Chapter 1

Local setting for trace formulas

In this chapter we establish almost all needed local setting for Selberg trace formula and Kuznetsov trace formula for $G := \mathrm{GL}_2(F)$ (where F is a local field) and their comparison.

Before go further, we need to fix some notation which we shall use in this chapter.

1.1 The p-adic case

In this section we work with a finite extension F of $\mathbb{Q}_p$ where p is a certain odd prime number. The field F is then the field of fractions of a discrete valuation ring $\mathcal{O}$. Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$ and $k = \mathcal{O}/\mathfrak{p}$ the residue field. Thus k is finite and of characteristic p. We shall denote the cardinality of k by q.

We choose one for all an uniformizer ϖ of $\mathfrak{p}$, that is, an element such that $\varpi\mathcal{O} = \mathfrak{p}$. Every element $x \in F^\times$ can be written uniquely in the form

$$x = u\varpi^n$$

with $u \in \mathcal{O}^\times$ and $n \in \mathbb{Z}$. (Note that the integer n does not depend on the choice of ϖ.) The integer number n is called the valuation of x over F and is denoted by $v_F(x)$ (we shall drop the subindex F when the field is clear). The absolute value $|.|_F : F \to \mathbb{R}$ defined by

$$|x|_F = q^{-v_F(x)},\ \forall x \in F^\times, \text{and } |0|_F = 0$$

gives a metric on F. In the metric space topology, F is a complete, locally compact, totally disconnected (that is no nonempty subsets are connected except singleton sets), Hausdorff topological field.

The matrix ring $M_2(F) \simeq F^4$ of 2×2 matrices with entries in F carries the product topology, relative to which it is a locally compact, totally disconnected, Hausdorff topological ring. Since $\det : M_2(F) \to F$ is a polynomial in the matrix entries, det is a continuous map. It implies that $\mathrm{GL}_2(F) = \det^{-1}(F^\times)$ ($F^\times = F \backslash \{0\}$ is an open subset of F) is an open subset of $M_2(F)$. We give $G = \mathrm{GL}_2(F)$ the topology it inherits as an open subset of $M_2(F)$. The inversion of matrices is continuous, so G is a locally compact, totally disconnected, Hausdorff topological group. In the terminology of [5] such a group is called an **ℓ-group**. From now on, we shall add ℓ-group beside G to indicate that a statement is true not only for $\mathrm{GL}_2(F)$ but also for any ℓ-group. The subgroups

$$K_0 = \mathrm{GL}_2(\mathcal{O}) := \{g \in M_2(\mathcal{O}) || \det(g)|_F = 1\}, \quad K_i = 1 + \varpi^i M_2(\mathcal{O}), \, \forall i \geq 1$$

are compact open, and give a fundamental system of open neighborhood of the identity in G.

1.1.1 Smooth representations of $\mathrm{GL}_2(F)$

A (continuous) **representation** (π, V) of an ℓ-group G consists of a topological $\mathbb{C}$-vector space V and a group homomorphism $\pi : G \to \mathrm{GL}(V)$ from G to the group of invertible linear operators on V such that for each $v \in V$, the map

$$G \to V : g \mapsto \pi(g)v$$

is continuous. The space V is called the representation space of G. We may refer to the representation as π (when V is clear from the context), or we may just say V (when the action π clear from the context). When V is equipped with the discrete topology, we obtain than a **smooth representation** of G. (Since the discrete topology on V is the finest topology on V, the smooth representation is continuous for any kind of topology on V.)

Lemma 1.1.1. *Let (π, V) be a representation of an ℓ-group G. The following conditions are equivalents:*

1. *The representation (π, V) is smooth.*
2. *For each $v \in V$, the map $\varphi_v : G \to V : g \mapsto \pi(g)v$ is **smooth**, i.e locally constant.*
3. *For each $v \in V$, the set $\mathrm{Stab}_G(v) := \{g \in G | \pi(g)v = v\}$ is open in G.*
4. *For each $v \in V$, there exist an open compact subgroup K_v (depends on v) of G such that $\pi(K_v)v = v$.*

Proof.

- (1) $\Leftrightarrow$ (2). Since V is equipped with the discrete topology, a function $\varphi_v : G \to V$ is smooth if and only if it is continuous.

- (2) $\Rightarrow$ (3). Since φ_v is locally constant, there exits an open neighborhood U of 1 (the unit element of G) such that $\pi(u)v = \pi(1)v = v$ for all $u \in U$. It implies that $U \subset \mathrm{Stab}_G(v)$. Let $g \in \mathrm{Stab}_G(v)$, we have $\pi(gu)v = \pi(g)(\pi(u)v) = \pi(g)v = v$ for all $u \in U$. Hence gU is an open neighborhood of g and is contained in $\mathrm{Stab}_G(v)$. So the set $\mathrm{Stab}_G(v)$ is open.

- (3) $\Rightarrow$ (4). Since $1 \in \mathrm{Stab}_G(v)$, the set $\mathrm{Stab}_G(v)$ is an open neighborhood of 1 in G. Since G is an ℓ-group, there exist an open compact subgroup $K_v \subset \mathrm{Stab}_G(v)$. For example for $\mathrm{GL}_2(F)$, we choose i large enough such that $K_v := K_i = 1 + \varpi^i M_2(\mathcal{O}) \subset \mathrm{Stab}_G(v)$. We have $\pi(K_v)v = v$.

- (4) $\Rightarrow$ (2). For all $g \in G$, gK_v is an open neighborhood of g. Set $g' = gu \in gK_v$, we have $\varphi_v(g') = \pi(gu)v = \pi(g)(\pi(u)v) = \pi(g)v = \varphi_v(g)$. Hence, φ_v is locally constant.

$\square$

Given a smooth representation (π, V) of an ℓ-group G, a subspace W of V is said to be G-**invariant** if for every $w \in W$ and every $g \in G$ we have $\pi(g)w \in W$.

If (π, V) and (π', V') are two representations of an ℓ-group G then we denote by $\mathrm{Hom}_G(\pi, \pi')$ the space of all linear maps $f : V \to V'$ such that $f(\pi(g)v) = \pi'(g)f(v)$ for all $v \in V$ and all $g \in G$. We say that π and π' are **equivalent** (or **isomorphic**) if $\mathrm{Hom}_G(\pi, \pi')$ contains an invertible element. In that case, we write $\pi \simeq \pi'$.

For every representation V of an ℓ-group G, a vector $v \in V$ is a **smooth vector** if its stabilizer $\mathrm{Stab}_G(v)$ is open in G. We shall denote by V^{sm} the G-invariant subspace consisting of smooth vectors of V. By Lemma 1.1.1, V^{sm} is a smooth representation of G.

Let (π, V) be a representation of G. We denote by V^* the space of all linear forms on V. For every $v^* \in V^*$ and $g \in G$, we define $\pi^*(g)v^* \in V^*$ by

$$(\pi^*(g)v^*)(u) = v^*(\pi(g^{-1})u), \quad \forall u \in V.$$

Clearly, (π^*, V^*) is a representation of G. The dual representation V^* might not be smooth even if V is smooth. Let $\widetilde{\pi}$ be the G-invariant subspace $\widetilde{V} = V^{*,\mathrm{sm}}$ of π^*. The representation $(\widetilde{\pi}, \widetilde{V})$ is called the **contragredient** of (π, V).

It doesn't like the representation theory of finite group; in general the representation $\widetilde{\widetilde{\pi}}$ is not equivalent to π. However, we shall soon see that this phenomena is true when we add some more condition to π.

A smooth representation (π, V) of an ℓ-group G is said to be **admissible** if for every compact open subgroup K of G, the subspace $V^K := \{v \in V | \pi(K)(v) = v\}$ is finite dimensional. In the case $G = \mathrm{GL}_2(F)$, because $V^{gKg^{-1}} = \pi(g)(V^K)$ and all the maximal compact subgroups of $\mathrm{GL}_2(F)$ are conjugate to K_0, a smooth representation V is admissible if and only if V^K is finite dimensional for every open subgroup K of K_0.

Proposition 1.1.2. *If a representation (π, V) of an ℓ-group G is admissible, then the representation $(\widetilde{\pi}, \widetilde{V})$ is also admissible. Futhermore, we have $\widetilde{\widetilde{\pi}} \simeq \pi$.*

Proof. Let K be a compact open subgroup of G. Since $\mathrm{Stab}_K(v) = \mathrm{Stab}_G(v) \cap K$ is open in K, we can consider V as a smooth representation of compact group K . Set

$$V(K) = \mathrm{Span}(\{\pi(g)(v) - v | g \in K, v \in V\}).$$

Observe that $V(K)$ and V^K are two K-invariant subspace of V. We make the following claim:

Claim: "$V = V^K \oplus V(K)$."

Assuming the claim for the time being we prove the proposition as follows.

Let $\widetilde{v} \in \widetilde{V}^K$. By definition of $\widetilde{V}^K$, we have

$$\widetilde{v}(\pi(g)u - u) = \widetilde{v}(\pi(g)(u)) - \widetilde{v}(u) = \widetilde{\pi}(g^{-1})(\widetilde{v})(u) - \widetilde{v}(u) = 0$$

for all $g \in K$ and $u \in V$. It implies that $\widetilde{v}_{|V(K)} = 0$. Thus $\widetilde{v} \in (V^K)^*$. By the admissibility of V, we have $\dim_{\mathbb{C}}((V^K)^*) = \dim_{\mathbb{C}}(V^K) < \infty$. Hence $\dim_{\mathbb{C}}(\widetilde{V}^K) < \infty$.

Now given $v^* \in (V^K)^*$, we extend $\widetilde{v}$ to an element of V^* by letting $\widetilde{v}$ equal to zero on $V(K)$. We shall prove that $\widetilde{v} \in \widetilde{V}^K$.

- Let $u \in V$. We have then $u = u^K + w$ where $u^K \in V^K$ and $w \in V(K)$. For all $g \in K$ we have

  $$\widetilde{\pi}(g)(\widetilde{v})(u) = \widetilde{v}(\pi(g^{-1})(u^K + w)) = \widetilde{v}(u^K + \pi(g^{-1})(w)) = \widetilde{v}(u^K) = \widetilde{v}(u).$$

  Thus $\widetilde{v}$ is invariant under the action of K.

- Assume that $g \in \mathrm{Stab}_{\pi^*}(\widetilde{v})$ (we use this notation to show that we are considering the action of G via π^*). Then gK is an open neighborhood of g which is contained in $\mathrm{Stab}_{\pi^*}(\widetilde{v})$. It implies that $\mathrm{Stab}_{\pi^*}(\widetilde{v})$ is open.

We have shown that $(\widetilde{V})^K = (V^K)^*$. It implies that

$$(\widetilde{\widetilde{V}})^K = ((\widetilde{V})^K)^* = ((V^K)^*)^* \simeq V^K.$$

For each $u \in V$, we consider the linear map $f_u : \widetilde{V} \to \mathbb{C}, \quad \widetilde{v} \mapsto \widetilde{v}(u)$. We have $\mathrm{Stab}_\pi(u) \subset \mathrm{Stab}_{(\widetilde{\pi})^*}(f_u)$. Because $\mathrm{Stab}_\pi(u)$ is open in G (since (π, V) is a smooth representation), it contains an open compact subgroup H of G. Let $g \in \mathrm{Stab}_{(\widetilde{\pi})^*}(f_u)$. Since gH is an open neighbourhood of g which is contained in $\mathrm{Stab}_{(\widetilde{\pi})^*}(f_u)$, then $\mathrm{Stab}_{(\widetilde{\pi})^*}(f_u)$ is open in G. Therefore f_u is an element of $\widetilde{\widetilde{V}}$.

We consider the natural map $\varphi : V \to \widetilde{\widetilde{V}}, \quad v \mapsto f_v$. As above $\varphi_{|V^K}$ is an isomorphism between V^K and $(\widetilde{\widetilde{V}})^K$ for any open compact subgroup K of G.

- Let f be any element of $\widetilde{\widetilde{V}}$. Since $\mathrm{Stab}_{(\widetilde{\pi})^*}(f)$ is open in G, then there exists an open compact subgroup $H \subset \mathrm{Stab}_{(\widetilde{\pi})^*}(f)$. It implies that $f \in (\widetilde{\widetilde{V}})^H$. Since $\varphi_{|V^H}$ is an isomorphism between V^H and $\widetilde{\widetilde{V}}^H$, there exist then $v \in V^H \subset V$ such that $\varphi(v) = f$. Hence φ is an epimorphism.

- Assume that $\varphi(v) = \varphi(v')$. Because $\mathrm{Stab}_\pi(v)$ and $\mathrm{Stab}_\pi(v')$ are two open subgroups of G, the subgroup $\mathrm{Stab}_\pi(v) \cap \mathrm{Stab}_\pi(v')$ is also open in G. There exists then an open compact subgroup $H \subset \mathrm{Stab}_\pi(v) \cap \mathrm{Stab}_\pi(v')$. We have $v, v' \in V^H$. Since $\varphi_{|V^H}$ is an isomorphism between V^H and $\widetilde{\widetilde{V}}^H$, we have $v = v'$. Hence φ is injective.

- We have

  $$\widetilde{\widetilde{\pi}}(g)(f_v)(\widetilde{u}) = f_v(\widetilde{\pi}(g^{-1})\widetilde{u}) = (\widetilde{\pi}(g^{-1})\widetilde{u})(v) = \widetilde{u}(\pi(g)v) = f_{\pi(g)v}(\widetilde{u}).$$

  It implies that $\varphi \circ \pi = \widetilde{\widetilde{\pi}} \circ \varphi$.

In conclusion, φ is an isomorphism between two representations (π, V) and $(\widetilde{\widetilde{\pi}}, \widetilde{\widetilde{V}})$. In other word,

$$\pi \simeq \widetilde{\widetilde{\pi}}.$$

It suffices now to prove the claim. Let v be any vector of V. Because $\mathrm{Stab}_K(v)$ is open in K and K is compact and totally disconnected, the set $S := K/\mathrm{Stab}_K(v)$ is finite. We have

$$v = \frac{1}{\#S}\sum_{g \in S}\pi(g)v - \frac{1}{\#S}\sum_{g \in S}(\pi(g)v - v).$$

It easy to check that in the right hand side, the first factor is a vector of V^K and the second one is a vector of $V(K)$. Hence $V = V^K + V(K)$.

Now we prove that $\sum_{g\in S}\pi(g)v = 0$ if $v \in V(K)$. By definition of $V(K)$, it suffices to prove for $v = \pi(g_0)u - u$ for some $g_0 \in K$ and $u \in V$. In fact, we have:

$$\sum_{g\in S}\pi(g)v = \sum_{g\in S}\pi(g)(\pi(g_0)u - u) = \sum_{g\in S}\pi(g)v - \sum_{g\in S}\pi(gg_0)u = 0.$$

The last equation is a consequence of the fact that gg_0 runs through all the equivalent classes of $K/\mathrm{Stab}_K(v)$. Therefore, if $v \in V^K \cap V(K)$, we have then

$$v = \frac{1}{\#S}\sum_{g\in S}\pi(g)v = 0.$$

Thus $V^K \cap V(K) = 0$. □

From the definition of a contragredient representation, it easy to check that the canonical non-degenerate bilinear form $\langle v, v^*\rangle = v^*(v)$ on $V \times \widetilde{V}$ satisfies

$$\langle \pi(v), \widetilde{\pi}(v^*)\rangle = \langle v, v^*\rangle.$$

A very natural question is that do a non-degenerate bilinear form invariant under the action of G defines a contragredient representation? The answer is yes in the case when π is admissible. More precisely, we have the following proposition.

Proposition 1.1.3. *Let (π, V) be an admissible representation. Assume that there exists an another admissible represenation (π', V') and a non-degenerate bilinear form $\phi : V \times V' \to \mathbb{C}$ such that*

$$\phi(\pi(g)(v), \pi'(g)(v')) = \phi(v, v').$$

Then $(\pi', V') \simeq (\widetilde{\pi}, \widetilde{V})$.

Proof. Denote $\varphi(v') = \phi(., v') \in V^*$ for all $v' \in V'$. We have

$$\widetilde{\pi}(g)(\varphi(v'))(v) = \phi(v')(\pi(g^{-1}v)) = \phi(\pi(g^{-1})v, v') = \phi(v, \pi'(g)v'). \quad (1.1.1)$$

Since ϕ is non-degenerate, then $\mathrm{Stab}_{\widetilde{\pi}}\varphi(v') = \mathrm{Stab}_{\pi'}(v')$. In other word, $\varphi(v')$ is a smooth vector in V^*.

