Continuous dependence of stationary distributions on parameters for stochastic predator-prey models

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Continuous dependence of stationary distributions on parameters for stochastic predator-prey models

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Abstract

This work studies the continuous dependence of the stationary distributions on the parameters for a stochastic predator-prey model with Beddington-DeAngelis functional response. We show that if the model is extinct (resp. permanent) for a parameter, it is still extinct (resp. permanent) in a neighbourhood of this parameter. In case of extinction, the Lyapunov exponent of predator quantity is negative and the prey quantity converges almost sure to the saturated situation where the predator is absent at an exponential rate. Under the condition permanence, the unique stationary distribution converges weakly to the degenerate measure concentrated at the unique limit cycle or at the globally asymptotical equilibrium when the diffusion term tends to 0.

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1 Introduction

sec:int

In ecology, a functional response is the intake rate of a consumer as a function of food density. It is associated with the numerical response, which is the reproduction rate of a consumer as a function of food density. Holling initiated the study of functional response in [8], where author introduced several types of such responses. The so-called Holling type II functional response is characterized by a decelerating intake rate following from the assumption that the consumer is limited by its capacity to process food. Similar to Holling-type functional response with an extra term describing mutual interference by predators, Beddington [2] and DeAngelis et al. [3] introduced the nowadays well-known Beddington-DeAngelis functional

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response; see also [19] and references therein. Such a model represents most of the qualitative features of the ratio-dependent models but avoids the "low densities problem."

As the building blocks of the bio - and ecosystems, the basic premise of the predator-prey models is that species compete, evolve, and disperse for the purpose of seeking resources to sustain their struggle and existence. Let $\alpha = (r, K, m, a, b, c, \beta, \gamma)$ be the vector of parameters whose components are appropriate positive constants. Then a deterministic predator-prey model with Beddington-DeAngelis functional response of the form

$$\begin{cases} dx_\alpha(t) = \left(rx_\alpha(t)\big(1 - \frac{x_\alpha(t)}{K}\big) - \dfrac{mx_\alpha(t)y_\alpha(t)}{a + by_\alpha(t) + cx_\alpha(t)} \right) dt \\ dy_\alpha(t) = y_\alpha(t) \left(-\gamma + \dfrac{\beta m x_\alpha(t)}{a + by_\alpha(t) + cx_\alpha(t)} \right) dt, \end{cases} \tag{1.1}$$

e1.0

where $x_\alpha(t)$ is the sizes of prey and $y_\alpha(t)$ is the size of predator at time t.

It is well-know that that the quadrants of the plan $\mathbb{R}^2_+ = \{(x, y) : x \geq 0, y \geq 0\}$ and its interior $\mathbb{R}^{2,o}_+ = \{(x, y) : x > 0, y > 0\}$ are invariant with respect to (1.1). Denote by $\Phi^\phi_\alpha(t) = (x_\alpha(t), y_\alpha(t))$ the unique solution of (1.1) with initial value $\phi = (x, y) \in \mathbb{R}^2_+$. Let

$$f(\phi, \alpha) = \left(rx(1 - \frac{x}{K}) - \frac{mxy}{a + by + cx}; -\gamma y + \frac{\beta mxy}{a + by + cx} \right)^\top.$$

Consider the Lyapunov function $V(\phi, \alpha) = \beta x + y$. It is seen that

$$\dot{V}(\phi, \alpha) = V_\phi(\phi, \alpha) f(\phi, \alpha) = \beta rx(1 - \frac{x}{K}) - \gamma y \leq \frac{\beta(r + \gamma)^2 K}{4r} - \gamma V(\phi, \alpha),$$

From this inequality, it is easy to prove that the set

$$\mathcal{R}(\alpha) := \left\{ (x, y) \in \mathbb{R}^2_+ : x + \beta y \leq \frac{\beta(r + \gamma)^2 K}{4r\gamma} \right\}$$

is also attractive set with respect to (1.1).

For any vector parameter $\alpha = (r, m, \beta, \gamma, a, b, c) \in \mathbb{R}^7_+$, construct the threshold value

$$\lambda_\alpha = -\gamma + \frac{\beta m K}{a + Kc}. \tag{1.2}$$

lambda_d

When $\lambda_\alpha > 0$, the system (1.1) has three nonnegative equilibriums $(0, 0)$, $(K, 0)$ and (x^*_α, y^*_α). The long term behavior of the model (1.1) has been classified (see [15, 20] for example) by using the threshold value λ_α as follows.

lem1.1 **Lemma 1.1** (see [20]). *Let λ_α be a threshold value that is defined by (1.2). Then,*

1. *If* $\lambda_\alpha \le 0$ *then the boundary equilibrium point* $(K, 0)$ *is globally asymptotically stable.*

2. *If* $\lambda_\alpha > 0$ *and* $b \ge \min\left\{\frac{c}{\beta}, \frac{m^2\beta^2 - c^2\gamma^2}{\gamma\beta(m\beta - c\gamma) + mr\beta^2}\right\}$ *then the positive equilibrium point* $\phi^*_\alpha = (x^*_\alpha, y^*_\alpha)$ *of the system* (1.1) *is globally asymptotically stable.*

3. *If* $\lambda_\alpha > 0$ *and* $b < \min\left\{\frac{c}{\beta}, \frac{m^2\beta^2 - c^2\gamma^2}{\gamma\beta(m\beta - c\gamma) + mr\beta^2}\right\}$ *then the positive equilibrium point* $\phi^*_\alpha = (x^*_\alpha, y^*_\alpha)$ *is unstable, and there is an exactly stable limit cycle* Γ_α.

Further, by the clarity of the positive equilibrium point formula and [7, Theorem 1.2, pp. 356], we have the following lemma.

lem1.2 **Lemma 1.2.** *Let* $\alpha = (r, m, \beta, \gamma, a, b, c)$ *be the parameters of* (1.1) *such that* $\lambda_\alpha > 0$. *Then,*

(i) *On the set* $b \ge \min\left\{\frac{c}{\beta}, \frac{m^2\beta^2 - c^2\gamma^2}{\gamma\beta(m\beta - c\gamma) + mr\beta^2}\right\}$, *the mapping* $\alpha \to (x^*_\alpha, y^*_\alpha)$ *is continuous.*

(ii) *On* $b < \min\left\{\frac{c}{\beta}, \frac{m^2\beta^2 - c^2\gamma^2}{\gamma\beta(m\beta - c\gamma) + mr\beta^2}\right\}$ *the mapping* $\alpha \to \Gamma_\alpha$ *is continuous in Hausdorff distance.*

Let $\mathcal{K} \subset \mathbb{R}^8_+$ be a compact set. Denote $\mathcal{R}(\mathcal{K}) = \cup_{\alpha \in \mathcal{K}} \mathcal{R}(\alpha)$. It is clear that $\overline{\mathcal{R}(\mathcal{K})}$ is a compact set. Since f has continuous partial derivatives $\nabla_\phi f$ and $\nabla_\alpha f$, these derivatives are uniformly bounded on $\overline{\mathcal{R}}(\mathcal{K}) \times \mathcal{K}$. As a consequence, there exists a positive constants L_1 such that

$$\|f(\phi_1, \alpha_1) - f(\phi_2, \alpha_2)\| \le L_1(\|\phi_1 - \phi_2| + \|\alpha_1 - \alpha_2\|), \quad \phi_1, \phi_2 \in \mathcal{R}(\mathcal{K}); \alpha_1, \alpha_2 \in \mathcal{K}. \tag{1.3}$$ Lip

Let $T > 0$. From the inequality (1.3), for any $\phi \in \mathcal{R}(\mathcal{K}); \alpha_1, \alpha_2 \in \mathcal{K}$, we have

$$\begin{aligned}\sup_{0\le t\le T} \|\Phi^\phi_{\alpha_1}(t) - \Phi^\phi_{\alpha_2}(t)\| &= \sup_{0\le t\le T} \left\| \int_0^t \left[f(\Phi^\phi_{\alpha_1}(s), \alpha_1) - f(\Phi^\phi_{\alpha_2}(s), \alpha_2)\right] ds \right\| \\ &\le \int_0^T \left\|f(\Phi^\phi_{\alpha_1}(s), \alpha_1) - f(\Phi^\phi_{\alpha_2}(s), \alpha_2)\right\| ds \\ &\le L_1 T\|\alpha_1 - \alpha_2\| + L_1 \int_0^T \sup_{0\le s\le t} \left\|\Phi^\phi_{\alpha_1}(s) - \Phi^\phi_{\alpha_2}(s)\right\| dt.\end{aligned}$$

