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To Professor Steven G. Krantz
Washington University in St Louis,
Saint Louis, Missouri,
United States of America
Editor in Chief of
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Dear Professors,

I would like to submit the copy of our paper entitled *"Robust stability for implicit differential equations with causal operators"* for a possible publication in Journal of Mathematical Analysis and Applications.

This manuscript has not been published elsewhere and that it has not been submitted simultaneously for publication elsewhere.

The manuscript has been prepared using AMS-LATEX2 and I have sent to you an a pdf file, enclosed with this letter.

Thank you very much in advance for paying attention on our paper.

Sincerely yours,

Nguyen Thu Ha

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ROBUST STABILITY FOR IMPLICIT DIFFERENTIAL EQUATIONS WITH CAUSAL OPERATORS

NGUYEN THU HA* AND TRAN MANH CUONG#

ABSTRACT. In this paper, we consider the robust stability of implicit differential equations where the leading term is singular and the driving term is a causal linear operator. We study the solvability of linear and nonlinear equations and then the robust stability under small perturbations is established. An L_p version of Bohl-Perron Theorems for these systems is also studied.

1. INTRODUCTION

The differential equations driving by the causal operators, named "aftereffect operators" (see [7]), are very important both in practice and theory because they are generalized from ordinary equations, partial differential equations, integral equations, delay or functional differential equations, which often are used to describe mathematical models in economy, industry, eco-systems... Therefore, the study of the quality and quantity, concerning with problems of stability or robust stability, of differential equations with causal operators has attracted many mathematical works. There are some ways to carry out these problems. One can use either the so-called stability radii, introduced by [19], to measure how is large possible for perturbation which still preserves the stability of an ordinary linear system [2, 4, 12, 22, 21, 23, 31] or the Lyapunov exponents, Bohl exponents to estimate the growth rate of solutions [2, 6, 11]. However, in general, the calculation of stability radii or Lyapunov exponents is a rather difficulty problem. Instead, one try to construct a Lyapunov function, via which, we know whenever a system is stable or unstable (see [7, 10]). In case the above mentioned methods are not applicable, researchers try to find a bound, under which the exponential stability or the boundedness of perturbed systems is preserved [2, 27, 26, 33].

An other aspect to consider the stability of a system is the Bohl-Perron type theorem which says that, for a differential/difference operator L if the solution of the equation $Lx(t) = f(t), t \geqslant 0$ is "good" for every function $f(\cdot)$ to be "rather good", then the solution of the corresponding homogeneous equation $Lx(t) = 0, t \geqslant 0$ is bounded or exponentially stable. The study of Bohl-Perron theorem is concerned with delay differential/difference equations can be founded in [3, 29, 30]; for infinite delay difference systems $x(n + 1) = (Lx)(n) + f(n)$ in [1, 3, 8, 20, 16].

An L_p-version for Bohl-Perron Theorem can be referred to [3, 15, 16]. However, almost works deal with just the case of bounded delay equations of integrability of kernels.

On the other hand, to describe the mathematical modeling in engineering and science such as in multibody and flexible body mechanics, electrical circuit design,

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incompressible fluids (see [14, 24, 32], a formulation of differential systems where the leading term of derivatives may be degenerate is presented. The simplest case is linear differential algebraic equations (DAEs) $A(t)x'(t) = B(t)x(t) + f(t), t \geqslant 0$ where $A(t)$ is singular for every t. There are many works dealing with the robust stability of these systems [4, 5, 24, 25]. Due to the fact that the dynamics of DAEs are constrained and combinate between differential and difference components, some extra difficulties appear in the analysis of stability as well numerical treatments of the implicit dynamic equations characterized by index concepts, see [5, 18, 24, 25].

In continuing to study these problems, it is natural to consider the robust stability and Bohl-Perron theorem for differential algebraic equations with the right hand to be a causal operator (say implicit differential equation)

$$A(t)x^{'}(t) = B(t)x(t) + \Sigma x(\cdot)(t) + f(t), \quad t \geqslant 0, \tag{1.1}$$

where A, B and Σ, f are specific late. The leading matrix $A(t)$ is supposed to be singular for all $t \geqslant 0$. It is known that the solvability of this equation depends strongly on the so-called index concepts. For some reasons, we concentrate in the index-1 equations and we focus on the study of L_p-stability version and the preservation of exponential stability of (1.1) under small perturbation. Also, we consider an L_p Bohl-Perron type theorem for this system.

The paper is organized as follows. In the next section, we introduce some preliminary notations, inequalities and the concept of causal operators. Section 3 deals with the solvability and the dependence on the initial condition of the solutions. In Section 4 we prove that if the linear homogeneous implicit equation is exponentially stable, then the perturbed system is either L_p-stable or exponentially stable depending on the assumption of perturbations. Section 5 presents the famous Bohl-Perron theorem for implicit differential systems in L_p-form.

2. Preliminary

For any $s < t$ and $p \geqslant 1$, let $L_p\big([s,t];\mathbb{R}^n\big)$ be the Banach space of all p–integrable functions $f : [s,t] \to \mathbb{R}^n$ with the norm

$$\|f\|_{L_p([s,t];\mathbb{R}^n)} = \left(\int_s^t \|f(\tau)\|^p d\tau\right)^{\frac{1}{p}}.$$

Denote by $L_p^{\mathrm{loc}}\left([0,\infty);\mathbb{R}^n\right)$ the space of all $f : [0,\infty) \to \mathbb{R}^n$ such that $f\big|_{[s,t]} \in L_p\big([s,t];\mathbb{R}^n\big)$ for all $t > s \geqslant 0$.

The truncated operators π_k at $k \in [0,\infty)$ and $[\,\cdot\,]_k$ on $L_p^{\mathrm{loc}}\big([0,\infty);\mathbb{R}^n\big)$ are defined by

$$\pi_k(u)(t) := \begin{cases} u(t), & \text{if } t \in [0,k] \\ 0, & \text{if } t \in (k,\infty), \end{cases} \quad \text{and} \quad [u(t)]_k = \begin{cases} 0 & \text{if } t \in [0,k) \\ u(t) & \text{if } t \in [k,\infty), \end{cases}$$

for $u \in L_p^{\mathrm{loc}}\big([0,\infty);\mathbb{R}^n\big)$.

Every element $f \in L_p\big([s,t];\mathbb{R}^n\big)$ can be considered as an element $\overline{f} \in L_p^{\mathrm{loc}}\left([0,\infty);\mathbb{R}^n\right)$ by putting $\overline{f} = \pi_t[f]_s$. Conversely, if $f \in L_p^{\mathrm{loc}}\left([0,\infty);\mathbb{R}^n\right)$, we can restrict the definition domain of f to obtain an element $\overline{f} \in L_p\big([s,t];\mathbb{R}^n\big)$. We will identify in the following f and $\overline{f}$ if there is no confusion.

Let

$$\mathscr{L} := \mathscr{L}\Big(L_p^{\mathrm{loc}}\left([0,\infty);\mathbb{R}^n\right), L_p^{\mathrm{loc}}\left([0,\infty);\mathbb{R}^n\right)\Big),$$

be the space of the linear operators Σ from $L_p^{\text{loc}}\big([0,\infty);\mathbb{R}^n\big)$ to $L_p^{\text{loc}}\big([0,\infty);\mathbb{R}^n\big)$ such that $\pi_t\Sigma\pi_t$ maps continuously from $L_p([0,t];\mathbb{R}^n)$ to $L_p([0,t];\mathbb{R}^n)$. Similarly as above, every continuous operator $\Sigma \in \mathscr{L}\big(L_p([s,t];\mathbb{R}^n), L_p([s,t];\mathbb{R}^n)\big)$ can be considered as

$$\overline{\Sigma} \in \mathscr{L}\big(L_p^{\text{loc}}([0,\infty);\mathbb{R}^n), L_p^{\text{loc}}([0,\infty);\mathbb{R}^n)\big),$$

by setting $\overline{\Sigma}(\cdot) = \pi_t\Sigma\pi_t[\cdot]_s$.

An operator $\Sigma \in \mathscr{L}$ is called to be *causal* if

$$\pi_t\Sigma\pi_t = \pi_t\Sigma, \quad \text{for every } t \geqslant 0. \tag{2.1}$$

To simplify notations, in the following, we write $L_p[s,t]; C[s,t]$ for $L_p([s,t];\mathbb{R}^n)$, $C([s,t];\mathbb{R}^n)$ respectively.

Lemma 2.1 (Gronwall-Bellman lemma, see [29])**.** *Let $f(t)$ be a non negative continuous function defined on $[s,\infty)$ and $k>0$. Assume that $f(t)$ satisfies the inequality*

$$f(t) \leqslant f_0 + k\int_s^t f(s)ds, \ \ \textit{for all} \ \ t \geqslant s.$$

Then, the relation $f(t) \leqslant f_0e^{k(t-s)}$ holds for all $t \geqslant s$.

Lemma 2.2 (Hardy's inequality [29])**.** *Let $1 \leqslant p < \infty$, there is a finite C for which*

$$\Big[\int_0^\infty \Big|U(x)\int_0^x f(t)dt\Big|^p dx\Big]^{\frac{1}{p}} \leqslant C\Big[\int_0^\infty \big|V(x)f(x)\big|^p dx\Big]^{\frac{1}{p}} \tag{2.2}$$

is true for real function f if and only if

$$B = \sup_{r>0}\Big[\int_r^\infty |U(x)|^p dx\Big]^{\frac{1}{p}}\Big[\int_0^r |V(x)|^{-q}dx\Big]^{\frac{1}{q}} < \infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $U(x),V(x)$ are weight functions. Furthermore, if C is the least constant for which (2.2) *holds, then $B \leqslant C \leqslant p^{\frac{1}{p}}q^{\frac{1}{q}}B$, for $1<p<\infty$ and $B=C$ if $p=1$ or ∞.*

Remark 2.3. *If we use $U(x)=V(x)=e^{-\alpha x}$ then*

$$B = \sup_{0\leqslant r\leqslant t}\Big[\int_r^t (e^{\alpha s})^p ds\Big]^{\frac{1}{p}}\Big[\int_s^r (e^{-\alpha s})^{-q}ds\Big]^{\frac{1}{q}} = \frac{1}{\alpha p^{\frac{1}{p}}q^{\frac{1}{q}}} \ \ \textit{and} \ \ \frac{1}{\alpha p^{\frac{1}{p}}q^{\frac{1}{q}}} \leqslant C \leqslant \frac{1}{\alpha}.$$

Lemma 2.4. *Let $\mathbb{U}: X \to Y$, $\mathbb{V}: Y \to X$ be the bounded linear operators in Banach spaces X, Y. Then the operator $I-\mathbb{U}\mathbb{V}$ is invertible if and only if $I-\mathbb{V}\mathbb{U}$ is invertible. Furthermore,*

$$(I-\mathbb{V}\mathbb{U})^{-1} = I + \mathbb{V}(I-\mathbb{U}\mathbb{V})^{-1}\mathbb{U}.$$

Proof. See [23]. □

We introduce some basic properties of linear algebra which are used later.

