log ||\tilde{f}(z)|| \\ &- H_Y(u) \log ||\tilde{F}(z)|| + O(uH_Y(u)),\end{aligned} \tag{31}$$

where the maximum is taken over all subsets $\mathcal{J} \subset \mathcal{L}$ with $\sharp \mathcal{J} = H_Y(u)$ and $\{L; L \in \mathcal{J}\}$ is linearly independent. From Theorem 3.1, we have

$$\frac{1}{uH_Y(u)} S_Y(u, \mathbf{c}_z) \geq \frac{1}{(k+1)\Delta} e_Y(\mathbf{c}_z) - \frac{(2k+1)\Delta}{u} \max_{1 \leq j \leq q} c_{j,z} \tag{32}$$

It is clear that

$$\max_{1\le j\le q} c_{j,z} \le \sum_{1\le j\le N+1} \log \frac{||\tilde{f}(z)||^d||}{|Q_{i_j}(\tilde{f})(z)|} + O(1),$$

where the term $O(1)$ does not depend on z. Combining (31), (32) and the above remark, we get

$$(33)\quad \begin{aligned} \frac{1}{(k+1)\Delta} e_Y(\mathbf{c}_z) \le & \frac{1}{uH_Y(u)}\left(\max_{\mathcal{J}\subset\mathcal{L}} \log \prod_{L\in\mathcal{J}} \frac{||\tilde{F}(z)||\cdot||L||}{|L(\tilde{F}(z))|} - H_Y(u)\log||\tilde{F}(z)||\right)\\ & + d\log||\tilde{f}(z)|| + \frac{(2k+1)\Delta}{u}\max_{1\le j\le q} c_{j,z} + O(1/u)\\ \le & \frac{1}{uH_Y(u)}\left(\max_{\mathcal{J}\subset\mathcal{L}} \prod_{L\in\mathcal{J}} \frac{||\tilde{F}(z)||\cdot||L||}{|L(\tilde{F}(z))|} - H_Y(u)\log||\tilde{F}(z)||\right)\\ & + d\log||\tilde{f}(z)|| + \frac{(2k+1)\Delta}{u}\sum_{1\le j\le N+1}\log\frac{||\tilde{f}(z)||^d|}{|Q_{i_j}(\tilde{f})(z)|} + O(1/m). \end{aligned}$$

Since $Q_{i_1}..., Q_{i_{N+1}}$ are in N-subgeneral with respect to X, By Lemma 3.2, we have

$$(34)\quad e_Y(\mathbf{c}_z) \ge \frac{\Delta}{N-k+1}(c_{1,z} + \cdots + c_{N+1,z})$$

Then, from (33) and (34) we have

$$(35)\quad \begin{aligned} & \frac{1}{N-k+1}\log\prod_{i=1}^{q}\frac{||\tilde{f}(z)||^d}{|Q_i(\tilde{f})(z)|}\\ & \le \frac{k+1}{uH_Y(u)}\left(\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{||\tilde{F}(z)||\cdot||L||}{|L(\tilde{F}(z))|} - H_Y(u)\log||\tilde{F}(z)||\right)\\ & \le \frac{k+1}{uH_Y(u)}\left(\sum_{\mathcal{J}\subset\mathcal{L}}\log^+\frac{\cdot|W(\tilde{F})|}{|\prod_{j=1}^{H_Y(u)-\sharp\mathcal{J}} H_j^{\mathcal{J}}(\tilde{F}(z))\prod_{L\in\mathcal{J}}L(\tilde{F}(z))|} - \log|W(\tilde{F})|\right)\\ & + d(k+1)\log||\tilde{f}(z)|| + \frac{(2k+1)(k+1)\Delta}{u}\sum_{1\le j\le q}\log\frac{||\tilde{f}(z)||^d}{Q_j(\tilde{f})(z)|} + O(1), \end{aligned}$$

where $W(\tilde{F})$ is the generalized Wronskian of the meromorphic mapping F, the term $O(1)$ does not depend on z, and $\{H_j^{\mathcal{J}}\}_{j=1}^{H_Y(u)-\sharp\mathcal{J}}$ is a family of linear forms chosen so that $\{H_j^{\mathcal{J}}\}_{j=1}^{H_Y(u)-\sharp\mathcal{J}} \cup \{L \in \mathcal{J}\}$ is independent.