Applying the Gronwall inequality we obtain

$$\sup_{0\le t\le T} \|\Phi^\phi_{\alpha_1}(t) - \Phi^\phi_{\alpha_2}(t)\| \le L_1 T \|\alpha_1 - \alpha_2\| e^{L_1 T}, \quad \forall \phi \in \mathcal{R}(\mathcal{K}), \alpha_1, \alpha_2 \in \mathcal{K}. \tag{1.4}$$ e2.8

We now deal with the evolution of prey-predator model (1.1) in random environment, that is some parameters are perturbed by noises (see: [6, 10, 11, 14, 17, 18]). Currently, one of the important ways to model the influence of the environmental fluctuations in biological systems is to assume that the white noises affect the growth rates. Thus, in random environment, the parameters r, γ become $r + \sigma_1 \dot{B}_1$ and $\gamma \hookrightarrow \gamma - \sigma_2 \dot{B}_2$, where B_1 and B_2 are two independent Brownian motions. Therefore, the equation (1.1) subjected to the environmental white noise can be rewritten

$$\begin{cases} dx_{\alpha,\sigma}(t) = \Big(r x_{\alpha,\sigma}(t)\big(1 - \frac{x_{\alpha,\sigma}(t)}{K}\big) - \dfrac{m x_{\alpha,\sigma}(t) y_{\alpha,\sigma}(t)}{a + b y_{\alpha,\sigma}(t) + c x_{\alpha,\sigma}(t)} \Big) dt + \sigma_1 x_{\alpha,\sigma}(t) dB_1(t), \\ dy_{\alpha,\sigma}(t) = y_{\alpha,\sigma}(t) \Big(-\gamma + \dfrac{\beta m x_{\alpha,\sigma}(t)}{a + b y_{\alpha,\sigma}(t) + c x_{\alpha,\sigma}(t)} \Big) dt + \sigma_2 y_{\alpha,\sigma}(t) dB_2(t), \end{cases} \tag{1.5}$$

me

where $\sigma = (\sigma_1, \sigma_2)$. Zou et al. in [20] has considered the existence of stationary distribution and the stochastic bifurcation for (1.5). They have proved that there is a critical point $b^*(\sigma_1; \sigma_2)$ which depends on σ_1 and σ_2 such that system (1.5) undergoes a stochastic Hopf bifurcation at $b^*(\sigma_1; \sigma_2)$. That is the shape of stationary distribution for system (1.5) changes from crater-like to peak-like. However, the conditions imposed on parameters are rather strict and some results in this paper need to be carefully discussed. In [6], Du et al. has constructed a threshold between the distinction and permanence (also the threshold of the existence of stationary distribution) to the system (1.5).

The main aim of this paper is to consider the robust of permanence and the continuous dependence of stationary distribution of the equation (1.5) on the data in case it exists. Precisely, we prove that if the model is extinct (resp. permanent) for a parameter, it is still extinct (resp. permanent) in a neighbourhood of this parameter. In case of extinction, the Lyapunov exponent of predator quantity is negative and the prey quantity converges almost sure to the saturated situation where the predator is absent. Further, if $\lambda_{\alpha_0} > 0$ and $\alpha \to \alpha_0$ and $\sigma \to 0$, the stationary distribution $\mu_{\alpha,\sigma}$ of the equation (1.5) will converge weakly to the degenerate distribution concentrated on $(x^*_{\alpha_0}, y^*_{\alpha_0})$ or on the limit cycle Γ_{α_0} of the system (1.1).

The paper is organized as follows. Next section deals with the main results. In section 3, we give an example to illustrate that when $(\alpha, \sigma) \to (\alpha_0, 0)$, the stationary distribution $\mu_{\alpha,\sigma}$ weakly converges to the degenerate distribution concentrated on $(x^*_{\alpha_0}, y^*_{\alpha_0})$.

2 Main result

sec:thr

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $B_1(t)$ and $B_2(t)$ be two mutually independent Brownian motions. It is well known that both R^2_+ and $R^{2;o}_+$ (the interior of $\mathbb{R}^2_+$)

are invariant to (1.5), i.e., for any initial value $\phi = (x(0), y(0)) \in \mathbb{R}^2_+$ (resp. in $\mathbb{R}^{2,\circ}_+$), there exists a unique global solution to (1.5) that remains in $\mathbb{R}^2_+$ (resp. $\in \mathbb{R}^{2,\circ}_+$) almost surely (see: [20]). Denote by $\Phi^{\phi}_{\alpha,\sigma}(t) = (x_{\alpha,\sigma}(t), y_{\alpha,\sigma}(t))$ the unique solution of (1.5) with initial value $\phi \in \mathbb{R}^2_+$. For $d > 0$ and $\delta > 0$, let

$$\begin{aligned}\mathcal{U}_\delta(\alpha) &= \{\alpha' = (r', m', \beta', \gamma', a', b', c') \in \mathbb{R}^7_+ : \ \|\alpha' - \alpha\| \le \delta\},\\ \mathcal{V}_\delta(\phi_0) &= \{\phi \in \mathbb{R}^2_+ : \ \|\phi - \phi_0\| \le \delta\}.\end{aligned}$$

the ball with the radius $\delta > 0$ and center α. Denote

$$\mathcal{R}_\delta(\alpha_0) = \bigcup_{\alpha \in \mathcal{U}_\delta(\alpha_0)} \mathcal{R}(\alpha).$$

For any $R > 0$, denote

$$\mathbf{B}_R = \{\phi = (x, y) \in \mathbb{R}^2_+ : \ \|\phi\| \le R\}.$$

Let $C^2(\mathbb{R}^2, \mathbb{R}_+)$ be the family of all non-negative functions $V(\phi)$ on $\mathbb{R}^2$ which are twice continuously differentiable in ϕ. For $V \in C^2(\mathbb{R}^2, \mathbb{R}_+)$, define the differential operator $\mathcal{L}V$ associated to the equation (1.5) is given by

$$\mathcal{L}V(\phi) = V_\phi(\phi) f(\phi, \alpha) + \frac{1}{2}\text{trace}\left[\mathrm{g}^\top(\phi, \sigma)\mathrm{V}_{\phi\phi}(\phi)\mathrm{g}(\phi, \sigma)\right] \tag{2.1}$$

where $V_\phi(\phi)$ and $V_{\phi\phi}(\phi)$ are the gradient and Hessian of $V(\cdot)$, g is the diffusion coefficient of (1.5) given by

$$g(\phi, \sigma) = \begin{pmatrix}\sigma_1 x & 0\\ 0 & \sigma_2 y\end{pmatrix}.$$

By virtue of the symmetry of Brownian motions, in the following we are interested only $\sigma_1 \ge 0, \sigma_2 \ge 0$.

lem2.1 **Lemma 2.1.** *Let $\mathcal{K} \in \mathbb{R}^{7,\circ}_+$ be a compact set and $\overline{\sigma} > 0$. Then for any $0 \le p \le \frac{2\gamma_*}{\overline{\sigma}^2}$ we have*

$$\mathbb{E}(V(\Phi^{\phi}_{\alpha,\sigma}(t))) \le e^{-H_1 t} V(\phi) + \frac{H_2}{H_1}, \ \forall\, \alpha \in \mathcal{K}, \ \|\sigma\| \le \overline{\sigma}, \ t \ge 0, \tag{2.2}$$ e2.2b

where $H_1 = \dfrac{(1+p)}{2}\Big(\gamma_* - \dfrac{p\overline{\sigma}^2}{2}\Big)$, $H_2 = \sup_{\alpha \in \mathcal{K}, |\sigma| \le \overline{\sigma}} \sup_{\phi \in \mathbb{R}^2_+} \{\mathcal{L}V(\phi) + H_1 V(\phi)\}$ *and* $\gamma_* = \inf\{\gamma : (r, m, \beta, \gamma, a, b, c) \in \mathcal{K}\}$ *and* $V(\phi) = (\beta x + y)^{1+p}$.