Lemma 2.5. *Let $\overline{A}$ and $\overline{B}$ be given $n\times n$ matrices, and $\overline{Q}$ be a projector onto* $\ker\overline{A}$, *i.e.,* $\overline{Q}^2 = \overline{Q}, \operatorname{im}\overline{Q} = \ker\overline{A}$. *Denote $S=\{x : \overline{B}x \in \operatorname{im}\overline{A}\}$. The following assertions are equivalent*

a) $S \cap \ker\overline{A} = \{0\}$.
b) *the matrix $\overline{G} = \overline{A} - \overline{B}\,\overline{Q}$ is nonsingular.*
c) $\mathbb{R}^n = S \oplus \ker\overline{A}$.

Proof. The proof of this lemma can be found in [28], Appendix 1, Lemma A1, p.329. □

Lemma 2.6. *$\overline{A}, \overline{B}$, $\overline{Q}, \overline{G}$ mentioned in Lemma 2.5 and suppose that the matrix $\overline{G}$ is nonsingular. Then, there hold the following relations:*

a) $\overline{P} = \overline{G}^{-1}\,\overline{A}$ *where* $\overline{P} = I - \overline{Q}$.
b) $-\overline{G}^{-1}\,\overline{B}\,\overline{Q} = \overline{Q}$.
c) $\widehat{\overline{Q}} := -\overline{Q}\,\overline{G}^{-1}\overline{B}$, *called canonical projector, is the projector onto* ker $\overline{A}$ *along S.*
d) $\overline{Q}\,\overline{G}^{-1}$ *does not depend on the choice of* $\overline{Q}$.

Proof. The results in this lemma are proved in [28], p.319. □

3. Solvability of differential algebraic equations with causal operators

3.1. Solvability of linear differential-algebraic equations.

Let $A(\cdot), B(\cdot)$ be two continuous functions valued in $\mathbb{R}^{n\times n}$ and $f \in L_p^{loc}\big([0,\infty);\mathbb{R}^n\big)$ Let $\Sigma \in \mathscr{L}$ be a causal operator. For ever $s \geqslant 0$, consider the differential-algebraic equation (DAE for short)

$$A(t)x'(t) = B(t)x(t) + \Sigma\big([x(\cdot)]_s\big)(t) + f(t),\ t \geqslant s. \tag{3.1}$$

Suppose that ker $A(\cdot)$ is smooth in the sense there exists a continuously differentiable projector $Q(t)$ onto ker $A(t)$, i.e., Q is continuously differentiable and $Q^2 = Q$, im $Q(t) = \ker A(t)$ for all $t \geqslant 0$. Set $P = I - Q$, then $P(t)$ is a projector along ker $A(t)$. By these notations, the equation (3.1) can be rewritten into the form

$$A(t)\big(Px\big)'(t) = \overline{B}(t)x(t) + \big(\Sigma[x(\cdot)]_s\big)(t) + f(t),\ t \geqslant s, \tag{3.2}$$

where $\overline{B} := B + AP'$. The homogeneous equation corresponding to (3.2) is

$$A(t)(Py)'(t) = \overline{B}(t)y(t) + \big(\Sigma[y(\cdot)]_s\big)(t),\ t \geqslant s. \tag{3.3}$$

From the equation (3.2) it is seen that the solution $x(\cdot)$, if it exists, is not necessarily differentiable but only $Px(\cdot)$. Moreover, since f is only measurable, we have to understand the solution x of the initial problem (3.2) in Caratheodory sense, i.e., $Px(\cdot)$ is continuous and it is differentiable almost every where with respect to Lebesgue measure on $[s,\infty)$. Therefore, we look for solutions $x(\cdot)$ of the equation (3.2) from elements of $\mathscr{H}^1(s)$ defined as

$$\mathscr{H}^1\big(s\big) = \left\{ \begin{array}{l} x(\cdot) \in L_p^{loc}\big([s,\infty),\mathbb{R}^n\big) : P(\cdot)x(\cdot) \text{ is continuous} \\ \text{and almost everywhere differentiable on } [s,\infty) \end{array} \right\}.$$

Next, we deal with the way to find these solutions based on the so-called index-1 concept of the equation (3.1). Consider the linear operators $G \in L_\infty^{loc}\big([0,\infty);\mathbb{R}^{n\times n}\big)$, $\widehat{G} \in \mathscr{L}\big(L^{\rm loc}([0,\infty);\mathbb{R}^n),\ L^{\rm loc}([0,\infty);\mathbb{R}^n)\big)$ defined by

$$G := A - \overline{B}Q \quad\text{and}\quad \widehat{G} := A - \big(\overline{B} + \Sigma\big)Q = \big(I - \Sigma Q G^{-1}\big)G.$$

Definition 3.1. *The DAEs* (3.1) *is said to be index-1 if $G(t)$ and $\widehat{G}$ are invertible for all $t \geqslant 0$.*

Remark 3.2. *Since $G(t)$ is invertible for all $t \geqslant 0$, the invertibility of $\widehat{G}$ is equivalent to the invertibility of $I - \Sigma Q G^{-1}$. Due to Lemma 2.4, it implies the invertibility of $H := I - QG^{-1}\Sigma$.*

Note that by Lemma 2.6, the index-1 property does not depend on the choice of projectors P, see [17].

In assuming that the equation (3.1) is index-1 we split the solution of (3.2) into $x(t) = P(t)x(t) + Q(t)x(t),\ t \geqslant s$ and try to solve $u(t) = P(t)x(t), v(t) = Q(t)x(t)$. In order to do that, let $t \geqslant s$, taking into account the equalities

$$G^{-1}A = P,\ G^{-1}\overline{B} = -Q + G^{-1}\overline{B}P,$$

we multiply both sides of (3.2) with $PG^{-1},\ QG^{-1}$ to obtain

$$u' = \big(P' + PG^{-1}\overline{B}\big)u + PG^{-1}\Sigma[x]_s + PG^{-1}f, \tag{3.4}$$

$$v = QG^{-1}\overline{B}u + QG^{-1}\Sigma[u+v]_s + QG^{-1}f. \tag{3.5}$$

Since $H = I - QG^{-1}\Sigma$ is invertible, from the (3.5) we get

$$v = H^{-1}QG^{-1}\big(\overline{B} + \Sigma\big)[u]_s + H^{-1}QG^{-1}[f]_s. \tag{3.6}$$

Combining with $H^{-1}QG^{-1}\Sigma = H^{-1} - H^{-1}(I - QG^{-1}\Sigma) = H^{-1} - I$ yields

$$v = -[u]_s + H^{-1}\widehat{P}[u]_s + H^{-1}QG^{-1}[f]_s,$$

where $\widehat{Q} = I - \widehat{P} = -QG^{-1}\overline{B}$ is the canonical projection onto $\ker A$ (see Lemma 2.6). Hence,

$$x(t) = \big(H^{-1}\widehat{P}[u]_s\big)(t) + \big(H^{-1}QG^{-1}[f]_s\big)(t). \tag{3.7}$$

Substituting this relation into (3.4) gets the inherent equation of (3.3)

$$u' = \big(P' + PG^{-1}\overline{B}\big)u + PG^{-1}\Sigma H^{-1}\big(\widehat{P}[u]_s + QG^{-1}[f]_s\big) + PG^{-1}[f]_s. \tag{3.8}$$

Lemma 3.3. *Let S be a function defined on $[s,T] \times L_p\big([s,T];\mathbb{R}^n\big)$, valued in $L_p\big([s,T];\mathbb{R}^n\big)$, such that $S(t,u)$ depends only the values of u on $[s,t]$ for every $u \in L_p\big([s,T];\mathbb{R}^n\big)$ and S satisfies the Lipschitz condition, i.e., there is a constant $k > 0$ such that*

$$\|S(t,u_1) - S(t,u_2)\|_{L_p[s,t]} \leqslant k\,\|u_1 - u_2\|_{L_p[s,t]}\,,$$

for all $s \leqslant t \leqslant T,\ u_1, u_2 \in L_p\big([s,T];\mathbb{R}^n\big)$. Then the equation

$$z' = (P' + PG^{-1}\overline{B})z + PG^{-1}S(t,z), \tag{3.9}$$

with the initial condition $z(s) = P(s)x_0$ has a unique solution $z(\cdot)$ satisfying:

i) *$z(t) \in \operatorname{im} P(t)$, for all $t \geqslant s$.*

ii) *there exists a constant c such that if $z_1(\cdot)$ and $z_2(\cdot)$ are two solutions of (3.9) then*

$$\|z_1(t) - z_2(t)\| \leqslant c\,\|z_1(s) - z_2(s)\|\,,\ \textit{for all}\ \ t \in [s,T]. \tag{3.10}$$

Proof. The existence of a unique solution with the initial condition $z(s) = Px_0$ can be referred to [7, Theorem 3.16].

Multiplying both sides of (3.9) with Q yields $Qz' = QP'z$, which implies that $(Qz)' = Q'Qz$. Thus, if $Q(s)z(s) = 0$ then $Q(t)z(t) = 0$ for all $t \geqslant s$. This means that $z(t) = P(t)z(t)$ or $z(t) \in \operatorname{im} P(t)$.

Finally, let z_1 and z_2 be two solutions of (3.9). Then,

$$\|z_1(t)-z_2(t)\| \leqslant \|z_1(s)-z_2(s)\| + \int_s^t \big\|(P'+PG^{-1}\overline{B})(\tau)(z_1(\tau)-z_2(\tau)\big\|\,d\tau$$
$$+\int_s^t \big\|PG^{-1}(\tau)\big(S(\tau,z_1)(\tau)-S(\tau,z_2)(\tau)\big)\big\|\,d\tau.$$

Therefore,

$$\|z_1(t)-z_2(t)\|^p \leqslant 3^{p-1}\Big[\|z_1(s)-z_2(s)\|^p + \Big(\int_s^t \big\|(P'+PG^{-1}\overline{B})(z_1(\tau)-z_2(\tau))\big\|\,ds\Big)^p$$
$$+\Big(\int_s^t \big\|PG^{-1}(\tau)(S(\tau,z_1)(\tau)-S(\tau,z_2)(\tau)\big\|\,d\tau\Big)^p\Big]$$
$$\leqslant 3^{p-1}\|z_1(s)-z_2(s)\|^p + K\int_s^t \|z_1(\tau)-z_2(\tau)\|^p\,d\tau,$$

where $K = 3^{p-1}T^{\frac{p}{q}}\big(\sup_{\tau\in[0,T]}\big\|(P'+PG^{-1}\overline{B})(\tau)\big\| + k\sup_{\tau\in[0,T]}\big\|(PG^{-1})(\tau)\big\|\big)$. By using Grownall-Belman inequality we get

$$\|z_1(t)-z_2(t)\| \leqslant \|z_1(s)-z_2(s)\|\,3e^{\frac{Kt}{p}}, \quad \text{for all } \ t\in[s,T].$$

Putting $c = 3e^{\frac{KT}{p}}$ gets (3.10). The proof of Lemma is complete. □

Thus, by virtue of Lemma 3.3, we can find the solution u with the initial condition $u(s) = P(s)x(s)$ from the equation (3.8). Next, we use (3.7) to get the solution $x(\cdot)$ of the equation (3.2).