Integrating both sides of the above inequality, we obtain

$$(36)\quad \begin{aligned} \| \ \frac{1}{N-k+1}\sum_{i=1}^{q} m_f(r,Q_i) \le & -\frac{k+1}{uH_Y(u)}N_{W(\tilde{F})}(r) + d(k+1)T_f(r)\\ & + \frac{(2k+1)(k+1)\Delta}{u}\sum_{1\le j\le q} m_f(r,Q_j) + O(\log^+(T_f(r))) + O(\log|1-r|^{-1}). \end{aligned}$$

We now estimate the quantity $N_{W(\tilde{F})}(r)$. Consider a point $z \in \mathbb{C}^m$ which is outside the indeterminacy locus of f. Without loss of generality, we may assume that $\nu^0_{Q_1(f)}(z) \geq \nu^0_{Q_2(f)}(z) \geq .... \geq \nu^0_{Q_q(f)}(z)$. Then we see that $\nu^0_{Q_i(f)}(z) = 0$ for all $i \geq N+1$, since $\{Q_1, ..., Q_q\}$ is in N-subgeneral position in V. Set $c_j = \max\{0, \nu^0_{Q_j}(z) - H_Y(u)\}$ and

$$\mathbf{c} = (c_1, \ldots, c_q) \in \mathbb{Z}^q_{\geq 0}.$$

Then there are

$$\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,q}), a_{i,j} \in \{1, ..., l_u\}$$

such that $\mathbf{y}^{\mathbf{a}_1}, ..., \mathbf{y}^{\mathbf{a}_{H_Y(u)}}$ is a basis of Y_u and

$$S_Y(m, \mathbf{c}) = \sum_{i=1}^{H_Y(u)} \mathbf{a}_i \mathbf{c}.$$

Similarly as above, we can write $\mathbf{y}^{\mathbf{a}_i} = L_i(v_1, ..., v_{H_Y(u)})$, where $L_1, ..., L_{H_Y(u)}$ are independent linear forms in variables y_i $(1 \leq i \leq q)$. By the property of the generalized Wronskian, we see that

$$W(\tilde{F}) = cW(L_1(\tilde{F}), ..., L_{H_Y(u)}(\tilde{F})),$$

where c is a nonzero constant. This yields that

$$\nu^0_{W(\tilde{F})}(z) = \nu^0_{W(L_1(\tilde{f}),...,L_{H_Y(u)}(\tilde{F}))} \geq \sum_{i=1}^{H_Y(u)} \max\{0, \nu^0_{L_i(\tilde{F})}(z) - n_u\}.$$

We also easily see that

$$\nu^0_{L_i(\tilde{F})}(z) = \sum_{1 \leq j \leq q} a_{i,j} \nu^0_{Q_j(\tilde{f})}(z),$$

and hence

$$\max\{0, \nu^0_{L_i(\tilde{F})}(z) - n_u\} \geq \sum_{i=1}^{H_Y(u)} a_{i,j} c_j = \mathbf{a}_i \cdot \mathbf{c}.$$

Thus, we have

$$\nu^0_{W(\tilde{F})}(z) \geq \sum_{i=1}^{H_Y(u)} \mathbf{a}_i \cdot \mathbf{c} = S_Y(u, \mathbf{c}). \tag{37}$$

Since $Q_1, ..., Q_{N+1}$ are in N-subgeneral position, then by Lemma 3.2, we have

$$e_Y(\mathbf{c}) \geq \frac{\Delta}{N+k-1} \cdot \sum_{j=1}^{N+1} c_{1,j} = \frac{\Delta}{N+k-1} \cdot \sum_{j=1}^{N+1} \max\{0, \nu^0_{Q_j(f)}(z) - n_u\}.$$

On the other hand, by Theorem 3.1, we have

$$\begin{aligned}S_Y(u,\mathbf{c}) &\geq \frac{uH_Y(u)}{(k+1)\Delta}e_Y(\mathbf{c}) - (2k+1)\Delta H_Y(u)\max_{1\leq j\leq q}c_j\\ &\geq \frac{uH_Y(u)}{(N-k+1)(k+1)}\sum_{j=1}^{N+1}\max\{0,\nu^0_{Q_j(\tilde{f})}(z)-n_u\}\\ &- (2k+1)\Delta H_Y(u)\sum_{1\leq j\leq q}\nu^0_{Q_j(\tilde{f})}(z).\end{aligned}$$

Combining this inequality and (37), we get

$$\begin{aligned}\frac{(N-k+1)(k+1)}{duH_Y(u)}\nu^0_{W(\tilde{F})}(z) \geq& \frac{1}{d}\sum_{i=1}^{q}\max\{0,\nu^0_{Q_i(\tilde{f})}(z)-n_u\}\\ &- \frac{(N-k+1)(2k+1)(k+1)\Delta}{du}\sum_{1\leq j\leq q}\nu^0_{Q_j(\tilde{f})}(z)\\ \geq& \frac{1}{d}\sum_{i=1}^{q}(\nu^0_{Q_i(\tilde{f})}(z)-\min\{\nu^0_{Q_i(\tilde{f})}(z),u\})\\ &- \frac{(N-k+1)(2k+1)(k+1)\Delta}{du}\sum_{1\leq j\leq q}\nu^0_{Q_j(\tilde{f})}(z).\end{aligned}$$

