Existence and Tikhonov-type regularization method for generalized affine variational inequality

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Abstract. In this paper, we present a sufficient condition for the solution existence of general affine variational inequality (GAVI) and use a Tikhonov-type regularization method to find a solution of GAVI. Under the suitable conditions, we characterize the unboundedness, closedness, and convexity of the solution set. The obtained results have extended and complemented previous ones.

Mathematics Subject Classification (2010). 90C33, 90C31, 54C60

Key Words. Generalized affine variational inequality, solution existence, Tikhonov-type regularization, convergence theorem, positive semi-definiteness

1 Introduction

Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be two given continuous maps and let K be a nonempty closed convex subset of $\mathbb{R}^n$. The *general variational inequality (GVI)*, denoted by $\mathrm{GVI}(F, G, K)$, is to determine a vector $x \in \mathbb{R}^n$ such that

$$G(x) \in K \text{ and } \langle F(x), y - G(x)\rangle \geq 0 \ \ \forall y \in K,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in real Euclidean space.

The above problem was first proposed by Noor [6] and has been received considerable attention in recent three decades. Many authors have been developed many numerical methods for GVI problems (see [4, 5, 7]). If both F and G are affine maps then $\mathrm{GVI}(F, G, K)$ reduces to the following *general affine variational inequality*: Find $x \in \mathbb{R}^n$ such that $Bx + b \in K$ and

$$\langle Ax + a, y - Bx - b\rangle \geq 0 \ \ \forall y \in K, \qquad (\mathrm{GAVI}(\omega, K))$$

with A, B being two matrices in $\mathbb{R}^{n\times n}$; $a, b \in \mathbb{R}^n$; and $\omega = (A, a, B, b)$. According to Proposition 2.1 (in Section 2), the *minimum of a quadratic function* on

K can be characterized by the problem GAVI(ω, K). It is well-known that the problem GAVI(ω, K) is closely related to a class of the *fixed point problems* (see Proposition 2.2 in Section 2). This connection allows us to approach the study of quadratic programming and of fixed point theory via GAVI. In particular, when $K \subset \mathbb{R}^n$ is a closed cone, GAVI(ω, K) reduces to the *general affine complementary problem* denoted by GACP(ω, K). In the special case where $Bx + b = x$ for every $x \in K$, GAVI(ω, K) reduces to the following *affine variational inequality*:

$$\text{Find } x \in K \text{ s.t. } \langle Ax + a, y - x\rangle \geq 0 \text{ for all } y \in K. \qquad (\mathrm{AVI}(A, a, K))$$

The above problem has been investigated in [2, 3, 11, 15] where the existence and stability for AVI(A, a, K) with K being polyhedral have been studied in detail.

Under the assumption that F is strongly monotone with respect to G on K and G is injective, Pang and Yao [9, Proposition 3.9] provided some sufficient conditions for the existence of solutions to GVI(F, G, K). So, there are two natural questions arising here:

Question 1 Whether the problem GAVI(ω, K) *has a solution provided that* $g(x) = Bx + b$ *is not injective*?

Question 2 How to compute a solution of GAVI(ω, K) *if it exists*?

In this paper, we present positive answers to *Questions 1 and 2*. In particular, we propose a sufficient condition for the solution existence of GAVI and use a Tikhonov-type regularization method to find a solution of GAVI(ω, K). By using positive semi-definiteness of matrices, exceptional family of elements, and recession cone of convex sets, we obtain an existence result for solutions of the problem GAVI. Our approach, which is motivated by Tikhonov regularization technique [13], is different from ones in [7] and references cited therein. Although Tikhonov-type regularization methods have been studied extensively in variational inequality theory (see, for instance, [1, 12]), as far as we know, there is no result on Tikhonov-type regularization method applying to GAVI.

The outline of the paper is as follows. Section 2 provides some preliminaries. In Section 3, we present a sufficient condition for the solution existence of GAVI. A Tikhonov-type regularization method including: the solution existence of the perturbing problem and the locally boundedness and upper semicontinuity of the solution map is proposed in Section 4. In Section 5, some properties of the solution set are discussed.

2 Preliminaries

Throughout this paper, for any positive integer n, $\mathbb{R}^n$ denotes a real Euclidean space equipped with the scalar product $\langle\cdot\,,\cdot\rangle$ and the induced norm $\|\cdot\|$. The set of all $n \times n$ real matrices is denoted by $\mathbb{R}^{n\times n}$. The superscript ^{T} denotes transposition. For any positive integer l, denote $[l] := \{1, \ldots, l\}$ and

$l! := l(l-1)\dots 1$. Let

$$f(x) := Ax + a \text{ and } g(x) := Bx + b.$$

For any nonempty closed convex set K of $\mathbb{R}^n$, the asymptotic (recession) cone of K is denoted by

$$K^\infty = \{v \in \mathbb{R}^n : x + tv \in K \ \forall t \geq 0\}$$

and let

$$H := \{v \in \mathbb{R}^n : \exists \alpha \in \mathbb{R}_+ \text{ such that } \alpha v + Bv \in K^\infty\},$$

where $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

For any cone $S \subset \mathbb{R}^n$, the dual of S is denoted by

$$S^* := \{y \in \mathbb{R}^n : \langle h, y\rangle \geq 0 \ \ \forall h \in S\}.$$

The open (closed) ball in $\mathbb{R}^n$ with center at 0 and radius ε is denoted by $B(0,\varepsilon)$ (resp., $\bar{B}(0,\varepsilon)$).

A multifunction $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be *locally bounded* at $\bar{x} \in \mathbb{R}^m$ if there exists $\varepsilon > 0$ such that

$$U_\varepsilon := \bigcup_{\tilde{x} \in B(0,\varepsilon)} S(\bar{x} + \tilde{x})$$

is bounded.

We recall the notion of upper semicontinuity of multifunctions. A multifunction $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be *upper semicontinuous* at $\bar{x} \in \mathbb{R}^m$ if for each open set V containing $S(\bar{x})$ there exists $\delta > 0$ such that $S(x) \subset V$ for every $x \in \mathbb{R}^m$ satisfying $\|x - \bar{x}\| < \delta$.

The solution set of GVI(F, G, K) (GAVI(ω, K), AVI(A, a, K)) is denoted by Sol(F, G, K) (resp., Sol(ω, K), Sol(A, a, K)).

The following proposition shows that the minimum of a quadratic function on K can be characterized by the problem GAVI(ω, K).

Proposition 2.1. *Let $p(x) = \frac{1}{2}\langle x, Qx\rangle + \langle q, x\rangle$ with $Q \in \mathbb{R}^{n\times n}$ being symmetric and $q \in \mathbb{R}^n$. Then, for some $\bar{x} \in \mathbb{R}^n$, if $B\bar{x} + b$ is the minimum of p on K then $\bar{x} \in \mathrm{Sol}(\omega, K)$ with $A = QB$ and $a = Qb + q$. The reverse is true if*

$$p(z + t(y - z)) \leq tp(y) + (1-t)p(z) \tag{1}$$

for every $y, z \in K$ and for every $t \in [0, 1]$.

Proof. Suppose that $B\bar{x} + b \in K$ is the minimum of p on K for some $\bar{x} \in \mathbb{R}^n$. Then, we have

$$p(B\bar{x} + b) \leq p(z) \ \ \forall z \in K.$$

Hence, for some $y \in K$ and $t \in (0, 1]$, we obtain $B\bar{x} + b + t(y - B\bar{x} - b) \in K$ and

$$p(B\bar{x} + b) \leq p(B\bar{x} + b + t(y - B\bar{x} - b)).$$

Dividing the above inequality by t and taking $t \to 0$, we have

$$\langle \nabla p(B\bar{x} + b), y - B\bar{x} - b \rangle = \langle A\bar{x} + a, y - B\bar{x} - b \rangle \geq 0,$$

where $\nabla p(B\bar{x} + b)$ denotes the gradient of p at $B\bar{x} + b$. This leads to $\bar{x} \in \mathrm{Sol}(\omega, K)$.