As a consequence,

$$\sup\left\{\mathbb{E}\|\Phi^{\phi}_{\alpha,\sigma}(t)\|^{1+p} : \alpha \in \mathcal{K}, \ \|\sigma\| \le \overline{\sigma}, \ t \ge 0\right\} < \infty. \tag{2.3}$$ e2.2bs

Proof. The differential operator $\mathcal{L}V(\phi)$ associated to the equation (1.5) is given by

$$\begin{aligned}\mathcal{L}V(\phi) &= (1+p)(\beta x+y)^p\Big(\beta rx\Big(1-\frac{x}{K}\Big)-\gamma y\Big)+\frac{p(1+p)}{2}(\beta x+y)^{p-1}(\beta^2\sigma_1^2x^2+\sigma_2^2y^2)\\ &\le (1+p)\Big(-\gamma+\frac{p\|\sigma\|^2}{2}\Big)(\beta x+y)^{1+p}+\beta r(1+p)(\beta x+y)^p x\Big(\frac{\gamma+r}{r}-\frac{x}{K}\Big)\\ &\le (1+p)\Big(-\gamma_*+\frac{p\overline{\sigma}^2}{2}\Big)(\beta x+y)^{1+p}+\beta r(1+p)(\beta x+y)^p x\Big(\frac{\gamma+r}{r}-\frac{x}{K}\Big)\\ &\le H_2-H_1V(\phi).\end{aligned}$$

where

$$H_1=\frac{1+p}{2}\Big(\gamma_*-\frac{p\overline{\sigma}^2}{2}\Big)$$

and

$$H_2=\sup_{\alpha\in\mathcal{K},|\sigma|\le\overline{\sigma}}\ \sup_{\phi\in\mathbb{R}^2_+}\{\mathcal{L}V(\phi)+H_1V(\phi)\}<\infty.$$

Thus,

$$\mathcal{L}V(\phi)\le H_2-H_1V(\phi). \qquad (2.4)$$ e2.9b1

By a standard argument as in [4, Lemma 2.3], from (2.4) it follows that

$$\mathbb{E}(e^{H_1t}V(\Phi^{\phi}_{\alpha,\sigma}(t)))\le V(\phi)+\frac{H_2(e^{H_1t}-1)}{H_1}.$$

Thus,

$$\mathbb{E}(V(\Phi^{\phi}_{\alpha,\sigma}(t)))\le e^{-H_1t}V(\phi)+\frac{H_2}{H_1},$$

i.e., we get (2.2).

By using the inequality $\|\phi\|^{1+p}\le\max\{1,\gamma_*^{-(1+p)}\}V(\phi)$ and (2.2), we have

$$\sup\big\{\mathbb{E}\|\Phi^{\phi}_{\alpha,\sigma}(t)\|^{1+p}:\alpha\in\mathcal{K},\ \|\sigma\|\le\overline{\sigma},\ t\ge 0\big\}<\infty.$$

Lemma is proved. □

When the predator is absent, the evolution of prey follows the stochastic logistic equation on the boundary,

$$d\varphi_{\alpha,\sigma}(t)=r\varphi_{\alpha,\sigma}(t)\left(1-\frac{\varphi_{\alpha,\sigma}(t)}{K}\right)dt+\sigma_1\varphi_{\alpha,\sigma}(t)dB_1(t). \qquad (2.5)$$ e2.1

Denote $\varphi_{\alpha,\sigma}(t)$ is a solution of the equation (2.5). By comparison theorem, it is seen that $x_{\alpha,\sigma}(t)\le\varphi_{\alpha,\sigma}(t)\ \forall t\ge 0$ a.s., provided that $x_{\alpha,\sigma}(0)=\varphi_{\alpha,\sigma}(0)>0$ and $y_{\alpha,\sigma}(0)\ge 0$.

lem2.1 **Lemma 2.2.** *Let $(x_{\alpha,\sigma}(t), y_{\alpha,\sigma}(t))$ be a solution of the equation(1.5) and $\varphi_{\alpha,\sigma}(t)$ be a solution of Equation (2.5). Then,*

(i) If $r < \frac{\sigma_1^2}{2}$ then the system is exponentially ruined in the sense that the Lyapunov exponents of $\varphi_{\alpha,\sigma}(\cdot), x_{\alpha,\sigma}(\cdot)$ and $y_{\alpha,\sigma}(\cdot)$ are negative.

(ii) In case $r - \frac{\sigma_1^2}{2} > 0$, the equation (2.5) has a unique stationary distribution $\nu_{\alpha,\sigma}$ with the density

$$p_{\alpha,\sigma}(x) = Cx^{\frac{2r}{\sigma_1^2}-2} e^{-\frac{2r}{\sigma_1^2 K}x}, x \geq 0.$$

Further $\nu_{\alpha,\sigma}$ weakly converges to $\delta_K(\cdot)$ as $\sigma_1 \to 0$. Where $\delta_K(\cdot)$ is the Dirac measure with mass at K.

Proof. It is easy to verify that with the initial condition $\varphi_{\alpha,\sigma}(0) = x_0$, the equation (2.5) has a unique solution

$$\varphi_{\alpha,\sigma}(t) = \frac{x_0 \exp\{(r - \frac{\sigma_1^2}{2})t + \sigma B_1(t)\}}{K + rx_0 \int_0^t \exp\{(r - \frac{\sigma_1^2}{2})s + \sigma B_1(s)\}ds}, \tag{2.6}$$ mdphi

(see [16] for example).

Therefore, by the law of iterated logarithm we see that if $r - \frac{\sigma_1^2}{2} < 0$ then

$$\lim_{t\to\infty} \frac{\ln \varphi_{\alpha,\sigma}(t)}{t} = r - \frac{\sigma_1^2}{2} < 0.$$

Using this estimate and the comparison theorem gets

$$\limsup_{t\to\infty} \frac{\ln x_{\alpha,\sigma}(t)}{t} \leq \lim_{t\to\infty} \frac{\ln \varphi_{\alpha,\sigma}(t)}{t} = r - \frac{\sigma_1^2}{2} < 0.$$

On other hand, it implied form the second equation in the system (1.5) that

$$\frac{\ln y_{\alpha,\sigma}(t)}{t} = \frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{\beta m x_{\alpha,\sigma}(s)}{a + by_{\alpha,\sigma}(s) + cx_{\alpha,\sigma}(s)}ds + \frac{\sigma_2 B_2(t)}{t}. \tag{2.7}$$ e2.5bs

By using strong law of large numbers and (2.7), we have

$$\limsup_{t\to\infty} \frac{\ln y_{\alpha,\sigma}(t)}{t} = -\gamma - \frac{\sigma_2^2}{2} < 0.$$

The item (i) is proved.

We now prove the item (ii). Consider the Fokker-Planck equation with respect to (2.5),

$$\frac{\partial p_{\alpha,\sigma}(x,t)}{\partial t} = -\frac{\partial\left[rx\left(1-\frac{x}{K}\right)p_{\alpha,\sigma}(x,t)\right]}{\partial x} + \frac{\sigma_1^2}{2}\frac{\partial^2\left[x^2 p_{\alpha,\sigma}(x)\right]}{\partial x^2}. \qquad (2.8)$$ e2.2

It is easy to see that for $r - \frac{\sigma_1^2}{2} > 0$ the equation (2.8) has a unique positive integrable solution

$$p_{\alpha,\sigma}(x) = Cx^{\frac{2r}{\sigma_1^2}-2}e^{-\frac{2r}{\sigma_1^2 K}x}, x \geq 0,$$

which is a stationary density of (2.5). Where C is the normalizing constant and defined by $C = \frac{1}{\Gamma(\frac{2r}{\sigma_1^2}-1)}\left(\frac{2r}{\sigma_1^2 K}\right)^{\frac{2r}{\sigma_1^2}-1}$ and $\Gamma(\cdot)$ is Gamma function. It means that the the equation (2.5) has a stationary distribution whose density is a Gamma distribution $\Gamma(\frac{2r}{\sigma_1^2}-1, \frac{2r}{\sigma_1^2 K})$. By direct calculation we have

$$\lim_{\sigma\to 0}\mathbb{E}(\varphi_{\alpha,\sigma}(t)) = \lim_{\sigma_1\to 0}\left[K - \frac{K\sigma_1^2}{2r}\right] = K,$$

and

$$\lim_{\sigma_1\to 0}\mathrm{Var}(\varphi_{\alpha,\sigma}(\mathrm{t})) = \lim_{\sigma_1\to 0}\left[\frac{\mathrm{K}^2\sigma_1^4}{4\mathrm{r}^2}\left(\frac{2\mathrm{r}}{\sigma_1^2}-1\right)\right] = 0.$$