Integrating both sides of this inequality, we obtain

$$\begin{aligned}\frac{(N-k+1)(k+1)}{duH_Y(u)}N_{W(\tilde{F})}(r) \geq& \frac{1}{d}\sum_{i=1}^{q}(N_{Q_i(f)}(r)-N^{[n_u]}_{Q_i(f)}(r))\\ &- \frac{(N-k+1)(2k+1)(k+1)\Delta}{du}\sum_{1\leq j\leq q}N_{Q_j(\tilde{f})}(r).\end{aligned}\tag{38}$$

Combining inequalities (36) and (38), we get

(39)

$$\begin{aligned}\Big|\Big|\ \big(q-(N-k+1)(k+1)\big)T_f(r) \leq& \sum_{i=1}^{q}\frac{1}{d}N^{[n_u]}_{Q_i(f)}(r) + \frac{(N-k+1)(2k+1)(k+1)q\Delta}{ud}T_f(r)\\ &+ O(\log^+(T_f(r))) + O(\log|1-r|^{-1}).\end{aligned}$$

Choose the smallest integer u such that

$$u > (N-k+1)(2k+1)(k+1)q\Delta\epsilon^{-1}.$$

Then $(N-k+1)(2k+1)(k+1)q\Delta\epsilon^{-1} < u \leq (N-k+1)(2k+1)(k+1)q\Delta\epsilon^{-1}+1$.

Form (39), we have

$$\Big|\Big|\ \big(q-(N-k+1)(k+1)-\epsilon\big)T_f(r) \leq \sum_{i=1}^{q}\frac{1}{d}N^{[n_u]}_{Q_i(f)}(r).\tag{40}$$

We note that $\deg Y = \Delta \leq d^k \deg(V)$. Then the number n_u is estimated as follows

$$n_u = H_Y(u) - 1 \leq \Delta \binom{k+u}{k} \leq d^k \deg(V) e^k \left(1 + \frac{u}{k}\right)^k$$

If $k = 1$, then $6Nq\Delta\epsilon^{-1} < u \leq 6Nq\Delta\epsilon^{-1} + 1$. Hence $u + 1 \leq 6Nq\Delta\epsilon^{-1} + 2$.
This implies that $\left(1 + \frac{u}{k}\right)^k = 1 + u \leq 2 + 6Nq\Delta\epsilon^{-1} < 7Nq\Delta\epsilon^{-1}$ if $\epsilon \in \left(0, \frac{Nq}{2}\right)$.
Hence $n_u \leq d \deg(V) e \left(7Nq\Delta\epsilon^{-1}\right) \leq 7 \deg(V)^2 e d^2 Nq\epsilon^{-1}$. Thus, we have

$$n_u \leq \left[7 \deg(V)^2 e d^2 Nq\epsilon^{-1}\right] = M_0$$

when $k = 1$.
If $k \geq 2$, then $1 + \frac{u}{k} \leq 1 + \frac{1}{k} + \frac{(N-k+1)(2k+1)(k+1)q\Delta\epsilon^{-1}}{k}$.
We now show that

$$1 + \frac{1}{k} + \frac{(N-k+1)(2k+1)(k+1)q\Delta\epsilon^{-1}}{k} \leq (N-k+1)(2k+4)q\Delta\epsilon^{-1}.$$

In fact, the above inequality is equivalent to

$$\begin{aligned} &\frac{k+1}{k} \leq (N-k+1)q\Delta\epsilon^{-1}\left((2k+4) - \frac{(2k+1)(k+1)}{k}\right) \\ \Leftrightarrow &\frac{k+1}{k} \leq (N-k+1)q\Delta\epsilon^{-1}\left(\frac{k-1}{k}\right). \end{aligned}$$

Therefore, we need to show that

$$k + 1 \leq (N-k+1)q\Delta\epsilon^{-1}(k-1).$$

Since $q > (N-k+1)(k+1), N-k+1 \geq 1, \Delta \geq 1, 0 < \epsilon < k-1$, it implies that

$$(N-k+1)q\Delta\epsilon^{-1}(k-1) > (N-k+1)^2(k+1)\Delta\epsilon^{-1}(k-1) \geq k+1.$$

Hence, we get

$$\begin{aligned} n_u = H_Y(u) - 1 \leq \Delta \binom{k+u}{k} &\leq d^k \deg(V) e^k \left(1 + \frac{u}{k}\right)^k \\ &\leq d^k \deg(V) e^k \left((N-k+1)(2k+4)q\Delta\epsilon^{-1}\right)^k \\ &\leq \deg(V)^{k+1} e^k d^{k^2+k} (N-k+1)^k (2k+4)^k q^k \epsilon^{-k}. \end{aligned}$$

It implies that

$$n_u \leq \left[\deg(V)^{k+1} e^k d^{k^2+k} (N-k+1)^k (2k+4)^k q^k \epsilon^{-k}\right] = M_0$$

when $k \geq 2$.