Conversely, suppose that $\bar{x} \in \mathrm{Sol}(\omega, K)$. For every $y \in K$ and for every $t \in (0, 1]$, by the assumption (1), we have

$$p(B\bar{x} + b + t(y - B\bar{x} - b)) \leq tp(y) + (1 - t)p(B\bar{x} + b).$$

This implies that

$$p(y) - p(B\bar{x} + b) \geq \frac{p(B\bar{x} + b + t(y - B\bar{x} - b)) - p(B\bar{x} + b)}{t}$$

Letting $t \to 0$, we have

$$p(y) - p(B\bar{x} + b) \geq \langle \nabla p(B\bar{x} + b), y - B\bar{x} - b \rangle = \langle A\bar{x} + a, y - B\bar{x} - b \rangle \geq 0.$$

Therefore, $B\bar{x} + b \in K$ is the minimum of p on K. $\square$

It is known that the problem $\mathrm{GAVI}(\omega, K)$ is equivalent to the fixed point problem. This relation is described in the following proposition.

Proposition 2.2. (see [8]) *For given $\bar{x} \in \mathbb{R}^n$, $\bar{x} \in \mathrm{Sol}(\omega, K)$ if and only if $\bar{x}$ satisfies the following relation*

$$\bar{x} = \Phi(\bar{x}),$$

where $\Phi(z) = z - Bz - b + P_K(Bz + b - Az - a)$ and P_K is the projection of $\mathbb{R}^n$ onto K.

Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be two continuous functions and let $\bar{x} \in \mathbb{R}^n$. A set of points $\{x^k\} \subset \mathbb{R}^n$ is called an *exceptional family of elements for the pair* (F, G) with respect to $\bar{x} \in \mathbb{R}^n$ if $\|x^k\| \to \infty$ as $k \to \infty$; and for each x^k, there exists a scalar $\alpha^k > 0$ such that $z^k := \alpha^k(x^k - \bar{x}) + G(x^k) \in K$ and

$$-\alpha^k(x^k - \bar{x}) - F(x^k) \in N_K(z^k),$$

where $N_K(z^k)$ is the normal cone of K at z^k.

The following is useful for our proofs.

Proposition 2.3. (see [14, Lemma 1]) *For two continuous mappings $F, G : \mathbb{R}^n \to \mathbb{R}^n$ and a nonempty, closed and convex set $K \subset \mathbb{R}^n$, there exists either a solution of* $\mathrm{GVI}(F, G, K)$ *or an exceptional family of elements with respect to any given $\bar{x} \in \mathbb{R}^n$ for the pair (F, G).*

The map F is called *generalized pseudo-monotone* with respect to G on K if $\langle F(x), G(y) - G(x) \rangle \geq 0$ for all $x, y \in \mathbb{R}^n$ satisfying $G(x), G(y) \in K$ implies that $\langle F(y), G(y) - G(x) \rangle \geq 0$ for all $x, y \in \mathbb{R}^n$.

3 Existence

In this section, we present a sufficient condition for the existence and uniqueness of solutions to the problem GAVI(ω, K).

Theorem 3.1. *Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. If the following conditions are satisfied:*

(i) $A + B$ and $A^T B$ are positive semidefinite on H;
(ii) $Sol((A, 0, B, 0), K^\infty) = \{0\}$,

then, for all $a, b \in \mathbb{R}^n$, GAVI(ω, K) *has a solution. In addition, if $A^T B$ is positive definite on $B^{-1}(K - K)$ then* GAVI(ω, K) *has a unique solution.*

Proof. On the contrary, suppose that GAVI(ω, K) has no solution. Then, it follows from Proposition 2.3 that there exist $\{x^k\} \subset \mathbb{R}^n$ satisfying $\|x^k\| \to \infty$ as $r \to \infty$ and $\alpha^k > 0$ such that $z^k := \alpha^k x^k + Bx^k + b \in K$ and

$$-\alpha^k x^k - Ax^k - a \in N_K(z^k).$$

By the definition of the normal cone, we obtain

$$\langle \alpha^k x^k + Ax^k + a, y - \alpha^k x^k - Bx^k - b \rangle \geq 0 \ \ \forall y \in K,$$

that is,

$$\langle \alpha^k x^k, y - Ax^k - a - Bx^k - b \rangle + \langle Ax^k + a, y - Bx^k - b \rangle - (\alpha^k)^2 \|x^k\|^2 \geq 0 \quad (2)$$

for every $y \in K$. Without loss of generality, we may assume that $\|x^k\| \neq 0$ for all k and $\frac{x^k}{\|x^k\|} \to \bar{h}$ for some $\bar{h} \in \mathbb{R}^n$. Dividing both sides of the inequality (2) by $\|x^k\|^2$, we obtain

$$\alpha^k \left\langle \frac{x^k}{\|x^k\|}, \frac{y - Ax^k - a - Bx^k - b}{\|x^k\|} \right\rangle + \left\langle \frac{Ax^k + a}{\|x^k\|}, \frac{y - Bx^k - b}{\|x^k\|} \right\rangle - (\alpha^k)^2 \geq 0. \quad (3)$$

Denote

$$u^k := \left\langle \frac{x^k}{\|x^k\|}, \frac{y - Ax^k - a - Bx^k - b}{\|x^k\|} \right\rangle$$

and

$$v^k := \left\langle \frac{Ax^k + a}{\|x^k\|}, \frac{y - Bx^k - b}{\|x^k\|} \right\rangle.$$

From (3) we have

$$\alpha^k u^k + v^k - (\alpha^k)^2 \geq 0. \quad (4)$$

It is obvious that

$$\lim_{k\to\infty} u^k = \lim_{k\to\infty} \left\langle \frac{x^k}{\|x^k\|}, \frac{y - Ax^k - a - Bx^k - b}{\|x^k\|} \right\rangle = -\langle \bar{h}, (A + B)\bar{h} \rangle$$

and

$$\lim_{k\to\infty} v^k = \lim_{k\to\infty} \left\langle \frac{Ax^k + a}{\|x^k\|}, \frac{y - Bx^k - b}{\|x^k\|} \right\rangle = \langle A\bar{h}, -B\bar{h}\rangle = -\langle \bar{h}, A^T B\bar{h}\rangle.$$

Consider the following two cases:

Case 1: $\{\alpha^k\}$ is unbounded. Dividing both sides of the inequality (4) by $(\alpha^k)^2$ and letting $k \to \infty$ yields $-1 \geq 0$, a contradiction.

Case 2: $\{\alpha^k\}$ is bounded. Then, without loss of generality we may assume that $\alpha^k \to \bar{\alpha}$ for some $\bar{\alpha} \in \mathbb{R}_+$. From (4), by passing to the limit, we obtain

$$-\bar{\alpha}\langle \bar{h}, (A+B)\bar{h}\rangle - \langle \bar{h}, A^T B\bar{h}\rangle - (\bar{\alpha})^2 \geq 0. \tag{5}$$

Since $\frac{1}{\|x^k\|} \to 0$, applying [10, Theorem 8.2] to $z^k = \alpha^k(x^k - \bar{x}) + Bx^k + b \in K$, we have

$$\frac{1}{\|x^k\|} z^k = \frac{1}{\|x^k\|}\big(\alpha^k(x^k - \bar{x}) + Bx^k + b\big) \to \bar{\alpha}\bar{h} + B\bar{h} \in K^\infty.$$

It follows that $\bar{h} \in H$. By (i) we have

$$\langle \bar{h}, (A+B)\bar{h}\rangle \geq 0 \tag{6}$$

and

$$\langle \bar{h}, A^T B\bar{h}\rangle \geq 0. \tag{7}$$

We claim that $\bar{\alpha} \neq 0$. Indeed, suppose to the contrary that $\bar{\alpha} = 0$. Fix $w \in K$. For every $h \in K^\infty$, putting $z := w + h\|x^k\|$, we have $z \in K$. From (2) it follows that

$$\langle \alpha^k x^k + Ax^k + a, z - \alpha^k x^k - Bx^k - b\rangle \geq 0,$$

that is,

$$\langle \alpha^k x^k + Ax^k + a, w + h\|x^k\| - \alpha^k x^k - Bx^k - b\rangle \geq 0.$$

Dividing both sides of last inequality by $\|x^k\|^2$ and letting $k \to +\infty$ yields:

$$\big\langle A\bar{h}, h - B\bar{h}\big\rangle \geq 0;$$

Hence, there exists $\bar{h} \neq 0$ such that $\bar{h} \in Sol((A, 0, B, 0), K^\infty)$. This contradicts to the assumption (ii). Therefore, we have $\bar{\alpha} \neq 0$.