3.2. **Variation of constants formula.**

We now try to give the variation of constants formula for the solution $x(\cdot)$ of (3.2). The homogeneous equation (3.3) can be rewritten as

$$\begin{cases}\overline{u}'(t) = \big(P'+PG^{-1}\overline{B}\big)\overline{u}(t) + PG^{-1}\Sigma H^{-1}\big(\widehat{P}[\overline{u}]_s(t)\big),\\ y(t) \ = \big(H^{-1}\widehat{P}[\overline{u}]_s\big)(t),\ t\geqslant s,\end{cases} \tag{3.11}$$

where $\overline{u} = Py$. Define by $\Phi(t,s), t\geqslant s\geqslant 0$ the Cauchy matrix generated by (3.3), i.e., it is the solution of the matrix equation

$$\begin{aligned}&A(t)\Phi'(t,s) = B(t)\Phi(t,s) + \Sigma\big([\Phi(\cdot,s)]_s\big)(t),\\ &P(s)\big(\Phi(s,s)-I\big) = 0,\ t\geqslant s\geqslant 0.\end{aligned} \tag{3.12}$$

Due to (3.11) and the invariant property mentioned in Lemma 3.3 we have

$$\Phi(t,s) = \big(H^{-1}\widehat{P}\,[\Phi_0(\cdot,s)]_s P(s)\big)(t),\ t\geqslant s\geqslant 0, \tag{3.13}$$

where, $\Phi_0(\cdot,\cdot)$ is the solution of the matrix equation

$$\begin{aligned}&\Phi_0'(t,s) = \big(P'+PG^{-1}\overline{B}\big)(t)[\Phi_0(t,s)]_s + PG^{-1}\Sigma H^{-1}\big(\widehat{P}[\Phi_0(\cdot,s)]_s\big)(t),\\ &\Phi_0(s,s) = I,\ t\geqslant s\geqslant 0.\end{aligned} \tag{3.14}$$

The variation of constants formula for the solution of (3.2) can be formulated as

Theorem 3.4. *The unique solution $x(\cdot)$ of the equation* (3.2) *with the initial condition $P(s)(x(s)-x_0)=0$ can be expressed as*

$$\begin{aligned}x(t)=\Phi(t,s)P(s)x_0+\int_s^t \Phi(t,\tau)PG^{-1}\left(\mathbb{T}[f]_s\right)(\tau)d\tau\\ +(H^{-1}QG^{-1}[f]_s)(t),\end{aligned} \tag{3.15}$$

for all $t\geqslant s$, where $\mathbb{T}:=I+\Sigma H^{-1}QG^{-1}$.

Proof. By directly differentiating both sides we see that the variation constants formula for the solution $u(\cdot)$ of (3.8) with the initial condition $u(s)=P(s)x_0$ is

$$\begin{aligned}u(t)&=\Phi_0(t,s)P(s)x_0+\int_s^t \Phi_0(t,\tau)PG^{-1}\big(I+\Sigma H^{-1}QG^{-1}\big)[f]_s(\tau)d\tau\\ &=\Phi_0(t,s)P(s)x_0+\int_s^t \Phi_0(t,\tau)PG^{-1}\big(\mathbb{T}[f]_s\big)(\tau)d\tau.\end{aligned} \tag{3.16}$$

By acting $H^{-1}\widehat{P}$ to both sides of (3.16) and paying attention to the expression (3.7) it is seen that the unique solution $x(\cdot)$ of (3.2) can be given by the variation of constants formula (3.15)

$$x(t)=\Phi(t,s)P(s)x_0+\int_s^t\Phi(t,\tau)PG^{-1}(\mathbb{T}[f]_s)(\tau)d\tau+(H^{-1}QG^{-1}[f]_s)(t).$$

The proof is complete. □

Remark 3.5. *From the expression* (3.15) *we see that although $P(\cdot)x(\cdot)$ is continuous, the solution $x(\cdot)$ of* (3.2) *may not be continuous.*

4. Preservation of stability under small nonlinear perturbations

Let $y(t,s,y_0), t\geqslant s$ denote a unique solution of the equation (3.3) with initial condition $P(s)(y(s)-y_0)=0$.

Definition 4.1.

i) *The differential algebraic equation* (3.3) *is uniformly stable if there exists a positive number $M_0>0$ such that for every $s\geqslant 0$ we have*

$$\|y(t,s,y_0)\|\leqslant M_0\,\|P(s)y_0\|\,,\quad t\geqslant s,\ y_0\in\mathbb{R}^n. \tag{4.1}$$

ii) *Let $\alpha>0$. The differential algebraic equation* (3.3) *is said to be α-exponentially stable if there exists a positive number M for every $s\geqslant 0$ we have*

$$\|y(t,s,y_0)\|\leqslant M\,\|P(s)y_0\|\,e^{-\alpha(t-s)},\ \ t\geqslant s,\ y_0\in\mathbb{R}^n. \tag{4.2}$$

Assumption 1. There exists a differentiable projector $Q(\cdot)$ onto ker $A(\cdot)$ such that $P=I-Q$ is bounded on $[0,\infty)$ by constant K_0.

The following characterizations of uniform stability and exponential stability are straightforward generalizations of the well-known results for ordinary differential equations, see the proof of (3.5), page 112 and (4.13), page 124 in [10]. Therefore, we omit the proof of the following theorem in details.

Theorem 4.2. *Let the assumption 1 hold. Then,*

i) *The differential algebraic equation* (3.3) *is uniformly stable if and only if there exists a positive number $M_0>0$ such that*

$$\|\Phi(t,s)\|\leqslant M_0,\ \ t\geqslant s\geqslant 0. \tag{4.3}$$

ii) *The differential algebraic equation* (3.3) *is α-exponentially stable if and only if there exists a positive number M such that*

$$\|\Phi(t,s)\| \leqslant Me^{-\alpha(t-s)},\ t \geqslant s \geqslant 0. \tag{4.4}$$

By Theorem (4.2) we can estimate $\|\Phi_0(t,s)P(s)\|$. Indeed, from (3.13), it is seen that

$$P(t)\Phi(t,s) = P(t)\big(H^{-1}\widehat{P}\,\Phi_0(\cdot,s)P(s)\big)(t) = \Phi_0(t,s)P(s),\ t \geqslant s. \tag{4.5}$$

Therefore, if the relation (4.4) and the Assumption 1 hold then

$$\|\Phi_0(t,s)P(s)\| = \|P(t)\Phi(t,s)\| \leqslant K_0 Me^{-\alpha(t-s)},\ t \geqslant s. \tag{4.6}$$

We are now in position to consider the stability of (3.3) when f is small perturbations.

Let F be a causal nonlinear operator from $L_p^{\text{loc}}\left([0,\infty);\mathbb{R}^n\right)$ to $L_p^{\text{loc}}\left([0,\infty);\mathbb{R}^n\right)$, i.e., the condition (2.1) for F is satisfied

$$\pi_t\, F\, \pi_t = \pi_t\, F \text{ for all } t \geqslant 0.$$

Suppose further that $F(\theta) = \theta$, where $\theta(t) = 0$, for all $t \geqslant 0$.

For every $s \geqslant 0$, consider the semi linear differential equation with causal operators

$$A(t)x'(t) = B(t)x(t) + \Sigma[x(\cdot)]_s(t) + F\big([x(\cdot)]_s\big)(t),\ t \geqslant s. \tag{4.7}$$

Since $F(\theta) = \theta$, the equation (4.7) has the trivial solution $x = \theta$.

A causal nonlinear operator $\Gamma : [0,\infty) \times L_p^{\text{loc}}\big([0,\infty);\mathbb{R}^n\big) \to L_p^{\text{loc}}\big([0,\infty);\mathbb{R}^n\big)$ is called locally Lipschitz with the function m if

- $\pi_t\Gamma(t,u) = \pi_t\Gamma(t,\pi_t u)$ for every $u \in L_p^{\text{loc}}\left([0,\infty);\mathbb{R}^n\right)$.
- there exists a positive continuous function $m.$ such that

$$\|\Gamma(t,x) - \Gamma(t,y)\|_{L_p[0,t]} \leqslant m_t\|x-y\|_{L_p[0,t]},$$

for all $t \in [0,\infty)$ and $x, y \in L_p^{\text{loc}}\big([0,\infty);\mathbb{R}^n\big)$

In case m is a constant function, we say simply that Γ is locally m-Lipschitz continuous.

Theorem 4.3. *Suppose that the function $PG^{-1}\mathbb{T}F\big(x\big)$ is locally Lipschitz continuous in x with the function $k.$ and $H^{-1}QG^{-1}F(x)$ is locally γ-Lipschitz continuous in x with $\gamma < 1$.*

Then the equation (4.7) *is solvable on $[s,\infty)$. Moreover, for any $T > 0$, there is a constant M_T such that*

$$\|x(\cdot)\|_{L_p[s,t]} \leqslant M_T\,\|P(s)x_0\|\,,\ \textit{for all}\ \ s \leqslant t \leqslant T, \tag{4.8}$$

where $x(\cdot)$ is the solution of (4.7) *with the initial condition $P(s)(x(s) - x_0) = 0$.*

Proof. To simplify notations, we take $s = 0$. Due to (3.15), x is the solution of the equation(4.7) if and oly if

$$\begin{aligned} x(t) = \ & \Phi(t,0)P(0)x_0 + \int_0^t \Phi(t,\rho)PG^{-1}\mathbb{T}F(x(\cdot))(\rho)d\rho \\ & + H^{-1}QG^{-1}F\big(x(\cdot)\big)(t). \end{aligned} \tag{4.9}$$