Then, the theorem is proved. ■

4. Pluri-subharmonic functions on complex manifolds

This subsection is devoted to prove a version of [9, Theorem A] in the case where M is Stein and u is a (not necessary continuous) plurisubharmonic function. Throughout this subsection M will denote an m-dimensional closed complex submanifold of $\mathbb{C}^n$ and the Kähler metric of M is induced from the canonical one of $\mathbb{C}^n$.

Definition 4.1. Let N be a complex manifold and f be a locally integrable real function in N. We say that f is *plurisubharmonic function* (or psh function, for brevity) if $\mathrm{dd^c} f \geq 0$ in the sense of currents.

Lemma 4.2. *(see* [9, Lemma, p.552]*) Let N_1 be a Kähler manifold and N_2 be a complex manifold. Let g be a holomorphic map of N_1 to N_2. Then for each C^2-psh function f in N_2, $f \circ g$ is subharmonic in N_1.*

Lemma 4.3. *The volume of M is infinite.*

Proof. Take a point $a \in M$. Let $B_M(a, R)$ be the ball centered at a of M and of radius R. Put $u = |z - a|$. Then u is a psh function on $\mathbb{C}^n$ and hence, it is a subharmonic function on M. Since the Kähler metric on M is induced from the canonical one of $\mathbb{C}^n$, it implies that $B_M(a, R) \subset B(a, R)$, where $B(a, R)$ is the usual ball centered at a and of radius R in $\mathbb{C}^n$. Therefore $u \leq R$ in $B_M(a, R)$. By [9, Theorem A], we get

$$\liminf_{r \to \infty} \frac{1}{R^2} \int_{B_M(a,R)} u^2 \, d\,\mathrm{vol} = \infty.$$

From this we deduce that $\int_M d\,\mathrm{vol} = \infty$. □

Proposition 4.4. *Let u be a psh function on M and K be a compact subset of M. For each open subset U of M such that $K \subset U \Subset M$, there exists a decreasing sequence of C^∞-psh functions u_k in U such that u_k converge to u, a.e in U. Moreover, if u is non-negative then u_k is non-negative.*

Proof. By [7, Chapter VIII, Theorem 8], there exists a holomorphic retraction α of an open subset V of $\mathbb{C}^n$ containing M to M, i.e α is holomorphic and $\alpha|_M = id_M$. Then $u \circ \alpha$ is a psh function on V. The conclusion now is deduced immediately from this fact. □

As a direct consequence, we get the following.

Corollary 4.5. *Let ξ be an increasing convex function in $\mathbb{R}$. Let u be a psh function on M. Then $\xi \circ u$ is a psh function. Specially, if u is non-negative then u^p $(p \geq 1)$ is in the Sobolev space $\mathcal{H}_0(M)$ of degree 0 of M.*

Theorem 4.6. *Let u be a non-negative psh function on M and p be a positive number greater than 1. Take a point $a \in M$. Let $B_M(a, R)$ (or $B(R)$ for brevity) be the ball centered at a of M and of radius R. Then one of the following two statements holds:*

(i)

$$\liminf_{r\to\infty} \frac{1}{R^2} \int_{B_M(a,R)} u^p \, d\,\mathrm{vol} = \infty.$$

(ii) *u is constant a.e in M.*

Proof. Suppose that u is not constant a.e in M and

$$\liminf_{r\to\infty} \frac{1}{R^2} \int_{B_M(a,R)} u^p \, d\,\mathrm{vol} = A < \infty. \tag{41}$$

Then, there exists a sequence $\{r_j\}$ such that

$$\frac{1}{r_j^2} \int_{B_M(a,r_j)} u^p \, d\,\mathrm{vol} = A.$$

By Proposition 4.4, there is a decreasing sequence u_k of C^∞-nonnegative functions such that u_k is psh in $B(r_{k+2})$ and u_k converge to $u, a.e$ in $B(r_{k+1})$. By the monotone convergence theorem, there exists a subsequence of u_k, without loss of generality we may assume that this subsequence is u_k, satisfying

$$\frac{1}{r_j^2} \int_{B(r_j)} u_j^p \, d\,\mathrm{vol} \le \frac{1}{r_j^2} \int_{B_M(a,r_j)} u^p \, d\,\mathrm{vol} + 1.$$