We consider a homormorphism

$$V' \to \widetilde{V} \quad v' \mapsto \varphi(v').$$

We now prove that this homomorphism is G-isomorphic.

- The identity (1.1.1) implies that

$$\widetilde{\pi}(g)(\varphi(v')) = \varphi(\pi'(g)v') \quad \forall g \in G, v' \in V'.$$

- Since ϕ is non-degenerate, $\varphi(v') = 0$ if and only if $v = 0$. Hence φ is injective.

- Let $\xi \in \widetilde{V}$. Take K be a compact subgroup contained in $\mathrm{Stab}_{\widetilde{\pi}}(\xi)$. We have then $\xi \in \widetilde{V}^K$. By the admissibility of π, π' and non-degenerateness of ϕ, we have $\dim_{\mathbb{C}}(V^K) = \dim_{\mathbb{C}}((V')^K) < \infty$. Using the proof Proposition 1.1.2, we also have $\dim_{\mathbb{C}}(V^K) = \dim_{\mathbb{C}}(\widetilde{V}^K) < \infty$. Thus $\varphi_{|(V')^K}$ is an isomorphism. In other word, there exists $\xi' \in V'$ such that $\varphi(\xi') = \xi$.

□

A smooth representation (π, V) of an ℓ-group G is said to be **irreducible** if the only G-invariant subspaces of V are 0 and V itself.

Lemma 1.1.4 (Schur's lemma)**.** *Let (π, V) be an irreducible admissible representation of an ℓ-group G. Then we have* $\dim_{\mathbb{C}}(\mathrm{Hom}_G(\pi, \pi)) = 1$.

Proof. Let $A \in \mathrm{Hom}_G(\pi, \pi)$. We take an arbitrary $v \in V$. Using Lemma 1.1.1, there exist an open compact subgroup K such that $v \in V^K$. By definition of A we have $\pi(g)(Au) = A(\pi(g)u) = Au$ for all $g \in K$ and all $u \in V^K$. Thus $A_{|V^K}$ is a linear homomorphism form V^K to itself. Moreover V^K is a finite dimensional space, since (π, V) is admissible. Thus, $A_{|V^K}$ is an automorphism of the finite dimensional space V^K. Let $\lambda \in \mathbb{C}$ be a proper value of A. Then there exist $v \neq 0$ such that $Av = \lambda.v$. Denote by $V' := \{v \in V | Av = \lambda.v\}$ the proper subspace w.r.t the proper value λ of V. It is easily seen that V' is a G-invariant subspace of V. By the irreducibility of V, we have $V' = V$. It follows that $A = \lambda \mathbf{1}_V$ (here $\mathbf{1}_V$ is the identity automorphism of V). □

Corollary 1.1.5. *Let $Z = \{g \in G | g'.g = g.g', \forall g' \in G\}$ be the center of G. If (π, V) is an irreducible admissible representation of an ℓ-group G, there exists then a **quasicharacter** (that is, a smooth one-dimensional representation) χ_π of Z such that $\pi(z) = \chi_\pi(z).\mathbf{1}_V$ for all $z \in Z$. (This χ_π is called the **central quasi-character** of π.)*

Proof. Let z be any element of Z. We have

$$\pi(g)\pi(z) = \pi(gz) = \pi(zg) = \pi(z)\pi(g)$$

for all $g \in G$. It implies that $\pi(z) \in \mathrm{Hom}_G(\pi, \pi)$. By the Schur's lemma, there exists $c_z \in \mathbb{C}$ such that $\pi(z) = c_z.\mathbf{1}_V$. We denote by

$$\chi_\pi : Z \to \mathbb{C}^\times, \quad z \mapsto c_z.$$

It is easy to check that χ_π is a group homormorphism and $\pi(z) = \chi_\pi(z).\mathbf{1}_V$. Finally, let v be a non-zero of V. By Lemma 1.1.1, there exists an open compact subgroup K of G such that $v \in V^K$. We have $v = \pi(z)v = \chi_\pi(z)v$ for all $z \in K \cap Z$. It implies that $\chi_\pi(z) = 1$ for all $z \in K \cap Z$. Thus for all $c \in \mathbb{C}$ the map $Z \to \mathbb{C} : z \mapsto \chi_\pi(z)c$ is locally constant. By loc. cit., χ_π is a smooth representation of Z. □

Lemma 1.1.6. *Let (π, V) be an admissible representation of an ℓ-group G and $(\widetilde{\pi}, \widetilde{V})$ its contragredient. Then (π, V) is irreducible if and only if $(\widetilde{\pi}, \widetilde{V})$ is irreducible.*

Proof. Assume that $0 \neq U$ is a G-invariant subspace of V. Let W be a subspace of V such that $V = U \oplus W$ (W does not need to be G-invariant). Each element $\lambda \in U^*$ can be extended to an element of V by letting $\lambda(w) = 0$ for all $w \in W$. In this sense, we can view U^* as a G-invariant subspace of V^*. Then $\widetilde{U}$ is a G-invariant subspace of $\widetilde{V}$. Moreover, by Proposition 1.1.2, $\widetilde{U} \neq 0$ (otherwise $0 = \widetilde{\widetilde{U}} \simeq U$). Thus $\widetilde{U}$ is a non-zero G-invariant subspace of $\widetilde{V}$. The lemma is a direct consequence of this argument and loc. cit.. For example, to prove that (π, V) is irreducible if $(\widetilde{\pi}, \widetilde{V})$ is irreducible, we do as follows.

Assume that (π, V) is reducible. There exists then a non-zero proper G-invariant subspace U of V. By irreducibility of $\widetilde{V}$, we have $\widetilde{U} = \widetilde{V}$. Now, using loc. cit., we have $U \simeq \widetilde{\widetilde{U}} = \widetilde{\widetilde{V}} \simeq V$ (contradictory).

□

One of the main goal of this chapter is to classify irreducible admissible representations of G. The finite dimensional admissible irreducible smooth representations of G are not very interesting. Each is a one dimensional space on which the element of G acts by scalar. This is the content of Proposition 1.1.8. The proof of this proposition requires the following lemma.

Lemma 1.1.7. *The matrices $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right)$ with $x, y \in F$ generate $\mathrm{SL}_2(F)$*

Proof. Every $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{SL}_2(F)$ can be written in the following form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}.$$

On other hand, if $c \neq 0$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & c^{-1}a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & 0 \\ 0 & -c^{-1} \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}.$$

Those identities imply that $\mathrm{SL}_2(F)$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the matrices $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$, with $x, y \in F$ and $z \in F^\times$.

Moreover, we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-1}-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix}$$

for all $z \in F^\times$. The proof of the lemma is a direct consequence of the above two identities. □

Proposition 1.1.8. *A finite admissible irreducible of G is one dimensional. Moreover, it is of the form $g \to \chi(\det(g))$ for some quasi-character χ of $F^\times$.*

Proof. Let (π, V) be a finite dimensional irreducible admissible representation of V. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. By Lemma 1.1.1, for each $i \in \{1, \ldots, n\}$ there exists an open compact subgroup $K_i \subset G$ that stabilises v_i. We denote by K the intersection of K_i. Then K is an open compact subgroup of G and fixes V. So the kernel $H := \ker(\pi)$ of the representation contains a compact open subgroup. In other word, H is a non-trivial open normal subgroup of G.

Now let $x \in F$ be arbitrary. We choose $b \in F$ such that $|bx|_F$ is sufficient small so that $\left(\begin{smallmatrix} 1 & bx \\ 0 & 1 \end{smallmatrix}\right) \in H$. Then

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & bx \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \in H.$$

Similarly, we also can show that $\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right) \in H$ for all $y \in F$. It immediately follows from Lemma 1.1.7 that $\mathrm{SL}_2(F) \subset H$. Note that $gg_1g^{-1}g_1^{-1} \in \mathrm{SL}_2(F)$ for all $g, g_1 \in G$. Thus $\pi(g)\pi(g_1) = \pi(g_1)\pi(g)$ for all $g, g_1 \in G$. It implies that $\pi(g) \in \mathrm{Hom}_G(\pi, \pi)$. By the Schur's lemma, there exists $\delta_g \in \mathbb{C}^\times$ such that $\pi(g) = \delta_g.\mathbf{1}_V$.

We consider the subspace $V_1 := \mathbb{C}v_1$ generated by the vector v_1 of V. For any $g \in G$, $k \in \mathbb{C}$, we have $\pi(g)(kv_1) = k\delta_g v_1 \in V_1$. It implies that V_1 is a G-invariant subspace of V. By the irreducibility of V, we have $V = V_1$. We have shown that the dimension of V is one.

Now we consider $\pi : G \to \mathbb{C}^\times$ being a smooth representation of G. We define an application $\chi : F^\times \to \mathbb{C}^\times$ by

$$\chi(z) = \pi((\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix})).$$

It is clear that χ is a quasi-character of $F^\times$ and

$$\pi(g) = \chi(\det(g)), \forall g \in G.$$

□

Corollary 1.1.9. *Any quasi-character of G is of the form $\phi \circ \det$, for some quasi-character ϕ of $F^\times$.*

1.1.2 Haar measures and the Hecke algebra

Let G be an ℓ-group. Let $C_c^\infty(G)$ be the space of functions $f : G \to \mathbb{C}$ which are locally constant and of compact support. The group G acts on $C_c^\infty(G)$ by *left* and *right translation* by the formulas

$$\ell_x f(y) = f(x^{-1}y), \text{ and } r_x f(y) = f(yx).$$

Local constancy and compactness of support of function in $C_c^\infty(G)$ imply that both of the G-representations $(C_c^\infty(G), \ell)$, $(C_c^\infty(G), r)$ are smooth.

Definition 1.1.10. A *left invariant distribution* on G is a linear form $\xi : C_c^\infty(G) \to \mathbb{C}$ such that $\xi(\ell_x f) = \xi(f)$ for all $x \in G$ and $f \in C_c^\infty(G)$.

A *left Haar distribution* on G is a non-zero left invariant distribution ξ such that $\xi(f) \geq 0$ whenever $f \geq 0$.

We can also define a *right invariant distribution* (resp. *right Haar distribution*) similarly, using right translation r instead of left translation ℓ.

Proposition 1.1.11. *There exists a left Haar distribution $I : C_c^\infty(G) \to \mathbb{C}$. Moreover, the space of left invariant distributions on G is one dimensional $\mathbb{C}$ -vector space.*

Proof. Let K be a compact open subgroup of G, we denote by $C_c^\infty(G)^K$ the space of functions in $C_c^\infty(G)$ that are right invariant under K. The $(C_c^\infty(G)^K, \ell)$ is then a smooth representation of G.

Lemma 1.1.12. *Viewing $\mathbb{C}$ as the trivial G-representation, we have*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_G(C_c^\infty(G/K), \mathbb{C})) = 1.$$

There exists a non-zero element $I_K \in \mathrm{Hom}_G(C_c^\infty(G/K), \mathbb{C})$ such that $I_K(f) \geq 0$ whenever $f \geq 0$. If 1_K is the characteristic function of K, then $I_K(1_K) > 0$.

Proof. The space $C_c^\infty(G/K)$ has a basis 1_{xK} consisting of characteristic function of right cosets xK. A linear form $\xi : C_c^\infty(G/K) \to \mathbb{C}$ is G-invariant if and only if $\xi(1_{xK}) = \xi(1_K)$ for all $x \in G$. In other words, the map $\mathrm{Hom}_G(C_c^\infty(G/K), \mathbb{C}) \to \mathbb{C}$ given by $\xi \mapsto \xi(1_K)$ is an isomorphism. In particular $\mathrm{Hom}_G(C_c^\infty(G/K), \mathbb{C})$ is one dimensional.

The linear form $I_K : 1_{xK} \mapsto 1$ has the required properties. □

We choose a descending sequence $\{K_i\}_{i\geq 1}$ of normal compact open subgroup K_i of G such that $\bigcap_i K_i = 1$ (due to van Dantzig's lemma, there always exists this kind of sequence - in the case when $G = \mathrm{GL}_2(F)$ we can choose $K_i = 1 + \varpi^i M_2(\mathcal{O})$ for all $i \geq 1$). We have then:

$$C_c^\infty(G) = \bigcup_{i\geq 1} C_c^\infty(G/K_i).$$

For each $i \geq 1$, there is a unique left G-invariant linear form $I_i : C_c^\infty(G/K_i) \to \mathbb{C}$ which maps the characteristic function of K_i to $(\#(K_1/K_i))^{-1}$. Since the restriction of I_{i+1} on $C_c^\infty(G/K_i)$ is I_i, the form $I : C_c^\infty(G) \to \mathbb{C}$ defined by $I(f) = I_i(f)$ whenever $f \in C_c^\infty(G/K_i)$ is well-defined. The statements of Proposition are immediate. □

Proposition 1.1.13. *content...*

Let H be a closed subgroup of G with module δ_H. Let $\theta : H \to \mathbb{C}^\times$ be a character of H. We consider the space $C_c^\infty(H\backslash G, \theta) = c - \mathrm{Ind}_H^G \theta$, i.e the space of functions $f : G \to \mathbb{C}$ which are G-smooth under right translation, compactly supported modulo H, and satisfy

$$f(hg) = \theta(h) f(g), \quad h \in H, g \in G.$$

Proposition 1.1.14. *Let $\delta_{H\backslash G}(h) = \delta_H(h)^{-1}\delta_G(h)$, $h \in H$. There exist a non-zero linear functional $I_{H\backslash G} : C_c^\infty(H\backslash G, \delta_{H\backslash G}) \to \mathbb{C}$ having the following two properties:*

(1) $I_{H\backslash G}(r_g(f)) = I_{H\backslash G}(f)$, for all $f \in C_c^\infty(H\backslash G, \delta_{H\backslash G})$ and all $g \in G$.

(2) If $g \in G$, K is a compact open subgroup of G, and $f \in C_c^\infty(H\backslash G, \delta_{H\backslash G})^K$ is supported on the double coset HgK, then $I_{H\backslash G}$ is a positive multiple of $f(g)$.

Proof. Let μ_G, μ_H be left Haar measures on G, H respectively. For each $f \in C_c^\infty(G)$, we define $\tilde{f} : G \to \mathbb{C}$ by

$$\tilde{f}(g) := \int_H \delta_G(h)^{-1} f(hg) d\mu_H(h).$$

By definition, we have

$$\begin{aligned}\tilde{f}(h_1g) &= \int_H \delta_G(h)^{-1}f(hh_1g)d\mu_H(h)\\ &= \delta_{H\backslash G}(h_1)\int_H \delta_G(hh_1)^{-1}f(hh_1g)\delta_H(h_1)d\mu_H(h)\\ &= \delta_{H\backslash G}(h_1)\int_H \delta_G(hh_1)^{-1}f(hh_1g)d\mu_H(hh_1)\\ &= \delta_{H\backslash G}(h_1)\tilde{f}(g)\end{aligned}$$

for all $h_1 \in H$. Since the support of f is compact, the support of $\tilde{f}$ is compact modulo H. If K is a compact open subgroup of G such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$, then $\tilde{f}(gk) = \tilde{f}(g)$ for all $g \in G$ and $k \in K$. Hence $\tilde{f} \in C_c^\infty(H\backslash G, \theta)$. Moreover, we have

$$\begin{aligned}r_{g_1}(\tilde{f})(g) &= \tilde{f}(gg_1) = \int_H \delta_G(h)^{-1}f(hgg_1)d\mu_H(h)\\ &= \widetilde{r_{g_1}(f)}(g).\end{aligned}$$

It implies that the map $(C_c^\infty(G), r) \to (C_c^\infty(H\backslash G, \delta_{H\backslash G}), r)$ which sends f to $\tilde{f}$ is a G-homomorphism.

This homomorphism satisfies

$$\begin{aligned}\widetilde{\ell_{h_1}(f)}(g) &= \int_H \delta_G(h)^{-1}f(h_1^{-1}hg)d\mu_H(h)\\ &= \delta_G(h_1)^{-1}\tilde{f}(g)\end{aligned}$$

for $h_1 \in H$ and $f \in C_c^\infty(G)$. We now prove that it is surjective.