From (6) and (7), we deduce that

$$-\bar{\alpha}\langle \bar{h}, (A+B)\bar{h}\rangle - \langle \bar{h}, A^T B\bar{h}\rangle - (\bar{\alpha})^2 < 0,$$

which is contrary to the inequality (11). Therefore, this case does not occur and the problem $GAVI(\omega, K)$ has a solution.

We now prove that GAVI(ω, K) has a unique solution. Indeed, suppose, on the contrary, that GAVI(ω, K) have two different solution $\bar{x}$ and $\hat{x}$. Then, $B(\hat{x} - \bar{x}) \in K - K$. We have

$$\langle A\bar{x} + a, B\hat{x} + b - B\bar{x} - b \rangle \geq 0,$$

and

$$\langle A\hat{x} + a, B\bar{x} + b - B\hat{x} - b \rangle \geq 0.$$

This follows that

$$\langle A(\hat{x} - \bar{x}), B(\hat{x} - \bar{x}) \rangle \leq 0,$$

which means

$$\langle (\hat{x} - \bar{x}), A^T B(\hat{x} - \bar{x}) \rangle \leq 0.$$

This contradicts the assumption that $A^T B$ is positive definite on $B^{-1}(K - K)$. Therefore, GAVI(ω, K) has a unique solution. $\square$

Remark 3.2. *Consider the problem* GVI(F, G, K) *with K being a nonempty closed convex subset of $\mathbb{R}^n$ and F, G being two continuous functions from $\mathbb{R}^n$ into itself, Pang and Yao [9, Proposition 3.9] showed that* GVI(F, G, K) *has a unique solution if G is injective and Lipschitz at u for some $u \in G^{-1}(K)$ and F is strongly monotone with respect to G on K. Clearly, the assumptions in Theorem 3.1 is weaker than ones in [9, Proposition 3.9] applied to* GAVI(ω, K). *This is illustrated by the following example.*

Example 3.3. *Consider the problem* GAVI(ω, K) *with $n = 2$,*

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & 1 \end{pmatrix}, \; a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and $K = \{(0, u) : u \in \mathbb{R}\}$ where $a_1, a_2, a_3 \in \mathbb{R}$. We show that the above problem has a solution for every $a_1, a_2, a_3 \in \mathbb{R}$. Indeed, we have K is a closed convex set and $K^\infty = K$. Then, $(K^\infty)^* = \{(v_1, v_2) \in \mathbb{R}^2 : v_2 = 0\}$ and

$$H = \{(v_1, v_2) \in \mathbb{R}^2 : \exists \alpha \in \mathbb{R}_+ \text{ s.t. } ((\alpha + 1)v_1, \alpha v_2) \in K^\infty\} = \{(0, v_2) : v_2 \in \mathbb{R}\}.$$

For any $a_1, a_2, a_3 \in \mathbb{R}$, we obtain that

$$\langle v, (A + B)v \rangle = v_2^2 \geq 0 \; \forall v = (v_1, v_2) \in H$$

and

$$\langle v, (A^T B)v \rangle = 0 \;\; \forall v = (v_1, v_2) \in H.$$

Hence, $A + B$ and $A^T B$ are positive semidefinite on H. The condition (i) follows. Solving the following system

$$Bv = (v_1, 0) \in K^\infty, \; Av = (a_1 v_1 + a_2 v_2, a_3 v_1 + v_2) \in (K^\infty)^* \text{ and } \langle Av, Bv \rangle = 0,$$

we obtain $v = (0, 0)$. Then, the assumption (ii) is satisfied. According to Theorem 3.1, the problem in this example has a solution for every $a_1, a_2, a_3 \in \mathbb{R}$. Note that the map G herein is not injective; hence, [9, Proposition 3.9] can not be applied to this problem.

The following example shows that $\mathrm{GAVI}(\omega, K)$ has no solution if the assumption on the positive semidefiniteness in Theorem 3.1 is violated.

Example 3.4. *We consider the problem* $\mathrm{GAVI}(\omega, \mathrm{K})$ *with* $n = 2$*,*

$$A = \begin{pmatrix} b_1 & b_2 \\ -2 & -1 \end{pmatrix}, \; a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and $K = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 \geq z_1^2, z_1 \geq 1\}$ *where* $b_1, b_2 \in \mathbb{R}$*. Then,* $K^\infty = \{(0, u) : u \in \mathbb{R}_+\}$ *and* $H = K^\infty$*. For any* $b_1, b_2 \in \mathbb{R}$*, we obtain that*

$$\langle v, (A^T B)v \rangle = -2v_2^2.$$

Thus, $A^T B$ *is not positive semidefinite on* H*.*

We prove that this problem has no solution for every $b_1, b_2 \in \mathbb{R}$*. Indeed, suppose that the above problem has a solution* $\bar{x} = (\bar{x}_1, \bar{x}_2)$*. Then, we have*

$$B\bar{x} = (\bar{x}_1, 2\bar{x}_2) \in K \tag{8}$$

and

$$\langle A\bar{x}, y - B\bar{x} \rangle \geq 0 \;\; \forall y \in K. \tag{9}$$

From (8) *it follows that* $\bar{x}_1 \geq 1$ *and* $\bar{x}_2 \geq \frac{1}{2}$*. Let* $\bar{y} = (\bar{x}_1, 2\bar{x}_2 + 2) \in K$*. Then,*

$$\langle A\bar{x}, \bar{y} - B\bar{x} \rangle = (b_1\bar{x}_1 + b_2\bar{x}_2, -2\bar{x}_1 - \bar{x}_2)^T (0, 2) = -4\bar{x}_1 - 2\bar{x}_2 < 0,$$

which contradicts to (9)*. Therefore, the above problem has no solution.*

In the case where $Bx + b = x$ for every $x \in K$, we have $H = K^\infty$. By Theorem 3.1, we obtain the following corollary.

Corollary 3.5. (see [2, Theorem 6.3]) *Consider the problem* $\mathrm{AVI}(\mathrm{A}, \mathrm{a}, \mathrm{K})$*. Let* $K \subset \mathbb{R}^n$ *be a nonempty closed convex set. If* A *is positive definite on* K^∞*, then for all* $a \in \mathbb{R}^n$*,* $\mathrm{AVI}(\mathrm{A}, \mathrm{a}, \mathrm{K})$ *has a solution.*

4 A Tikhonov-type regularization method

In this section, we use a Tikhonov-type regularization method which is motivated by Tikhonov regularization technique [13] to find a solution of the problem $\mathrm{GAVI}(\omega, K)$. We prove that any sequence generated by the algorithm converges to a solution of the problem $\mathrm{GAVI}(\omega, K)$. To do this, for each $\varepsilon > 0$, let

$$f_\varepsilon(x) := Ax + \varepsilon x + a$$

and

$$g_\varepsilon(x) := Bx + \varepsilon x + b.$$

Consider the following perturbed problem: Find $x \in \mathbb{R}^n$ such that $g_\varepsilon(x) \in K$ and

$$\langle f_\varepsilon(x), y - g_\varepsilon(x) \rangle \geq 0 \ \ \forall y \in K. \qquad (\mathrm{GAVI}(f_\varepsilon, g_\varepsilon, K))$$

Let $x_\varepsilon \in \mathrm{Sol}(f_\varepsilon, g_\varepsilon, K)$. Then, the sequence $\{x_\varepsilon : \varepsilon > 0\}$ is called the *Tikhonov-type trajectory* of the problem $\mathrm{GAVI}(\omega, K)$. We show the convergence of the Tikhonov-type trajectory $\{x_\varepsilon : \varepsilon > 0\}$ under some checkable conditions.

4.1 Convergence theorem

The main result is presented as follows.

Theorem 4.1. *Let $A^T B$ and $A+B$ be two positive semidefinite matrices. Then, the following statements are valid:*

(a) *For each $\varepsilon > 0$, $\mathrm{GAVI}(f_\varepsilon, g_\varepsilon, K)$ has a unique solution;*

(b) *If $A+B$ is a positive definite matrix then, for the Tikhonov-type trajectory $\{x_\varepsilon : \varepsilon > 0\}$, the following three properties are equivalent:*
(b_1) $\lim_{\varepsilon\to 0} x_\varepsilon$ *exists;*
(b_2) $\limsup_{\varepsilon\to 0} \|x_\varepsilon\| < +\infty$*;*
(b_3) $\mathrm{Sol}(\omega, K)$ *is nonempty.*
Moreover, if any one of the statements (b_1)–(b_3) holds, the limit $\lim_{\varepsilon\to 0} x_\varepsilon$ is not only a solution of the problem $\mathrm{GAVI}(\omega, 0, K)$ but also the unique solution of the problem $\mathrm{AVI}((A+B)^T, \hat{K})$, where $\hat{K}$ is the convex hull of $\mathrm{Sol}(\omega, K)$.