For each $j \ge 1$, let φ_j be a Lipschitz continuous function such that $\varphi_j(x) \equiv 1$ on $B(a, r_j)$ and $\varphi_j(x) \equiv 0$ in $M \setminus B(a, r_{j+1})$ and $\mathrm{grad}\varphi_j \le \frac{C}{r-s}$ a.e on M, where C is a constant which does not depend on index j (see [9, Lemma 1]). Put

$$I_j^N(\epsilon) = \int_{B(r_{j+1})} \varphi_j^2 (u_N^2 + \epsilon)^{\frac{p-2}{2}} ||\mathrm{grad}\, u_N||^2 \, d\,\mathrm{vol}, N \ge j$$

and $I_j^N = \lim_{\epsilon\to 0} I_j^N(\epsilon)$. By the proof of [9, Theorem 2.1], $I_j^N < +\infty$ (by 41) and there exists a constant $C = C(p) > 0$ such that

$$I_{j+1}^N I_j^N \le (I_{j+1}^N)^2 \le C(I_{j+1}^N - I_j^N), \; 1 \le j \le N. \tag{42}$$

The rest of the proof is proceeded as in the one of [9, Theorem 2.4]. For convenience we sketch it here.

It is easy to see that for some j, $I_j^N > 0$ for an infinite number of values of N. Since $I_j^N \le I_k^N$ for $j \le k \le N$, it follows that there exist an index j_0 and a sequence $N_k \to +\infty$ such that for each $m \ge j_0, N_k \ge m$, $I_m^{N_k} > 0$. Divide (42) by $I_{j+1}^{N_k} I_j^{N_k}, m \le j \le N_k$ and summing over j (from m to N_k), we obtain $1/I_m^{N_k} \ge C(N_k - m)$ for a constant C. Hence,

$$\lim_{k\to+\infty} \int_{B(r_m)} (u_{N_k}^2 + \epsilon)^{\frac{p-2}{2}} ||\mathrm{grad}\, u_{N_k}||^2 \, d\,\mathrm{vol} = 0.$$

Now, let $\varphi \in C^2_o(M)$ and q be the smallest integer greater than $(p-2)/2$. Then,

$$\begin{aligned}\int_M u^{q+1}\Delta\varphi d\,\mathrm{vol} &= \lim_{k\to+\infty} u^{q+1}_{N_k}\Delta\varphi d\,\mathrm{vol}\\ &= -(q+1)\lim_{k\to+\infty} u^q_{N_k} < \mathrm{grad}\, u_{N_k}, \mathrm{grad}\,\varphi >\\ &= 0.\end{aligned}$$

In the other words, $\Delta u = 0$ in the sense of currents. Hence, $u^{q+1} \in C^\infty$ by the regularity theorem (so that "$\mathrm{grad}\, u^{q+1}$" makes sense). Put $X = \mathrm{grad}\, u^{q+1}$. Then,

$$\begin{aligned}\int ||X||^2\varphi d\,vol &= \int < \mathrm{grad}\, u^{q+1}, \varphi X > d\,\mathrm{vol}\\ &= -\int u^{q+1}\mathrm{div}(\varphi X) d\,\mathrm{vol}\\ &= -\lim_{k\to+\infty}\int u^{q+1}_{N_k}\mathrm{div}(\varphi X) d\,\mathrm{vol}\\ &= (q+1)\lim_{k\to+\infty}\int u^{q+1}_{N_k} < \mathrm{grad}\, u_{N_k}, \varphi X > d\,\mathrm{vol}\\ &= 0.\end{aligned}$$

Therefore, $X = 0$. That means u is constant, a contradiction. □

Corollary 4.7. *Let u be a psh function on M. Then*

$$\int_M e^u\, d\,\mathrm{vol} = \infty.$$

5. The proof of Theorem 1.3 for defect relation with no truncation

First of all, we will give a modification of [3, Theorem 2'].

Proposition 5.1. *Let $M, \delta_1, \delta_2 > 0$ and $q, n \in \mathbb{N}, q > 2n$. Then, there is a number $\alpha = \alpha(\delta_1, \delta_2, M, q, n) > 0$ with the following property: If $u, u_1, \cdots, u_q$ are subharmonic functions in an open neighborhood of $\overline{\Delta}_1 \subset \mathbb{C}$ with Riesz charges $\nu, \nu_1, \cdots, \nu_q$, respectively such that*

$$(\nu + \sum_{i=1}^q \nu_i)(\overline{\Delta}_1) \le M,$$

then

$$\int_{\overline{\Delta}_1} |\sup_{1\le k\le n+1} u_{i_k} - u| dxdy \le \alpha, \text{ for all } 1 \le i_1 < \cdots < i_{n+1} \le q.$$