We show that the integral equation (4.9) has a unique solution on every interval $[0,T]$ for fixed $T > 0$. Denote

$$\mathbb{S} = I - H^{-1}QG^{-1}F$$

to be a function from $L_p^{loc}[0,\infty) \to L_p^{loc}[0,\infty)$. Since $H^{-1}QG^{-1}F$ is γ–Lipschitz and $F(\theta)=\theta$, it follows that $\left\|H^{-1}QG^{-1}F\right\|_{L_p[0,t]} < \gamma < 1$ for all $t \geqslant 0$. Hence, $\mathbb{S}$ is invertible and

$$\mathbb{S}^{-1} = \sum_{k=0}^{\infty}(H^{-1}QG^{-1}F)^k.$$

Moreover, for any $z_1, z_2 \in L_p^{loc}[0,\infty)$ we have

$$\left\|\mathbb{S}^{-1}z_1 - \mathbb{S}^{-1}z_2\right\|_{L_p[0,t]} \leqslant \sum_{k=0}^{\infty}\gamma^k \|z_1 - z_2\|_{L_p[0,t]} = \frac{1}{1-\gamma}\|z_1 - z_2\|_{L_p[0,t]}, \tag{4.10}$$

for all $t \geqslant 0$. This means that $\mathbb{S}^{-1}$ is a Lipschitz operator.The integral equation (4.9) can be rewritten under an equivalent form

$$x(t) = \mathbb{S}^{-1}\Big(\Phi(\cdot,0)P(0)x_0 + \int_0^{\cdot} \Phi(\cdot,\rho)PG^{-1}\mathbb{T}F(x(\cdot))(\rho)d\rho\Big)(t),. \tag{4.11}$$

Consider the operator $\mathscr{U}(\cdot) : L_p[0,T] \to L_p[0,T]$ given by

$$\big(\mathscr{U}(x)\big)(t) = \mathbb{S}^{-1}\Big(\Phi(\cdot,0)P(0)x_0 + \int_0^{\cdot} \Phi(\cdot,\rho)PG^{-1}\mathbb{T}F(x(\cdot))(\rho)d\rho\Big)(t),$$

for $t \in [0,T]$ and $x \in L_p[0,T]$. Let $d \in \mathbb{N}$ and $\delta = \frac{T}{d}$. We prove that the restriction of $\mathscr{U}$ on $[0,\delta]$ is a contractive mapping when d is large enough. Indeed, put $N_T = \sup_{0\leqslant s\leqslant t\leqslant T}\|\Phi(t,s)\|$. For any $y,z \in L_p[0,\delta]$, using (4.10) has

$$\begin{aligned}
\|\mathscr{U}(x) - \mathscr{U}(y)\|_{L_p[0,\delta]} &= \Big\|\mathbb{S}^{-1}\Big(\Phi(\cdot,0)P(0)x_0 + \int_0^{\cdot} \Phi(\cdot,\rho)PG^{-1}\mathbb{T}F(y)(\rho)d\rho\Big) \\
&\quad - \mathbb{S}^{-1}\Big(\Phi(\cdot,0)P(0)x_0 + \int_0^{\cdot} \Phi(\cdot,\rho)PG^{-1}\mathbb{T}F(z)(\rho)d\rho\Big)\Big\|_{L_p[0,\delta]} \\
&\leqslant \frac{1}{1-\gamma}\Big(\int_0^{\delta}\Big\|\int_0^t \Phi(t,\rho)PG^{-1}\mathbb{T}\big(F(y)-F(z)\big)(\rho)d\rho\Big\|^p dt\Big)^{1/p} \\
&\leqslant \frac{N_T}{1-\gamma}\Big(\int_0^{\delta}\Big(\int_0^t \big\|PG^{-1}\mathbb{T}\big(F(y)-F(z)\big)(\rho)\big\|\,d\rho\Big)^p dt\Big)^{1/p} \\
&\overset{\text{Hölder}}{\leqslant} \frac{N_T\delta^{\frac{1}{q}}}{1-\gamma}\Big(\int_0^{\delta}\Big(\int_0^t \big\|PG^{-1}\mathbb{T}\big(F(y)-F(z)\big)(\rho)\big\|^p d\rho\Big)dt\Big)^{1/p} \\
&\leqslant \frac{N_T\delta^{\frac{1}{q}}}{1-\gamma}\Big(\int_0^{\delta}\Big(\int_0^{\delta} \big\|PG^{-1}\mathbb{T}\big(F(y)-F(z)\big)(\rho)\big\|^p d\rho\Big)dt\Big)^{1/p} \\
&\leqslant \frac{N_T\delta^{\frac{1}{q}+\frac{1}{p}}}{1-\gamma}\Big(\int_0^{\delta} \big\|PG^{-1}\mathbb{T}\big(F(y)-F(z)\big)(\rho)\big\|^p d\rho\Big)^{1/p} \\
&= \frac{N_T\delta}{1-\gamma}\left\|PG^{-1}\mathbb{T}\big(F(y)-F(z)\big)\right\|_{L_p[0,\delta]} = \frac{N_Tk_T}{1-\gamma}\delta\,\|y-z\|_{L_p[0,\delta]}.
\end{aligned}$$

By choosing d such that $\frac{N_Tk_T}{1-\gamma}\delta < 1$ it is seen that $\mathscr{U}$ is contractive. As a consequence, there exists an $x \in L_p[0,\delta]$ with $P(0)\big(x(0)-x_0\big) = 0$ such that

$$x(t) = \mathscr{U}(x)(t), \quad \text{for all } t \in [0,\delta],$$

i.e., x is a solution of (4.9) on $[0,\delta]$. We now extend the solution x to the interval $[0,2\delta]$. Every function $\xi \in L_p[\delta,2\delta]$ can be considered as a function $\overline{\xi} \in L_p[0,2\delta]$ where $\overline{\xi}(t) = x(t)$ for $t \in [0,\delta]$ and $\overline{\xi}(t) = \xi(t)$ for $t \in (\delta,2\delta]$. Thus we can define

$\mathscr{U}$ on $L_p[\delta, 2\delta]$ by putting $\mathscr{U}(\xi) = \mathscr{U}(\bar{\xi})$. A similar way as above says that $\mathscr{U}$ is contractive, which implies that the solution x can be extended to the interval $[\delta, 2\delta]$. Continuing this way, we can solve $x(\cdot)$ on $[0, T]$.

We prove (4.8). It is seen that

$$\begin{aligned}\|x\|_{L_p[s,t]} &= \Big\|\mathbb{S}^{-1}\Big(\Phi(\cdot,0)P(s)x_0 + \int_s^{\cdot}\Phi(\cdot,\rho)PG^{-1}\mathbb{T}F([x(\cdot)]_s)(\rho)d\rho\Big)\Big\|_{L_p[s,t]}\\ &\leqslant \frac{1}{1-\gamma}\left(\|\Phi(\cdot,s)P(s)x_0\|_{L_p[s,t]} + \Big\|\int_s^{\cdot}\Phi(\cdot,\rho)PG^{-1}\mathbb{T}F([x]_s)(\rho)d\rho\Big\|_{L_p[s,t]}\right)\\ &\leqslant \frac{N_T}{1-\gamma}\left[\Big(\int_s^t \|P(s)x_0\|^p\, ds\Big)^{1/p} + \Big(\int_s^t \Big\|\int_s^{\tau} PG^{-1}\mathbb{T}F([x]_s)d\rho\Big\|^p d\tau\Big)^{1/p}\right]\\ &\leqslant \frac{N_T}{1-\gamma}\left[T^{\frac{1}{p}}\|P(s)x_0\| + \Big(\int_s^t \tau^{\frac{p}{q}}\|PG^{-1}\mathbb{T}F([x]_s)\|^p_{L_p[s,\tau]}\, d\tau\Big)^{1/p}\right].\end{aligned}$$

Thus,

$$\|x\|_{L_p[s,t]} \leqslant \frac{N_T T^{\frac{1}{p}}}{1-\gamma}\|P(s)x_0\| + \frac{N_T T^{\frac{1}{q}}}{1-\gamma}\Big(\int_s^t k_0^p\|x(\cdot)\|^p_{L_p[s,\tau]}d\tau\Big)^{1/p}. \tag{4.12}$$

By using the inequality in [9, Theorem 37] we have

$$\|x(\cdot)\|_{L_p[s,t]} \leqslant M_T\|P(s)x_0\|,\ s \leqslant t \leqslant T,$$

for a certain $M_T > 0$. The proof is complete. □

In the following, let Assumptions 1 hold.

Definition 4.4 (See [23]). *The trivial solution θ of (4.7) is said to be uniformly L_p-stable if there exist constants $M_1, M_2 > 0$ such that*

$$\begin{aligned}&\|P(t)x(t;s,x_0)\|_{\mathbb{R}^n} \leqslant M_1\|P(s)x_0\|_{\mathbb{R}^n},\ t \geqslant s,\\ &\|x(\cdot;s,x_0)\|_{L_p[s,\infty)} \leqslant M_2\|P(s)x_0\|_{\mathbb{R}^n}.\end{aligned} \tag{4.13}$$

Theorem 4.5. *Assume that the DAE (3.3) is index-1, exponentially stable and $H^{-1}QG^{-1}F(x)$ is locally γ-Lipschitz continuous in x with $\gamma < 1$. Further,*

1. *There exists a continuous function $m : [0, \infty) \to \mathbb{R}_+$ such that*

$$\big\|PG^{-1}\big[\mathbb{T}F(x)\big]_t - PG^{-1}\big[\mathbb{T}F(y)\big]_t\big\|_{L_p[0,T]} \leqslant m_t\|x-y\|_{L_p[0,T]}, \tag{4.14}$$

for $0 < t \leqslant T < \infty$ and $x, y \in L_p^{loc}\big([0,\infty);\mathbb{R}^n\big)$.

2. *$\limsup_{t\to\infty} m_t < \frac{1-\gamma}{MC}$, with $C = C(\alpha)$ is defined in Remark 2.3 corresponding to the function $V(t) = U(t) = e^{-\alpha t}$ and α, M are defined in Theorem 4.2.*

Then, the solution θ of the perturbed equation (4.7) is uniformly L_p-stable.