Let $\varphi \in C_c^\infty(H\backslash G, \theta)$. Then there exists a compact open subgroup K of G such that $\varphi \in C_c^\infty(H\backslash G, \theta)^K$ (the subspace of $C_c^\infty(H\backslash G, \theta)$ which is invariant under the action of K). Since φ has compact support modulo H, there exist $g_1, \ldots, g_n \in G/K$ such that $\varphi(g) = 0$ if $g \notin \bigsqcup_{i=1}^n Hg_iK$. We define a function $f : G \to \mathbb{C}$ as follows

$$f(g_ik) = \mathrm{vol}(H \cap g_iKg_i^{-1})\varphi(g_i)$$

and $f(g) = 0$ for $g \notin \bigsqcup_{i=1}^n g_iK$. By definition, $f \in C_c^\infty(G)$ and

$$\begin{aligned}\tilde{f}(h_1g_ik) &= \theta(h_1)\tilde{f}(g_ik) = \theta(h_1)\int_H (\theta\delta_H)(h)^{-1}f(hg_ik)d\mu_H(h)\\ &= \theta(h_1)\int_{H\cap g_iKg_i^{-1}} (\theta\delta_H)(h)^{-1}\mathrm{vol}(H \cap g_iKg_i^{-1})\varphi(g_i)d\mu_H(h).\end{aligned}$$

□

1.1.3 Parabolic induction and Jacquet module

One of the way to construct representations of G is to induce representations from smaller subgroups. In this section, we induce the representations of B which are trivial on its nilpotent subgroup N. Non-trivial characters on N (Whittaker functionals) are also interesting. They will be studied in Section 1.1.6.

Definition 1.1.15. Let (σ, W) be a smooth representation of T. We consider the space $\mathrm{Ind}_B^G W$ of smooth functions $f : G \to W$ which satisfy

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \left|\frac{a_1}{a_2}\right|^{1/2} \sigma\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} g\right) f(g).$$

We define a homomorphism $\mathrm{Ind}_B^G\sigma : G \to \mathrm{Aut}_{\mathbb{C}}\left(\mathrm{Ind}_B^G W\right)$ by

$$\mathrm{Ind}_B^G\sigma(g) f : x \mapsto f(xg), \quad x, g \in G.$$

The pair $(\mathrm{Ind}_B^G\sigma, \mathrm{Ind}_B^G W)$ provides a smooth representation of G. It is called the *(normalized) parabolic induction* of σ.

Remark 1.1.16. (1) Due to Iwasawa decomposition $G = BK_0$, the subspace $c - \mathrm{Ind}_B^G\sigma$ of smooth functions $f \in \mathrm{Ind}_B^G\sigma$ which are *compactly supported modulo* B (this means that the image of the support of f in $B\backslash G$ is compact) is the whole space $\mathrm{Ind}_B^G\sigma$. In other word, $\mathrm{Ind}_B^G\sigma$ is also the compact induction of σ.

(2) Let $\chi = \chi_1 \otimes \chi_2$ be a quasi-character of T. The representation $\mathrm{Ind}_B^G\chi$ is called the *principal series representation* of G.

Lemma 1.1.17. *The principal series* $\mathrm{Ind}_B^G\chi$ *is admissible.*

Proof. Let K be a compact open subgroup of G. We may assume that $K \subset K_0$ (since all the maximal compact subgroups of G are conjugate to K_0). Since $G = BK_0$ (Iwasawa decomposition) and K_0/K is finite, the set of double cosets $B\backslash G/K$ is also finite. By definition, a function $f \in \mathrm{Ind}_B^G\chi$ is defined uniquely by its image over the set of double cosets $B\backslash G/K$. Hence,

$$\dim_{\mathbb{C}}((\mathrm{Ind}_B^G\chi)^K) < \infty.$$

□

The character $\delta_B^{1/2} : \mathrm{diag}(a_1, a_2) \mapsto \left|\frac{a_1}{a_2}\right|^{1/2}$ was introduced so that $\mathrm{Ind}_B^G\chi$ preserves unitarity.

Proposition 1.1.18. *If χ is unitary then* $\mathrm{Ind}_B^G\chi$ *has a natural G-invariant Hermitian inner product, defined by* $||f||^2 = \int_{K_0} |f(k)|^2 dk$.

Definition 1.1.19. Let (V, π) be a smooth representation of G. Let

$$V(N) := \mathrm{Span}(\{\pi(n)v - v | n \in N, v \in V\}).$$

This $V(N)$ is an N-invariant subspace of V. Let $V_N = V/V(N)$ the largest quotient of V on which N acts trivially. Because N is invariant under T, the V_N inherits a representation π_N of $B/N = T$ (can be also viewed as a representation of B which is trivial on N), which is smooth. The *(normalized) Jacquet module* $\mathrm{Jac}_B^G\pi$ or $\mathrm{Jac}_B^G(V)$ is the representation $(\pi_N \otimes \delta_B^{-1/2}, V_N)$ of B.

Theorem 1.1.20 (Frobenius reciprocity). *For any smooth representation (π, V) (resp. (σ, W)) of G (resp. T) we have a natural isomorphism*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G\sigma) \simeq \mathrm{Hom}_T(\mathrm{Jac}_B^G\pi, \sigma).$$

Corollary 1.1.21. *Let (π, V) be an irreducible smooth representation of G. If* $\mathrm{Jac}_B^G(V) \neq 0$ *(equivalently, $V_N \neq 0$) then V is embeds in a principal series representation of G (i.e in a* $\mathrm{Ind}_B^G\chi$ *for some quasi-character χ of T).*

Let w_0 be the longest Weyl element of G, i.e

$$w_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For each smooth representation σ of T, we define the representation $\sigma^{w_0} : t \mapsto \sigma(w_0 t w_0^{-1})$, and view it as a representation of B which is trivial on N.

Lemma 1.1.22 (Restriction-Induction Lemma). *Let (σ, W) be a smooth representation of T. There is an exact sequence of representations of T:*

$$0 \to \delta_B^{-1/2} \otimes \sigma^{w_0} \to \mathrm{Jac}_B^G \mathrm{Ind}_B^G \sigma \xrightarrow{\alpha_\sigma} \delta_B^{1/2} \otimes \sigma \to 0,$$

where α_σ is the canonical T-map $\mathrm{Jac}_B^G(\mathrm{Ind}_B^G(W)) \to W$ *defined by $f \mapsto f(1)$.*

Theorem 1.1.23 (Irreducibility Criterion). *Let $\chi = \chi_1 \otimes \chi_2$ be a quasicharacter of T.*

(1) The representation $\mathrm{Ind}_B^G\chi$ *is irreducible unless* $\chi_1\chi_2^{-1} = |.|^{\pm 1}$.

(2) If $\chi_1\chi_2^{-1} = |.|$ then $\mathrm{Ind}_B^G\chi$ *contains an irreducible admissible G-subspace of codimension 1.*

(3) *If* $\chi_1\chi_2^{-1} = |.|^{-1}$ *then* $\mathrm{Ind}_B^G\chi$ *contains a 1-dimensional* G*-subspace whose quotient is irreducible.*

Theorem 1.1.24 (Classification theorem)**.** *Let* π *be an irreducible admissible representation of* G*. Then* π *is equivalent to one of the following disjoint types:*

(1) *the irreducible induced representations* $\mathrm{Ind}_B^G\chi$*, where* $\chi \neq \phi \otimes \delta_B^{\pm 1/2}$ *for any quasi-character* ϕ *of* $F^\times$*;*

(2) *the special representations* $\chi \otimes \mathrm{St}_G$*, where* χ *ranges over the quasi-characters of* $F^\times$*;*

(3) *the cuspidal representations;*

(4) *the 1-dimensional representations* $\chi \circ \det$*, where* χ *ranges over the quasi-characters of* $F^\times$*.*

Moreover

(a) *in* (1)*, we have* $\mathrm{Ind}_B^G\chi \simeq \mathrm{Ind}_B^G\psi$ *if and only if* $\psi = \chi$ *or* χ^ω*;*

(b) *in* (2)*, we have* $\chi \otimes \mathrm{St}_G \simeq \chi' \otimes \mathrm{St}_G$ *if and only if* $\chi = \chi'$*;*

(c) *in* (4)*, we have we have* $\chi \circ \det \simeq \chi' \circ \det$ *if and only if* $\chi = \chi'$*.*

1.1.4 Cuspidal representations

Let E/F be a separable quadratic extension of local field F. We fix a non-trivial additive character $\psi = \psi_F : F \to \mathbb{C}^\times$. Then $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$ is a non-trivial additive character of E. Let $C_c^\infty(E)$ be the space of complex valued smooth functions of compact support on E. Given $f \in C_c^\infty(E)$, define the *Fourier transform* $\hat{f} \in C_c^\infty(E)$ by

$$\hat{f}(y) = \int_E f(x)\psi_E(xy)dx,$$

where dx is the self-dual measure with respect to ψ_E on E (i.e dx is the normalized Haar measure so that $\hat{\hat{f}}(x) = f(-x)$). Since dx is self-dual, we have then the *Fourier inversion formula*

$$f(x) = \int_E \hat{f}(y)\widetilde{\psi_E(xy)}dy.$$

Lemma 1.1.25 (Weil constant)**.** *There exist a constant* $\gamma(\psi_F, E)$ *such that for every function* $\phi \in C_c^\infty(E)$

$$\int_E (\phi * f)(x)\psi_E(xy)dx = \gamma(\psi_F, E)f^{-1}(\iota(y))\hat{f}(y).$$

1.1.5 Kloosterman integrals and Shalika germs

In this section, we shall prove the existence of Shalika germs for **orbital (Kloosterman) integrals** which are appeared in the geometric side of Kuznetsov trace formula for GL_2. The main reference for this subsection is [10, 12]. (In these loc. cit. Jacquet and Ye proved the existence of Shalika germs for a more general orbital integrals which are appeared in the geometric side of Kuznetsov trace formula for GL_r).

For a convenience, we recall the definition of the orbital integral. Let G be the group GL_2 viewed as an algebraic group over F. We often write G for $G(F)$. We denote by $C_c^\infty(G)$ the space of complex valued, locally constant functions of compact support on G. Let Z be the center of G. Let W be the Weyl group of G. Let T be the subgroup of diagonal matrices of G and N the subgroup of upper-triangular matrices with unit diagonal. We fix a non-trivial additive quasi-character ψ of F and define a character $\theta : N \to \mathbb{C}^\times$ by the formula

$$\theta(u) = \psi(n_{1,2}),$$

where $n = \left(\begin{smallmatrix} 1 & n_{1,2} \\ 0 & 1 \end{smallmatrix}\right)$.

The Kloosterman integrals of a function $f \in C_c^\infty(G)$ which we want to study are the functions:

$$I(g, f) = \int f({}^t n_1 g n_2)\theta(n_1 n_2) dn_1 dn_2.$$

Here g is a *relevant element*, i.e g satisfies a condition that $\theta(n_1 n_2) = 1$ if ${}^t n_1 g n_2 = g$. The integral is taken over the quotient of $N(F) \times N(F)$ by the subgroup N^g of elements (n_1, n_2) of $N(F) \times N(F)$ satisfying ${}^t n_1 g n_2 = g$.

Lemma 1.1.26. *Let $N \times N$ operate on G by $(n_1, n_2).g = {}^t n_1 g n_2$. Then any relevant orbit of $N \times N$ contains a unique representative of the form wt with $w \in R(G) := \left\{ e := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, w_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ and*

$$t \in T_w := \begin{cases} T & \text{if } w = e, \\ Z & \text{if } w = w_0. \end{cases}$$

Suppose that $w \in R(G)$. Let M_w be the standard Levi subgroup such that w is the longest element of $M_w \cap W$. Let $P_w = M_w U_w$ be the standard parabolic subgroup which has Levi factor M_w. Set $V_w = N \cap M_w$. For every $t \in T_w$, (by an elementary matrix calculation) we have

$$N^{wt} = N^w = \{(n_1, n_2) \in V_w^2 | n_2 = w {}^t n_1^{-1} w\}.$$

Lemma 1.1.27. *Any point of the orbit of wt under the action of $N(F) \times N(F)$ can be uniquely written in the following form*

$$\mu(u_1, v, u_2) = u_1 w \mathbf{t} v u_2$$

with $u_i \in U_w$ and $v \in V_w$.

Proof. Since $N \subset P_w$, by using the Levi decomposition for elements of N, we can rewrite any element of the orbit of wt under the action of $N(F) \times N(F)$ as below:

$$\begin{aligned} {}^t n_1 w \mathbf{t} n_2 &= {}^t u_1 {}^t v_1 w t v_2 u_2 \\ &= {}^t u_1 [{}^t v_1 w t (w {}^t v_1^{-1} w)][(w {}^t v_1 w) v_2] u_2 \\ &= {}^t u_1 w \mathbf{t} v u_2. \end{aligned}$$

Here $u_i \in U_w$, $v_i \in V_w$, $v \in V_w$ such that $v_1 u_1 = n_1$, $v_2 u_2 = n_2$ and $v = (w {}^t v_1 w) v_2$. The last identity follows $({}^t v_1, w {}^t v_1^{-1} w) \in N^w$.

Suppose that ${}^t u_1 w t v u_2 = {}^t u_1' w \mathbf{t} v' u_2'$ with $u_i, u_i' \in U_w$ and $v, v' \in V_w$. We have then

$${}^t(u_1^{-1} u_1') w t (v' u_2' u_2^{-1} v^{-1}) = wt.$$

Hence, $v' u_2' u_2^{-1} v^{-1} \in V_w$ and ${}^t(u_1^{-1} u_1') \in V_w$. It implies that $\{u_2' u_2^{-1}, u_1^{-1} u_1'\} \subset U_w \cap V_w = \{e\}$. Thus $u_i = u_i'$ for all $i \in \{1, 2\}$. As a consequence, we have $v = v'$. □

Since the orbits of $N(F) \times N(F)$ are closed, the map μ is an isomorphism of $U_w(F) \times V_w(F) \times U_w(F)$ onto the orbit of wt. Recall that we let dx be the self-dual Haar measure on F with respect to the fixed non-trivial additive character ψ. If α is a root let X_α be the corresponding root vector in the Lie algebra of N (entry at α is 1, the other entries are 0). If U is a subgroup of N generated by a set of roots S (i.e $U = \{u = 1 + \sum_{\alpha \in S} x_\alpha X_\alpha\}$) we set $du = \otimes_{\alpha \in S} dx_\alpha$. We take for invariant measure on the orbit of wt the product measure $du_1 dv du_2$. Thus

$$I(wt, f) = \int_{U_w(F) \times V_w(F) \times U_w(F)} f({}^t u_1 w t v u_2) \theta(u_1 u_2) \theta(v) du_1 dv du_2. \tag{1.1.2}$$

Since the orbit is closed, for $f \in C_c^\infty(G)$, the integral on the right hand side has compact support. Thus the integral converges and define a smooth function on $T_w(F)$ which send $t \in T_w(F)$ to $I(wt, f)$.

We denote by $T_w^{w_0} := \{t \in T_{w_0} | \det(w_0 t) = \det(w)\}$ for each $w \in R(G)$. For instance, the set $T_e^{w_0}$ is the set of matrices of the form

$$\begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix}.$$

Theorem 1.1.28. *There is a locally constant function $K_e^{w_0}$ on $T_e^{w_0}$ satisfying the following properties. For each function $f \in C_c^\infty(G)$, there is a function $\omega \in C_c^\infty(T_e)$ such that*

$$I(et, f) = \omega(t) + \sum_{(b,c)} K_e^{w_0}(b) I(w_0 c, f).$$

The sum is taken over the finite set

$$\{(b, c) \in T_e^{w_0} \times T_{w_0} | bc = t\}.$$

Proof. Let $G_1 = \{g \in G | \det(g) = \det(w_0)\}$. We have $w_0 T_{w_0} \cap G_1 = w_0 T_{w_0}^{w_0}$, a finite set. If $w_0 t$ where $t \in T_{w_0}$ is in G_1, then the scalar matrix $t = \mathrm{diag}(z, z)$ verifies $z^2 = 1$. We can choose $f_0 \in C_c^\infty(G)$ such that $I(w_0, f_0) = 1$ and $I(w_0 \mathrm{diag}(z, z), f_0) = 0$ if $z \neq 1$ and z is a square-root of 1 in F. (For example, we can choose $f_0 = \phi_m$ with m large enough as in Lemma 1.1.29 below.)

We define a function $K_e^{w_0}$ on $T_e^{w_0}$ by

$$K_e^{w_0}(t) = I(et, f_0).$$

We define a function f_1 on G by the formula

$$f_1(g) = \sum_{(g_1,c)} f_0(g_1) I(w_0 c, f),$$

where the sum is over the finite set

$$S_g := \{(g_1, c) \in G_1 \times T_{w_0} | g_1 c = g\}.$$

It is a smooth function on G.