Moreover, there exists $r \in [1-\delta_1, 1]$ such that

$$\Big(\sum_{i=1}^q \nu_i - (q-2n)\nu\Big)(\Delta_r) > -\delta_2.$$

Proof. Suppose the theorem is false. Then there are a number $\delta > 0$ and a sequence $(u^j, u_1^j, \cdots, u_q^j), j \in \mathbb{N}$ with Riesz charge ν^j, ν_i^j respectively such that

$$(\nu^j + \sum_{i=1}^{q} \nu_i^j)(\overline{\Delta}_1) \leq M,$$

and for all $r \in [1 - \delta_1, 1]$ one has

$$\Big(\sum_{i=1}^{q} \nu_i - (q - 2n)\nu)\Big)(\Delta_r) \leq -\delta_2.$$

Hence, by passing to a subsequence if necessary, we can assume

$$\nu_i^j \to \nu_i, \ \nu^j \to \nu,$$

Let $G{*}\lambda$ be the Green potential of the charge λ in the disk Δ_1. By the Riesz representation formula, we have

$$u_i^j - u^j = h_i^j + G * (\nu_i^j - \nu^j),$$

where h_i^j is harmonic in Δ_1. By the proof of [3, Theorem 2'], we get the followings:

(i) $G * \nu_i^j \to G * \nu_i, \ G * \nu^j \to G * \nu$ in $L^1(\Delta_1)$.

(ii) $h_i^j \to h_i$ uniformly on compact subsets, some of h_i may be identical $-\infty$. We can suppose $h_i \not\equiv -\infty$ for $1 \leq i \leq q'$ and $h_i = -\infty$ for $i > q'$. Note that $q' - q \leq n$.

(iii) Put $u_i = h_i + G * \nu_i \ (1 \leq i \leq q'), u = G * \nu, n' = n - (q - q')$. Then $u = \sup_{1 \leq k \leq n+1} u_{i_k}$ for all $1 \leq i_1 < \cdots < i_{n+1} \leq q'$.

Hence, by [3, Theorem 2], we get

$$\sum_{i=1}^{q'} \nu_i - (q' - 2n')\nu \geq 0.$$

Consequently,

$$\begin{aligned} \kappa &:= \sum_{i=1}^{q} \nu_i - (q - 2n)\nu \\ &= \sum_{i=1}^{q'} \nu_i - (q' - 2n)\nu + \sum_{i=q'+1}^{q} \nu_i + \{(q' - 2n') - (q - 2n)\}\nu. \end{aligned}$$

Obviously, the expression in the braces is nonnegative. Therefore, $\kappa \geq 0$. Finally, for a Radon measure λ in a neighborhood of $\overline{\Delta}_1$, we have $\lambda(\partial\Delta_r) = 0$ for all r outside a countable subset of $[0, 1]$. Thus, we can choose a sequence r_n increasingly tending to 1 such that $\nu^j(\partial\Delta_{r_n}) = 0$. Hence,

$$\sum_{i=1}^{q} \nu_i^j(\Delta_{r_n}) - (q - 2n)\nu^j(\Delta_{r_n}) \to \kappa(\Delta_{r_n})$$

as $j \to \infty$. From these, we get a contradiction. □

Corollary 5.2. *Let $M, \delta > 0$ and $q, n \in \mathbb{N}, q > 2n$. Let $u, u_1, \cdots, u_q$ be subharmonic functions in an open neighborhood of $\overline{\Delta}_R \subset \mathbb{C}$ with Riesz charges $\nu, \nu_1, \cdots, \nu_q$, respectively such that the following two statements satisfied*

(i) $\nu(\Delta_r) \to +\infty$ *as r tends to R,*

(ii) *For all $1 \leq i_1 < \cdots < i_{n+1} \leq q$, we have*

$$\frac{1}{r^2}\int_{\overline{\Delta}_r} |\sup_{1\leq k\leq n+1} u_{i_k} - u| dxdy = O(1)$$

as r tends to R.

Then for each $\delta > 0$,

$$\Big(\sum_{i=1}^{q} \nu_i - (q-2n)\nu\Big)(\Delta_r) \geq -\delta\Big(\nu(\Delta_r) + \sum_{i=1}^{q} \nu_j(\Delta_r)\Big)$$

for r close enough to R.

Proof. Put

$$w(z) = \frac{u(rz)}{\nu(\Delta_r) + \sum_{i=1}^{q} \nu_j(\Delta_r)},\ w_i(z) = \frac{u_i(rz)}{\nu(\Delta_r) + \sum_{i=1}^{q} \nu_j(\Delta_r)}$$

for z in a neighborhood of $\overline{\Delta}_1$ and $r < R$. By the condition (i) and Proposition 5.1, we obtain the assertion. □

By the Jensen formula and Corollary 5.2, we have the Eremenko-Sodin second main theorem.