These equalities imply that the process $\varphi_{\alpha,\sigma}(t)$ converges to K in L_2 as $\sigma_1 \to 0$. The item (ii) of Lemma is proved. □

By the item (i) of Lemma 2.2, from now on, we are interested only on the case $r > \frac{\sigma_1^2}{2}$. For any $\alpha \in \mathbb{R}_+^{7,o}$ and $\sigma \geq 0$ we define a threshold

$$\lambda_{\alpha,\sigma} = -\gamma - \frac{\sigma_2^2}{2} + \int_0^\infty \frac{m\beta x}{a+cx}p_{\alpha,\sigma}(x)dx. \qquad (2.9)$$ ld

It is noted that when $\sigma = 0$ we have $\lambda_{\alpha,0} = \lambda_\alpha$ to be defined by (1.2).

lem2.0 **Lemma 2.3.** *The mapping $(\alpha,\sigma) \to \lambda_{\alpha,\sigma}$ is continuous on the domain*

$$\mathcal{D} := \left\{\alpha = (r,K,m,a,b,c,\beta,\gamma) : \alpha \in \mathbb{R}_+^8,\ r > \frac{\sigma_1^2}{2}, \sigma_1 \geq 0\right\}.$$

Proof. The integrality of the function $x^{\frac{2r}{\sigma_1^2}-2}e^{-\frac{2r}{\sigma_1^2 K}x}, x \geq 0$ depends on two singular points 0 and ∞. For $\alpha_0 = (r_0, K_0, m_0, a_0, b_0, c_0, \beta_0, \gamma_0) \in \mathcal{D}$ and σ_0 with $\sigma_{0,1} > 0$, we can find a sufficiently small $\eta > 0$ such that the function

$$h(x) = \begin{cases} x^{\frac{2r_0-\eta}{\sigma_{0,1}^2+\eta}-2}e^{-\frac{2r_0-\eta}{\sigma_{0,1}^2 K_0+\eta}x} & \text{for } 0 < x < 1 \\ x^{\frac{2r_0+\eta}{\sigma_{0,1}^2-\eta}-2}e^{-\frac{2r_0-\eta}{\sigma_{0,1}^2 K+\eta}x} & \text{for } 1 \leq x. \end{cases}$$

is integrable on $\mathbb{R}_+$. Further, for all α to be closed to α_0 and σ to be closed to σ_0, we have $p_{\alpha,\sigma}(x) \le h(x)\ \forall\ x \in \mathbb{R}_+$. Paying attention that the function $\frac{m\beta x}{a+cx}$ is bounded, we can use the Lebesgue dominated convergent theorem to get $\lim_{\alpha\to\alpha_0,\sigma\to\sigma_0} \lambda_{\alpha,\sigma_0} = \lambda_{\alpha,\sigma_0}$.

The case $\sigma_0 = 0$ follows from the item (ii) of Lemma 2.2.

Lemma is proved. □

thm2.1 **Theorem 2.4.** *If $\lambda_{\alpha,\sigma} < 0$ then $y_{\alpha,\sigma}(t)$ has the Lyapunov exponent $\lambda_{\alpha,\sigma}$ and $x_{\alpha,\sigma}(t) - \varphi_{\alpha,\sigma}(t)$ converges almost surely to 0 as $t \to \infty$ at an exponential rate.*

Proof. Since the function $h(u) = \dfrac{u}{A+u}$ is increasing in $u > 0$, it follows from (2.7) and comparison theorem that

$$\begin{aligned}\frac{\ln y_{\alpha,\sigma}(t)}{t} &= \frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{\beta m x_{\alpha,\sigma}(s)}{a + by_{\alpha,\sigma}(s) + cx_{\alpha,\sigma}(s)} ds + \frac{\sigma_2 B_2(t)}{t} \\ &\le \frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{\beta m \varphi_{\alpha,\sigma}(s)}{a + c\varphi_{\alpha,\sigma}(s)} ds + \frac{\sigma_2 B_2(t)}{t}.\end{aligned}$$

Letting $t \to \infty$ and applying the law of large number to the process $\varphi_{\alpha,\sigma}$ obtain

$$\limsup_{t\to\infty} \frac{\ln y_{\alpha,\sigma}(t)}{t} \le -\gamma - \frac{\sigma_2^2}{2} + \int_0^\infty \frac{\beta m x}{a+cx} p_{\alpha,\sigma}(x)dx = \lambda_{\alpha,\sigma}. \qquad (2.10)$$ E2.13bs2

We now prove that the process $x_{\alpha,\sigma}(t) - \varphi_{\alpha,\sigma}(t)$ converges almost surely to 0 by estimating the rate of convergence $\varphi_{\alpha,\sigma}(t) - x_{\alpha,\sigma}(t)$ when $t \to \infty$. Using the Itô's formula we get

$$\ln x_{\alpha,\sigma}(t) = \ln x_0 + \int_0^t \left(r\left(1 - \frac{x_{\alpha,\sigma}(s)}{K}\right) - \frac{m y_{\alpha,\sigma}(s)}{a + by_{\alpha,\sigma}(s) + cx_{\alpha,\sigma}(s)} - \frac{\sigma_1^2}{2}\right) ds + \sigma_1 B_1(t), \qquad (2.11)$$ E2.13bs

and

$$\ln \varphi_{\alpha,\sigma}(t) = \ln \varphi_{\alpha,\sigma}(0) + \int_0^t \left(r\left(1 - \frac{\varphi_{\alpha,\sigma}(s)}{K}\right) - \frac{\sigma_1^2}{2}\right) ds + \sigma_1 B_1(t). \qquad (2.12)$$ E2.13bs1

From (2.11), (2.12) and inequalities $\dfrac{my}{a+by+cx} \le \frac{my}{a}$, $\forall x, y > 0$ and $x_{\alpha,\sigma}(t) \le \varphi_{\alpha,\sigma}(t), t \ge 0$, we have

$$\begin{aligned}0 \le \limsup_{t\to\infty} \frac{\ln \varphi_{\alpha,\sigma}(t) - \ln x_{\alpha,\sigma}(t)}{t} &\le \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(\frac{r}{K}(x_{\alpha,\sigma}(s) - \varphi_{\alpha,\sigma}(s)) + \frac{m}{a} y_{\alpha,\sigma}(s)\right) ds \\ &\le \frac{m}{a} \limsup_{t\to\infty} \frac{1}{t}\int_0^t y_{\alpha,\sigma}(s)ds = 0. \qquad (2.13)\end{aligned}$$ bs

From (2.6), it is easy to see that

$$\lim_{t\to\infty} \frac{\ln \varphi_{\alpha,\sigma}(t)}{t} = 0,\ a.s. \qquad (2.14)$$ lnphi

Combining (2.14) and (2.13) we obtain

$$\lim_{t\to\infty} \frac{\ln x_{\alpha,\sigma}(t)}{t} = 0,\ a.s. \tag{2.15}$$ X

Hence, from (2.10)

$$\limsup_{t\to\infty} \frac{\ln y_{\alpha,\sigma}(t) - \ln x_{\alpha,\sigma}(t)}{t} \le \lambda_{\alpha,\sigma} < 0. \tag{2.16}$$ lny-lnx

Otherwise,

$$d\Big(\frac{1}{x_{\alpha,\sigma}(t)}\Big) = \Big(\frac{\sigma_1^2 - r}{x_{\alpha,\sigma}(t)} + \frac{r}{K} + \frac{my_{\alpha,\sigma}(t)}{x_{\alpha,\sigma}(t)(a+by_{\alpha,\sigma}(t)+cx_{\alpha,\sigma}(t))}\Big)dt - \frac{\sigma_1}{x_{\alpha,\sigma}(t)}dB_1(t), \tag{2.17}$$ ito_1/x

and

$$d\Big(\frac{1}{\varphi_{\alpha,\sigma}(t)}\Big) = \Big(\frac{\sigma_1^2 - r}{\varphi_{\alpha,\sigma}(t)} + \frac{r}{K}\Big)dt - \frac{\sigma_1}{\varphi_{\alpha,\sigma}(t)}dB_1(t). \tag{2.18}$$ ito_1/phi