Proof. (a) Suppose that $A+B$ and $A^T B$ are positive semidefinite matrices. Let $A_\varepsilon := A + \varepsilon I$ and $B_\varepsilon := B + \varepsilon I$, where $I \in \mathbb{R}^{n\times n}$ is the unit matrix. Clearly, $A_\varepsilon + B_\varepsilon$ and $(A_\varepsilon)^T(B_\varepsilon)$ are positive definite matrices. Suppose to the contrary that $\mathrm{GAVI}(f_\varepsilon, g_\varepsilon, K)$ has no solution. Then, it follows from Proposition 2.3 that there exist $\{x^r\} \subset \mathbb{R}^n$ satisfying $\|x^r\| \to \infty$ as $r \to \infty$ and $\sigma^r > 0$ such that $z^r := \sigma^r x^r + g_\varepsilon(x^r) \in K$ such that

$$\langle \sigma^r x^r + A_\varepsilon x^r + a, y - \sigma^r x^r - B_\varepsilon x^r - b \rangle \geq 0 \ \ \forall y \in K.$$

Multiplying both sides of the above inequality by $\|x^r\|^{-2}$ yields

$$\sigma^r \left\langle \frac{x^r}{\|x^r\|}, \frac{y - A_\varepsilon x^r - a - B_\varepsilon x^r - b}{\|x^r\|} \right\rangle + \left\langle \frac{A_\varepsilon x^r + a}{\|x^r\|}, \frac{y - B_\varepsilon x^r - b}{\|x^r\|} \right\rangle - (\sigma^r)^2 \geq 0.$$

It implies

$$\sigma^r \alpha^r + \gamma^r - (\sigma^r)^2 \geq 0, \tag{10}$$

where $\alpha^r := \left\langle \frac{x^r}{\|x^r\|}, \frac{y - A_\varepsilon x^r - a - B_\varepsilon x^r - b}{\|x^r\|} \right\rangle$ and $\gamma^r := \left\langle \frac{A_\varepsilon x^r + a}{\|x^r\|}, \frac{y - B_\varepsilon x^r - b}{\|x^r\|} \right\rangle$. Without loss of generality, we assume that $x^r/\|x^r\| \to \bar{v}$ for some $\bar{v} \in \mathbb{R}^n \setminus \{0\}$. Then,

$$\lim_{r\to\infty} \alpha^r = \lim_{r\to\infty} \left\langle \frac{x^r}{\|x^r\|}, \frac{y - A_\varepsilon x^r - a - B_\varepsilon x^r - b}{\|x^r\|} \right\rangle = -\langle \bar{v}, (A_\varepsilon + B_\varepsilon)\bar{v} \rangle$$

and

$$\lim_{r\to\infty} \gamma^r = \lim_{r\to\infty} \left\langle \frac{A_\varepsilon x^r + a}{\|x^r\|}, \frac{y - B_\varepsilon x^r - b}{\|x^r\|} \right\rangle = \langle A_\varepsilon \bar{v}, -B_\varepsilon \bar{v}\rangle = -\langle \bar{v}, A_\varepsilon^T B_\varepsilon \bar{v}\rangle.$$

If $\{\sigma^r\}$ is unbounded then dividing both sides of the inequality (10) by $(\sigma^r)^2$ and letting $r \to \infty$ yields $-1 \geq 0$, a contradiction. Thus, $\{\sigma^r\}$ is bounded. There exists a subsequence $\{\sigma^{r_j}\} \subset \{\sigma^r\}$ such that $\sigma^{r_j} \to \bar{\sigma}$ as $j \to \infty$ for some $\bar{\sigma} \in \mathbb{R}_+$. Passing (10) to the limit as $j \to \infty$ gives

$$-\bar{\sigma}\langle \bar{v}, (A_\varepsilon + B_\varepsilon)\bar{v}\rangle - \langle \bar{v}, A_\varepsilon^T B_\varepsilon \bar{v}\rangle - (\bar{\sigma})^2 \geq 0. \tag{11}$$

This contradicts to the fact that $A_\varepsilon + B_\varepsilon$ and $(A_\varepsilon)^T(B_\varepsilon)$ are positive definite matrices. Hence, this case does not occur and the problem GAVI$(f_\varepsilon, g_\varepsilon, K)$ has a solution.

Suppose to the contrary that GAVI$(f_\varepsilon, g_\varepsilon, K)$ has two different solution $\bar{x}$ and $\hat{x}$. Then, $\langle A_\varepsilon \bar{x} + a, B_\varepsilon \hat{x} + b - B_\varepsilon \bar{x} - b\rangle \geq 0$, and $\langle A_\varepsilon \hat{x} + a, B_\varepsilon \bar{x} + b - B_\varepsilon \hat{x} - b\rangle \geq 0$. This follows that $\langle A_\varepsilon(\hat{x} - \bar{x}), B_\varepsilon(\hat{x} - \bar{x})\rangle \leq 0$, which contradicts to the assumption that $A_\varepsilon^T B_\varepsilon$ is positive definite. Therefore, GAVI$(f_\varepsilon, g_\varepsilon, K)$ has a unique solution.

(b) Suppose that $A + B$ is a positive definite matrix and $\{x_\varepsilon : \varepsilon > 0\}$ is the Tikhonov trajectory of the problem GAVI(ω, K). We show that (b_1), (b_2), and (b_3) are equivalent.

$(b_1) \Rightarrow (b_2)$ This is obvious.

$(b_2) \Rightarrow (b_3)$ By the assumption that $\limsup_{\varepsilon\to 0} \|x_\varepsilon\| < +\infty$, there exists a subsequence $\{x_{\varepsilon^j}\} \subset \{x_\varepsilon\}$ such that $x_{\varepsilon^j} \to \hat{x}$ for some $\hat{x} \in \mathbb{R}^n$. Since $x_{\varepsilon^j} \in$ Sol$(f_{\varepsilon^j}, g_{\varepsilon^j}, K)$, we have

$$Bx_{\varepsilon^j} + \varepsilon^j x_{\varepsilon^j} + b \in K$$

and

$$\langle Ax_{\varepsilon^j} + \varepsilon^j x_{\varepsilon^j} + a, y - Bx_{\varepsilon^j} - \varepsilon^j x_{\varepsilon^j} - b\rangle \geq 0.$$

Passing these relations to the limits as $j \to \infty$ yields

$$B\hat{x} + b \in K \text{ and } \langle A\hat{x} + a, y - B\hat{x} - b\rangle \geq 0.$$

This follows that $\hat{x} \in$ Sol(ω, K); hence, Sol(ω, K) is nonempty.

$(b_3) \Rightarrow (b_1)$ Suppose that (b_3) holds. Let any $\bar{x} \in$ Sol(ω, K). Since $x_\varepsilon \in$ Sol$(f_\varepsilon, g_\varepsilon, K)$ and $\bar{x} \in$ Sol(ω, K), we have

$$\langle A\bar{x} + a, Bx_\varepsilon + \varepsilon x_\varepsilon + b - B\bar{x} - b\rangle \geq 0 \tag{12}$$

and

$$\langle Ax_\varepsilon + \varepsilon x_\varepsilon + a, B\bar{x} + b - Bx_\varepsilon - \varepsilon x_\varepsilon - b\rangle \geq 0. \tag{13}$$

By (12) and (13) one obtains

$$\langle A(x_\varepsilon - \bar{x}) + \varepsilon x_\varepsilon, B(x_\varepsilon - \bar{x}) + \varepsilon x_\varepsilon\rangle \leq 0.$$

This implies that

$$\langle (A+B)(x_\varepsilon - \bar{x}), x_\varepsilon \rangle \le 0 \tag{14}$$

since $A^T B$ is a positive semidefinite matrix. By (14) and the assumption that $A+B$ is a positive definite matrix, there exists $\sigma > 0$ such that

$$\sigma \langle x_\varepsilon, x_\varepsilon \rangle \le \langle (A+B)x_\varepsilon, x_\varepsilon \rangle \le \langle (A+B)\bar{x}, x_\varepsilon \rangle.$$