Proof. Let $s \geqslant 0$. By (4.14) and the locally γ-Lipschitz continuity of $H^{-1}QG^{-1}F(x)$ in x with $\gamma < 1$, it is seen that for any $x_0 \in \mathbb{R}^n$, there exists the solution $x(\cdot) = x(\cdot, s)$ defined on $[s, \infty)$ with the initial condition $P(t_0)(x(s) - x_0) = 0$. The second assumption of Theorem implies the existence of $\xi \geqslant s$ such that

$$m_t < \frac{1-\gamma}{MC}, \quad \text{for all } t \geqslant \xi. \tag{4.15}$$

By the variation of constants formula (4.9), when $t \geqslant t_0$ one has

$$x(t) = \Gamma_0(t) + \int_s^t \Phi(t,\rho)PG^{-1}\big[\mathbb{T}F\big([x]_s\big)\big]_\xi(\rho)d\rho + H^{-1}QG^{-1}F([x]_s)(t), \tag{4.16}$$

where

$$\Gamma_0(t) = \Phi(t,s)P(s)x_0 + \int_s^t \Phi(t,\rho)PG^{-1}\pi_\xi \mathbb{T}F([x]_s)(\rho)d\rho.$$

The assumption of exponential stability of (3.3) says $\|\Phi(t,\rho)\| \leqslant Me^{-\alpha(t-\rho)}$, for all $0 \leqslant \rho \leqslant t$. Therefore,

$$\|\Gamma_0(t)\| \leqslant Me^{-\alpha(t-s)}\|P(s)x_0\| + M\int\limits_s^t e^{-\alpha(t-\rho)}\left\|PG^{-1}\pi_\xi \mathbb{T}F([x]_s)(\rho)\right\|d\rho.$$

Hence,

$$\|\Gamma_0(\cdot)\|_{L_p[s,\infty)} \leqslant \frac{M\|P(s)x_0\|}{(p\alpha)^{\frac{1}{p}}} + M\left[\int\limits_s^\infty\left(e^{-\alpha\tau}\int\limits_s^\tau e^{\alpha\rho}\left\|PG^{-1}\pi_\xi\mathbb{T}F([x]_s)(\rho)\right\|d\rho\right)^p d\tau\right]^{\frac{1}{p}}.$$

By Hardy's inequality in Lemma 2.2 with the weight functions $U(\tau) = V(\tau) = e^{-\alpha\tau}$ we have

$$\begin{aligned}\|\Gamma_0(\cdot)\|_{L_p[s,\infty)} &\leqslant \frac{M\|P(s)x_0\|}{(p\alpha)^{\frac{1}{p}}} + MC\left[\int\limits_s^\infty\left\|PG^{-1}\pi_\xi\mathbb{T}F([x]_s)(\rho)\right\|^p d\rho\right]^{\frac{1}{p}}\\ &\leqslant \frac{M\|P(s)x_0\|}{(p\alpha)^{\frac{1}{p}}} + m_0MC\left\|x(\cdot)\right\|_{L_p[s,\xi]} \leqslant M_3\left\|P(s)x_0\right\|,\end{aligned} \tag{4.17}$$

where $M_3 = \frac{M}{(p\alpha)^{\frac{1}{p}}} + m_0M_\xi MC$ and M_ξ in (4.8).

On the other hand, from (4.16) it follows that

$$x(t) = \mathbb{S}^{-1}\Big(\Gamma_0(\cdot) + \int_s^\cdot \Phi(\cdot,\rho)PG^{-1}[\mathbb{T}F([x]_s)]_\xi(\rho)d\rho\Big)(t),$$

which implies

$$\begin{aligned}\|x(\cdot)\|_{L_p[s,t]} \leqslant \frac{1}{1-\gamma}\Bigg(&\|\Gamma_0(.)\|_{L_p[s,t]}\\ &+ M\left[\int\limits_s^t\left(e^{-\alpha\tau}\int\limits_s^\tau e^{\alpha\rho}\left\|\left(PG^{-1}[\mathbb{T}F([x]_s)]_\xi\right)(\rho)\right\|d\rho\right)^p d\tau\right]^{\frac{1}{p}}\Bigg).\end{aligned}$$

Using Hardy's inequality in Lemma 2.2 with the weight functions $U(s) = V(s) = e^{-\alpha s}$ and (4.17) yields

$$\begin{aligned}(1-\gamma)\|x(\cdot)\|_{L_p[s,t]} &\leqslant \|\Gamma_0(t)\|_{L_p[s,\infty)} + MC\left(\int\limits_s^t\left\|\left(PG^{-1}[\mathbb{T}F([x]_s)]_\xi\right)(\rho)\right\|^p d\rho\right)^{\frac{1}{p}}\\ &\leqslant M_3\left\|P(s)x(s)\right\| + MCm_\xi\left\|x(\cdot)\right\|_{L_p[s,t]}.\end{aligned}$$

Thus,

$$\|x(\cdot)\|_{L_p[s,t]} \leqslant \frac{M_3}{1-\gamma-MCm_\xi}\left\|P(s)x_0)\right\|, \quad t \geqslant s.$$

From this inequality it is seen that

$$\|x(\cdot)\|_{L_p[s,\infty)} \leqslant M_2\left\|P(s)x_0\right\|_{\mathbb{R}^n}, \tag{4.18}$$

where $M_2 = \frac{M_3}{1-\gamma-MCm_\xi}$.

We now estimate $u = Px$. Combiniing (3.16) with (4.6) we have

$$\begin{aligned}\|u(t)\| &\leqslant MK_0\|u(s)\| + \int_s^t MK_0 e^{-\alpha(t-\rho)}\left\|PG^{-1}\mathbb{T}F([x]_s)(\rho)\right\|d\rho \\ &\overset{\text{Hölder}}{\leqslant} MK_0\|u(s)\| + \frac{MK_0}{(\alpha q)^{\frac{1}{q}}}\left(\int_s^t \left\|PG^{-1}\mathbb{T}F([x]s)(\rho)\right\|^p d\rho\right)^{\frac{1}{p}} \\ &\leqslant MK_0\|u(s)\| + m_s\frac{MK_0}{(\alpha q)^{\frac{1}{q}}}\left(\int_s^t \|x(\rho)\|^p d\rho\right)^{\frac{1}{p}} \leqslant M_1\|u(s)\|,\end{aligned}$$

for all $t \geqslant s$, where $M_1 = MK_0\big(1 + \frac{M_2 m_s}{(\alpha q)^{\frac{1}{q}}}\big)$. Thus, we get (4.13). The proof is complete. □

In general, we can not expect the preservation of exponential stability under small perturbation without some further assumptions. Indeed, we consider the example

Example 4.6. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } \Sigma(x,y)(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{1}{1+t}\int_0^t y(s)ds.$$

Consider the equation

$$A\begin{pmatrix} x' \\ y' \end{pmatrix} = B\begin{pmatrix} x \\ y \end{pmatrix} + \Sigma(x,y). \tag{4.19}$$

We see that

$$H(x,y) = (x, H_1(y))^\top \text{ where } H_1(y) = y + \frac{1}{1+t}\int_0^t y(s)ds,$$

and $H_1^{-1}(v)(t) = v(t) - \frac{1}{(1+t)^2}\int_0^t (1+s)v(s)ds$. The solution of the homogeneous equation of (4.19) is $(e^{-t}x_0, 0)$ which is exponentially stable. However, for $f(x,y) = \delta(0,x)^\top$, the perturbed system

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{1}{1+t}\int_0^t y(s)ds + \begin{pmatrix} 0 \\ \delta x \end{pmatrix}.$$

satisfies the equation

$$\begin{aligned}\dot{x}(t) &= -(1+\delta)x(t) + \frac{\delta}{(1+t)^2}\int_0^t (1+s)x(s)ds \\ y(t) &= -\delta x(t) + \frac{\delta}{(1+t)^2}\int_0^t (1+s)x(s)ds.\end{aligned}$$

It is easy to show that the solution of this system is uniformly L_p-stable for $-1 < \delta < 0$ but not exponentially stable.

From Example 4.6 it is seen that to study the preservation of exponential stability, we need to add some further assumptions.

For any $\lambda \in \mathbb{R}$, we endow $L_p[s,t]$ with the new norm $\|\cdot\|_{L_p^\lambda[s,t]}$

$$\|z\|^p_{L_p^\lambda[s,t]} = \int_s^t \|e^{\lambda\tau}z(\tau)\|^p d\tau.$$

Denote by $\mathscr{H}^1_{p,\lambda}(s)$ the set of the functions $z \in \mathscr{H}^1(s)$ with $\|z\|_{L_p^\lambda[s,\infty)} < \infty$ and by $\mathscr{B}_\lambda(s)$ the set of the functions $z \in \mathscr{H}^1(s)$ satisfying $\sup_{t\geqslant s} e^{\lambda t}\|z(t)\| < \infty$.

Theorem 4.7. *Suppose that the DAE* (3.3) *is index-1, α–exponentially stable and there exists $0<\lambda<\alpha$ such that $H^{-1}QG^{-1}F(x)$ is locally γ-Lipschitz continuous in x with $\gamma<1$ in the norm $\|\cdot\|_{L_p^\lambda[0,t]}$. Suppose further that*

1. *There exists a continuous function $m : [0,\infty) \to \mathbb{R}_+$ such that*

$$\big\|PG^{-1}\big[\mathbb{T}F(x)\big]_t - PG^{-1}\big[\mathbb{T}F(y)\big]_t\big\|_{L_p^\lambda[0,T]} \leqslant m_t\|x-y\|_{L_p^\lambda[0,T]}, \tag{4.20}$$

for all $0<t\leqslant T<\infty$ and $x,y \in L_p^{loc}([0,\infty);\mathbb{R}^n)$.

2. $\limsup\limits_{t\to\infty} m_t < \frac{1-\gamma}{MC}$, *where $C = C(\alpha-\lambda)$ is defined in Remark 2.3 corresponding to the function $V(t) = U(t) = e^{-(\alpha-\lambda)t}$ and α, M are defined in Theorem 4.2.*
3. *The function $\mathbb{S}^{-1} = (I - H^{-1}QG^{-1}F)^{-1}$ acts continuously from $\mathscr{H}^1_{p,\lambda}(0)$ to $\mathscr{B}_\lambda(0)$.*

Then, the perturbed equation (4.7) *is exponentially λ-stable, i.e., there is positive constant M such that*

$$\|x(t)\| \leqslant Ke^{-\lambda(t-s)}\|P(s)x_0\|, \quad t \geqslant s \geqslant 0,$$

where $x(\cdot)$ is the solution of (4.7) *with the initial condition $P(s)(x(s)-x_0) = 0$.*

Proof. Let $x(t), t \geqslant s$ be a solution of (4.7) with the initial condition $P(s)(x(s)-x_0) = 0$. For any $t \geqslant s$, we put $\varphi(t) = e^{\lambda(t-s)}x(t)$. Since $x \in \mathscr{H}^1(s)$, so is φ. Moreover,

$$A(t)\varphi'(t) = \big(\lambda A(t) + B(t)\big)\varphi(t) + \widetilde{\Sigma}[\varphi]_s(t) + \widetilde{F}([\varphi]_s)(t), \tag{4.21}$$

where $\widetilde{\Sigma}[\varphi]_s(t) = e^{\lambda t}\Sigma\big(e^{-\lambda\cdot}[\varphi]_s\big)(t)$ and $\widetilde{F}([\varphi]_s)(t) = e^{\lambda t}F\big(e^{-\lambda\cdot}[\varphi]_s\big)(t)$.