Put $z(t) = \dfrac{1}{x_{\alpha,\sigma}(t)} - \dfrac{1}{\varphi_{\alpha,\sigma}(t)}$. We see that $z(t) \ge 0$ for all $t \ge 0$ and

$$dz(t) = \bigg((\sigma_1^2 - r)z(t) + \frac{my_{\alpha,\sigma}(t)}{x_{\alpha,\sigma}(t)(a+by_{\alpha,\sigma}(t)+cx_{\alpha,\sigma}(t))}\bigg)\,dt - \sigma_1 z(t)dB_1(t).$$

In view of the variation of constants formula [13, Theorem 3.1, pp.96], it yields that

$$z(t) = c_1\Psi(t)\int_0^t \Psi^{-1}(s)\frac{my_{\alpha,\sigma}(s)}{x_{\alpha,\sigma}(s)(a+by_{\alpha,\sigma}(s)+cx_{\alpha,\sigma}(s))}ds, \tag{2.19}$$ z

where

$$\Psi(t) = e^{\left(\frac{\sigma_1^2}{2}-r\right)t-\sigma_1 B_1(t)}. \tag{2.20}$$ Z_phi

It is easy to see that

$$\lim_{t\to\infty} \frac{\ln\Psi(t)}{t} = \frac{\sigma_1^2}{2} - r. \tag{2.21}$$ Z_phi'

Let $0 < \overline{\lambda} < \max\{\frac{r-\sigma_1^2}{2}, -\lambda_{\alpha,\sigma}\}$ be arbitrary and let us choose $\varepsilon > 0$ such that $0 < \overline{\lambda} + 3\varepsilon < \max\{r - \frac{\sigma_1^2}{2}, -\lambda_{\alpha,\sigma}\}$. From (2.21), there are two positive random variables η_1, η_2 such that

$$\eta_1 e^{(\frac{\sigma_1^2}{2}-r-\varepsilon)t} \le \Psi(t) \le \eta_2 e^{(\frac{\sigma_1^2}{2}-r+\varepsilon)t},\ \forall t \ge 0\ a.s. \tag{2.22}$$ est_phi

Further, follows from (2.16), there exists a positive random variable η_3 satisfies

$$\frac{y_{\alpha,\sigma}(t)}{x_{\alpha,\sigma}(t)} \le \eta_3 e^{(\lambda_{\alpha,\sigma}+\varepsilon)t},\ \forall t \ge 0\ a.s. \tag{2.23}$$ est_yx

Combining (2.19), (2.22) and (2.23), gets

$$e^{\overline{\lambda}t}z(t) \le \frac{c_1 m\eta_2\eta_3}{a\eta_1}e^{(\overline{\lambda}+\frac{\sigma_1^2}{2}-r+\varepsilon)t}\int_0^t e^{(r-\frac{\sigma_1^2}{2}+\lambda_{\alpha,\sigma}+2\varepsilon)s}ds.$$

Thus,

$$0 \le \lim_{t\to\infty} e^{\overline{\lambda}t} z(t) \le \frac{c_1 m\eta_2\eta_3}{a\eta_1(r - \frac{\sigma_1^2}{2} + \lambda_{\alpha,\sigma} + 2\varepsilon)} \lim_{t\to\infty} \left(e^{(\lambda_{\alpha,\sigma}+\overline{\lambda}+3\varepsilon)t} - e^{(\overline{\lambda}+\frac{\sigma_1^2}{2}-r+\varepsilon)t} \right) = 0.$$

As a result

$$\lim_{t\to\infty} e^{\overline{\lambda}t} z(t) = \lim_{t\to\infty} e^{\overline{\lambda}t} \left(\frac{1}{x_{\alpha,\sigma}(t)} - \frac{1}{\varphi_{\alpha,\sigma}(t)} \right) = 0. \tag{2.24}$$ bs3

Since $\lim_{t\to\infty} \dfrac{\ln \varphi_{\alpha,\sigma}(t)}{t} = 0$, $\lim_{t\to\infty} e^{-\frac{\overline{\lambda}t}{2}} \varphi^2_{\alpha,\sigma}(t) = 0$. Using (2.24) obtains

$$\begin{aligned} \lim_{t\to\infty} e^{\frac{\overline{\lambda}t}{2}} (\varphi_{\alpha,\sigma}(t) - x_{\alpha,\sigma}(t)) &= \lim_{t\to\infty} e^{\frac{\overline{\lambda}t}{2}} \varphi_{\alpha,\sigma}(t) x_{\alpha,\sigma}(t) \left(\frac{1}{x_{\alpha,\sigma}(t)} - \frac{1}{\varphi_{\alpha,\sigma}(t)} \right) \\ &= \lim_{t\to\infty} e^{-\frac{\overline{\lambda}t}{2}} \varphi_{\alpha,\sigma}(t) x_{\alpha,\sigma}(t) e^{\overline{\lambda}t} \left(\frac{1}{x_{\alpha,\sigma}(t)} - \frac{1}{\varphi_{\alpha,\sigma}(t)} \right) = 0. \end{aligned}$$

It means, $x_{\alpha,\sigma}(t) - \varphi_{\alpha,\sigma}(t)$ converges almost surely to 0 as $t \to \infty$ at an exponential rate.

We now turn to the estimate of Lyapunov exponent of $y_{\alpha,\sigma}$. From (2.7) we get

$$\begin{aligned} \frac{\ln y_{\alpha,\sigma}(t)}{t} &= \frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{\beta m x_{\alpha,\sigma}(s)}{a + by_{\alpha,\sigma}(s) + cx_{\alpha,\sigma}(s)} ds + \frac{\sigma_2 B_2(t)}{t} \\ &= \frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{\beta m \varphi_{\alpha,\sigma}(s)}{a + c\varphi_{\alpha,\sigma}(s)} ds + \frac{\sigma_2 B_2(t)}{t} \\ &\quad + \frac{1}{t}\int_0^t \beta m \left(\frac{x_{\alpha,\sigma}(s)}{a + by_{\alpha,\sigma}(s) + cx_{\alpha,\sigma}(s)} - \frac{\varphi_{\alpha,\sigma}(s)}{a + c\varphi_{\alpha,\sigma}(s)} \right) ds. \end{aligned}$$

Since $\lim_{t\to\infty}(x_{\alpha,\sigma}(t) - \varphi_{\alpha,\sigma}(t)) = 0$ and $\lim_{t\to\infty} y_{\alpha,\sigma}(t) = 0$, it is easy to see that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \beta m \left(\frac{x_{\alpha,\sigma}(s)}{a + by_{\alpha,\sigma}(s) + cx_{\alpha,\sigma}(s)} - \frac{\varphi_{\alpha,\sigma}(s)}{a + c\varphi_{\alpha,\sigma}(s)} \right) ds = 0.$$

Thus,

$$\lim_{t\to\infty} \frac{\ln y_{\alpha,\sigma}(t)}{t} = \lim_{t\to\infty} \left(\frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{\beta m \varphi_{\alpha,\sigma}(s)}{a + c\varphi_{\alpha,\sigma}(s)} ds + \frac{\sigma_2 B_2(t)}{t} \right) = \lambda_{\alpha,\sigma}.$$

The proof is complete. □

The following lemma is similar to one in [5].

lem2.3 **Lemma 2.5.** *For $T, \varsigma > 0$ and $\alpha_0 \in \mathbb{R}_+^{7,o}$, there exists a number κ and δ such that for all $\alpha \in \mathcal{U}_\delta(\alpha_0)$ and $0 < \|\sigma\| < \kappa$,*

$$\mathbb{P}\left\{ \|\Phi^\phi_{\alpha,\sigma}(t) - \Phi^\phi_{\alpha_0}(t)\| \ge \varsigma, \textit{ for some } t \in [0,T] \right\} < \exp\left\{ -\frac{\kappa}{\|\sigma\|^2} \right\}, \quad \phi \in \mathcal{R}_\delta(\alpha_0). \tag{2.25}$$ e2.23