This follows that

$$\|x_\varepsilon\| \le \frac{1}{\sigma} \|(A+B)\bar{x}\|.$$

By the boundedness of $\{x_\varepsilon\}$, there exists a subsequence $\{x_{\varepsilon^j}\} \subset \{x_\varepsilon\}$ converging to $\hat{x}$ for some $\hat{x} \in \mathbb{R}^n$. By the fact that $Bx_{\varepsilon^j} + \varepsilon^j x_{\varepsilon^j} + b \in K$ and the closedness of K, we have $B\hat{x} + b \in K$. Since $x_{\varepsilon^j} \in \mathrm{Sol}(f_{\varepsilon^j}, g_{\varepsilon^j}, K)$, for each $y \in K$, we have

$$\langle Ax_{\varepsilon^j} + \varepsilon^j x_{\varepsilon^j} + a, y - Bx_{\varepsilon^j} - \varepsilon^j x_{\varepsilon^j} - b \rangle \ge 0.$$

Letting $\varepsilon^j \to 0$ in the inequality above, we get $\langle A\hat{x} + a, y - B\hat{x} - b \rangle \ge 0$. Hence, $\hat{x} \in \mathrm{Sol}(\omega, K)$.

Furthermore, passing the inequality (14) to the limit as $\varepsilon \to 0$ yields

$$\langle (A+B)^T \hat{x}, \bar{x} - \hat{x} \rangle \ge 0. \tag{15}$$

Since $\bar{x} \in \mathrm{Sol}(\omega, K)$ is chosen arbitrarily, the inequality (15) holds for every $\bar{x} \in \mathrm{Sol}(\omega, K)$.

Suppose that $\hat{K}$ is the convex hull of $\mathrm{Sol}(\omega, K)$. Then, for each $x \in \hat{K}$, there exist $\bar{x}_1, \ldots, \bar{x}_p$ in $\mathrm{Sol}(\omega, K)$ and nonnegative real numbers $\alpha_1, \ldots, \alpha_p$ satisfying $\alpha_1 + \ldots + \alpha_p = 1$ such that $x = \alpha_1 \bar{x}_1 + \ldots + \alpha_p \bar{x}_p$. By (15), we have

$$\langle (A+B)^T \hat{x}, x - \hat{x} \rangle = \sum_{i=1}^{p} \alpha_i \langle (A+B)^T \hat{x}, \bar{x}_i - \hat{x} \rangle \ge 0.$$

Combining this with the fact that $\hat{x} \in \hat{K}$, we obtain

$$\hat{x} \in \mathrm{Sol}((A+B)^T, 0, \hat{K}).$$

By the positive definiteness of $A+B$ and the result in part (a), the problem $\mathrm{AVI}((A+B)^T, 0, \hat{K})$ has a unique solution. Therefore, $\{x_{\varepsilon^j}\} \equiv \{x_\varepsilon\}$ and $\lim_{\varepsilon \to 0} x_\varepsilon$ exists. The proof is complete. □

4.2 A Tikhonov-type Regularization Algorithm

By the above arguments, we obtain a *Tikhonov-type Regularization Algorithm* (TTRA) as follows.

Algorithm TTRA

Step 1 Taken $x^0 \in \mathbb{R}^n$ satisfying $g(x^0) = Bx^0 + b \in K$.
Step 2 Given x^k, if x^k solves GAVI(ω, K) then $x^{k+p} = x^k$ for all $p \geq 1$ and the algorithm stops, otherwise go to Step 3.
Step 3 Calculate a point $x^{k+1} \in \mathrm{Sol}(f_{\varepsilon^k}, g_{\varepsilon^k}, K)$ with $\varepsilon^k \downarrow 0$ and go to Step 2 with $k := k+1$.

The following corollary follows from Theorem 4.1.

Corollary 4.2. *Let $A + B$ be positive definite and $A^T B$ be positive semidefinite. Then, if* $\mathrm{Sol}(\omega, K)$ *is nonempty then the approximate solution x^{k+1} obtained from the above algorithm TTRA converges to a solution $\bar{x}$ of the problem* GAVI(ω, K).

Example 4.3. *Consider the problem* GAVI(ω, K) *with $n = 3$, $K = \mathbb{R}^3_+$, and*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \; a = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}.$$

It is not difficult to check that $A + B$ is positive definite and $A^T B$ is positive semidefinite. For each $\varepsilon > 0$, the problem GAVI$(f_\varepsilon, g_\varepsilon, K)$ *reduces to the following generalized complementary problem: Finding $x \in \mathbb{R}^n_+$ such that*

$$Ax + \varepsilon x + a \geq 0, Bx + \varepsilon x + b \geq 0, \langle Ax + \varepsilon x + a, Bx + \varepsilon x + b \rangle = 0,$$

that is,

$$\begin{pmatrix} (1+\varepsilon)x_1 \\ (1+\varepsilon)x_2 \\ (2+\varepsilon)x_3 \end{pmatrix} \geq 0, \; \begin{pmatrix} (1+\varepsilon)x_1 - 1 \\ \varepsilon x_2 + 1 \\ (1+\varepsilon)x_3 - 2 \end{pmatrix} \geq 0, \; \begin{pmatrix} (1+\varepsilon)x_1 \\ (1+\varepsilon)x_2 \\ (2+\varepsilon)x_3 \end{pmatrix}^T \begin{pmatrix} (1+\varepsilon)x_1 - 1 \\ \varepsilon x_2 + 1 \\ (1+\varepsilon)x_3 - 2 \end{pmatrix} = 0.$$

Since $\varepsilon x_2 + 1 > 0$, we have $x_2 = 0$. If $x_1 = 0$ then $(1+\varepsilon)x_1 - 1 < 0$. Hence, $x_1 = \frac{1}{1+\varepsilon}$. Similarly, if $x_3 = 0$ then $(1+\varepsilon)x_3 - 2 < 0$. It follows $x_3 = \frac{2}{1+\varepsilon}$. Thus,

$$\mathrm{Sol}(f_\varepsilon, g_\varepsilon, K) = \left\{ x_\varepsilon = \left(\frac{1}{1+\varepsilon}, 0, \frac{2}{1+\varepsilon} \right) \right\}.$$

We have $x_\varepsilon \to \bar{x} = (1, 0, 2)$ as $\varepsilon \to 0$. By the above algorithm TTRA, we conclude that $\bar{x} \in \mathrm{Sol}(\omega, K)$.

4.3 Semicontinuity of the Tikhonov-type trajectory generated by Algorithm TTRA

Let $\mathrm{Sol}(\cdot) : \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ be a multifunction defined by

$$\mathrm{Sol}(\varepsilon) := \mathrm{Sol}(f_\varepsilon, g_\varepsilon, K)$$

for every $\varepsilon \geq 0$. In this section, we characterize the upper/lower semicontinuity of the map $\mathrm{Sol}(\cdot)$.

Theorem 4.4. *Let* $\mathrm{Sol}(\varepsilon)$ *be nonempty for every* $\varepsilon \geq 0$*. If* A^TB *and* $A+B$ *are positive semidefinite on* $B^{-1}(K-K)$ *then the solution map* $\mathrm{Sol}(\cdot)$ *is lower semicontinuous on* $\mathbb{R}_+$*.*

Proof. We will show that $\mathrm{Sol}(\cdot)$ is lower semicontinuous on $\mathbb{R}_+$. Indeed, suppose to the contrary that there exists $\varepsilon \geq 0$ such that $\mathrm{Sol}(\cdot)$ is not lower semicontinuous at ε, that is, there exist $\bar{x} \in \mathrm{Sol}(\varepsilon)$ and a sequence $\{\varepsilon^k\} \subset \mathbb{R}_+$ satisfying $\varepsilon^k \to \varepsilon$ such that, for any $z^k \in \mathrm{Sol}(\varepsilon^k)$ satisfying $z^k \to \bar{z}$, one has $\bar{z} \neq \bar{x}$.