Corollary 5.3. *Let the notations and the hypothesis be as in Corollary 5.2. Then for each $\delta > 0$,*

$$\int_0^{2\pi}\Big(\sum_{i=1}^{q} u_i(re^{it}) - (q-2n)u(re^{it})\Big)dt > -\delta\int_0^{2\pi} u(re^{it})dt + O(1)$$

for all r close enough to R. Here the term $O(1)$ is a constant as $r \to R$, but depends on u_i, u.

In high dimension, we have

Corollary 5.4. *Let $M, \delta > 0$ and $q, n \in \mathbb{N}, q > 2n$. Let $u, u_1, \cdots, u_q$ be psh functions in an open neighborhood of $\overline{\Delta}_R \subset \mathbb{C}^l$ such that the following two statements are satisfied*

(i) $\int_{\partial'\Delta_r} u\, dt_1 \cdots dt_l \to +\infty$ *as r tends to R,*

(ii) *For all $1 \leq i_1 < \cdots < i_{n+1} \leq q$,*

$$|\sup_{1\leq k\leq n+1} u_{i_k}(z) - u(z)| = O(1)$$

as $|z|$ tends to R.

Then for each $\delta > 0$,

$$\int_{\partial'\Delta_r}\Big(\sum_{i=1}^q u_i-(q-2n)u\Big)dt_1\cdots dt_l > -\delta\int_{\partial'\Delta_r} u dt_1\cdots dt_l + O(1)$$

for r *close enough to* R.

Proof. Put

$$w(\cdot)=\int_0^{2\pi}\cdots\int_0^{2\pi} u(\cdot, r_2e^{it_2},\cdots, r_le^{it_l})dt_2\cdots dt_l.$$

And we define w_i $(1\le i\le q)$ in the similar manner. The radius $r=(r_1,\cdots,r_n)$ is chosen close enough to R. It is easy to see that w, w_i satisfy conditions in Corollary 5.2. □

Theorem 5.5. *Let* V *be a plurisubharmonic function on an open neighborhood of a polydisc* Δ *of* $\mathbb{C}^n$. *Let* $g_{\Delta,a}$ *be a pluricomplex Green function of* Δ *with pole at* $a = (a_1,\cdots,a_n)\in\Delta$. *Then*

$$\int_{\partial'\Delta} V\prod_{j=1}^n \frac{R_j^2-|a_j|^2}{|a_j-R_je^{it}|^2}\,dt_1\cdots dt_n-(2\pi)^nV(a)=$$

$$\int_{-\infty}^0\int_{\{g_{\Delta,a}<t\}} \mathrm{dd^c}V\wedge(\mathrm{dd^c}g_{\Delta,a})^{n-1}\,dt$$

Proof. By the Lelong-Jensen formula (see [10, Chapter 6, Section 6.5] and [2, Chapter III, Section 6]) and the fact that $(\mathrm{dd^c}g_{\Delta,a})^n=(2\pi)^n\delta_{\{a\}}$, we obtain for $r<0$

$$\mu_r(V)-\int_{\{g_{\Omega,a}<r\}} V(\mathrm{dd^c}g_{\Delta,a})^n=\int_{-\infty}^r\int_{\{g_{\Delta,a}<t\}}\mathrm{dd^c}V\wedge(\mathrm{dd^c}g_{\Delta,a})^{n-1}\,dt.$$

It is clear that the right-handed side converges to

$$\int_{-\infty}^0\int_{\{g_{\Delta,a}<t\}}\mathrm{dd^c}V\wedge(\mathrm{dd^c}g_{\Delta,a})^{n-1}\,dt$$

as r tends to 0. Now suppose that V is continuous. Since the supports of $\mu_r\subset\Delta$ and μ_r weakly converge to $\mu_{\Omega,a}$, we get $\mu_r(V)\to\mu_{\Omega,a}(V)$ as r tends to 0. In general, by taking a decreasing sequence of continuous plurisubharmonic functions V_n converging to V, we get the desired equality. Notice that $\mu_{\Omega,a}(V)$ is finite and

$$\int_\Delta V(\mathrm{dd^c}g_{\Delta,a})^n=(2\pi)^n\delta_{\{a\}}(V)=(2\pi)^nV(a)$$

is finite or $-\infty$. □

Proof of Theorem 1.3. We now suppose on the contrary. By definition of the non-integrated defect, there exist $\eta_i\ge 0$ $(1\le i\le q)$ and nonnegative functions h_i such that

$$\eta_iF^*\mathrm{dd^c}\log(|z_1|^2+\cdots+|z_{m+1}|^2)+\mathrm{dd^c}\log h_i^2\ge F^*H_i$$

and $1-\eta_i \leq \bar{\delta}_{F,\mathcal{H}_m}(H_i) \leq 1$. Put $\eta = \sum_{i=1}^{q}(1-\eta_i)$. Therefore,

$$(q-\eta)\mathrm{dd^c}\log||F||^2 + \mathrm{dd^c}\log h'^2 \geq \sum_{i=1}^{q} F^*H_i,$$

where h' is measurable and bounded. Subtracting $(q-2n)\mathrm{dd^c}\log||F||^2$ from the two sides of the above inequality, we get

$$\mathrm{dd^c}\log h'^2 \geq \left(\sum_{i=1}^{q} F^*H_i - (q-2n)\mathrm{dd^c}\log||F||^2\right) + (\eta-2n)\mathrm{dd^c}\log||F||^2.$$