Proof. Let $0 < \delta \le \frac{\varsigma}{2L_1Te^{L_1T}}$. From (1.4) we have

$$\sup_{0\le t\le T} \|\Phi_\alpha^\phi(t) - \Phi_{\alpha_0}^\phi(t)\| \le \frac{\varsigma}{2}, \;\; \forall\, \alpha \in \mathcal{U}_\delta(\alpha_0), \phi \in \mathcal{R}_\delta(\alpha_0). \tag{2.26}$$ e2.24bs

Let $R > 0$ be large enough such that $\mathcal{R}_\delta(\alpha_0) \subset \mathbf{B}_R$ and let $h_R(\cdot)$ be a twice differentiable function such that

$$h_R(\phi) = \begin{cases} 1 & \text{if } \|\phi\| \le R \\ 0 & \text{if } \|\phi\| > R+1 \end{cases}, \quad 0 \le h_R \le 1.$$

Put

$$f_h(\phi,\alpha) = h(\phi)f(\phi,\alpha) = \begin{pmatrix} h(\phi)f^{(1)}(\phi,\alpha) \\ h(\phi)f^{(2)}(\phi,\alpha) \end{pmatrix},$$

and

$$g_h(\phi) = h(x,y)g(x,y) = h(x,y)\begin{pmatrix} \sigma_1 x & 0 \\ 0 & \sigma_2 y \end{pmatrix}.$$

It is seen that $f_h(\phi,\alpha)$ is a Lipschitz function, i.e., there exist $M > 0$ such that

$$\|f_h(\phi_1,\alpha) - f_h(\phi_2,\alpha)\| \le M\|\phi_2 - \phi_1\|, \;\; \forall\, \alpha \in \mathcal{U}_\delta(\alpha_0), \phi_1,\phi_2 \in \mathbb{R}^2_+. \tag{2.27}$$ Lip1

If we choose $M \ge 2R^2$ we also have

$$\|g_h(\phi)g_h^\top(\phi)\| \le M\|\sigma\|^2, \;\; \forall\, \phi \in \mathbb{R}^2_+. \tag{2.28}$$ Bound

Let $\widetilde{\Phi}_{\alpha,\sigma}^\phi(t)$ be the solution starting at $\phi \in \mathcal{R}_\delta(\alpha_0)$ of the equation

$$d\widetilde{\Phi}(t) = f_h(\widetilde{\Phi}(t),\alpha)dt + g_h(\widetilde{\Phi}(t))dB(t), \tag{2.29}$$

where $B(t) = (B_1(t), B_2(t))^\top$. Define the stopping time

$$\tau_R^\phi = \inf\{t \ge 0 : \|\Phi_{\alpha,\sigma}^\phi(t)\| \ge R\}. \tag{2.30}$$

It is easy to see that $\widetilde{\Phi}_{\alpha,\sigma}^\phi(t) = \Phi_{\alpha,\sigma}^\phi(t)$ up to time τ_R^ϕ. Since the state space $\mathcal{R}_\delta(\alpha_0)$ of (1.1) is contained in $\mathbf{B}_R$, the solution $\Phi_\alpha^\phi(\cdot)$ of (1.1) is also the solution of the equation

$$d\Phi(t) = f_h(\Phi(t),\alpha)dt, \; t \ge 0$$

with initial value $\phi \in \mathcal{R}_\delta(\alpha_0)$. For all $t \in [0,T]$, using Itô's formula we have

$$\begin{aligned} \|\widetilde{\Phi}_{\alpha,\sigma}^\phi(t) - \Phi_\alpha^\phi(t)\|^2 = 2\int_0^t &\left(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s) - \Phi_\alpha^\phi(s)\right)^\top \left(f_h(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s),\alpha) - f_h(\Phi_\alpha^\phi(s),\alpha)\right) ds \\ &+ \int_0^t h(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s))\mathrm{trace}\left(\mathrm{g_h}(\widetilde{\Phi}_{\alpha,\sigma}^\phi(\mathrm{s}))\mathrm{g_h^\top}(\widetilde{\Phi}_{\alpha,\sigma}^\phi(\mathrm{s}))\right) \mathrm{ds} \\ &+ 2\int_0^t \left(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s) - \Phi_\alpha^\phi(s)\right)^\top g_h(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s))dB(t). \end{aligned} \tag{2.31}$$

By the exponential martingale inequality, for $T, \varsigma > 0$, there exists a number $\kappa = \kappa(T,\varsigma)$ such that $\mathbb{P}(\widetilde{\Omega}) \geq 1 - \exp\{-\frac{\kappa}{\|\sigma\|^2}\}$, where

$$\widetilde{\Omega} = \Bigg\{ \sup_{t\in[0,T]} \bigg[\int_0^t \left(\widetilde{\Phi}^\phi_{\alpha,\sigma}(s) - \Phi^\phi_\alpha(s)\right)^\top g_h(\widetilde{\Phi}^\phi_{\alpha,\sigma}(s))dB(t)$$
$$-\frac{1}{2\|\sigma\|^2}\int_0^t \left(\widetilde{\Phi}^\phi_{\alpha,\sigma}(s) - \Phi^\phi_\alpha(s)\right)^\top g_h(\widetilde{\Phi}^\phi_{\alpha,\sigma}(s))g_h^\top(\widetilde{\Phi}^\phi_{\alpha,\sigma}(s))\left(\widetilde{\Phi}^\phi_{\alpha,\sigma}(s) - \Phi^\phi_\alpha(s)\right)ds\bigg] \leq \kappa, \phi \in \mathcal{R}_\delta(\alpha_0)\Bigg\}.$$

From (2.27) and (2.28), it implies that for all $\omega \in \widetilde{\Omega}$,

$$\begin{aligned}\|\widetilde{\Phi}^\phi_{\alpha,\sigma}(t) - \Phi^\phi_\alpha(t)\|^2 &\leq 2\int_0^t \|\widetilde{\Phi}^\phi_{\alpha,\sigma}(s) - \Phi^\phi_\alpha(s)\|\|f_h(\widetilde{\Phi}^\phi_{\alpha,\sigma}(s),\alpha) - f_h(\Phi^\phi_\alpha(s),\alpha)\|ds \\ &\quad + \int_0^t h(\widetilde{\Phi}^\phi_{\alpha,\sigma}(s))\mathrm{trace}\left(\mathrm{g_h}(\widetilde{\Phi}^\phi_{\alpha,\sigma}(\mathrm{s}))\mathrm{g_h^\top}(\widetilde{\Phi}^\phi_{\alpha,\sigma}(\mathrm{s}))\right)\mathrm{ds} \\ &\quad + \frac{1}{\|\sigma\|^2}\int_0^t \left\|\widetilde{\Phi}^\phi_{\alpha,\sigma}(s) - \Phi^\phi_\alpha(s)\right\|^2 \left\|g_h(\widetilde{\Phi}^\phi_{\alpha,\sigma}(s))g_h^\top(\widetilde{\Phi}^\phi_{\alpha,\sigma}(s))\right\|ds + 2\int_0^t \kappa ds \\ &\leq 3M\int_0^t \|\widetilde{\Phi}^\phi_{\alpha,\sigma}(s) - \Phi^\phi_\alpha(s)\|^2ds + (M\|\sigma\|^2 + 2\kappa)T. \end{aligned} \quad (2.32)$$

For all $t \in [0,T]$, an application of Gronwall-Belmann's inequality implies that

$$\|\widetilde{\Phi}^\phi_{\alpha,\sigma}(t) - \Phi^\phi_\alpha(t)\| \leq \sqrt{(M\|\sigma\|^2 + 2\kappa)T\exp\{3MT\}}$$

in the set $\widetilde{\Omega}$. Choosing κ is sufficiently small such that $(M\kappa^2 + 2\kappa)T\exp\{3MT\} \leq \frac{\varsigma^2}{4}$ implies

$$\|\widetilde{\Phi}^\phi_{\alpha,\sigma}(t) - \Phi^\phi_\alpha(t)\| \leq \frac{\varsigma}{2}, \text{ for all } \alpha \in \mathcal{U}_\delta(\alpha_0), 0 < \|\sigma\| < \kappa \,, \omega \in \widetilde{\Omega}. \quad (2.33)$$ e2.31bs