Since $z^k \in \mathrm{Sol}(\varepsilon^k)$, we conclude that

$$Bz^k + \varepsilon^k z^k + b \in K \text{ and } \langle Az^k + \varepsilon^k z^k + a, z - Bz^k - \varepsilon^k z^k - b\rangle \geq 0 \ \ \forall z \in K. \quad (16)$$

For each $z \in K$, passing the two relations in (16) to the limit as $k \to \infty$ gives

$$B\bar{z} + \varepsilon\bar{z} + b \in K \text{ and } \langle A\bar{z} + \varepsilon\bar{z} + a, z - B\bar{z} - \varepsilon\bar{z} - b\rangle \geq 0. \quad (17)$$

Since $\bar{x} \in \mathrm{Sol}(\varepsilon)$, we have $B\bar{x} + \varepsilon\bar{x} + b \in K$. Substituting $z = B\bar{x} + \varepsilon\bar{x} + b$ into (17) yields

$$\langle A\bar{z} + \varepsilon\bar{z} + a, B\bar{x} + \varepsilon\bar{x} + b - B\bar{z} - \varepsilon\bar{z} - b\rangle \geq 0. \quad (18)$$

From $B\bar{z} + \varepsilon\bar{z} + b \in K$ it follows that

$$\langle A\bar{x} + \varepsilon\bar{x} + a, B\bar{z} + \varepsilon\bar{z} + b - B\bar{x} - \varepsilon\bar{x} - b\rangle \geq 0. \quad (19)$$

Combining (18) with (19), we obtain $(B + \varepsilon I)(\bar{x} - \bar{z}) \in K - K$ and

$$\langle (A + \varepsilon I)(\bar{x} - \bar{z}), (B + \varepsilon I)(\bar{x} - \bar{z})\rangle \leq 0. \quad (20)$$

By the assumption that A^TB and $A+B$ are positive semidefinite on $B^{-1}(K-K)$, we see that

$$(A + \varepsilon I)^T(B + \varepsilon I) = A^TB + \varepsilon(A + B) + \varepsilon^2 I$$

is positive definite on $B^{-1}(K-K)$. This contradicts the inequality (20). Therefore, the solution map $\mathrm{Sol}(\cdot)$ is lower semicontinuous on $\mathbb{R}_+$. The proof is complete. □

Denote $G := \{\varepsilon \in \mathbb{R}_+ : \mathrm{Sol}(A_\varepsilon, 0, B_\varepsilon, 0, K^\infty) = \{0\}\}$ with $A_\varepsilon = A + \varepsilon I$ and $B_\varepsilon = B + \varepsilon I$. We have the following lemma.

Lemma 4.5. G *is open in* $\mathbb{R}_+$*.*

Proof. Suppose to the contrary G is not open in $\mathbb{R}_+$. Then, there exists $\{\rho^k\} \subset \mathbb{R}_+ \backslash G$ converging to $\rho \in S$. For each ρ^k, there exists $v^k \in \mathbb{R}^n$ such that $\|v^k\| = 1$ and

$$Bv^k + \rho^k v^k \in K^\infty, Av^k + \rho^k v^k \in (K^\infty)^*, \langle Av^k + \rho^k v^k, Bv^k + \rho^k v^k\rangle = 0. \quad (21)$$

Without loss of generality, we may assume that the sequence $\{v^k\}$ itself converges to $\hat{v}$ for some $\hat{v} \in \mathbb{R}^n$. Taking the limits in (21) as $k \to \infty$ yields

$$\|\hat{v}\| = 1, B\hat{v} + \rho\hat{v} \in K^\infty, A\hat{v} + \rho\hat{v} \in (K^\infty)^* \text{ and } \langle A\hat{v} + \rho\hat{v}, B\hat{v} + \rho\hat{v}\rangle = 0.$$

This implies that $0 \neq \hat{v} \in \mathrm{Sol}(A_\rho, 0, B_\rho, 0, K^\infty)$ and $\rho \notin G$, which contradicts to the fact that $\rho \in G$. The proof is complete. □

The following theorem characterize the upper semicontinuity of the solution map $\mathrm{Sol}(\cdot)$.

Theorem 4.6. *For each $\varepsilon \in \mathbb{R}_+$, if $\mathrm{Sol}(A_\varepsilon, 0, B_\varepsilon, 0, K^\infty) = \{0\}$ and $\mathrm{Sol}(f_\varepsilon, g_\varepsilon, K)$ is nonempty, then the solution map $\mathrm{Sol}(\cdot)$ is upper semicontinuous at ε.*

Proof. By Lemma 4.5, there exists $\rho > 0$ such that $\varepsilon + \bar{B}(0, \rho) \subset G$. We claim that $\mathrm{Sol}(\cdot)$ is locally bounded at ε, that is,

$$U_\rho := \bigcup_{\tilde{\rho} \in B(0,\rho)} Sol(\varepsilon + \tilde{\rho}) \tag{22}$$

is bounded. Indeed, suppose to the contrary that U_ρ is unbounded. Then, there exist $\rho^k \in B(0, \rho)$ and $z^k \in Sol(\varepsilon + \rho^k)$ such that $\|z^k\| \to \infty$. Since $B(0, \rho)$ is bounded, we assume that $\rho^k \to \bar{\rho}$ for some $\bar{\rho} \in \bar{B}(0, \rho)$ and $\varepsilon + \bar{\rho} \in \varepsilon + \bar{B}(0, \rho) \subset G$, that is, $\mathrm{Sol}(A + (\varepsilon + \bar{\rho})I, 0, B + (\varepsilon + \bar{\rho})I, 0, K^\infty) = \{0\}$. Without loss of generality, we may assume that $\|z^k\| \neq 0$ for all k and $\|z^k\|^{-1}z^k \to \bar{v}$ for some $\bar{v} \in \mathbb{R}^n$ with $\|\bar{v}\| = 1$. Since $z^k \in Sol(\varepsilon + \rho^k)$, we have $Bz^k + b + (\rho^k + \varepsilon)z^k \in K$ and

$$\langle Az^k + a + (\varepsilon + \rho^k)z^k, z - Bz^k - (\varepsilon + \rho^k)z^k - b\rangle \geq 0. \tag{23}$$

Since $\frac{1}{\|z^k\|} \to 0$, applying [10, Theorem 8.2] to $Bz^k + b + (\varepsilon + \rho^k)z^k \in K$, we have

$$(B + (\varepsilon + \bar{\rho})I)\bar{v} \in K^\infty.$$

Fix any $\bar{x} \in K$. For every $h \in K^\infty$, one has $\bar{x} + \|z^k\|h \in K$. Substituting $\bar{x} + \|z^k\|h$ for z in (23), we obtain

$$\langle Az^k + a + (\varepsilon + \rho^k)z^k, \bar{x} + \|z^k\|h - Bz^k - (\varepsilon + \rho^k)z^k - b\rangle \geq 0. \tag{24}$$

Dividing both sides of the inequality (24) by $\|z^k\|$ and letting $k \to \infty$ yields

$$\big\langle (A + (\varepsilon + \bar{\rho})I)\bar{v}, h - (B + (\varepsilon + \bar{\rho})I)\bar{v}\big\rangle \geq 0.$$

This leads to the following

$$0 \neq \bar{v} \in \mathrm{Sol}(A + (\varepsilon + \bar{\rho})I, 0, B + (\varepsilon + \bar{\rho})I, 0, K^\infty) = \{0\},$$

which is a contradiction. Hence, $\mathrm{Sol}(\cdot)$ is locally bounded at ε.

Suppose that Sol($\cdot$) is not upper semicontinuous at ε. Then, there exist a nonempty open set U which contains Sol(ε) and $x^k \in Sol(\varepsilon^k)$ with $\varepsilon^k \to 0$ satisfying

$$x^k \in Sol(\varepsilon^k) \setminus U. \tag{25}$$

Since Sol($\cdot$) is locally bounded at ε, the sequence $\{x^k\}$ is bounded. Without loss of generality we may assume that $x^k \to \hat{x}$ and $\hat{x} \in Sol(\varepsilon)$. Hence, $\hat{x} \in U$. This contradicts the fact that U is open and (25) holds. Therefore, Sol($\cdot$) is upper semicontinuous at ε. □

5 Properties of the set of solutions

In this section, some properties of the solution set Sol(ω, K) of the problem GAVI(ω, K) are investigated. In Subsection 3.1, we show that Sol(ω, K) is the union of finitely many polyhedral convex sets. A necessary and sufficient condition for unboundedness of Sol(ω, K) is also discussed. In Subsection 3.2, we characterize the closedness and convexity under the assumption on generalized pseudo-monotonicity.