It is seen that

$$\overline{G}(t) = A(t) - \big(\lambda A(t) + B(t)\big)Q(t) = A(t) - B(t)Q(t) = G(t),$$

and the invertibility of $H = I - QG^{-1}\Sigma$ is equivalent to the invertibility of $\widetilde{H} = I - QG^{-1}\widetilde{\Sigma}$. Moreover,

$$\widetilde{H}^{-1}z(t) = e^{\lambda t}H^{-1}(e^{-\lambda\cdot}z(\cdot))(t), \quad t \geqslant 0, z \in L_p^{loc}[0,\infty),$$

which implies that the equation (4.21) is index-1. The homogeneous equation corresponding to (4.21) is

$$A(t)y'(t) = \big(\lambda A(t) + B(t)\big)y(t) + \big(\widetilde{\Sigma}[y]_s\big)(t). \tag{4.22}$$

Denote by $\Psi(t,s)$ the Cauchy matrix of the equation (4.22). By the α–exponential stability of (3.3), we see that

$$\|\Psi(t,s)\| \leqslant Me^{-(\alpha-\lambda)(t-s)}, \quad t \geqslant s \geqslant 0.$$

Let $\widetilde{\mathbb{T}} := I + \widetilde{\Sigma}\widetilde{H}^{-1}QG^{-1}$. The variation of constant formula says that

$$\begin{aligned}\varphi(t) = \Psi(t,s)P(s)x_0 + \int_s^t \Psi(t,\tau)PG^{-1}\widetilde{\mathbb{T}}\widetilde{F}([\varphi(\cdot)]_s)(\tau)d\tau \\ + \widetilde{H}^{-1}QG^{-1}\widetilde{F}\big([\varphi(\cdot)]_s\big)(t), \quad t \geqslant s.\end{aligned} \tag{4.23}$$

By the assumption of Theorem, the function $H^{-1}QG^{-1}F(\cdot)$ is locally γ-Lipschitz in the norm $\|\cdot\|_{L_p^\lambda[s,t]}$, i.e., for all $z_1, z_2 \in L_p^{loc}[0,\infty)$,

$$\|H^{-1}QG^{-1}(F(e^{-\lambda\cdot}z_1) - F(e^{-\lambda\cdot}z_2))\|_{L_p^\lambda[0,t]} \leqslant \gamma\|e^{-\lambda\cdot}(z_1-z_2)\|_{L_p^\lambda[0,t]}$$

$$=\gamma \|z_1 - z_2\|_{L_p[0,t]}.$$

This inequality implies that

$$\begin{aligned}\|\widetilde{H}^{-1}QG^{-1}(\widetilde{F}(z_1) - \widetilde{F}(z_2))\|_{L_p[0,t]} &= \|H^{-1}QG^{-1}(F(e^{-\lambda\cdot}z_1) - F(e^{-\lambda\cdot}z_2))\|_{L_p^\lambda[0,t]}\\ &\leqslant \gamma \|z_1 - z_2\|_{L_p[0,t]},\end{aligned}$$

for all $z_1, z_2 \in L_p^{loc}[0,\infty)$, i.e., $\widetilde{H}^{-1}QG^{-1}\widetilde{F}$ is γ-locally Lipschitz.

Further, for any $0 \leqslant t \leqslant T$ we have

$$\begin{aligned}&\left\|PG^{-1}[\mathbb{T}F(e^{-\lambda\cdot}z)]_t\right\|^p_{L_p^\lambda[0,T]} = \int_t^T \left\|e^{\lambda\tau}PG^{-1}\left[\left(I + \Sigma H^{-1}QG^{-1}\right)F(e^{-\lambda\cdot}z)\right]_t(\tau)\right\|^p d\tau\\ &= \int_0^T \left\|PG^{-1}\left[\left(I + e^{\lambda\cdot}\Sigma\left(e^{-\lambda\cdot}[e^{\lambda\cdot}H^{-1}(e^{-\lambda\cdot}QG^{-1})]\right)\right)e^{\lambda\cdot}F(e^{-\lambda\cdot}z)\right]_t(\tau)\right\|^p d\tau\\ &= \int_0^T \left\|PG^{-1}\left[\left(I + \widetilde{\Sigma}\widetilde{H}^{-1}QG^{-1}\right)e^{\lambda\cdot}F(e^{-\lambda\cdot}z)\right]_t(\tau)\right\|^p d\tau = \left\|PG^{-1}\left[\widetilde{\mathbb{T}}\widetilde{F}(z)\right]_t\right\|^p_{L_p[0,T]}.\end{aligned}$$

Combining with (4.20), we get

$$\|PG^{-1}[\widetilde{\mathbb{T}}\widetilde{F}(z_1)]_t - PG^{-1}[\widetilde{\mathbb{T}}\widetilde{F}(z_2)]_t\|_{L_p[0,T]} \leqslant m_t \|z_1 - z_2\|_{L_p[0,T]}, \tag{4.24}$$

for all $z_1, z_2 \in L_p^{loc}[0,\infty)$.

Thus, all assumptions of Theorem 4.5 are satisfied, which implies that

$$\|\varphi(\cdot)\|_{L_p[s,\infty)} \leqslant M_2\|P(s)\varphi(s)\|. \tag{4.25}$$

By using (4.24) and (4.25) we have

$$\begin{aligned}&\int_s^\infty \left\|\int_s^t \Psi(t,\tau)PG^{-1}\widetilde{\mathbb{T}}\widetilde{F}([\varphi]_s)(\tau)d\tau\right\|^p dt \leqslant \int_s^\infty \left[Me^{-(\alpha-\lambda)(t-\tau)}\int_s^t \|PG^{-1}\widetilde{\mathbb{T}}\widetilde{F}([\varphi]_s)(\tau)\|d\tau\right]^p dt\\ &\overset{\text{Hardy}}{\leqslant} CM^p \int_s^\infty \|PG^{-1}\widetilde{\mathbb{T}}\widetilde{F}([\varphi]_s)(\tau)\|d\tau\Big\|^p dt \leqslant C(m_s M M_2)^p \|P(s)\varphi(s)\|^p,\end{aligned}$$

where $C = C(\alpha - \lambda)$ is defined in Remark 2.3 corresponding to the function $V(t) = U(t) = e^{-(\alpha-\lambda)t}$. This means that the function

$$h(t) = \Psi(t,s)P(s)x_0 + \int_s^t \Psi(t,\tau)PG^{-1}\widetilde{\mathbb{T}}\widetilde{F}([\varphi]_s)(\tau)d\tau, t \geqslant s$$

belongs to $\mathscr{H}_p^1[s,\infty)$ and

$$\|h\|_{L_p[s,\infty)} \leqslant \left[m_s M M_2 C^{\frac{1}{p}} + \frac{1}{p^{\frac{1}{p}}(\alpha - \lambda)^{\frac{1}{p}}}\right]\|P(s)\varphi(s)\| := N\|P(s)\varphi(s)\|.$$

From (4.23), it is clear that when $t \geqslant s$

$$\begin{aligned}\widetilde{\mathbb{S}}(\varphi)(t) :&= \varphi(t) - \widetilde{H}^{-1}QG^{-1}\widetilde{F}([\varphi]_s)(t)\\ &= \varphi(t) - e^{\lambda t}H^{-1}QG^{-1}F(e^{-\lambda\cdot}[\varphi]_s)(t) = h(t)\\ &\Longleftrightarrow e^{-\lambda t}\varphi(t) - H^{-1}QG^{-1}F(e^{-\lambda\cdot}[\varphi]_s)(t) = e^{-\lambda t}h(t)\\ &\Longleftrightarrow (I - H^{-1}QG^{-1}F)(e^{-\lambda\cdot}[\varphi]_s)(t) = e^{-\lambda t}h(t)\\ &\Longleftrightarrow \mathbb{S}(e^{-\lambda\cdot}[\varphi]_s)(t) = e^{-\lambda t}h(t) \Longleftrightarrow \varphi(t) = e^{\lambda t}\mathbb{S}^{-1}(e^{-\lambda\cdot}[h]_s)(t).\end{aligned}$$

Note that if $h \in \mathscr{H}^1_p(s)$ then $e^{-\lambda \cdot}h(\cdot) \in \mathscr{H}^1_{p,\lambda}(s)$. Since $\mathbb{S}^{-1}$ acts continuously from $\mathscr{H}^1_p(0)$ to $\mathscr{B}_\lambda(0)$, it follows that

$$\begin{aligned}\sup_{t\geqslant s}\|\varphi(t)\| &= \sup_{t\geqslant 0}\left\|e^{\lambda t}\mathbb{S}^{-1}e^{-\lambda\cdot}[h]_s(t)\right\| \\ &= \left\|\mathbb{S}^{-1}e^{-\lambda\cdot}[h]_s\right\|_{\mathscr{B}_\lambda(0)} \leqslant C_1\left\|e^{-\lambda\cdot}[h]_s\right\|_{L_p^\lambda[0,\infty)} \\ &= C_1\|h\|_{L_p[s,\infty)} \leqslant C_1 N\|P(s)\varphi(s)\| := K\|P(s)\varphi(s)\|,\end{aligned}$$

where $C_1 = \left\|\mathbb{S}^{-1}\right\|$. Thus,

$$\|x(t)\| \leqslant Ke^{-\lambda(t-s)}\|P(s)x(s)\|, \;\; t \geqslant s \geqslant 0.$$

The proof is complete. □

Example 4.8. Consider the equation

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x'(t)\\ y'(t)\end{pmatrix} = \begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} + \Sigma(x,y)(t),$$

where

$$\Sigma(x,y)(t) = \begin{pmatrix}0\\ \int_0^t y(s)ds\end{pmatrix}.$$

The solution $(0,0)$ of this equation is α-exponentially stable with $\alpha = 1, M = 1$. Let $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a Lipschitz function with the Lipschitz coefficient δ and $F(0,0) = 0$. Consider the perturbed equation

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x'(t)\\ y'(t)\end{pmatrix} = \begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} + \Sigma(x,y)(t) + F(x,y)(t), \;\; t \geqslant 0. \tag{4.26}$$