From (2.33), (2.26) and triangle inequality, we have

$$\|\widetilde{\Phi}^\phi_{\alpha,\sigma}(t) - \Phi^\phi_{\alpha_0}(t)\| \leq \varsigma \;\; \forall \alpha \in \mathcal{U}_\delta(\alpha_0), t \in [0,T], \omega \in \widetilde{\Omega}.$$

It also follows from this inequality that when $\omega \in \widetilde{\Omega}$ we have $\tau_R > T$, which implies

$$\mathbb{P}\left\{\|\Phi^\phi_{\alpha,\sigma}(t) - \Phi^\phi_{\alpha_0}(t)\| < \varsigma, \;\text{ for all } t \in [0,T]\right\} \geq \mathbb{P}(\Omega_1) \geq 1 - \exp\left\{\frac{\kappa}{\|\sigma\|^2}\right\}$$

holds for all $0 < \|\sigma\| < \kappa$ and $\alpha \in \mathcal{U}_\delta(\alpha_0)$. The proof is complete. □

thm2.4 **Theorem 2.6.** *Let $\alpha_0 = (r_0, m_0, \beta_0, \gamma_0, a_0, b_0, c_0)$ be a vector of parameters of the system (1.1) such that $\lambda_{\alpha_0} > 0$ and $b_0 < \min\left\{\frac{c_0}{\beta_0}, \frac{m_0^2\beta_0^2 - c_0^2\gamma_0^2}{\gamma_0\beta_0(m_0\beta_0 - c_0\gamma_0) + m_0r_0\beta_0^2}\right\}$. Then, there exist $\delta_1 > 0$ and $\overline{\sigma} > 0$ such that for all $\alpha \in \mathcal{U}_{\delta_1}(\alpha_0)$ and $0 < \|\sigma\| \leq \overline{\sigma}$, the Markov process $\Phi^\phi_{\alpha,\sigma}(t) =$*

$(x_{\alpha,\sigma}(t), y_{\alpha,\sigma}(t))$ has a unique stationary distribution $\mu_{\alpha,\sigma}$. Further, $\mu_{\alpha,\sigma}$ is concentrated on $\mathbb{R}_+^{2,\circ}$ and has the density with respect to Lebesgue measure. Besides, for any open set $\mathcal{V}$ containing Γ_{α_0} we have

$$\lim_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(\mathcal{V}) = 1, \tag{2.34}$$

e2.16

where Γ_{α_0} is limit cycle of the system (1.1) corresponding to the parameter α_0.

Proof. Since $\lambda_{\alpha_0} = -\gamma_0 + \frac{Km_0\beta_0}{a_0+Kc_0} > 0$ and $b_0 < \min\left\{\frac{c_0}{\beta_0}, \frac{m_0^2\beta_0^2 - c_0^2\gamma_0^2}{\gamma_0\beta_0(m_0\beta_0 - c_0\gamma_0) + m_0r_0\beta_0^2}\right\}$, we can use Lemma 2.3 to find $\delta_1 > 0$ and $\overline{\sigma}$ such that

$$\lambda_{\alpha,\sigma} = -\gamma - \frac{\sigma_2^2}{2} + \int_0^\infty \frac{m\beta x}{a+cx} p_{\alpha,\sigma}(x)dx > 0,$$
$$b < \min\left\{\frac{c}{\beta}, \frac{m^2\beta^2 - c^2\gamma^2}{\gamma\beta(m\beta - c\gamma) + mr\beta^2}\right\}$$

hold for all $\alpha \in \mathcal{U}_{\delta_1}(\alpha_0)$ and $0 \le \|\sigma\| \le \overline{\sigma}$. By virtue of [6, Theorem 2.3, p. 192], the Markov process $\Phi^\phi_{\alpha,\sigma}(t) = (x_{\alpha,\sigma}(t), y_{\alpha,\sigma}(t))$ has a unique stationary distribution $\mu_{\alpha,\sigma}$ with support $\mathbb{R}^{2,\circ}$. Further, by [1, 12], the stationary distribution $\mu_{\alpha,\sigma}$ has the density with respect to Lebesgue measure on $\mathbb{R}^{2,\circ}$ by the non degenerate property of $g(\phi)$.

On the other hand, it follows from (2.3) that for any $\varepsilon > 0$ there exist $R = R(\varepsilon)$ such that $\mu_{\alpha,\sigma}(\mathbf{B}_R) \ge 1 - \varepsilon$ for every $\alpha \in \mathcal{U}_{\delta_1}(\alpha_0)$ and $0 \le \|\sigma\| \le \overline{\sigma}$.

By assumption of Theorem 2.6, it is seen from the item (2) of Lemma 1.1 that $\phi^*_{\alpha_0}$ is a source point, i.e., two eigenvalues of the matrix $Df(\phi^*_{\alpha_0}, \alpha_0)$ have the positive real parts. Therefore, the Lyapunov equation

$$Df(\phi^*_{\alpha_0}, \alpha_0)^\top P + PDf(\phi^*_{\alpha_0}, \alpha_0) = I$$

has a positively definite solution P. Since $Df(\phi, \alpha)$ is continuous in ϕ, α, there exist a positive constants $0 < \delta_2 < \delta_1$, $0 < \delta_3$ and c such that

$$\dot{V}(\phi, \alpha) = V_\phi(\phi, \alpha)f(\phi, \alpha) > c\|\phi - \phi^*_\alpha\|, \tag{2.35}$$

e2.23

$$\text{trace}\left(\mathrm{g}^\top(\phi)\mathrm{P}\mathrm{g}(\phi)\right) \ge \mathrm{c}\|\sigma\|^2, \tag{2.36}$$

for all $\phi \in \mathcal{V}_{\delta_3}(\phi^*_{\alpha_0}), \alpha \in \mathcal{U}_{\delta_2}(\alpha_0)$. Where $V(\phi, \alpha) = (\phi - \phi^*_\alpha)^\top P(\phi - \phi^*_\alpha)$. Hence,

$$\mathcal{L}V(\phi, \alpha) = V_\phi(\phi, \alpha)f(\phi, \alpha) + \text{trace}\left(\mathrm{g}^\top(\phi)\mathrm{P}\mathrm{g}(\phi)\right) \ge \mathrm{c}\|\sigma\|^2 \tag{2.37}$$

e2.24

for all $\phi \in \mathcal{V}_{\delta_3}(\phi^*_{\alpha_0}), \alpha \in \mathcal{U}_{\delta_2}(\alpha_0)$.

Denoting $S = \mathcal{V}_{\delta_3/2}(\phi^*_{\alpha_0})$ and $\mathcal{Z} = \mathcal{V}_{\delta_3}(\phi^*_{\alpha_0})$ we now prove that

$$\lim_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(S) = 0. \tag{2.38}$$ e2.24b

First, we note from (2.35) that there is T^* such that if $\phi \in \mathbf{B}_R \setminus S$ then $\Phi^\phi_\alpha(t) \in \mathbf{B}_R \setminus \mathcal{Z}$ for all $t \geq T^*$. Further, for any $T > 0$ we have

$$\begin{aligned}\mu_{\alpha,\sigma}(S) &= \int_{\mathbb{R}^2_+} P_{\alpha,\sigma}(T,\phi,S)\mu_{\alpha,\sigma}(d\phi)\\ &= \int_{\mathbf{B}_R\setminus S} P_{\alpha,\sigma}(T,\phi,S)\mu_{\alpha,\sigma}(d\phi) + \int_S P_{\alpha,\sigma}(T,\phi,S)\mu_{\alpha,\sigma}(d\phi) + \int_{\mathbf{B}^c_R} P_{\alpha,\sigma}(T,\phi,S)\mu_{\alpha,\sigma}(d\phi),\end{aligned}$$

where $P_{\alpha,\sigma}(T,\phi,\cdot) = \mathbb{P}(\Phi^\phi_{\alpha,\sigma}(T) \in \cdot)$ is the transition probability of the Markov process $\Phi^\phi_{\alpha,\sigma}$. It is clear that

$$\int_{\mathbf{B}^c_R} P_{\alpha,\sigma}(T,\phi,S)\mu_{\alpha,\sigma}(d\phi) \leq \mu_{\alpha,\sigma}(\mathbf{B}^c_R) \leq \varepsilon. \tag{2.39}$$