5.1 Unboundedness of the set of solutions

Let K be a nonempty polyhedral convex set defined by $K := \{x \in \mathbb{R}^n : Cx + d \le 0\}$ with C being a $m \times n$ real matrix and $d \in \mathbb{R}^m$. The following lemma is useful to investigate the properties of the solution set.

Lemma 5.1. *A vector $\bar{x} \in \mathbb{R}^n$ is a solution of* GAVI(ω, K) *if and only if there exists $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)$ such that*

$$\begin{cases} A\bar{x} + C^T\bar{\lambda} + a = 0, \\ CB\bar{x} + Cb + d \le 0, \ \bar{\lambda} \ge 0, \\ \bar{\lambda}^T(CB\bar{x} + Cb + d) = 0. \end{cases} \tag{26}$$

Proof. Necessity: Denote by C_i the i-th row of C and denote by d_i the i-th component of d. For each $i \in [m]$, put $c_i := C_i^T$. For any $\bar{x} \in \mathrm{Sol}(\omega, K)$, denote

$$I_0 := \{i \in [m] : \langle c_i, B\bar{x} + b\rangle + d_i = 0\} \text{ and } I_1 := [m] \setminus I_0.$$

Let any $h \in \mathbb{R}^n$ satisfying $\langle c_i, h\rangle \le 0$ for all $i \in I_0$. Put $x_t = B\bar{x} + b + th$. Then, there exists $\varepsilon > 0$ such that $\langle c_i, x_t\rangle + d_i \le 0$ for every $i \in [m]$ for every $t \in (0, \varepsilon)$; hence, $x_t \in K$. Since $\bar{x} \in \mathrm{Sol}(\omega, K)$, we have

$$0 \le \langle A\bar{x} + a, x_t - B\bar{x} - b\rangle = t\langle A\bar{x} + a, h\rangle.$$

It follows that $\langle -A\bar{x} - a, h\rangle \le 0$ for every $h \in \mathbb{R}^n$ satisfying $\langle c_i, h\rangle \le 0$ for all $i \in I_0$. According to Farkas Lemma [10, p. 200], there exist $\bar{\lambda}_i \ge 0$ and $i \in I_0$, such that

$$\sum_{i\in I_0} \bar{\lambda}_i c_i = -A\bar{x} - a.$$

For each $i \in I_1$, let $\bar{\lambda}_i = 0$. Choose $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_m)$ and the system (26) follows.

Sufficiency: Suppose that there exists $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_m)$ satisfying (26). For every $z \in K$ we have $B\bar{x} + b \in K$ and

$$\begin{aligned}
\langle A\bar{x} + a, z - B\bar{x} - b\rangle &= \langle -C^T\bar{\lambda}, z - B\bar{x} - b\rangle \\
&= -\langle \bar{\lambda}, (Cz + d) - (C(B\bar{x} + b) + d)\rangle \\
&= -\bar{\lambda}^T(Cz + d) + \bar{\lambda}^T(C(B\bar{x} + b) + d) \\
&= -\bar{\lambda}^T(Cz + d) \\
&\geq 0.
\end{aligned}$$

Therefore, $\bar{x} \in \mathrm{Sol}(\omega, K)$. □

The following property shows that $\mathrm{Sol}(\omega, K)$ is the union of many polyhedral convex sets.

Theorem 5.2. *The set* $\mathrm{Sol}(\omega, K)$ *is the union of* N *polyhedral convex sets with*

$$N = \sum_{p=1}^{m} \frac{m!}{p!(m-p)!}.$$

Proof. According to Lemma 5.1, a point $x \in \mathbb{R}^n$ is a solution of $\mathrm{GAVI}(\omega, K)$ if and only if there exists $\lambda = (\lambda_1, \dots, \lambda_m)$ such that

$$\begin{cases} Ax + C^T\lambda + a = 0, \\ Qx + q \leq 0, \ \lambda \geq 0, \\ \lambda^T(Qx + q) = 0 \end{cases} \tag{27}$$

with $Q = CB$ and $q = Cb + d$. For each $x \in \mathrm{Sol}(\omega, K)$, let

$$I := \{i \in [m] : Q_i x + q_i = 0\},$$

where Q_i is the i-th row vector of the matrix Q and q_i is the i-th component of the vector q. Let $I^c := [m] \setminus I$. By the fact that $\lambda^T(Qx + q) = 0$, we have $\lambda_{I^c} = 0$. Then, x is a solution of $\mathrm{GAVI}(\omega, K)$ if and only if there exists $\lambda \in \mathbb{R}^m_+$ such that $(x, \lambda) \in \Delta_I$ with Δ_I being the set of solutions of the following system

$$\begin{cases} Ax + C^T\lambda + a = 0, \\ Q_I x + q_I = 0, \ \lambda_I \geq 0, \\ Q_{I^c} x + q_{I^c} \leq 0, \ \lambda_{I^c} = 0. \end{cases} \tag{28}$$

This leads to

$$\mathrm{Sol}(\omega, K) = \bigcup_{I \subset [m]} P_{\mathbb{R}^n}(\Delta_I)$$

with $P_{\mathbb{R}^n}(x,\lambda) = x$. Since Δ_I is a polyhedral convex set and $P_{\mathbb{R}^n}$ is a linear operator, $P_{\mathbb{R}^n}(\Delta_I)$ is also a polyhedral convex set. Therefore, $\mathrm{Sol}(\omega, K)$ is the union of N polyhedral convex sets with

$$N = \sum_{p=1}^{m} \frac{m!}{p!(m-p)!}.$$

□

The following corollary follows immediately from Theorem 5.2.

Corollary 5.3. *The following statements hold:*

(*i*) *If* $\mathrm{Sol}(\omega, K)$ *is unbounded, then it contains a solution ray, that is, there exist* $\bar{x} \in \mathrm{Sol}(\omega, K)$ *and* $\bar{v} \in \mathbb{R}^n \setminus \{0\}$ *such that* $\bar{x} + t\bar{v} \in \mathrm{Sol}(\omega, K)$ *for every* $t \geq 0$*;*
(*ii*) *If* $\mathrm{Sol}(\omega, K)$ *is bounded and infinite, then it contains a solution interval, that is, there exist* $\alpha > 0$, $\bar{x} \in \mathrm{Sol}(\omega, K)$, *and* $\bar{v} \in \mathbb{R}^n \setminus \{0\}$ *such that* $\bar{x} + t\bar{v} \in \mathrm{Sol}(\omega, K)$ *for every* $t \in [0, \alpha]$*;*
(*iii*) *If* $\mathrm{Sol}(\omega, K)$ *is convex, then it is a polyhedral convex set.*

A necessary and sufficient condition for the unboundedness of $\mathrm{Sol}(\omega, K)$ is proposed in the following theorem.

Theorem 5.4. *The set* $\mathrm{Sol}(\omega, K)$ *is unbounded if and only if there exists a pair* $(u, v) \in \mathrm{Sol}(\omega, K) \times \setminus\{(0,0)\}$ *satisfying the following three conditions:*

(*i*) $Bv \in K^\infty$, $Av \in (K^\infty)^*$, $\langle Av, Bv\rangle = 0$*;*
(*ii*) $\langle Au + a, Bv\rangle = 0$*;*
(*iii*) $\langle Av, z - Bu - b\rangle \geq 0 \;\; \forall z \in K$*.*

In particular, if $\mathrm{Sol}((A, 0, B, 0), K^\infty) = \{0\}$, *then* $\mathrm{Sol}(\omega, K)$ *is bounded.*

Proof. Necessity: Suppose that $\mathrm{Sol}(\omega, K)$ is unbounded. Arguing similarly as in the proof of Theorem 5.2, we can show that there exists a subset $I \subset [m]$ such that $P_{\mathbb{R}^n}(\Delta_I)$ defined by (28) is unbounded. Then, there exist $u \in P_{\mathbb{R}^n}(\Delta_I)$ and $v \in \mathbb{R}^n \setminus \{0\}$ such that

$$u + tv \in P_{\mathbb{R}^n}(\Delta_I) \subset \mathrm{Sol}(\omega, K) \tag{29}$$

for every $t \geq 0$. Since $g(u + tv) \in K$, we have

$$0 \geq C(B(u + tv) + b) + d = C(Bu) + Cb + tC(Bv)$$

for every $t \geq 0$. It implies that $C(Bv) \leq 0$; hence, $Bv \in K^\infty$. From (29), one has

$$\langle f(u + tv), z - g(u + tv)\rangle \geq 0 \; \forall z \in K. \tag{30}$$

For any z fixed, we have

$$\left\langle \frac{Au+a}{t} + Av, \frac{z-Bu-b}{t} - Bv \right\rangle \geq 0 \ \ \forall t > 0.$$