For $t \in T_e$, we consider all possible decompositions

$${}^t n_1 e t n_2 = g_1 c,$$

with $g_1 \in G_1$ and $c \in T_{w_0}$. Since c is in the centre of G, we can write

$$g_1 = {}^t n_1 e t c^{-1} n_2 = {}^t n_1 e b n_2$$

where $b = tc^{-1} \in T_e^{w_0}$ (since $g_1 \in G_1$). Thus

$$f_1({}^t n_1 e t n_2) = \sum_{(b,c)} f_0({}^t n_1 e b n_2) I(w_0 c, f)$$

where the sum is over the finite set

$$\{(b,c) \in T_e^{w_0} \times T_{w_0} | bc = t\}.$$

Since c is in the centre of G, we have $N^{et} = N^{eb}$. After integrating two side of above identity over the quotient of $N(F) \times N(F)$ by the subgroup N^{et}, we obtain then

$$I(et, f_1) = \sum_{(b,c)} I(eb, f_0) I(w_0 c, f) = \sum_{(b,c)} K_e^{w_0}(b) I(w_0 c, f).$$

We define a function ω on T_e by the formula

$$\omega(t) = I(et, f) - I(et, f_1) = I(et, f - f_1).$$

It is a smooth function on T_e and we have

$$I(et, f) = \omega(t) + \sum_{(b,c)} K_e^{w_0}(b) I(w_0 c, f).$$

□

Lemma 1.1.29. *Let* $t = \mathrm{diag}(z, z)$ *with* $z^2 = 1$ *and* ϕ_m *a product of the characteristic function of the congruence group* K_m *and the scalar* $\mathrm{vol}(\mathfrak{p}^m)^{-1}$. *For* m *large enough, we have then*

$$I(w_0 t, \phi_m) = \begin{cases} 1, & \text{if } z = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Firstly, we calculate the integral $I(w_0, \phi)$. This integral has the form (cf. the formula (1.1.2))

$$I(w_0, \phi) = \int_F \phi\left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \psi(x) dx$$

Now we take ϕ be a product of the characteristic function of $w_0 K_m$ and the scalar $\mathrm{vol}(\mathfrak{p}^m)^{-1}$, this integral is equal to

$$\mathrm{vol}(\mathfrak{p}^m)^{-1} \int_{\mathfrak{p}^m} \psi(x) dx.$$

For m large enough (for example m is larger then the level of ψ), we have $\psi(x) = 1$. It implies that

$$\int_{\mathfrak{p}^m} \psi(x) dx = \mathrm{vol}(\mathfrak{p}^m).$$

In consequence, the first assertion is proved.

Choosing m large enough such that $z \notin K_m$ for all z which satisfy $z^2 = 1$ and $z \neq 1$. We have then ${}^t n_1 w_0 t n_2 \notin w_0 K_m$ for all $(n_1, n_2) \in N_r(F) \times N_r(F)$. The second assertion follows. □

Proposition 1.1.30. *The germ $K_e^{w_0}$ is given, for $|z|$ small enough, by*

$$K\begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix} = \left|\frac{1}{2z}\right|^{1/2} \psi\left(\frac{2}{z}\right)\gamma\left(\frac{2}{z}, \psi\right).$$

Proof. Let $f = \phi_m$ as in Lemma 1.1.29. The relation defining germ $K_e^{w_0}$ reads

$$I(t, \phi_m) = \omega_{\phi(m)}(t) + \sum_{(b,c)} K_e^{w_0}(b) I(w_0 c, \phi_m). \tag{1.1.3}$$

Since ω_{ϕ_m} is of compact support, for $|z|$ small enough we have

$$\omega_{\phi_m}\begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix} = 0.$$

Substituting $t = \begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix}$ with $|z|$ small enough to (1.1.3) and using Lemma 1.1.29, we have then

$$\begin{aligned} K_e^{w_0}\begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix} &= I\left(\begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix}, \phi_m\right) \\ &= \int_{F\times F} \phi_m\begin{pmatrix} z & zx_1 \\ zx_2 & -z^{-1} + x_1 x_2 z \end{pmatrix} \psi(x_1 + x_2) dx_1 dx_2. \end{aligned}$$

After changing x_1 to x_1/z and x_2 to x_2/z, the germ $K_e^{w_0}(\mathrm{diag}(z, -z^{-1}))$ is equal to

$$|z|^{-2} \int_{F\times F} \phi_m\begin{pmatrix} z & x_1 \\ x_2 & -z^{-1} + z^{-1}x_1 x_2 \end{pmatrix} \psi\left(\frac{x_1 + x_2}{z}\right) dx_1 dx_2.$$

The integral is 0 unless $z \in \mathfrak{p}^m$. We can choose $|z|$ small enough such that $z \in \mathfrak{p}^m$. We see that then the integral is equal to

$$|z|^{-2}\mathrm{vol}(\mathfrak{p}^m)^{-1} \int \psi\left(\frac{x_1 + x_2}{z}\right) dx_1 dx_2$$

integrated over the domain defined by:

$$x_i \equiv 1 \mod \mathfrak{p}^m \text{ for } i = 1, 2,$$

$$x_1 x_2 \equiv 1 \mod z\mathfrak{p}^m.$$

We change variables and set

$$x_2 = t x_1^{-1},$$

where now the domain of integration is defined by:

$$x_1 \equiv 1 \mod \mathfrak{p}^m,\ t \equiv 1 \mod z\mathfrak{p}^m.$$

(Since $z \in \mathfrak{p}^m$, the two conditions on x_1 and t guarantee that $t/x_1 \equiv 1 \mod \mathfrak{p}^m$.) After integrating over t the integral becomes

$$|z|^{-1} \int_{x_1 \equiv 1 \mod \mathfrak{p}^m} \psi\left(\frac{\phi}{z}\right) dx_1,$$

where the phase function ϕ is given by:

$$\phi = x_1 + \frac{1}{x_1}.$$

We set $x_1 = 1 + v$ with $v \in \mathfrak{p}^m$. The phase function takes the form

$$\phi = 1 + v + \frac{1}{1+v}.$$

The Taylor expansion of this function at the origin has the form

$$2 + v^2 + \text{higher degree terms.}$$

By the principle of the stationary phase there is a compact neighborhood Ω of 0 in F such that, for $|z|$ small enough, the integral is equal to

$$|z|^{-1} \int_\Omega \psi\left(\frac{2+v^2}{z}\right) dv = \left|\frac{1}{2z}\right|^{1/2} \psi\left(\frac{2}{z}\right) \gamma\left(\frac{2}{z}, \psi\right).$$

□

1.1.6 Kirillov models and Whittaker models

We fix a non-trivial character ψ of the additive group F. Let (π, V) is a representation of $G(F)$. Let $N = \{n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} | x \in F\}$, ψ defines a character ψ_N of N by

$$\psi_N(n) = \psi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \psi(x).$$

Definition 1.1.31.

- A **Kirillov model** for (π, V) is a sub-$\mathbb{C}$-vector space $\mathcal{K}(\pi, \psi)$ of the space of $\mathbb{C}$-valued on $F^\times$, and an action $\pi_{\mathfrak{k}}$ of $G(F)$ on $\mathcal{K}(\pi, \psi)$ with the property that

$$\pi_{\mathfrak{k}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} (f)(x) = \psi(bx)f(ax) \quad \forall a, x \in F^\times, b \in F, f \in \mathcal{K}(\pi, \psi),$$

  such that the representation V and $\mathcal{K}(\pi, \psi)$ are isomorphic.

- A **Whittaker model** for (π, V) is a sub-$\mathbb{C}$-vector space $\mathcal{W}(\pi, \psi)$ of the space of locally constant $\mathbb{C}$-valued functions on G satisfying

$$f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)f(g), \quad \forall g \in G, x \in F,$$

  and an action of $G(F)$ on $\mathcal{W}(\pi, \psi)$ defined by a right translation, i.e. $(g.f)(g') = f(g'g)$ such that the representation V and $\mathcal{W}(\pi, \psi)$ are isomorphic.

Theorem 1.1.32 ([9, Theorem 1, p. 1.3])**.** *If (π, V) is an irreducible admissible infinite-dimesional representation of $G(F)$ then (π, V) has a unique Kirillov model $\mathcal{K}(\pi, \psi)$. Furthermore, every $\kappa \in \mathcal{K}(\pi, \psi)$ is a locally constant function on $F^\times$ and vanishes outside some compact subset of F. The space $C_c^\infty(F^\times)$ of locally constant functions on $F^\times$ with compact support is a subspace of finite codimension of $\mathcal{K}(\pi, \psi)$.*

Proof. Assume that (π, V) has a Kirillov model $\mathcal{K}(\pi, \psi)$. Then the subspace $\mathcal{K}_0$ of $\mathcal{K}(\pi, \psi)$ consisting of f such that $f(1) = 0$ has codimension 1. □

Corollary 1.1.33. *If (π, V) is an irreducible admissible infinite-dimesional representation of $G(F)$ then (π, V) has a unique Whittaker model.*

Proof. Let $\mathcal{K}(\pi, \psi)$ be a Kirillov model for (π, V). For every $\kappa \in \mathcal{K}(\pi, \psi)$, we consider the function

$$W_\kappa(g) = \pi_{\mathfrak{k}}(g)(\kappa)(e).$$

The vector space generated by $\{W_\kappa | \kappa \in \mathcal{K}(\pi, \psi)\}$ is a Whittaker model for (π, V).

Let $\mathcal{W}(\pi, \psi)$ be a Whittaker model for (π, V). The vector space generated by $\{\kappa_W(x) = W(\mathrm{diag}(x, 1)) | W \in \mathcal{W}(\pi, \psi)\}$ is a Kirillov model for (π, V).

Using the existence and uniqueness of the Kirillov model for irreducible admissible infinite-dimensional representation (cf. Theorem 1.1.32), we obtain then a proof for this corollary.

□

Definition 1.1.34. Let (π, V) be a representation of G. A ψ **Whittaker functional** on (π, V) is non-zero linear form $L : V \to \mathbb{C}$ such that

$$L(\pi(n)v) = \psi(n)L(v)$$

for all $n \in N$ and $v \in V$.

We have a relation between Whittaker functional and Whittaker model as follows: given a Whittaker model $\mathcal{W}(\pi, \psi)$ define L by $L(v) = W_v(e)$ where e is the neutral element of G, W_v is the image of v via the G-isomorphism $V \to \mathcal{W}(\pi, \psi)$, and given a Whittaker functional L define $\mathcal{W}(\pi, \psi)$ as the space of $W_v : G \to \mathbb{C}$ defined by $g \mapsto L(\pi(g)v)$ when v runs through V. In other word, we have that to give a Whittaker functional (up to scalar multiples) is to give a Whittaker model and vice-versa.

Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Now let π be an irreducible admissible infinite-dimensional representation of $G(F)$ and $\mathcal{K}(\pi, \psi)$ its corresponding Kirillov model. Since $\mathcal{K}(\pi, \psi)$ is irreducible, it is generated by $\pi_{\mathfrak{k}}(g)C_c^\infty(F^\times)$. Moreover, $C_c^\infty(F^\times)$ is stable under the action of Borel subgroup of G, and $\pi_{\mathfrak{k}}(n(t)w)\kappa - \pi_{\mathfrak{k}}(w)\kappa$ belongs to $C_c^\infty(F^\times)$ for every $\kappa \in \mathcal{K}(\pi, \psi)$ and every $t \in F$. Using Bruhat's decomposition, we obtain then

$$\mathcal{K}(\pi, \psi) = C_c^\infty(F^\times) + \pi_{\mathfrak{k}}(w)C_c^\infty(F^\times).$$

Theorem 1.1.35 ([9, Theorem 2, p. 1.18])**.** *Let (π, V) be an infinite-dimensional irreducible admissible of $G(F)$. Then the contragredient $\widetilde{\pi}$ of π is equivalent to $\chi_\pi^{-1} \otimes \pi$, where χ_π is the central character of π, and the Kirillov space $\mathcal{K}(\widetilde{\pi}, \psi^{-1})$ is the set of function $x \mapsto \chi_\pi(x)^{-1}\kappa(x)$ with $\kappa \in \mathcal{K}(\pi, \psi)$. Furthermore the invariant duality between $\mathcal{K}(\pi, \psi)$ and $\mathcal{K}(\widetilde{\pi}, \psi^{-1})$ is given by the bilinear form $\langle \kappa, \eta \rangle$ such that*

$$\langle \kappa, \eta \rangle = \int \kappa_1(x)\eta(-x)d^\times x + \int \kappa_2(x)\widetilde{\pi}_{\mathfrak{k}}(w)\eta(-x)d^\times x$$

if $\kappa = \kappa_1 + \pi_{\mathfrak{k}}(w)\kappa_2$ with $\kappa_1, \kappa_2 \in C_c^\infty(F^\times)$ and $\eta \in \mathcal{K}(\widetilde{\pi}, \psi^{-1})$.

1.1.7 Bessel distributions and Bessel functions

Let (π, V) be an infinite-dimensional irreducible admissible representation of G. Due to Corollary 1.1.33, there exists an unique (up to scalar multiples) ψ Whittaker functional $L : V \to \mathbb{C}$. Let $\widetilde{L}$ be a ψ^{-1} Whittaker functional on the representation contragredient $(\widetilde{\pi}, \widetilde{V})$ to (π, V). It follows from Theorem 1.1.35, we normalize $\widetilde{L}$ so that if $v \in V$ and $\widetilde{v} \in \widetilde{V}$ are such that either

$x \mapsto L(\pi\,(\mathrm{diag}(x,1))\,v)$ or $x \mapsto \widetilde{L}(\widetilde{\pi}\,(\mathrm{diag}(x,1))\,\widetilde{v})$ has compact support in $F^\times$ then

$$\widetilde{v}(v) = \langle v, \widetilde{v}\rangle = \int_{F^\times} L(\pi\,(\mathrm{diag}(x,1))\,v)\widetilde{L}(\widetilde{\pi}\,(\mathrm{diag}(x,1))\,\widetilde{v})d^\times x.$$

(Note that $L(\pi\,(\mathrm{diag}(x,1))\,v) \in \mathcal{K}(\pi,\psi)$ and $\widetilde{L}(\widetilde{\pi}\,(\mathrm{diag}(x,1))\,\widetilde{v}) \in \mathcal{K}(\widetilde{\pi},\psi^{-1})$).

For $f \in C_c^\infty(G)$ we define the linear functional $\rho(f)\widetilde{L} : \widetilde{V} \to \mathbb{C}$ by

$$(\rho(f)\widetilde{L})(\widetilde{v}) = \int_G f(g)\widetilde{L}(\widetilde{\pi}(g^{-1})\widetilde{v})dg, \quad \widetilde{v} \in \widetilde{V}. \tag{1.1.4}$$

It clear that $\rho(f)\widetilde{L} \in \widetilde{\widetilde{V}}$ (i.e a smooth linear functional). Using the canonical isomorphism $\widetilde{\widetilde{\pi}} \simeq \pi$ (cf. Proposition 1.1.2), we can identify $\rho(f)\widetilde{L}$ with a vector $v_{f,\widetilde{L}} \in V$.

Definition 1.1.36 (Bessel distribution)**.** Let (π, V) be an infinite-dimensional irreducible admissible representation of G. The (Gelfand-Kazhdan) Bessel distribution of π is the distribution $J_\pi : C_c^\infty(G) \to \mathbb{C}$ defined by

$$J_\pi(f) = L(v_{f,\widetilde{L}}).$$

Our main theorem in this section is the following:

Theorem 1.1.37. *There exists a locally integrable function j_π on G such that*

$$J_\pi(f) = \int_G j_\pi(g)f(g)dg, \quad f \in C_c^\infty(G).$$

The strategy to prove this Theorem is that:

- We firstly define the function j_π via the uniqueness of Whittaker model for π on the open Bruhat cell. (We follow the work of Soudry in [17]). This function is the *Bessel function* of π.

- We then prove that j_π is a locally integrable function and $J_\pi(f) = \tilde{J}_\pi(f) := \int_G j_\pi(g)f(g)dg$ for all $f \in C_c^\infty(G)$. (We follow the work of Baruch in [1]).

Let N_m be the subgroup of N defined by

$$N_m := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| |x| \leq q^m \right\}.$$

Let $\mathcal{W}(\pi,\psi)$ be the Whittaker model of (π, V). Let $W \in \mathcal{W}(\pi,\psi)$. We define $W_m : G \to \mathbb{C}$ by

$$W_m(g) := \int_{N_m} W(gn)\psi^{-1}(n)dn. \tag{1.1.5}$$

Since W smooth and N_m compact, this function is well defined. We can easily verify that

$$W_m(ng) = \psi(n)W_m(g), \quad \forall n \in N, g \in G.$$

Lemma 1.1.38. *We have* $W_m(\mathrm{diag}(y,1)) \in C_c^\infty(F^\times) \subset \mathcal{K}(\pi,\psi)$.