Note that by $\overline{f(M)} \cap B(E) = \varnothing$, we have

$$|\log||F||^2 - \max_{1\leq j\leq N+1}\log|H_{i_j}(F)|^2| = O(1) \tag{43}$$

for all $1 \leq i_1 < \cdots i_{N+1} \leq q$.

Claim 1.

$$\int_{\Delta_R} \nu\overline{\nu}\omega^n = +\infty$$

Indeed, denote by $|\nu|^2$ the trivialization of $\nu\overline{\nu}$ on Δ_R. If $|\nu|^2$ is not integrable over M then we are done. Otherwise, it is integrable hence since ν holomorphic $|\nu|^2$ is plurisubharmonic function on M. Taking into account of Theorem 4.6 ones get Claim 1.

Claim 2.

$$\lim_{r\to R}\int_{\partial'\Delta_r} \log||F(z)|| = +\infty.$$

Applying Theorem 5.5 for $\frac{l}{d}\log||F||$ and $\nu\overline{\nu}\omega^n$, and integrating in a over Δ_{r_0} (for some r_0 fixed) and the inequality in the hypothesis of Theorem 1.3 we get

$$\int_{\partial'\Delta'_R} \log||F(z)|| \geq \frac{d}{l}\int_{\partial'\Delta'_R} \nu\overline{\nu}\omega^n + C,$$

where $R' < R$ and C is a constant that does not depend on R'. On the other hand,

$$+\infty = \int_{\Delta_R} \nu\overline{\nu}\omega^n = \int_0^{R_1} r_1 dr_1 \cdots \int_0^{R_n} r_n dr_n \int_{\partial'\Delta_r} \nu\overline{\nu}\omega^n$$

Hence, $\int_{\partial'\Delta'_R} \nu\overline{\nu}\omega^n$ tends to $+\infty$ as r closes to R. That yields the claim 2. Now, by applying Corollary 5.4 for $u = \log||F(z)||, u_i = H_i(F)$ and Jensen's formula, we get the desired conclusion. □

We recall the following version of the Bloch-Cartan theorem which plays an essential role in Geometric Function Theory.

Theorem 5.6. *(see* [11, Corollary 3.10.8, p.137]*) If a holomorphic map $f : \mathbb{C} \to \mathbb{P}^m(\mathbb{C})$ misses $2m+1$ or more hyperplanes in general position, then it is a constant map.*

The Bloch-Cartan theorem is generalized to hypersurfaces in general position in $\mathbb{P}^m(\mathbb{C})$ by Babets and Eremenko-Sodin

Theorem 5.7. *(see [1] and [3]) If a holomorphic map $f : \mathbb{C} \to \mathbb{P}^m(\mathbb{C})$ misses $2m+1$ or more hypersurfaces in general position, then it is a constant map.*

From Theorem 1.3, we have the following Bloch-Cartan theorem for meromorphic mappings from $\mathbb{C}^l$ to a smooth algebraic variety V in $\mathbb{P}^m(\mathbb{C})$ missing $2N+1$ or more hypersurfaces in N-general position.

Corollary 5.8. *Let f be a meromorphic mapping of $\mathbb{C}^l$ to a smooth algebraic variety V in $\mathbb{P}^m(\mathbb{C})$. Let $D_1, \cdots, D_{2N+1}$ be hypersurfaces of $\mathbb{P}^m(\mathbb{C})$ such that $V \not\subset D_j$ and $D_j \cap V$ are in N-subgeneral position in V. Assume that f omits D_j $(1 \leq j \leq 2N+1)$. Then f is constant.*

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Pham Nguyen Thu Trang,
Department of Mathematics, Hanoi National University of Education, 136 XuanThuy str., Hanoi, Vietnam
Email address: trangpnt@hnue.edu.vn

Do Duc Thai,
Department of Mathematics, Hanoi National University of Education, 136 XuanThuy str., Hanoi, Vietnam
Email address: doducthai@hnue.edu.vn

Pham Ngoc Mai,
Basic Sciences Faculty, Foreign Trade University, 91 ChuaLang Str., Hanoi, Vietnam
Email address: phamngocmai@ftu.edu.vn