By direct calculation we see that $Q = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}; G^{-1} = \begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix};$

$$\begin{aligned}&H(x,y) = (x, H_1(y))^\top, \;\; \text{where} \qquad H_1 y = y + \int_0^t y(s)ds,\\ &H_1^{-1}(v) = v - e^{-t}\int_0^t e^s v(s)ds; \qquad H^{-1}\begin{pmatrix}x\\ y\end{pmatrix} = \big(x, H_1^{-1}(y)\big),\\ &H^{-1}QG^{-1}F = (0, -H_1^{-1}F_2); \qquad PG^{-1}\mathbb{T}F = (F_1 - H_1^{-1}F_2, 0)^\top.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|H^{-1}QG^{-1}F(x,y)\right\|_{L_p^{\frac{1}{2}}[0,t]} &= \left(\int_0^t |e^{\frac{s}{2}}H_1^{-1}F_2(x,y)(s)|^p ds\right)^{\frac{1}{p}}\\ &= \left(\int_0^t \left|e^{\frac{s}{2}}F_2(x,y)(s)\right|^p ds\right)^{\frac{1}{p}} + \left(\int_0^t \left|e^{-\frac{s}{2}}\int_0^s e^\tau F_2(x,y)(\tau)d\tau\right|^p ds\right)^{\frac{1}{p}}\\ &\leqslant \delta\left(\int_0^t \left(e^{\frac{s}{2}}\left\|(x,y)(s)\right\|\right)^p ds\right)^{\frac{1}{p}} + \delta\left[\int_0^t \left(e^{-\frac{s}{2}}\int_0^s e^\tau\left\|(x,y)(\tau)\right\|d\tau\right)^p ds\right]^{\frac{1}{p}}\\ &\overset{\text{Hardy}}{\leqslant} \delta\big(1 + C\big)\left\|\begin{pmatrix}x\\ y\end{pmatrix}\right\|_{L_p^{\frac{1}{2}}[0,t]},\end{aligned}$$

where $C = C(\frac{1}{2})$ is defined in Hardy inequality (2.2) corresponding to the function $V(t) = U(t) = e^{-\frac{t}{2}}$. Furthermore

$$
\begin{aligned}
&\left\|PG^{-1}[\mathbb{T}F(x,y)]_t\right\|_{L_p^{\frac{1}{2}}[0,T]} = \left(\int_t^T \left\|e^{\frac{s}{2}}PG^{-1}[\mathbb{T}F(x,y)(s)]_t\right\|^p ds\right)^{\frac{1}{p}} \\
&= \left(\int_0^T |e^{\frac{s}{2}}F_1(x,y)(s)|^p ds\right)^{\frac{1}{p}} + \left(\int_0^T \left(e^{\frac{s}{2}}|H_1^{-1}F_2(x,y)(s)|\right)^p ds\right)^{\frac{1}{p}} \\
&\leqslant \delta(2+C)\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_{L_p^{\frac{1}{2}}[0,T]}.
\end{aligned}
$$

Thus, with δ such that $\delta < \frac{1}{2C(2+C)}$, all assumptions of Theorem 4.7 are satisfied, which implies that (4.26) is exponentially stable.

5. Bohl-Perron type theorem

We now pass to the study of the Bohl-Perron's Theorem for implicit differential equations with causal operators. That is we investigate the relation between the exponential stability of homogeneous equation (3.3) in Lyapunov sense and the boundedness of solutions of non homogeneous equation (3.2). We keep Assumption 1 to this section. For any $\beta \geqslant 0$ and $s \geqslant 0$, define the weighted space

$$
\mathcal{L}_p^\beta(s) = \left\{ \begin{array}{l} q \in L_p^{loc}([s,\infty)); \mathbb{R}^n) : \int_s^\infty \|e^{\beta t}PG^{-1}\mathbb{T}[q]_s(t)\|^p dt < \infty \\ \text{ and } \int_s^\infty \|e^{\beta t}QG^{-1}\mathbb{T}[q]_s(t)\|^p dt < \infty \end{array} \right\}.
$$

Endow $\mathcal{L}_p^\beta(s)$ with the norm

$$
\|q\|^p_{\mathcal{L}_p^\beta(s)} = \int_s^\infty \left(\|e^{\beta t}PG^{-1}\mathbb{T}[q]_s(t)\|^p + \|e^{\beta t}QG^{-1}\mathbb{T}[q]_s(t)\|^p\right) dt.
$$

Lemma 5.1. *For any $\beta \geqslant 0$ and $s \geqslant 0$, $\mathcal{L}_p^\beta(s)$ is a Banach space.*

Proof. We need only proving the positive definiteness of the norm $\|\cdot\|^p_{\mathcal{L}_p^\beta(s)}$. By noting $I + H^{-1}QG^{-1}\Sigma = H^{-1}$, it follows that $\mathbb{T} := I + \Sigma H^{-1}QG^{-1}$ is invertible by Lemma 2.4. Let $q \in \mathcal{L}_p^\beta(s)$ with $\|q\|^p_{\mathcal{L}_p^\beta(s)} = 0$, which implies $G^{-1}\mathbb{T}[q]_s(t) = 0$ for almost everywhere $t \in [s,\infty)$. Since G and $\mathbb{T}$ are invertible, $q = 0$.

The proof is complete. □

For any $s \geqslant 0$, by formula (3.15), the solution of (3.2) with the initial $P(s)x(s) = 0$ is given by $x(t) = \mathscr{F}_s f$, where

$$
\mathscr{F}_s f(t) = \int_s^t \Phi(t,s)PG^{-1}\mathbb{T}[f]_s(s)\,ds + (H^{-1}QG^{-1}[f]_s)(t),\ t \geqslant s. \tag{5.1}
$$

Theorem 5.2. *Suppose that $\beta < \alpha$. If the system (3.3) is α-exponentially stable, then for every $f \in \mathcal{L}_p^\beta(0)$, the solution of (3.2) with the initial $P(0)x(0) = 0$ is in $\mathscr{H}^1_{p,\beta}(0)$.*

Proof. From the exponential stability of (3.3) we have

$$
\left\|\int_0^\bullet \Phi(\bullet,s)PG^{-1}\mathbb{T}f(s)\,ds\right\|_{L_p^\beta[0,\infty)} = \left(\int_0^\infty \left\|e^{\beta t}\int_0^t \Phi(t,s)PG^{-1}(\mathbb{T}f)(s)\,ds\right\|^p dt\right)^{\frac{1}{p}}
$$

$$\leqslant M\left(\int_0^\infty\left[e^{\beta t}\int_0^t e^{-\alpha(t-s)}\left\|PG^{-1}\mathbb{T}f(s)\right\|\,ds\right]^p dt\right)^{\frac{1}{p}}$$
$$= M\left(\int_0^\infty\left[e^{-(\alpha-\beta)t}\int_0^t e^{\alpha s}\left\|PG^{-1}\mathbb{T}f(s)\right\|\,ds\right]^p dt\right)^{\frac{1}{p}}.$$

By using again Hardy's inequality with $U(t)=V(t)=e^{-(\alpha-\beta)t}$ we obtain

$$\left\|\int_0^\bullet \Phi(\bullet,s)PG^{-1}\mathbb{T}f(s)\,ds\right\|_{L_p^\beta[0,\infty)} \leqslant MC\left(\int_0^\infty\left\|e^{-(\alpha-\beta)s}e^{\alpha s}PG^{-1}\mathbb{T}f(s)\right\|^p ds\right)^{\frac{1}{p}}$$
$$\leqslant MC\left(\int_0^\infty\left\|e^{\beta s}PG^{-1}\mathbb{T}f(s)\right\|^p ds\right)^{\frac{1}{p}}<\infty.$$

Next, note that

$$\begin{aligned}QG^{-1}\mathbb{T} &= QG^{-1}\big(I+\Sigma H^{-1}QG^{-1}\big)=QG^{-1}+QG^{-1}\Sigma H^{-1}QG^{-1}\\ &= QG^{-1}-(I-QG^{-1}\Sigma)H^{-1}QG^{-1}+H^{-1}QG^{-1}=H^{-1}QG^{-1}.\end{aligned}$$

Therefore, the assumption $f\in\mathcal{L}^\beta(s)$ implies that

$$\left\|H^{-1}QG^{-1}f\right\|_{L_p^\beta[0,\infty)}=\Big(\int_0^\infty\left\|e^{\beta t}QG^{-1}\mathbb{T}f(t)\right\|^p dt\Big)^{\frac{1}{p}}<\infty.$$

Thus, $x\in\mathscr{H}^1_{p,\beta}[0,\infty)$. The proof is complete. □

Denote by $\mathscr{L}(s)$ the space of linear continuous operators from $\mathcal{L}_1^\gamma(s)$ to $L_p^\beta[s,\infty)$, $s\geqslant 0$. To prove the inverse relation, we need the following lemma.

Lemma 5.3. *Let $\gamma,\beta\geqslant 0$. If $\mathscr{F}_0\in\mathscr{L}(0)$ then $\mathscr{F}_s\in\mathscr{L}(s)$ for any $s\geqslant 0$. Further,*

$$\sup_{s\geqslant 0}\|\mathscr{F}_s\|_{\mathscr{L}(s)}:=k<\infty. \tag{5.2}$$

Proof. Firstly, we show that $\mathscr{F}_0$ is bounded. By assumption, for any $f\in\mathcal{L}_1^\gamma(0)$, the solution $x(t)$ associated to f of (3.2) with the initial condition $P(0)x(0)=0$ is in $L_p^\beta(0)$. We define a family of operators $\{V_t\}_{t\geqslant 0}$ as following:

$$\begin{aligned}V_t:\mathcal{L}_1^\gamma(0)&\longrightarrow L_p^\beta(0)\\ f\ &\longmapsto V_t(f)=\pi_t\mathscr{F}_0(f).\end{aligned}$$

From the assumption of Lemma, we have

$$\sup_{t\geqslant 0}\|V_tf\|_{L_p^\beta(0)}=\sup_{t\geqslant 0}\Big(\int_0^t\left\|e^{\beta\tau}\mathscr{F}_0f(\tau)\right\|^p d\tau\Big)^{\frac{1}{p}}=\Big(\int_0^\infty\left\|e^{\beta t}\mathscr{F}_0f(t)\right\|^p dt\Big)^{\frac{1}{p}}<\infty,$$

for any $f\in\mathcal{L}_1^\gamma(0)$. Using Uniform Boundedness Principle, we see that

$$\|\mathscr{F}_0\|_{\mathscr{L}(0)}=\sup_{t\geqslant 0}\|V_t\|_{\mathscr{L}(0)}:=k<\infty. \tag{5.3}$$

Let f be arbitrary function in $\mathcal{L}_1^\gamma(s)$. By variation of constants formula, the solution $x(t), t\geqslant s$ of the Cauchy problem (3.2) with the initial condition $P(s)x(s)=0$ corresponding to f is of the form

$$x(t)=\int_s^t\Phi(t,\tau)PG^{-1}\mathbb{T}[f]_s(\tau)d\tau+\big(H^{-1}QG^{-1}[f]_s\big)(t),\ t\geqslant s.$$

It is easy to see that the function

$$z(t) = [x(t)]_s, \quad t \geqslant 0,$$

is the solution of the Cauchy problem (3.2) associated with $[f]_s$ with the initial condition $P(0)z(0) = 0$. Therefore, from (5.3) we have

$$\|x(\cdot)\|_{L_p^\beta[s,\infty)} = \|z(\cdot)\|_{L_p^\beta[0,\infty)} \leqslant k\|[f]_s\|_{\mathcal{L}_1^\gamma(0)} = k\|f\|_{\mathcal{L}_1^\gamma(s)}.$$

Thus,

$$\|\mathscr{F}_s\|_{\mathscr{L}(s)} \leqslant k, \quad \text{for all } s \geqslant 0.$$