Proof. It easy to see that W_m is a smooth function.

We have

$$W\left(\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\right) = W\left(\begin{pmatrix} 1 & xy \\ 0 & 1\end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\right) = \psi(xy)W\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}.$$

It implies that

$$W_m(\mathrm{diag}(y,1)) = W(\mathrm{diag}(y,1)).\int_{\varpi^{-m}\mathcal{O}} \psi(xy)dx.$$

Since

$$\int_{\varpi^{-m}\mathcal{O}} \psi(xy)dx = \begin{cases} q^m & \text{if } |y| \le q^{-m-c}, \\ 0 & \text{otherwise,}\end{cases}$$

where c is the conductor of ψ, W_m has a compact support. □

As a consequence of Lemma 1.1.38, we have $W_m \in \mathcal{W}(\pi,\psi)$.

Lemma 1.1.39. *If* $g \in Bw_0B$ *then there exists* $m_0 = m_{0,g}$ *such that* $W_m(g) = W_{m_0}(g)$ *for all* $m \ge m_0$.

Proof. We note that for any $W \in \mathcal{W}(\pi,\psi)$, $W(\mathrm{diag}(y,1)) \in \mathcal{K}(\pi,\psi) = C_c^\infty(F^\times) + \pi_{\mathfrak{k}}(w)C_c^\infty(F^\times)$. Assume first that $W(\mathrm{diag}(y,1)) \in C_c^\infty(F^\times)$, then for a fixed $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in Bw_0B$ (i.e $c \neq 0$), the function $W\left(g\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}\right)$ has a compact support in z. Indeed, let $|z|$ be so large that

$$\pi\begin{pmatrix} 1 & 0 \\ -(z+\frac{d}{c})^{-1} & 1\end{pmatrix} W = W$$

then

$$\begin{aligned} W\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) &= W\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -(z+\frac{d}{c})^{-1} & 1 \end{pmatrix}\right) \\ &= W\begin{pmatrix} \frac{\det(g)}{cz+d} & az+b \\ 0 & cz+d \end{pmatrix} \\ &= \chi_\pi(cz+d)\psi\left(\frac{az+b}{cz+d}\right) W\begin{pmatrix} \frac{\det(g)}{(cz+d)^2} & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

By our assumption, there exist m_0 (depending on g) such that

$$W\begin{pmatrix} \frac{\det(g)}{(cz+d)^2} & 0 \\ 0 & 1 \end{pmatrix} = 0$$

if $|z| \geq q^{m_0}$. It implies that $W_m(g) = W_{m_0}(g)$ for all $m \geq m_0$.

Now let W be any function in $\mathcal{W}(\pi, \psi)$. Fix an integer $m_1 > 0$. Let $m \geq m_1$, and $g \in Bw_0B$, we have

$$W_m(g) = \int_{N_m} W_{m_1}(gn)\psi^{-1}(n)dn.$$

Using Lemma 1.1.38 and above argument, we obtain then a proof for this Lemma. □

For $g \in Bw_0B$, we define $L_g(W) = \lim_{m\to\infty} W_m(g)$. Due to Lemma 1.1.39, this limit converges. For each $v \in V$, assume that W_v is the image of v via the isomorphism $V \to \mathcal{W}(\pi, \psi)$. We abuse the notation of L_g to define a function from V to $\mathbb{C}$: $L_g(v) := L_g(W_v)$. It is easily to check that L_g is a Whittaker functional on (π, V). From the uniqueness of Whittaker functional, there exists a function $j_\pi : Bw_0B \to \mathbb{C}$ independent of v, such that

$$L_g(v) = j_\pi(g)W_v(e), \quad g \in Bw_0B,\ v \in V.$$

Lemma 1.1.40. *Assume that $g = n_1 z \mathrm{diag}(x,1)w_0 n_2 \in Bw_0B$ with $n_1, n_2 \in N$, $z \in Z(G)$ and $x \in F^\times$. We have then*

$$j_\pi(g) = \psi(n_1)\psi(n_2)\chi_\pi(z)j_\pi(\mathrm{diag}(x,1)w_0).$$

Proof. By definition, we have

$$\begin{aligned}
L_g(v) &= \lim_{m\to\infty}\int_{N_m} W_v(gn)\psi^{-1}(n)dn \\
&= \lim_{m\to\infty}\int_{N_m} W_v(n_1 z\mathrm{diag}(x,1)w_0 n_2 n)\psi^{-1}(n)dn \\
&= \lim_{m\to\infty}\int_{N_m} \psi(n_2)W_v(n_1 z\mathrm{diag}(x,1)w_0 n_2 n)\psi^{-1}(n_2 n)dn \\
&= \lim_{m\to\infty}\int_{N_m} \psi(n_2)W_v(n_1 z\mathrm{diag}(x,1)w_0 n)\psi^{-1}(n)dn \quad \text{(changing variable)} \\
&= \lim_{m\to\infty}\int_{N_m} \psi(n_2)\psi(n_1)W_v(\mathrm{diag}(x,1)w_0 n z)\psi^{-1}(n)dn \\
&= \lim_{m\to\infty}\int_{N_m} \psi(n_2)\psi(n_1)W_{\pi(z)(v)}(\mathrm{diag}(x,1)w_0 n)\psi^{-1}(n)dn \\
&= \lim_{m\to\infty}\int_{N_m} \psi(n_2)\psi(n_1)W_{\chi_\pi(z).v}(\mathrm{diag}(x,1)w_0 n)\psi^{-1}(n)dn \\
&= \lim_{m\to\infty}\int_{N_m} \psi(n_1)\psi(n_2)\chi_\pi(z)W_v(\mathrm{diag}(x,1)w_0 n)\psi^{-1}(n)dn \\
&= \psi(n_1)\psi(n_2)\chi_\pi(z)L_{\mathrm{diag}(x,1)w_0}(v) \\
&= \psi(n_1)\psi(n_2)\chi_\pi(z)j_\pi(\mathrm{diag}(x,1)w_0)W_v(e).
\end{aligned}$$

The last identity implies that $j_\pi(g) = \psi(n_1)\psi(n_2)\chi_\pi(z)j_\pi(\mathrm{diag}(x,1)w_0)$. □

Lemma 1.1.41. *For $|x|$ large enough, we have then*

$$j_\pi\left(\mathrm{diag}(x,1)w_0\right) = \int_{F^\times} I\left(\mathrm{diag}(z,xz), 1_{w_0K_0}\right)\chi_\pi(z)^{-1}d^\times z.$$

(Recall that $I(wt,f)$ is the orbital integral defined in Section 1.1.5.)

Proof. Let n_0 be an arbitrary non-negative integer. Take $W = W_0$ in $\mathcal{W}(\pi,\psi)$ such that the function $W_0(\mathrm{diag}(x,1))$ is the characteristic function of $1+\varpi^{n_0}\mathcal{O}$.

Since W_0 is smooth, there exists m such that if $|z| \geq q^m$ then

$$\pi\begin{pmatrix} 1 & 0 \\ -z^{-1} & 1 \end{pmatrix}(W_0) = W_0,$$

and then

$$\begin{aligned}
W_0\left(\begin{pmatrix}0 & x\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right) &= W_0\left(\begin{pmatrix}0 & x\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -z^{-1} & 1\end{pmatrix}\right)\\
&= W_0\begin{pmatrix}\frac{-x}{z} & x\\ 0 & y\end{pmatrix} = W_0\left(z\begin{pmatrix}1 & \frac{x}{z}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{-x}{z^2} & 0\\ 0 & 1\end{pmatrix}\right)\\
&= \chi_\pi(z)\psi\left(\frac{x}{z}\right)W_0\begin{pmatrix}\frac{-x}{z^2} & 0\\ 0 & 1\end{pmatrix}.
\end{aligned}$$

It implies that

$$\begin{aligned}
L_{\mathrm{diag}(x,1)w_0}(W_0) &= \int_{|z|\le q^m} W_0\left(\mathrm{diag}(x,1)w_0\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(z)dz\\
&+ \int_{|z|>q^m}\chi_\pi(z)\psi\left(\frac{x}{z}-z\right)W_0\begin{pmatrix}\frac{-x}{z^2} & 0\\ 0 & 1\end{pmatrix}dz
\end{aligned}$$

Let C_{π,n_0} be such that

$$W_0\left(\mathrm{diag}(x,1)w_0\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right) = 0$$

for all $|z|\le q^m$ and all $|x|\ge C_{\pi,n_0}$ and using $W_0(e)=1$ we obtain

$$j_\pi(\mathrm{diag}(x,1)w_0) = L_{\mathrm{diag}(x,1)w_0}(W_0) = \int_{xz^{-2}+1\in\varpi^{n_0}\mathcal{O}}\chi_\pi(z)\psi\left(\frac{x}{z}-z\right)dz \tag{1.1.6}$$

for all $|x|\ge C_{\pi,n_0}$.

On the other hand,

$$I(\mathrm{diag}(z,zx),1_{w_0K_{n_0}}) = \int 1_{w_0K_{n_0}}\begin{pmatrix}z & zx_2\\ zx_1 & zx_1x_2+xz\end{pmatrix}\psi(x_1+x_2)dx_1dx_2.$$

This is 0 unless $z\in\varpi^{n_0}\mathcal{O}=\mathfrak{p}^{n_0}$. We change x_1 to x_1/z and x_2 to x_2/z. We obtain then

$$I(\mathrm{diag}(z,xz),1_{w_0K_{n_0}}) = |z|^{-2}\int\psi\left(\frac{x_1+x_2}{z}\right)dx_1dx_2,$$

integrated over the domain defined by:

$$\begin{aligned}
x_i &\equiv 1 \mod \mathfrak{p}^{n_0} \qquad \text{for } i=1,2,\\
x_1x_2 &\equiv -xz^2 \mod z\mathfrak{p}^{n_0}.
\end{aligned}$$

This domain is empty unless $-xz^2 \equiv 1 \mod \mathfrak{p}^{n_0}$. We change variables and set

$$x_2 = tx_1^{-1},$$

where now the domain of integration is defined by:

$$x_1 \equiv 1 \mod \mathfrak{p}^{n_0},\ t \equiv -xz^2 \mod z\mathfrak{p}^{n_0}.$$

Choose n_0 large enough such that $\psi(u) = 1$ for all $u \in \mathfrak{p}^{n_0}$, after integrating over t the integral becomes

$$|z|^{-2}\mathrm{vol}(\mathfrak{p}^{n_0}) \int_{x_1 \equiv 1 \mod \mathfrak{p}^{n_0}} \psi\left(\frac{\phi}{z}\right) dx_1,$$

where the phase function ϕ is given by:

$$\phi = x_1 + \frac{-xz^2}{x_1}.$$

We set $x_1 = 1 + v$ with $v \in \mathfrak{p}^{n_0}$. The phase function takes the form

$$\phi = 1 + v + \frac{-xz^2}{1+v}.$$

The Taylor expansion of this function at the origin has the form

$$(1 - xz^2) + (1 + xz^2)v - (xz^2)v^2 + \text{higher degree terms}.$$

By the principle of the stationary phase there is a compact neighborhood Ω of 0 in F such that, for $|z|$ small enough, the integral is equal to

$$|z|^{-1} \int_{\Omega} \psi\left(\frac{2+v^2}{z}\right) dv = \left|\frac{1}{2z}\right|^{1/2} \psi\left(\frac{2}{z}\right) \gamma\left(\frac{2}{z}, \psi\right).$$

We note that for $|x| > q^{\frac{n_0}{2}}$, the condition $-xz^2 \equiv 1 \mod \mathfrak{p}^{n_0}$ implies that $z \in \mathfrak{p}^{n_0}$. Hence

$$\int_{F^\times} I(\mathrm{diag}(z, xz), 1_{w_0 K_0})\chi_\pi(z)^{-1} \quad = \quad \int_{xz^2+1 \in \mathfrak{p}^{n_0}} \tag{1.1.7}$$

□

Lemma 1.1.42. *Let $W \in \mathcal{W}(\pi, \psi)$ be such that the function $W(\mathrm{diag}(x,1))$ belongs to $C_c^\infty(F^\times)$. Then*

$$W(g) = \int_{F^\times} j_\pi(g.\mathrm{diag}(x^{-1}, 1))W(\mathrm{diag}(x,1))d^\times x$$

for all $g \in Bw_0B$.

Proof. We put

$$\phi_{W,g}(z) = W\left(g\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right)$$

then

$$\widehat{\phi_{W,g}}(1) = \int_F \phi_{W,g}(z)\psi^{-1}(z)dz = L_g(W) = j_\pi(g)W(e).$$

We have

$$\begin{aligned}
\widehat{\phi_{W,g}}(y) &= \int_F \phi_W(z)\psi^{-1}(yz)dz \\
&= \int_F |y|^{-1}\phi_{W,g}(y^{-1}z)\psi^{-1}(z)dz \quad \text{(changing variable)} \\
&= \int_F |y|^{-1} W\left(g\begin{pmatrix}1 & y^{-1}z\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(z)dz \\
&= \int_F |y|^{-1} W\left(g\begin{pmatrix}y^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(z)dz \\
&= |y|^{-1}\int_F \phi_{\pi\left(\begin{smallmatrix}y & 0\\ 0 & 1\end{smallmatrix}\right)(W),g\left(\begin{smallmatrix}y^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)}(z)\psi^{-1}(z)dz \\
&= |y|^{-1}\widehat{\phi_{\pi\left(\begin{smallmatrix}y & 0\\ 0 & 1\end{smallmatrix}\right)(W),g\left(\begin{smallmatrix}y^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)}}(1) \\
&= |y|^{-1} j_\pi(g.\mathrm{diag}(y^{-1},1)).\pi\left(\begin{smallmatrix}y & 0\\ 0 & 1\end{smallmatrix}\right)(W)(e) \\
&= |y|^{-1} j_\pi(g.\mathrm{diag}(y^{-1},1))W(\mathrm{diag}(y,1))
\end{aligned}$$

and hence

$$\begin{aligned}
W(g) &= \phi_{W,g}(0) = \widehat{\widehat{\phi_{W,g}}}(0) = \int_F \widehat{\phi_{W,g}}(y)dy \\
&= \int_F |y|^{-1} j_\pi(g.\mathrm{diag}(y^{-1},1))W(\mathrm{diag}(y,1))dy \\
&= \int_{F^\times} j_\pi(g.\mathrm{diag}(y^{-1},1))W(\mathrm{diag}(y,1))d^\times y.
\end{aligned}$$

□

Lemma 1.1.43. *Let $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \psi^{-1})$ be such that the function $\widetilde{W}(\mathrm{diag}(x,1))$ belongs to $C_c^\infty(F^\times)$. Then*

$$\widetilde{W}(g^{-1}) = \int_{F^\times} j_\pi\left(\mathrm{diag}(x,1)g\right)\widetilde{W}(\mathrm{diag}(x,1))d^\times x$$

for all $g \in Bw_0B$.