Letting $t \to \infty$ yields

$$\langle Av, Bv \rangle \leq 0. \tag{31}$$

Choosing $z = g(u) + t^2 Bv \in K$, by (30) we have

$$\langle A(u+tv) + a, (t^2 - t)Bv \rangle \geq 0 \ \forall t > 1. \tag{32}$$

Dividing both sides of the last inequality by $t(t^2 - t)$ and letting $t \to \infty$, we obtain

$$\langle Av, Bv \rangle \geq 0. \tag{33}$$

Combining (31) with (33) gives

$$\langle Av, Bv \rangle = 0.$$

Choosing $z = g(u) \in K$, by (30) one has

$$\langle A(u+tv) + a, -tBv \rangle \geq 0.$$

It implies that $\langle Au + a, Bv \rangle \leq 0$. Thanks to (32), we get $\langle Au + a, Bv \rangle \geq 0$. Hence, condition (ii) is valid. Then, (30) implies that

$$0 \leq \langle A(u+tv)+a, z - B(u+tv) - b \rangle = \langle Au+a, z - Bu - b \rangle + t\langle Av, z - Bu - b \rangle$$

for every $z \in K$ for every $t > 0$. This leads to $\langle Av, z - Bu - b \rangle \geq 0$ and condition (iii) is satisfied. For each $h \in K^\infty$, choosing $z = g(u) + h \in K$, from the last inequality, we have $\langle Av, h \rangle \geq 0$. Thus, $Av \in (K^\infty)^*$ and condition (i) follows.

Sufficiency: Suppose that there exists a pair $(u, v) \in \mathrm{Sol}(\omega, K) \times \mathbb{R}^n \setminus \{(0,0)\}$ satisfying (i)–(iii). For each $t > 0$, let $z_t = u + tv$. Then,

$$g(z_t) = B(u+tv) + b = Bu + b + tBv \in K$$

since $Bu + b \in K$ and $Bv \in K^\infty$. For any $z \in K$, we obtain that

$$\begin{aligned}
&\langle f(z_t), z - g(z_t) \rangle = \langle A(u+tv) + a, z - B(u+tv) - b \rangle \\
&= \langle Au + a + tAv, z - Bu - b - tBv \rangle \\
&= \langle Au + a, z - Bu - b \rangle - t\langle Au + a, Bv \rangle + t\langle Av, z - Bu - b \rangle - t^2 \langle Av, Bv \rangle \\
&\geq \ 0.
\end{aligned}$$

It follows that $z_t \in \mathrm{Sol}(\omega, K)$ for every $t > 0$; hence, $\mathrm{Sol}(\omega, K)$ is unbounded.

Finally, if $\mathrm{Sol}((A, 0, B, 0), K^\infty) = \{0\}$ then there is no (u, v) satisfying the condition (i). Therefore, $\mathrm{Sol}(\omega, K)$ is bounded. $\square$

5.2 Convexity of the set of solutions

Consider the following *generalized Minty variational inequality*: Find $\bar{x} \in \mathbb{R}^n$ such that

$$B\bar{x} + b \in K \text{ and } \langle Ay + a, y - B\bar{x} - b\rangle \geq 0. \qquad (\mathrm{GAVI}^M(\omega, K))$$

Denote by $\mathrm{Sol}^M(\omega, K)$ the set of solutions of $\mathrm{GAVI}^M(\omega, K)$.

The following theorem describes the relation between the problem $\mathrm{GAVI}(\omega, K)$ and the problem $\mathrm{GAVI}^M(\omega, K)$ and characterizes the closedness and convexity of $\mathrm{Sol}(\omega, K)$.

Theorem 5.5. *Suppose that $g^{-1}(y) \neq \emptyset$ for every $y \in K$, and $f(g(x)) = f(x)$ for all $x \in g^{-1}(K)$. The following two statements are valid:*

(*i*) $\mathrm{Sol}^M(\omega, K) \subset \mathrm{Sol}(\omega, K)$*;*

(*ii*) *If f is generalized pseudo-monotone with respect to g on K then*

$$\mathrm{Sol}(\omega, K) \subset \mathrm{Sol}^M(\omega, K)$$

and $\mathrm{Sol}(\omega, K)$ *is a closed convex set.*

Proof. (*i*) Fix any $\bar{x} \in \mathrm{Sol}^M(\omega, K)$. For each $y \in K$, since $g^{-1}(y) \neq \emptyset$ for every $y \in K$, there exists $z \in \mathbb{R}^n$ such that $g(z) = y$. Let $z(t) := \bar{x} + t(z - \bar{x})$ for each $t \in (0, 1)$. The convexity of K implies that $g(z(t)) = (1 - t)g(\bar{x}) + tg(z) \in K$. In addition, since $f(g(x)) = f(x)$ for all $x \in g^{-1}(K)$, we have $z(t) \in g^{-1}(K)$ and $f(g(z(t))) = f(z(t))$. From the fact that $\bar{x} \in \mathrm{Sol}^M(\omega, K)$ and $g(z(t)) \in K$, one gets

$$0 \leq \langle f(g(z(t))), g(z(t)) - g(\bar{x})\rangle = \langle f(z(t)), t(g(z) - g(\bar{x}))\rangle$$

for every $t > 0$. Then,

$$\langle f(z(t)), y - g(\bar{x})\rangle \geq 0.$$

Letting $t \to 0$ yields

$$\langle f(\bar{x}), y - g(\bar{x})\rangle \geq 0. \tag{34}$$

This deduces that $\bar{x} \in \mathrm{Sol}(\omega, K)$.

(*ii*) Let any $\bar{x} \in \mathrm{Sol}(\omega, K)$. Then, for each $y \in K$, we have $g(\bar{x}) \in K$ and $\langle f(\bar{x}), y - g(\bar{x})\rangle \geq 0$. For each $y \in K$, since $g^{-1}(y) \neq \emptyset$, there exists $z \in K$ such that $g(z) = y$. Combining this with the fact that $f(g(x)) = f(x)$ for all $x \in g^{-1}(K)$, we have $f(y) = f(g(z)) = f(z)$. Since f is generalized pseudo-monotone with respect to g on K, by (34) it follows that

$$\langle f(z), g(z) - g(\bar{x})\rangle \geq 0.$$

Hence, $\langle f(y), y - g(\bar{x})\rangle \geq 0$. This proves that $\bar{x} \in \mathrm{Sol}^M(\omega, K)$.

We now show that $\mathrm{Sol}(\omega, K)$ is a polyhedral convex set. Indeed, for each $y \in K$, we denote by $S(y)$ the set of all $\bar{x}$ satisfying $\langle f(y), y - g(\bar{x})\rangle \geq 0$. Then, $S(y)$ is a polyhedral convex set. Since f is generalized pseudo-monotone with respect to g on K and $\mathrm{Sol}^M(\omega, K) = \mathrm{Sol}(\omega, K)$, we obtain that

$$\mathrm{Sol}(\omega, K) = \bigcap_{y \in K} S(y)$$

is a closed convex set. $\square$

6 Conclusions

In the present paper, we have investigated the general affine variational inequalities and have presented the following contributions:

(i) A sufficient condition for the solution existence of GAVI (Theorem 3.1);

(ii) A Tikhonov-type regularization method to find a solution of the problem GAVI, including: algorithm, convergence theorem, and semicontinuity of Tikhovov-type trajectory have been proposed (Theorems 4.1–4.6);

(iii) Under the suitable conditions, we have characterized unboundedness, closedness, and convexity of the set of solutions (Theorems 5.2–5.5).

The obtained results have provided useful information to further study on theory, algorithms, and practical applications for general variational inequalities.

Acknowledgments

A part of this work was done at Vietnam Institute for Advanced Study in Mathematics (VIASM). The authors would like to thank the VIASM for hospitality during our recent stay at the Institute.

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