The proof is complete. □

Theorem 5.4. *Suppose that for $p\gamma \leqslant \beta$ we have*

(1) *The unique solution of the Cauchy problem* (3.2) *with the initial condition $P(s)x(s) = 0$, associated with every $f \in \mathcal{L}_1^\gamma(s)$ is in $L_p^\beta(s)$.*
(2) *The operator $QH^{-1}\widehat{P}$ acts continuously on $\mathscr{B}_\gamma(0)$.*
(3) *The operator $PG^{-1}\Sigma$ satisfies the condition*

- $$\left\|e^{-\varepsilon\cdot} PG^{-1}\Sigma H^{-1}\widehat{P}z(\cdot)\right\|_{L_p^\beta[0,t]} \leqslant K_1 \|z(\cdot)\|_{L_p^\beta[0,t]}, \quad \text{for all } z \in L^{loc}[0,\infty), \quad (5.4)$$
- $$\sup_{t\geqslant 0} e^{-\varepsilon t} \left\|\left(P' + PG^{-1}\overline{B}\right)(t)\right\| \leqslant K_1, \quad \text{where } \varepsilon = \beta - p\gamma, K_1 > 0. \quad (5.5)$$

Then, the index-1 DAE (3.3) *is exponentially stable.*

Proof. With an arbitrary $s \geqslant 0$, for any $\sigma > 0$ and $v \in \mathbb{R}^n$, let

$$f(t) = e^{-\gamma t}A(t)v\mathbf{1}_{[s,s+\sigma]}(t).$$

It is clear that $QG^{-1}[f]_s = 0$, $\mathbb{T}[f]_s = [f]_s$ and

$$\|f\|_{\mathcal{L}_1^\gamma(s)} = \int_s^\infty e^{\gamma t}\left\|PG^{-1}\mathbb{T}e^{-\gamma\cdot}\mathbf{1}_{[s,s+\sigma]}(\cdot)A(\cdot)v(t)\right\|dt = \int_s^{s+\sigma}\|P(t)v\|\,dt \leqslant \sigma K_0\|v\|_{\mathbb{R}^n}.$$

This means that $f \in \mathcal{L}_1^\gamma(s)$. Let $x(t)$ be the solution of (3.2) associated to f with the initial condition $P(s)x(s) = 0$. It follows from (3.15) that for all $t \geqslant s$

$$\begin{aligned} x(t) &= \int_s^t \Phi(t,\tau)PG^{-1}\mathbb{T}[f]_s(\tau)d\tau = \int_s^t \Phi(t,\tau)PG^{-1}[f]_s(\tau)d\tau \\ &= \int_s^{(s+\sigma)\wedge t} e^{-\gamma\tau}\Phi(t,\tau)P(\tau)vd\tau, \end{aligned}$$

Thus, by (4.5) we have

$$P(t)x(t) == \int_s^{(s+\sigma)\wedge t} e^{-\gamma\tau}\Phi_0(t,\tau)P(\tau)vd\tau,$$

Hence, combining with (5.2) yields

$$\begin{aligned} \Big(\int_s^\infty \Big\|e^{\beta t}\int_s^{(s+\sigma)\wedge t} e^{-\gamma\tau}\Phi_0(t,\tau)P(\tau)vd\tau\Big\|^p dt\Big)^{\frac{1}{p}} &= \|Px(\cdot)\|_{L_p^\beta(s)} \\ &\leqslant K_0\|x(\cdot)\|_{L_p^\beta(s)} \leqslant kK_0\|f\|_{\mathcal{L}_1^\gamma(s)} \leqslant k\sigma K_0^2\|v\|. \end{aligned}$$

Since $\Phi_0(t,s), t \geqslant s$ is the solution of (3.14), $\lim_{\tau\downarrow s}\Phi_0(t,\tau) = \Phi_0(t,s)$. Therefore, dividing both sides of this inequality by σ and letting $\sigma \to 0$ obtain

$$\left(\int_s^\infty \left\|e^{\beta t}\Phi_0(t,s)P(s)v\right\|^p dt\right)^{\frac{1}{p}} \leqslant kK_0^2 e^{\gamma s}\|P(s)v\|. \quad (5.6)$$

Denote by $y(\cdot)$ the solution of the equation (3.3) with the initial condition $P(s)(y(s)-x_0)=0$. Put $u=Py$ then $u(\cdot)=\Phi_0(\cdot,s)P(s)x_0$. By virtue of the first equation of (3.11) we have

$$\left\|e^{-\varepsilon\cdot}u'(\cdot)\right\|_{L_p^\beta[s,t]} \leqslant \left\|e^{-\varepsilon\cdot}(P'+PG^{-1}\overline{B})u(\cdot)\right\|_{L_p^\beta[s,t]} + \left\|e^{-\varepsilon\cdot}PG^{-1}\Sigma H^{-1}\widehat{P}[u]_s\right\|_{L_p^\beta[s,t]}.$$

It is seen that

$$\begin{aligned}\|e^{-\varepsilon\cdot}(P'+PG^{-1}\overline{B})u(\cdot)\|_{L_p^\beta([s,t])} &= \left(\int_s^t \left\|e^{\beta\tau}e^{-\varepsilon\tau}(P'+PG^{-1}\overline{B})u(\tau)\right\|^p d\tau\right)^{\frac{1}{p}}\\ &\leqslant K_1\left(\int_s^t \left\|e^{\beta\tau}u(\tau)\right\|^p d\tau\right)^{\frac{1}{p}} \leqslant kK_0^2K_1e^{\gamma s}\left\|P(s)x_0\right\|,\end{aligned}$$

and

$$\begin{aligned}\left\|(e^{-\varepsilon\cdot}PG^{-1}\Sigma H^{-1}\widehat{P}[u(\cdot)]_s\right\|_{L_p^\beta([0,t])} &\leqslant K_1\left\|[u(\cdot)]_s\right\|_{L_p^\beta([0,t])}\\ &= K_1\left(\int_s^t \left\|e^{\beta\tau}\Phi_0(\tau,s)P(s)x_0\right\|^p d\tau\right)^{\frac{1}{p}} \leqslant kK_0^2K_1e^{\gamma s}\left\|P(s)x_0\right\|.\end{aligned}$$

Thus,

$$\left\|e^{-\varepsilon\cdot}u'(\cdot)\right\|_{L_p^\beta[s,\infty)} \leqslant K_2e^{\gamma s}\left\|P(s)x_0\right\|. \tag{5.7}$$

where $K_2=2kK_0^2K_1$. On the other hand,

$$\|u'(t)\| = \lim_{h\downarrow 0}\left\|\frac{u(t+h)-u(t)}{h}\right\| \geqslant \lim_{h\downarrow 0}\left|\frac{\|u(t+h)\|-\|u(t)\|}{h}\right| = \left|\frac{d\,\|u(t)\|}{dt}\right|.$$

Therefore, from $\gamma=\beta-\varepsilon$ and $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\begin{aligned}&\left\|e^{\gamma t}u(t)\right\|^p = \|e^{\gamma s}u(s)\|^p + \int_s^t \frac{d\,\|e^{\gamma\tau}u(\tau)\|^p}{d\tau}d\tau\\ &= \|e^{\gamma s}u(s)\|^p + p\gamma\int_s^t \|e^{\gamma\tau}u(\tau)\|^p d\tau + \int_s^t e^{p\gamma\tau}\|u(\tau)\|^{p-1}\frac{d\,\|u(\tau)\|}{d\tau}d\tau\\ &\leqslant \|e^{\gamma s}u(s)\|^p + p\gamma\int_s^t \|e^{\gamma\tau}u(\tau)\|^p d\tau + \int_s^t e^{(\beta-\varepsilon)\tau}\|u(\tau)\|^{p-1}\|u'(\tau)\|\,d\tau\\ &= \|e^{\gamma s}u(s)\|^p + p\gamma\int_s^t \|e^{\gamma\tau}u(\tau)\|^p d\tau + \int_s^t e^{\frac{\beta}{q}\tau}\|u(\tau)\|^{p-1}e^{\frac{\beta}{p}\tau}\left\|e^{-\varepsilon\tau}u'(\tau)\right\|d\tau\\ &\overset{\text{Hölder}}{\leqslant} \|e^{\gamma s}u(s)\|^p + p\gamma\int_s^t \|e^{\gamma\tau}u(\tau)\|^p d\tau\\ &\quad + \left(\int_s^t e^{\beta\tau}\|u(\tau)\|^{q(p-1)}d\tau\right)^{\frac{1}{q}}\left(\int_s^t e^{\beta\tau}\left\|e^{-\varepsilon\tau}u'(\tau)\right\|^p d\tau\right)^{\frac{1}{p}}.\end{aligned}$$

By using (5.6) and (5.7) we get

$$\begin{aligned}\left\|e^{\gamma t}u(t)\right\|^p &\leqslant \left(1+p\gamma k^pK_0^{2p}\right)\|e^{\gamma s}u(s)\|^p + \left(kK_0^2e^{\gamma s}\|u(s)\|\right)^{\frac{p}{q}}K_2e^{\gamma s}\|u(s)\|\\ &= (K_3)^pe^{p\gamma s}\|u(s)\|^p,\end{aligned}$$

where $(K_3)^p=1+p\gamma k^pK_0^{2p}+(kK_0^2)^{\frac{p}{q}}K_2$. Thus

$$\|u(t)\| \leqslant K_3e^{-\gamma(t-s)}\|u(s)\|,\ \text{for all } t\geqslant s. \tag{5.8}$$

On the other hand, by assumption, the operator $QH^{-1}\widehat{P}$ acts continuously on $\mathscr{B}_\gamma(0)$. Therefore, with $v = Qy$ we have

$$\begin{aligned}\sup_{t\geqslant s} e^{\gamma t}\,\|v(t)\| &= \sup_{t\geqslant s} e^{\gamma t}\left\|QH^{-1}\widehat{P}[u(\cdot)]_s(t)\right\| \leqslant \left\|QH^{-1}\widehat{P}\right\| \sup_{t\geqslant s} e^{\gamma t}\,\|u(t)\| \\ &= K_3\left\|QH^{-1}\widehat{P}\right\| e^{\gamma s}\,\|u(s)\| \text{ for all } t\geqslant s.\end{aligned}$$

Or,

$$\|v(t)\| \leqslant K_3\left\|QH^{-1}\widehat{P}\right\| e^{-\gamma(t-s)}\,\|u(s)\|. \tag{5.9}$$

Combining (5.8) and (5.9) obtains

$$\|y(t)\| \leqslant K_4 e^{-\gamma(t-s)}\,\|P(s)x_0\|\,, \text{ for all } t\geqslant s,$$

where $K_4 = K_3(\|QH^{-1}\widehat{P}\| + 1)$.

The proof is complete. □

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