Proof. We define W by

$$W(g) = \widetilde{W}(w_0 g^* w_0)$$

where $g^* = (g^t)^{-1}$. Since $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \psi^{-1})$, W is a locally constant $\mathbb{C}$-valued function on G and

$$\begin{aligned} W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) &= \widetilde{W}\left(w_0 \left(g^t \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right)^{-1} w_0\right) \\ &= \widetilde{W}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} w_0 (g^t)^{-1} w_0\right) \\ &= \psi(x)\widetilde{W}(w_0(g^t)^{-1}w_0) = \psi(x)W(g). \end{aligned}$$

Moreover,

$$\begin{aligned} W(\mathrm{diag}(x,1)) &= \widetilde{W}(w_0 \mathrm{diag}(x,1)^* w_0) = \widetilde{W}(\mathrm{diag}(1,x^{-1})) \\ &= \widetilde{W}(x\mathrm{diag}(1,x^{-1})) = \chi_{\widetilde{\pi}}(x^{-1})\widetilde{W}(\mathrm{diag}(x,1)) \\ &= \chi_\pi(x)\widetilde{W}(\mathrm{diag}(x,1)) \end{aligned}$$

belongs to $C_c^\infty(F^\times) \subset \mathcal{K}(\pi,\psi)$. (Due to Theorem 1.1.35, we have $\chi_{\widetilde{\pi}}(x) = \chi_\pi(x)^{-1}$.) Hence W satisfies the condition of Lemma 1.1.42. By using Lemma 1.1.42 for W and $g = \mathrm{diag}(y,1)w_0$, we have then

$$\begin{aligned} \widetilde{W}(g^{-1}) &= \widetilde{W}(w_0\mathrm{diag}(y,1)^{-1}) = W(w_0(g^{-1})^* w_0) = W(\mathrm{diag}(y,1)w_0) \\ &= \int_{F^\times} j_\pi(\mathrm{diag}(y,1)w_0\mathrm{diag}(x^{-1},1))W(\mathrm{diag}(x,1))d^\times x \\ &= \int_{F^\times} j_\pi(\mathrm{diag}(y,1)w_0\mathrm{diag}(x^{-1},1))\chi_\pi(x)\widetilde{W}(\mathrm{diag}(x,1))d^\times x \\ &= \int_{F^\times} j_\pi(x\mathrm{diag}(y,1)w_0\mathrm{diag}(x^{-1},1))\widetilde{W}(\mathrm{diag}(x,1))d^\times x \\ &= \int_{F^\times} j_\pi(\mathrm{diag}(x,1)\mathrm{diag}(y,1)w_0)\widetilde{W}(\mathrm{diag}(x,1))d^\times x \qquad (1.1.8) \end{aligned}$$

Now for $g = n_1 z\mathrm{diag}(y,1)w_0 n_2$ we have

$$\begin{aligned} \widetilde{W}(g^{-1}) &= \widetilde{W}(n_2^{-1}z^{-1}w_0\mathrm{diag}(y,1)^{-1}n_1^{-1}) \\ &= \psi^{-1}(n_2^{-1})\chi_{\widetilde{\pi}}(z^{-1})\widetilde{\pi}(n_1^{-1})\widetilde{W}(w_0\mathrm{diag}(y,1)^{-1}) \\ &= \end{aligned}$$

□

Corollary 1.1.44 (cf. [1, Corollary 4.2])**.** *There exist constants $C = C_\pi$ and $D = D_\pi$ such that for $|x| > C$,*

$$|j_\pi(\mathrm{diag}(x,1)w_0)| \leq D|\chi_\pi(x)|^{1/2}|x|^{1/4}.$$

Proof. We denote by ζ a square root of $\frac{-1}{x}$. Another square root of $\frac{-1}{x}$ is then $-\zeta$. Using germ expansion (cf. Theorem 1.1.28), for any $f \in C_c^\infty(G)$ and $z \in F^\times$ we obtain then

$$I(\mathrm{diag}(z,xz),f) = \omega_f(\mathrm{diag}(z,xz)) + K_e^{w_0}\begin{pmatrix}\zeta & 0\\ 0 & -\zeta^{-1}\end{pmatrix} I\left(\frac{z}{\zeta}w_0, f\right)$$
$$+K_e^{w_0}\begin{pmatrix}-\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix} I\left(\frac{-z}{\zeta}w_0, f\right). \quad (1.1.9)$$

□

Proposition 1.1.45 (cf. [1, Proposition 4.3]). *Let $f \in C_c^\infty(G)$.*

(a) There exists a positive constant $M = M_f$ such that for $|x| < M$ we have
$$\int |f(n_1 w_0 \mathrm{diag}(x,1) z n_2)\, \chi_\pi(z)|\, d^\times z = 0.$$

(b) There exist positive constants $C = C_f$ and $D = D_f$ such that for $|x| > C$ we have

$$\int |f(n_1 w_0 \mathrm{diag}(x,1) z n_2)\, \chi_\pi(z)|\, d^\times z \leq D|\chi_\pi(x)|^{1/2}|x|^{1/2}.$$

Proof. We let
$$\tilde{f}(g) := \int_{NZ} |f(nzg)\chi_\pi(z)| dn d^\times z.$$
Since f is smooth and compactly supportted, $\tilde{f}$ is well-defined. Moreover, $\tilde{f}$ is smooth on the right (i.e there exists a compact open subgroup K of G such that $\tilde{f}(gk) = \tilde{f}(g)$ for all $g \in G$), compactly supported modulo NZ

(a)

□

Theorem 1.1.46. *The function j_π is locally integrable.*

Proof of Theorem 1.1.37. We define the distribution on $C_c^\infty(G)$ to be

$$\tilde{J}_\pi(f) := \int_G j_\pi(g) f(g), \quad f \in C_c^\infty(G).$$

By Theorem 1.1.46, $\tilde{J}_\pi$ is well defined. We shall prove that $\tilde{J}_\pi = J_\pi$.

Let $f \in C_c^\infty(G)$. Since $(C_c^\infty(G), \ell)$ smooth, there exist an integer m such that $\ell_{\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)} f = f$ for all $x \in K_m$. Let $\widetilde{v} \in \widetilde{V}$ be such that

$$\widetilde{L}(\widetilde{\pi}(\mathrm{diag}(x,1))\widetilde{v}) = q^m 1_{K_m}(x) \in C_c^\infty(F^\times)$$

for all $x \in F^\times$. We have

$$\begin{aligned}
\tilde{J}_\pi(f) &= \int_{F^\times} \tilde{J}_\pi(\ell_{\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)} f)\widetilde{L}(\widetilde{\pi}(\mathrm{diag}(x,1))\widetilde{v})d^\times x \\
&= \int_{F^\times} \left(\int_G j_\pi(g) f\left(\mathrm{diag}(x^{-1},1)g\right) dg\right) \widetilde{L}(\widetilde{\pi}(\mathrm{diag}(x,1))\widetilde{v})d^\times x \\
&= \int_{F^\times} \left(\int_G j_\pi\left(\mathrm{diag}(x,1)g\right) f\left(g\right) dg\right) \widetilde{L}(\widetilde{\pi}(\mathrm{diag}(x,1))\widetilde{v})d^\times x \\
&= \int_G f(g) \left(\int_{F^\times} j_\pi\left(\mathrm{diag}(x,1)g\right) \widetilde{L}(\widetilde{\pi}(\mathrm{diag}(x,1))\widetilde{v})d^\times x\right) dg \\
&= \int_G f(g)\widetilde{L}(\widetilde{\pi}(g^{-1})\widetilde{v})dg \quad (\text{ cf. Lemma 1.1.43(1)}) \\
&= (\rho(f)\widetilde{L})(\widetilde{v}) \quad (\text{ cf. (1.1.4)}). \qquad (1.1.10)
\end{aligned}$$

In other hand, we have:

$$\begin{aligned}
J_\pi(f) &= \int_{F^\times} J_\pi(\ell_{\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)} f)\widetilde{L}(\widetilde{\pi}(\mathrm{diag}(x,1))\widetilde{v})d^\times x \\
&= \int_{F^\times} L(\pi(\mathrm{diag}(x,1))v_{f,\widetilde{L}})\widetilde{L}(\widetilde{\pi}(\mathrm{diag}(x,1))\widetilde{v})d^\times x \quad (\text{by definition of } J_\pi) \\
&= \langle v_{f,\widetilde{L}}, \widetilde{v}\rangle \quad (\text{By the normalization of } \widetilde{L}) \\
&= (\rho(f)\widetilde{L})(\widetilde{v}). \qquad (1.1.11)
\end{aligned}$$

Combining (1.1.10) and (1.1.9), we obtain then

$$J_\pi(f) = \tilde{J}_\pi(f).$$

□

The rest of this section is devoted to calculate the Bessel function j_π.

Bessel functions for the principal series of G. (We will follow the work of Baruch and Mao in [2].) Now let π be the infinite dimensional irreducible component of $\mathrm{Ind}_B^G \chi$ where $\chi = \chi_1 \otimes \chi_2$ and χ_1, χ_2 are two multiplicative quasi-characters on $F^\times$.

For a smooth representation (π, V) of N, we denote by $V_\psi(N)$ the subspace generated by all vectors in V of the form

$$\pi(n)(v) - \psi(n)v$$

where $n \in N$ and $v \in V$. We set $V_{\psi,N} := V/V_\psi(N)$. This space can be viewed as Jacquet space of the twisted N-representation $\psi^{-1} \otimes V$. The group N acts on $V_{\psi,N}$ by $\psi : \pi(n)(v) = \psi(n)v$.

Lemma 1.1.47. *We have*

$$V_\psi(N) = \left\{ v \in V | \int_{N_m} \pi(n)(v)\psi^{-1}(n)dn = 0 \textit{ for some } m \right\}.$$

Proof. Let e_m be $\mathrm{vol}(N_m)^{-1}$ times the characteristic function of N_m. By definition we have

$$\int_{N_m} \pi(n)(v)\psi^{-1}(n)dn = \int_N e_m(n)\pi(n)(v)\psi^{-1}(n)dn.$$

Let $v \in V$, $n \in N$. There exist some $m \in \mathbb{Z}$ such that $n \in N_m$. Because N_m is a group and $n \in N_m$, we have $e_m(n'n^{-1}) = e_m(n')$ for all $n' \in N$, so

$$\begin{aligned} \int_{N_m} \pi(n_1)(\pi(n)(v))\psi^{-1}(n_1)dn_1 &= \int_N e_m(n_1)\pi(n_1)(\pi(n)(v))\psi^{-1}(n_1)dn_1 \\ &= \int_N e_m(n_2n^{-1})\pi(n_2)(v)\psi^{-1}(n_2n^{-1})dn_2 \\ &= \psi(n)\int_{N_m} \pi(n_2)(v)\psi^{-1}(n_2)dn_2. \end{aligned}$$

This implies $\int_{N_m} \pi(n_1)(\pi(n)(v) - \psi(n)v)\psi^{-1}(n_1)dn_1 = 0$. Thus

$$V_\psi(N) \subset \left\{ v \in V | \int_{N_m} \pi(n)(v)\psi^{-1}(n)dn = 0 \text{ for some } m \right\}.$$

Suppose $v \in V$ and $\int_{N_m} \pi(n)(v)\psi^{-1}(n)dn = 0$ for some m. Let $N_{m,v} = \{n \in N_m | \pi(n)v = v\} \cap \ker(\psi)$. Then $N_{m,v}$ is an open subgroup of the compact group N_m. Thus $N_m/N_{m,v}$ is finite and

$$\int_{N_m} \pi(n)(v)\psi^{-1}(n)dn = |N_m/N_{m,v}|^{-1} \sum_{k \in N_m/N_{m,v}} \pi(k)(v)\psi^{-1}(k).$$

This implies

$$\begin{aligned} v &= v - \int_{N_m} \pi(n)(v)\psi^{-1}(n)dn \\ &= -|N_m/N_{m,v}|^{-1} \sum_{k \in N_m/N_{m,v}} \psi^{-1}(k)(\pi(k)(v) - \psi(k)v). \end{aligned}$$

□

Proposition 1.1.48. *The functor $V \to V_{\psi,N}$ (viewing as a functor in the category of N-modules) is exact.*

Corollary 1.1.49. *Let $f \in \mathrm{Ind}_B^G\chi$. We can then always write f as*

$$f = f' + f'',$$

where f' is in $V_\psi(N)$ and f'' has support in Bw_0N.

Proof. Let V be subspace of $\mathrm{Ind}_B^G\chi$ contains all the functions have support in Bw_0N. We have then the following exact sequence (of N-modules):

$$0 \to V \to \mathrm{Ind}_B^G\chi \to \mathbb{C} \to 0.$$

Note that $\mathbb{C}_{\psi,N} = 0$. Using Proposition 1.1.48, we obtain then $V_{\psi,N} \simeq (\mathrm{Ind}_B^G\chi)_{\psi,N}$. □

Corollary 1.1.50. *Let $f \in \mathrm{Ind}_B^G\chi$. Then the integral*

$$L_m := \int_{N_m} f(w_0 n)\psi^{-1}(n)dn$$

converges when m tends to ∞. Moreover $L = \lim_{m\to\infty} L_m$ is a Whittaker functional on $\mathrm{Ind}_B^G\chi$.

Proof. Denotes

$$I_m := \int_{N_m} f(w_0 n)\psi^{-1}(n)dn.$$

We shall prove that there exists m_0 such that $I_m = I_{m_0}$ for all $m \geq m_0$.

Using Corollary 1.1.49, the function f can be written as

$$f = f' + f''$$

where $f' \in (\mathrm{Ind}_B^G\chi)_\psi(N)$ and f'' has support in Bw_0N.

Due to Proposition 1.1.48, there exist $m_1 \in \mathbb{Z}$ such that

$$\int_{N_m} f'(w_0 n)\psi^{-1}(n)dn = 0$$

for all $m \geq m_1$.

Furthermore, the function $n \mapsto f''(w_0 n)$ has a compact support in N. Indeed, let $|z|$ be so large that

$$f''\begin{pmatrix} 1 & 0 \\ \frac{1}{z} & 1 \end{pmatrix} = f''\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

then

$$\begin{aligned} f''\left(w_0\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right) &= f''\left(\begin{pmatrix}\frac{-1}{z} & 1\\ 0 & z\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{z} & 1\end{pmatrix}\right)\\ &= \chi_1\left(\frac{-1}{z}\right)\chi_2(z)\left|\frac{-1}{z^2}\right|^{\frac{1}{2}} f''\begin{pmatrix}1 & 0\\ \frac{1}{z} & 1\end{pmatrix} = 0. \end{aligned}$$

It implies that there exists $m_2 \in \mathbb{Z}$ such that

$$\int_{N_m} f''(w_0 n)\psi^{-1}(n)dn = \int_{N_m} f''(w_0 n)\psi^{-1}(n)dn$$

for all $m \geq m_2$.

Take $m_0 = \max\{m_1, m_2\}$, we obtain then our claim.

The second assertion of this corollary is obvious. □

We can now describe the Whittaker model associated to $\mathrm{Ind}_B^G\chi$. Let $f(g) \in \mathrm{Ind}_B^G\chi$. We define

$$W_f(g) = L(r_g(f)) = \lim_{m\to\infty}\int_{N_m} f(w_0 n g)\psi^{-1}(n)dn.$$

Since

$$w_0\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\mathrm{diag}(a,1)w_0 = \begin{pmatrix}-\frac{a}{z} & 1\\ 0 & z\end{pmatrix} w_0 \begin{pmatrix}1 & \frac{a}{z}\\ 0 & 1\end{pmatrix},$$

we have

$$W_{f_m}(\mathrm{diag}(a,1)w_0) = \lim_{n\to\infty}\int_{\left|\frac{a}{z}\right|\leq q^m}^{|z|\leq q^n} \chi_1\left(-\frac{a}{z}\right)\chi_2(z)\left|\frac{a}{z^2}\right|^{\frac{1}{2}}\psi\left(\frac{a}{z}-z\right)dz.$$

Theorem 1.1.51. *Let π be the infinite dimensional irreducible component of $\mathrm{Ind}_B^G\chi$. We have*

$$j_\pi(g) = \psi(n_1 n_2)\chi_1(z)\chi_2(z)\int^{+,-}\chi_1\left(\frac{-a}{x}\right)\chi_2(x)\left|\frac{a}{x^2}\right|^{\frac{1}{2}}\psi\left(\frac{a}{x}-x\right)dx$$

if $g = n_1 z\mathrm{diag}(a,1)w_0 n_2$ with $n_1, n_2 \in N$, $z \in Z(G)$ and $j_\pi(g) = 0$ otherwise. Here

$$\int^{+,-}\phi(x)dx = \lim_{m\to\infty}\int_{q^{-m}\leq|x|\leq q^m}\phi(x)dx,$$

if the limit exists.

Bessel functions for cuspidal representations of G. (We will follow the work of Baruch and Snitz in [4]). We have known that (for p is odd) all cuspidal representations are given by the construction of Jacquet and Langlands (cf. Section 1.1.4). For a convenience, we recall their construction. Let E be a quadratic extension of the p-adic field F. Let β be a quasi-character of $E^\times$ which does not factor through the norm, i.e there does not exist a quasi-character α of $F^\times$ such that $\beta(z) = \alpha(N(z))$ for all $z \in E^\times$. Let τ be the non-trivial quadratic character defined on $F^\times/N(E^\times)$ and extended to $F^\times$. Let $C_c^\infty(E)$ be the Schwartz space of locally constant and compactly supported functions on E. Let $S_\beta(E)$ be the subspace of functions $f \in C_c^\infty(E)$ such that

$$f(xz) = \beta(z^{-1})f(x) \tag{1.1.12}$$

for all $z \in E^1 := \{z \in E|N(z) = 1\}$. Let G_+ be the subgroup of matrices in G whose determinant is a norm. Let $a \in F$ be a norm. Then there exists $z_a \in E$ such that $N(z_a) = a$. The group G_+ acts on $S_\beta(E)$ as follows:

$$\begin{aligned} (n(x)f)(y) &:= \psi(xy^2)f(y), \\ (\mathrm{diag}(a,1)f)(y) &:= |z_a|_E^{1/2}\beta(z_a)f(yz_a), \\ (\mathrm{diag}(b,b^{-1})f)(y) &:= \tau(b)|b|_E^{1/2}f(by), \end{aligned} \tag{1.1.13}$$

and

$$(wf)(y) := \gamma(\psi, E)\hat{f}(\overline{y})$$

where $\gamma(\psi, E)$ is the Weil constant defined in Lemma 1.1.25. We denote by r_β the cuspidal representation attached to β of G via the construction of Jacquet and Langlands. Then r_β is the representation of G induced from the above representation of G_+. In other word, the space of r_β is given by

$$V_{r_\beta} := \{\mathfrak{f} : G \to S_\beta(E)|\mathfrak{f}(hx) = h\mathfrak{f}(x), h \in G_+\},$$

and G acts by right translation: $(r_\beta(g)\mathfrak{f})(x) = \mathfrak{f}(xg)$.

Before stating our formula for Bessel functions for cuspidal representations of G, we need to fix some Haar measures. Let dr be a self dual measure on F with respect to ψ. We let $d^\times r = dr/|r|_F$ be a multiplicative Haar measure on $F^\times$. Let dz be an additive Haar measure on E. Let $\{\epsilon_1, \ldots, \epsilon_\ell\}$ be a set of representatives of $F^\times/(F^\times)^2$. Then, $E^\times$ is the disjoint union of E_{ϵ_i} $(i = 1, \ldots, \ell)$, where

$$E_{\epsilon_i} := \{z \in E|\exists r_z \in K^\times, N(z) = r_z^2\epsilon_i\}.$$

Note that E_{ϵ_i} is empty if ϵ_i is not a norm, and r_z is defined up to a sign. If E_{ϵ_i} is non-empty, we define a measure on E_{ϵ_i} to be the restriction of dz to the open sets E_{ϵ_i}. Assume that ϵ_i is a norm and choose $z_{\epsilon_i} \in E$ such that $N(z_{\epsilon_i}) = \epsilon_i$. Then every element $z \in E_{\epsilon_i}$ can be written in the form (unique up to the sign of r_z and α) $z = z_{\epsilon_i} r_z \alpha$ with $\alpha \in E^1$. We define then a Haar measure $d\alpha$ on E^1 such that

$$dz = |z_{\epsilon_i}|_E |r_z|_E^{1/2} dr d\alpha.$$

It is easy to check that this measure does not depend on ϵ_i.

For $x \in K$, we define $E^x := \{z \in E | N(z) = x\}$. It is easy to see that E^x is empty when x is not a norm. If E^x is non-empty, then $E^x = zE^1$, where z is any element satisfying $N(z) = x$. We define a measure $d_x\alpha$ on E^x by $d_x\alpha = |z|_E^{1/2} d\alpha$. It is clear that this measure does not depend on the choice of z.

Theorem 1.1.52. *Let β be a quasi-character of $E^\times$ which does not factor through the norm form E to F. Let r_β be the cuspidal representation of $\mathrm{GL}_2(F)$ attached to β. We have*

$$j_{r_\beta}(g) = \psi(n_1 n_2)\beta(z)\gamma(\psi, E) \int_{E^a} \beta(\alpha)\psi(\mathrm{tr}(\alpha)) d_a\alpha$$

if $g = n_1 z \mathrm{diag}(a, 1) w n_2$ with $n_1, n_2 \in N$, $z \in Z(G)$, a is a norm and $j_\pi(g) = 0$ otherwise.

Proof. We consider a Whittaker functional $L : V_{r_\beta} \to \mathbb{C}$ defined by $L(\mathfrak{f}) := \mathfrak{f}(I)(1)$, where I is unit matrix of G and 1 is the unit element of E. The corresponding Whittaker function is then $W_{\mathfrak{f}}(g) := L(r_\beta(g)\mathfrak{f})$. Using the standard way, we obtain then the Kirillov functions

$$\phi_{\mathfrak{f}}(b) := W_{\mathfrak{f}}(\mathrm{diag}(b, 1)) = L(r_\beta(\mathrm{diag}(b, 1))\mathfrak{f}). \tag{1.1.14}$$

It follows from the definition that the mapping $\mathfrak{f} \to \phi_{\mathfrak{f}}$ is one to one and the space of all such function $\phi_{\mathfrak{f}}$ is $C_c^\infty(F^\times)$. Due to Lemma 1.1.42, the Bessel function j_{r_β} can be calculated by calculating $\phi_{r_\beta(w)\mathfrak{f}}(b) = L(r_\beta(\mathrm{diag}(b, 1)w)\mathfrak{f})$.

Since $\{\epsilon_1, \dots, \epsilon_\ell\}$ is a set of representatives of $F^\times/(F^\times)^2$, there exist $r_b \in F^\times$ and $j \in \{1, \dots, \ell\}$ such that $b = r_b^2 \epsilon_j$. We can write then

$$\mathrm{diag}(b, 1)w = \mathrm{diag}(r_b^2, 1)\mathrm{diag}(\epsilon_j, \epsilon_j) w \mathrm{diag}(\epsilon_j^{-1}, 1),$$

and $\phi_{r_\beta(w)\mathfrak{f}}(b)$ becomes

$$L(r_\beta(\mathrm{diag}(r_b^2, 1)\mathrm{diag}(\epsilon_j, \epsilon_j) w \mathrm{diag}(\epsilon_j^{-1}, 1))\mathfrak{f}).$$

Now r_b^2 is the norm of the element $r_b \in F$ viewed as a vector in E, and the scalar matrix $\mathrm{diag}(\epsilon_j, \epsilon_j)$ acts by the central character. So we get (cf. (1.1.12))

$$\begin{aligned}\phi_{r_\beta(w)\mathfrak{f}}(b) &= |r_b|_E^{1/2}\beta(r_b)\beta(\epsilon_j)\mathfrak{f}(w\mathrm{diag}(\epsilon_j^{-1},1))(r_b)\\ &= |r_b|_E^{1/2}\beta(r_b\epsilon_j)\gamma(\psi,E)\widehat{\mathfrak{f}(\mathrm{diag}(\epsilon_j^{-1},1))}(r_b)\\ &= |r_b|_E^{1/2}\beta(r_b\epsilon_j)\gamma(\psi,E)\int_E \mathfrak{f}(\mathrm{diag}(\epsilon_j^{-1},1))(y)\psi(\mathrm{tr}(r_b y))dy.\end{aligned}$$

Recall that E is the disjoint union of E_{ϵ_i} $(i = 1, \ldots, \ell)$. Therefore, the integral over E breaks up into a sum of integrals over the sets E_{ϵ_i}, i.e,

$$\phi_{r_\beta(w)\mathfrak{f}}(b) = \sum_{i=1}^{\ell} I_{\epsilon_i}(b,\mathfrak{f}) \tag{1.1.15}$$

where

$$I_{\epsilon_i}(b,\mathfrak{f}) := r_b|_E^{1/2}\beta(r_b\epsilon_j)\gamma(\psi,E)\int_{E_{\epsilon_i}} \mathfrak{f}(\mathrm{diag}(\epsilon_j^{-1},1))(y)\psi(\mathrm{tr}(r_b y))dy.$$

If $E_{\epsilon_i} =$ (is equivalent to that ϵ_i is not a norm), we set $I_{\epsilon_i}(b,\mathfrak{f}) = 0$. Recall that if E_{ϵ_i} is non-empty, then every element $y \in E_{\epsilon_i}$ can be written in the form $y = z_{\epsilon_i} r_y \alpha$ with $\alpha \in E^1$, $r_y \in F^\times$ and $z_{\epsilon_i} \in E$ such that $N(z_{\epsilon_i}) = \epsilon_i$. So, I_{ϵ_i} can be written as a double integral

$$|r_b|_E^{1/2}\beta(r_b\epsilon_j)\gamma\int_{F^\times}\int_{E^1} \mathfrak{f}(\mathrm{diag}(\epsilon_j^{-1},1))(z_{\epsilon_i} r_y\alpha)\psi(\mathrm{tr}(r_b z_{\epsilon_i} r_y\alpha))d\alpha|z_{\epsilon_i} r_y|_E d^\times r_y.$$

Using relation (1.1.11), I_{ϵ_i} is then

$$|r_b|_E^{1/2}\beta(r_b\epsilon_j)\gamma\int_{F^\times} \mathfrak{f}(\mathrm{diag}(\epsilon_j^{-1},1))(z_{\epsilon_i} r_y)|z_{\epsilon_i} r_y|_E\int_{E^1}\beta(\alpha^{-1})\psi(\mathrm{tr}(r_b z_{\epsilon_i} r_y\alpha))d\alpha d^\times r_y.$$

Now using equations (1.1.13) and (1.1.12), we have

$$\phi_{\mathfrak{f}}(r_y^2\epsilon_i\epsilon_j^{-1}) = L(r_\beta(\mathrm{diag}(r_y^2\epsilon_i\epsilon_j^{-1}))\mathfrak{f}) = |r_y z_{\epsilon_i}|_E^{1/2}\beta(r_y z_{\epsilon_i})\mathfrak{f}(\mathrm{diag}(\epsilon_j^{-1},1))(r_y z_{\epsilon_i}),$$

so

$$I_{\epsilon_i}(b,\mathfrak{f}) = |r_b|_E^{1/2}\beta(r_b\epsilon_j)\gamma\int_{F^\times}\phi_{\mathfrak{f}}(r_y^2\epsilon_i\epsilon_j^{-1})|z_{\epsilon_i} r_y|_E^{1/2}\beta(z_{\epsilon_i} r_y)^{-1}\times$$
$$\int_{E^1}\beta(\alpha^{-1})\psi(\mathrm{tr}(r_b z_{\epsilon_i} r_y\alpha))d\alpha d^\times r_y.$$

We define

$$J_{\epsilon_i}(b, \epsilon_i\epsilon_j^{-1}r_y^2) = \gamma|r_b z_{\epsilon_i} r_y|_E^{1/2}\beta(r_b\epsilon_j z_{\epsilon_i}^{-1} r_y^{-1})\int_{E^1}\beta(\alpha^{-1})\psi(\mathrm{tr}(r_b z_{\epsilon_i} r_y \alpha))d\alpha \tag{1.1.16}$$

if ϵ_i is a norm and $J(b, \epsilon_i\epsilon_j^{-1}r_y^2) = 0$ otherwise. We have then

$$I_{\epsilon_i}(b, \mathfrak{f}) = \int_{F^\times}\phi_{\mathfrak{f}}(r_y^2\epsilon_i\epsilon_j^{-1})J(b, \epsilon_i\epsilon_j^{-1}r_y^2)d^\times r_y.$$

We change the variable of integration to $x = r_y^2\epsilon_i\epsilon_j^{-1}$ and integrate over the set $\epsilon_i\epsilon_j^{-1}(F^\times)^2$, and we get

$$I_{\epsilon_i}(b, \mathfrak{f}) = \int_{\epsilon_i\epsilon_j^{-1}(F^\times)^2} J_{\epsilon_i}(b, x)\phi_{\mathfrak{f}}(x)d^\times x. \tag{1.1.17}$$

For any $x \in F^\times$, there exists uniquely $i \in \{1, \ldots, \ell\}$ such that the square class of bx is ϵ_i. Recall that $b = \epsilon_j r_b^2$. So there exist uniquely (up to a sign) $r_y \in K^\times$ such that $x = \epsilon_i\epsilon_j^{-1}r_y^2$. We define

$$J(b, x) = J_{\epsilon_i}(b, \epsilon_i\epsilon_j^{-1}r_y^2).$$

Combining equations (1.1.14), (1.1.16) and definition of $J(b, x)$, we get

$$\phi_{r_\beta(w)\mathfrak{f}}(b) = \int_{F^\times} J(b, x)\phi_{\mathfrak{f}}(x)d^\times x. \tag{1.1.18}$$

Let $z = r_b r_y z_{\epsilon_i}\alpha$. As α varies over E^1, z varies over E^{bx}. Recall that

$$dz = |r_b r_y z_{\epsilon_i}|_E^{1/2}d\alpha,$$

so we can write J as (cf. equation (1.1.15))

$$J(b, x) = \gamma\beta(b)\int_{E^{bx}}\beta(z^{-1})\psi(\mathrm{tr}(z))dz = \gamma\beta(x^{-1})\int_{E^{bx}}\beta(bxz^{-1})\psi(\mathrm{tr}(z))dz$$

Since $N(z) = z\overline{z} = bx$, we have $bxz^{-1} = z\overline{z}z^{-1} = \overline{z}$. Moreover $\mathrm{tr}(\overline{z}) = \mathrm{tr}(z)$, so

$$J(b, x) = \gamma\beta(x^{-1})\int_{E^{bx}}\beta(z)\psi(\mathrm{tr}(z))dz.$$

Combine above equation with (1.1.17), we obtain then

$$\phi_{r_\beta(w)\mathfrak{f}}(b) = \int_{F^\times}\phi_{\mathfrak{f}}(x)\beta(x^{-1})\left[\gamma\int_{E^{bx}}\beta(z)\psi(\mathrm{tr}(z))dz\right]d^\times x.$$

It implies that $j_{r_\beta}(w) =$ □

1.1.8 Orbital integrals

We denote by G_{rs} the set of semi-simple regular elements of G, i.e the set of matrix has a separable characteristic polynomial. Let T' be a maximal torus of G , we shall denote by $T'_{G-\mathrm{reg}} = T' - Z$ the subset of regular elements. (T' can be a centralizer of an *elliptic* element which has an irreducible (in $F[X]$) characteristic polynomial or the "standard" split torus T.)

For $g \in G$ we denote $D(g) = 4 - \det(g)^{-1}\mathrm{tr}(g)^2$.

Proposition 1.1.53 (Orbital integrals)**.** *Let $\gamma \in G$ and $f \in C_c^\infty(G)$. Then $\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)d\dot{g}$ where G_γ is the centralizer of $\gamma \in G$ converges absolutely. The integral*

$$O_\gamma(f) := \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)d\dot{g}$$

*is called **orbital integral** of f at γ.*

Proof. If γ is central, then $G_\gamma = G$. So the statement is trivial.

Now we look at the case when γ is a *hyperbolic* (or *split*) semi-simple regular element (which is conjugated to $\mathrm{diag}(x, y) \in T$ for $x \neq y$). We can assume that $\gamma \in T$, so that $G_\gamma = T$. Using Iwasawa decompostion $G = T \times N \times K_0$ we have

$$O_\gamma(f) = \int_{N\times K_0} f(k^{-1}n^{-1}\gamma nk)dndk.$$

Denote $\gamma = \mathrm{diag}(x, y)$ and $n = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$. Then $n^{-1}\gamma n = \gamma \left(\begin{smallmatrix} 1 & (1-y/x)u \\ 0 & 1 \end{smallmatrix}\right)$ and we have

$$\begin{aligned} O_\gamma(f) &= |1 - y/x|^{-1} \int_{K_0\times N} f(k^{-1}\gamma nk)dkdn \\ &= |D(\gamma)|^{-1/2}\delta_B^{1/2}(\gamma) \int_{K_0\times N} f(k^{-1}\gamma nk)dkdn. \qquad (1.1.19) \end{aligned}$$

Since $f \in C_c^\infty(G)$, there exists $K \subset K_0$ an open compact subgroup of G such that f is bi-K-invariant. □

Theorem 1.1.54 (Germ expansion)**.** *Let γ be an elliptic element in G which is sufficiently close to e. Write E for the splitting field of the quadratic torus T'-determined uniquely up to conjugation in G by γ. Then*

$$O_\gamma(f) = -\frac{2}{q-1}\mathrm{vol}(K_0)O_e(f) + \kappa_{T'}c(T')|D(\gamma)|^{-1/2}O_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}(f),$$

where $\kappa_{T'} = \begin{cases} 2 & \text{if } E/F \text{ is ramified} \\ \frac{q+1}{q} & \text{if } E/F \text{ is unramified,} \end{cases}$ *and* $c(T') = c(E)$ *is the square root of the absolute value of a generator of the discriminant of the splitting field* E *over* F.

Proof. First we need to describe γ A local field F has the form $\mathbb{F}_q((\varpi))$, power series in the variable ϖ over the field $\mathbb{F}_q$ where q is a power of an odd prime number p. Its ring of integers $\mathcal{O} = \mathbb{F}_q[[\varpi]]$, has the maximal ideal $\varpi\mathcal{O}$, and group of units $\mathcal{O}^\times = \mathcal{O} - \varpi\mathcal{O}$.

The ramified quadratic separable extension of F are $E = F(r)$ where r is a root of x^2 □

Corollary 1.1.55. *Let* C *be a compact subset of* G/Z. *Then there is* $c = c(C) > 0$ *such that*

$$O_\gamma(1_C) \le c|D(\gamma)|^{-1/2}c(E)$$

for every $\gamma \in G_{\text{rs}}^{\text{ell}}$ *where* 1_C *is the characteristic function of* C *in* G/Z *and* $E = F(\gamma)$.

Proof. Using Germ expansion (cf. Theorem 1.1.54) for $f = 1_C$ and taking

$$c = 2\left|O_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}(1_C)\right|$$

we obtain then the Corollary. □

Theorem 1.1.56 (Change of variable formula)**.** *Let* $\phi : X \to Y$ *be a morphism between p-adic manifolds of constant dimensions such that the differential of* ϕ *is everywhere invertible (in particular,* $\dim(X) = \dim(Y)$*). Assume that the fibers of* ϕ *have bounded cardinality, and denote* $c_\phi : Y \to \mathbb{Z}_{\ge 0}$, $y \mapsto \#(\phi^{-1}(\{y\}))$. *Then for any differential form* ω *on* Y *and any function* $f : Y \to \mathbb{C}$ *that is integrable with respect to* $|\omega|$, *we have*

$$\int_X f \circ \phi|\phi^*\omega| = \int_Y fc_\phi|\omega|.$$

1.1.9 Harish-Chandra characters

Theorem 1.1.57. *Let* (π, V) *be an irreducible representation of* G. *Then there is a unique smooth function* $\Theta_\pi : G_{\text{rs}} \to \mathbb{C}$ *such that* Θ_π *is locally integrable on* G, *and for any* $f \in \mathcal{H}(G)$ *we have*

$$\text{tr}\pi(f) = \int_G f(g)\Theta_\pi(g)dg.$$

Definition 1.1.58. Let (π, V) be a smooth representation of G. Assume that χ_π is its central quasi-character. We say that π is **square-integrable** (or part of the **discrete series**) if χ_π is unitary and for any $v \in V$ and $\widetilde{v} \in \widetilde{V}$,

$$\int_{G/Z} |\langle \pi(g)v, \widetilde{v}\rangle|^2 dg < +\infty.$$

We say that π is **essentially square-integrable** if there exists $s \in \mathbb{R}_+$ such that $|\det|^s \otimes \pi$ is square-integrable.

Lemma 1.1.59. *Any irreducible square-integrable representation is unitarizable, i.e admits a G-invariant hermitian inner product. Moreover the G-invariant hermitian inner product is unique up to $\mathbb{R}_+$.*

Proof. Let (π, V) be an irreducible square-integrable representation of G with central character χ_π. We denote by $L^2(G, \chi_\pi)$ the space of measurable functions $G \to \mathbb{C}$ such that $f(zg) = \chi_\pi(z)f(g)$ for all $z \in Z$, $g \in G$ and $\int_{G/Z} |f(g)|^2 dg < +\infty$. This space has a canonical Hermitian form

$$H_0(f, f') = \int_{G/Z} f(g)\overline{f'}(g)dg.$$

For each $\widetilde{v} \in \widetilde{V} - \{0\}$, the function $g \mapsto \langle \pi(g)v, \widetilde{v}\rangle$ is belong to $L^2(G, \chi_\pi)$ and the map $v \mapsto (g \mapsto \langle \pi(g)v, \widetilde{v}\rangle)$ gives a G-equivariant embedding of V into $L^2(G, \chi_\pi)$. Thus V admits a G-invariant hermitian inner product.

Let $H(.,.)$ be any G-invariant hermitian inner form on V. We denote by $\overline{V}$ the $\mathbb{C}$-vector space V but with the product $\mathbb{C} \times V \to V : (c, v) \mapsto \bar{c}v$. Then $H(.,.)$ is a G-invariant non-degenerate bilinear form on $V \times \overline{V}$. Using Proposition 1.1.3, $H(.,.)$ can be see as a G-isomorphism $\varphi_H : \overline{V} \to \widetilde{V}$. More precisely,

$$H(v_1, v_2) = \langle v_1, \varphi_H(v_2)\rangle.$$

Since $\overline{V}$ is irreducible, by Schur's lemma, the G-invariant hermitian inner product is unique up to a scalar. Due to positive-definiteness of inner product, this scalar should be in $\mathbb{R}_+$.

□

Proposition 1.1.60 (Formal degree)**.** *Let (π, V) be an irreducible essentially square-integrable representation of G. Then for any $u, v \in V$ and $\widetilde{u}, \widetilde{v} \in \widetilde{V}$ the integral*

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle \langle v, \widetilde{\pi}(g)\widetilde{v}\rangle$$

converges absolutely and there exists a unique $d_\pi \in \mathbb{R}_+$, *called the* ***formal degree*** *of* π *such that*

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle \langle v, \widetilde{\pi}(g)\rangle = \frac{1}{d_\pi}\langle u, \widetilde{v}\rangle\langle v, \widetilde{u}\rangle.$$

Proof. Assume that χ_π is the central quasi-character of π. Since $\widetilde{\pi}$ is equivalent to $\chi_\pi^{-1} \otimes \pi$ (cf. Theorem 1.1.32), the absolute convergence of

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle \langle v, \widetilde{\pi}(g)\widetilde{v}\rangle dg$$

is equivalent to the absolute convergence of

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle \langle \pi(g)v, \widetilde{v}\rangle \chi_\pi^{-1}(\det(g))dg.$$

Moreover, since (π, V) is essentially square-integrable we have

$$\begin{aligned}\int_{G/Z} |\langle \pi(g)u, \widetilde{u}\rangle \langle \pi(g)v, \widetilde{v}\rangle \chi_\pi^{-1}(\det(g))|dg &\leq \frac{1}{2}\int_{G/Z} |\langle \pi(g)u, \widetilde{u}\rangle|^2 + \langle \pi(g)v, \widetilde{v}\rangle|^2 dg \\ &< +\infty.\end{aligned}$$

Now we fix $\widetilde{u} \in \widetilde{V}$ and $v \in V$, then the integral

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle \langle v, \widetilde{\pi}(g)\widetilde{v}\rangle dg$$

is a G-invariant non-degenerate bilinear form on $V \times \widetilde{V}$. Since $\widetilde{V}$ is irreducible (cf. 1.1.6), using Proposition 1.1.3 and Schur's lemma, it is a complex number times the canonical non-degenerate bilinear form on $V \times \widetilde{V}$. In other word, there exist a function $c_\pi : V \times \widetilde{V} \to \mathbb{C}$ such that

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle \langle v, \widetilde{\pi}(g)\widetilde{v}\rangle dg = c_\pi(v, \widetilde{u})\langle u, \widetilde{v}\rangle \quad \forall (u, \widetilde{v}) \in V \times \widetilde{V}.$$

Fix $u \in V$ and $\widetilde{v} \in \widetilde{V}$. The integral

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle \langle v, \widetilde{\pi}(g)\widetilde{v}\rangle dg$$

is also a G-invariant non-degenerate bilinear form on $V \times \widetilde{V}$. It implies that $c_\pi(v, \widetilde{u})$ is a G-invariant non-degenerate bilinear form on $V \times \widetilde{V}$. Using Proposition 1.1.3 and Schur's lemma again, there exist $c_\pi \in \mathbb{C}$ such that

$$c_\pi(v, \widetilde{u}) = c_\pi \langle v, \widetilde{u}\rangle.$$

Hence

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle\langle v, \widetilde{\pi}(g)\widetilde{v}\rangle dg = c_\pi\langle u, \widetilde{v}\rangle\langle v, \widetilde{u}\rangle. \qquad (1.1.20)$$

It remains to show that $c_\pi \in \mathbb{R}_+$. Up to twisting by a character, we can assume that π is square-integrable. Pick any G-invariant hermitian inner product $H(.,.)$ on V, which is equivalent to an isomorphism $\varphi_H : \overline{V} \to \widetilde{V}$ (cf. proof of Lemma 1.1.59). Taking $\widetilde{u} = \varphi_H(v)$ and $\widetilde{v} = \varphi_H(u)$ for arbitrary $u, v \in V - \{0\}$, the LHS of (1.1.19) is equal to

$$\int_{G/Z} H(\pi(g)u, v)H(v, \pi(g)u)dg = \int_{G/Z} |H(\pi(g)u, v)|^2 dg$$

which is non-negative and not identically vanishing, and the RHS of (1.1.19) is equal to $c_\pi H(u,u)H(v,v)$, therefore $c_\pi \in \mathbb{R}_+$. □

Theorem 1.1.61 (Weyl integration formula)**.** *Fix a set $\mathcal{T}$ of representatives of conjugacy classes of tori in $G(F)$. Let f be a measurable function on G. Then*

$$\int_G f(g)dg = \sum_{T' \in \mathcal{T}} \frac{1}{2} \int_{T'_{G-reg}} |D(t)| O_t(f) dt$$

if one side is absolutely convergent.

Proposition 1.1.62. *Let $\chi : T \to \mathbb{C}^\times$ be a quasicharacter of T, and consider $\pi = \mathrm{Ind}_B^G \chi$. Then Theorem 1.1.57 holds for π, and Θ_π is the unique G-invariant function on G_{rs} which vanishes identically on G_{rs}^{ell} and such that for any $t \in T_{G-reg}$ we have*

$$\Theta_\pi(t) = |D(t)|^{-1/2}(\chi(t) + \chi^w(t)).$$

Proof. Due to the definition of $\mathrm{Ind}_B^G\chi$ and the Iwasawa decomposition, $\mathrm{Ind}_B^G\chi$ may be regarded as a space of function in $\phi \in C^\infty(K_0)$ which satisfies

$$\phi(bk) = \delta_B^{1/2}(b)\chi(b)\phi(k) = \chi(b)\phi(k)$$

for all $b \in B \cap K_0$ and $k \in K_0$. To evaluate $\mathrm{tr}\pi(f)$ we observe that if ϕ be a such function, $f \in \mathcal{H}(G)$ and $k_1 \in K_0$, and using Iwasawa decomposition we have then

$$\begin{aligned}
\pi(f)(\phi)(k_1) &= \int_G f(g)\phi(k_1 g)dg = \int_G \phi(g) f(k_1^{-1}g)dg \\
&= \int_{K_0 \times B} \phi(bk_2) f(k_1^{-1}bk_2)dk_2 db \\
&= \int_{K_0} \phi(k_2) \int_B \chi(b)\delta_B^{1/2}(b) f(k_1^{-1}bk_2)dbdk_2 \\
&= \int_{K_0} \phi(k_2)\psi(k_1,k_2)dk_2
\end{aligned}$$

where $\psi(k_1,k_2)=\int_B \mu(b)\delta_B^{1/2}(b)f(k_1^{-1}bk_2)db$ is a smooth function on $K_0\times K_0$. We denote by $I(\psi)$ the integral operator on $C^\infty(K_0)$ defined by

$$\phi \mapsto I(\psi)(\phi)(.)=\int_{K_0}\phi(k)\psi(.,k)dk.$$

Then $\pi(f)$ coincides with $I(\psi)$ on $\mathrm{Ind}_B^G\chi$. Moreover, we can easily check that $I(\psi)(\phi)$ belongs to $\mathrm{Ind}_B^G\chi$ for all $\phi\in C^\infty(K_0)$. (In fact, for $k_1,k_2\in K_0$ and $b_1\in B\cap K_0$ we have

$$\begin{aligned}\psi(b_1k_1,k_2) &= \int_B \chi(b)\delta_B^{1/2}(b)f(k_1^{-1}b_1^{-1}bk_2)db\\ &= \int_B \chi(b_1(b_1^{-1}b))\delta_B^{1/2}(b_1(b_1^{-1}b))f(k_1^{-1}(b_1^{-1}b)k_2)d(b_1^{-1}b)\\ &= \chi(b_1)\psi(k_1,k_2).)\end{aligned}$$

Hence

$$\mathrm{tr}\pi(f)=\mathrm{tr}I(\psi)=\int_{K_0}\psi(k,k)dk$$

and so

$$\begin{aligned}\mathrm{tr}\pi(f) &= \int_{K_0}\int_B \chi(b)\delta_B^{1/2}(b)f(k^{-1}bk)dbdk\\ &= \int_{K_0}\int_{T\times N}\chi(t)\delta_B^{1/2}(t)f(k^{-1}tnk)dtdndk\\ &= \int_T \chi(t)|D(t)|^{1/2}O_t(f)dt \quad \text{(c.f. (1.1.18))}. \end{aligned}\tag{1.1.21}$$

In other hand, using Weyl integration formula (c.f Theorem 1.1.61) and the definition of Θ_π, we have then

$$\begin{aligned}\int_G f(g)\Theta_\pi(g)dg &= \frac{1}{2}\int_{T_{G-reg}}|D(t)|\Theta_\pi(t)O_t(f)dt\\ &= \frac{1}{2}\int_{T_{G-reg}}|D(t)|^{1/2}(\chi(t)+\chi^w(t))O_t(f)dt\\ &= \frac{1}{2}\Big[\int_T |D(t)|^{1/2}\chi(t)O_t(f)dt\\ &\quad+\int_T |D(w_0^{-1}tw_0)|^{1/2}\chi(w_0^{-1}tw_0)O_{w_0^{-1}tw_0}(f)d(w_0^{-1}tw_0)\Big]\\ &= \int_T \chi(t)|D(t)|^{1/2}O_t(f)dt.\end{aligned}\tag{1.1.22}$$

Combining (1.1.20) and (1.1.21), we obtain then

$$\mathrm{tr}\pi(f) = \int_G f(g)\Theta_\pi(g)dg$$

with Θ_π is defined as in the Proposition. □

Bibliography

[1] E. M. Baruch, *On Bessel distribution for* GL_2 *over p-adic field*, Journal of number theory 67, 190-202 (1997).

[2] E. M. Baruch, Z. Mao, *Bessel identities in the Waldspurger Correspondence over p-adic field*, American Journal of Mathematics, vol. 125, No.2 (2003), 225-288.

[3] E. M. Baruch, Z. Mao, *Bessel identities in the Waldspurger Correspondence over the real numbers*, Israel journal of Mathematics 145 (2005), 1-81.

[4] E. M. Baruch, K. Snitz, *A note on Bessel functions for supercuspidal representations of* $\mathrm{GL}(2)$ *over a p-adic field*, Algebra Colloquium 18 (spec 1) (2011) 733-738.

[5] I. N. Bernstein, A. V. Zelevinsky, *Representation theory of* $\mathrm{GL}(n, F)$ *where* F *is a non-Archimedean local field*, Russian Math. Survey, 31:3 (1976), 1-68.

[6] C. J. Bushnell, G. Henniart, *The local Langlands conjecture for* $\mathrm{GL}(2)$, Grundlehren der mathematischen Wissenschaften 335, Springer, 2006.

[7] H. Jacquet, *Kloosterman integrals for* $\mathrm{GL}(2, \mathbb{R})$, Pure and applied mathematics quarterly, volume 1 (2005), 257-289.

[8] I. M. Gelfand, D. Kazhdan, *Representations of the group* $\mathrm{GL}(n, K)$ *where* K *is a local field, Lie Groups and Their Representations*, Halsted, Newyork, (1975), 95-118.

[9] R. Godement, *Notes on Jacquet-Langlands Theory*, The Institute for Advanced Study (1970).

[10] H. Jacquet, *Germs for Kloosterman integrals, a review*, Contemporary Mathematics, vol. 664, (2016), 173-185.

[11] H. Jacquet, R. P.Langlands, *Automorphic forms on* GL(2), LNM 114, Springer, 1970.

[12] H. Jacquet, Y. Ye, *Distinguished representations and quadratics base change for* GL(3), Proc. AMS, vol 348, no. 3, (1996), 913-939.

[13] A. Knightly, C. Li, *Traces of Hecke operator*, Math. Sur. and Mono., vol 133, AMS (2006).

[14] R. P. Langlands, *Base change for* GL(2), Annals of Mathematics Study 96 (1980).

[15] B. C. Ngo, *Harmonic analysis and representation theory of p-adic reductive groups*, updated on 19th May 2016.

[16] A. J. Silberger, PGL_2 *over the p-adics: its representations, spherical functions, and Fourrier analysis*, LNM 166, Springer, 1970.

[17] D. Soudry, *The L and γ factors for generic representations of* $\mathrm{GSp}(4,k) \times \mathrm{GL}(2,k)$ *over a local non-archimedean field k*, Duke Math. Journal 51, no. 2, (1984), 355-394.