For $\phi \in S$, let $\tau^\phi_{\alpha,\sigma}$ be the exit time of $\Phi^\phi_{\alpha,\sigma}(\cdot)$ from $\mathcal{Z}$, i.e.,

$$\tau^\phi_{\alpha,\sigma} = \inf\{t \geq 0 : \Phi^\phi_{\alpha,\sigma}(t) \notin \mathcal{Z}\}. \tag{2.40}$$

By Itô's formula, we have

$$\begin{aligned}\theta \geq \mathbb{E}V(\Phi^\phi_{\alpha,\sigma}(\tau^\phi_{\alpha,\sigma} \wedge t)) - V(\phi) &= \mathbb{E}\int_0^{\tau^\phi_{\alpha,\sigma}\wedge t} \mathcal{L}V(\Phi^\phi_{\alpha,\sigma}(s))ds &(2.41)\\ \geq c\|\sigma\|^2\mathbb{E}\left[\tau^\phi_{\alpha,\sigma} \wedge t\right] &\geq c\|\sigma\|^2 t\, \mathbb{P}(\tau^\phi_{\alpha,\sigma} \geq t), \quad t \geq 0, &(2.42)\end{aligned}$$

where $\theta = \sup\{V(\phi,\alpha) : \phi \in \mathcal{Z}, \alpha \in \mathcal{U}_{\delta_2}(\alpha_0)\}$. Choosing $T_\sigma = \max\left\{\frac{2\theta}{c\|\sigma\|^2}, T^*\right\}$ obtains

$$\mathbb{P}(\tau^\phi_{\alpha,\sigma} \geq T_\sigma) \leq \frac{1}{2}, \tag{2.43}$$ e2.19

for all $\phi \in S, \alpha \in \mathcal{U}_{\delta_2}(\alpha_0)$. Thus,

$$\begin{aligned}P_{\alpha,\sigma}(T_\sigma,\phi,S) &= \mathbb{P}(\Phi^\phi(T_\sigma) \in S)\\ &= \mathbb{P}(\Phi^\phi(T_\sigma) \in S, \tau^\phi_{\alpha,\sigma} \geq T_\sigma) + \mathbb{P}(\Phi^\phi(T_\sigma) \in S, \tau^\phi_{\alpha,\sigma} < T_\sigma) &(2.44)\\ &\leq \frac{1}{2} + \mathbb{P}(\Phi^\phi(T_\sigma) \in S, \tau^\phi_{\alpha,\sigma} < T_\sigma).\end{aligned}$$ e2.42

By using strong Markov property of the solution, one gets

$$\begin{aligned}&\mathbb{P}(\Phi^\phi(T_\sigma) \in S, \tau^\phi_{\alpha,\sigma} < T_\sigma)\\ &\quad= \int_0^{T_\sigma}\left[\int_{\partial\mathcal{Z}} \mathbb{P}\left\{\Phi^\psi_{\alpha,\sigma}(T_\sigma - t) \in S\right\}\mathbb{P}\left\{\Phi^\phi_{\alpha,\sigma}(t) \in d\psi\right\}\right] P\left\{\tau^\phi_{\alpha,\sigma} \in dt\right\}.\end{aligned}$$

It remarks that when $\psi \in \partial\mathcal{Z}$ we have $\Phi_\alpha^\psi(T_\sigma - t) \notin \mathcal{Z}$ for all $t \geq 0$. Therefore, by Lemma 2.5 with $\varsigma = \delta_3/2$ we can find $\overline{\sigma} > \kappa > 0$ and $0 < \delta_4 \leq \delta_2$ such that $\mathbb{P}(\Phi_{\alpha,\sigma}^\psi(T_\sigma - t) \in S) \leq \exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\}$ for $0 \leq t \leq T_\sigma,\ 0 < \|\sigma\| < \kappa$ and $\alpha \in \mathcal{U}_{\delta_4}(\alpha_0)$. Hence,

$$\mathbb{P}\left\{\Phi^\phi(T_\sigma) \in S, \tau_{\alpha,\sigma}^\phi < T_\sigma\right\} \leq T_\sigma \exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\},\ \alpha \in \mathcal{U}_{\delta_4}(\alpha_0),\ 0 < \|\sigma\| < \kappa. \qquad (2.45)$$ e2.43

Combining (2.44) and (2.45) obtains

$$\int_S P\big(T_\sigma, \phi, S\big)\mu_{\alpha,\sigma}(d\phi) \leq \frac{1}{2}\mu_{\alpha,\sigma}(S) + T_\sigma \exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\},\ \alpha \in \mathcal{U}_{\delta_4}(\alpha_0),\ 0 < \|\sigma\| < \kappa.$$

On the other hand, when $\phi \in \mathbf{B}_R \setminus S$ we see that $\Phi_\alpha^\phi(T_\sigma) \notin \mathcal{Z}$. Therefore, using (2.35) again we get

$$\int_{\mathbf{B}_R \setminus S} P\big(T_\sigma, \phi, S\big)\mu_{\alpha,\sigma}(d\phi) < \exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\},\ \alpha \in \mathcal{U}_{\delta_4}(\alpha_0),\ 0 < \|\sigma\| < \kappa.$$

Summing up we have

$$\mu_{\alpha,\sigma}(S) \leq \frac{1}{2}\mu_{\alpha,\sigma}(S) + (T_\sigma + 1)\exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\} + \varepsilon,\ \alpha \in \mathcal{U}_{\delta_4}(\alpha_0),\ 0 < \|\sigma\| < \kappa.$$

Noting that $\lim_{(\alpha,\sigma)\to(\alpha_0,0)} (T_\sigma + 1)\exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\} = 0$ we can pass the limit to get

$$\limsup_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(S) \leq \frac{1}{2} \limsup_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(S) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\limsup_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(S) = 0.$$

We now prove (2.34). Let $\mathcal{V}$ be an open set containing Γ_{α_0}. It suffices to show that for any compact set B not intersecting $\mathcal{V}$ nor S we have

$$\lim_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(B) = 0.$$

Let $3d = \text{dist}\,(\partial\mathcal{V}, \Gamma_{\alpha_0})$ and

$$\mathcal{V}_d(\Gamma_{\alpha_0}) = \{x : \text{dist}\,(x, \Gamma_{\alpha_0}) < d\}.$$

From item (ii) of Lemma 1.2, there exists $0 < \delta_5 < \delta_4$ such that $\Gamma_\alpha \subset \mathcal{V}_d(\Gamma_{\alpha_0})$ for all $\alpha \in \overline{\mathcal{U}}_{\delta_5}(\alpha_0)$. It is clear that

$$\text{dist}\,(B, \mathcal{V}_d(\Gamma_{\alpha_0})) > 2d.$$

Let $\varepsilon > 0$ and $R = R(\varepsilon) > 0$ such that $S, B \subset \mathbf{B}_R$ and $\mu_{\alpha,\sigma}(\mathbf{B}_R^c) \le \varepsilon$. Since Γ_{α_0} is asymptotically stable, we can find an open neighbourhood U of Γ_{α_0} such that $U \subset \mathcal{V}_d(\Gamma_{\alpha_0})$ and if $\phi \in U$ then $\Phi^{\phi}_{\alpha_0}(t) \in \mathcal{V}_d(\Gamma_{\alpha_0})$ for all $t \ge 0$. Further, the simple property of the limit cycle Γ_{α_0} implies that

$$\lim_{t\to\infty} \text{dist}\,(\Phi^{\phi}_{\alpha_0}(t), \Gamma_{\alpha_0}) = 0 \text{ for all } \phi \notin S.$$

This means that for every $\phi \in \overline{\mathbf{B}}_R \setminus S$, there exists a T^{ϕ} such that $\Phi^{\phi}_{\alpha_0}(t) \in U$ for all $t \ge T^{\phi}$.

By the continuity of solutions on the initial condition, there exist $\delta_{\phi} > 0$ such that $\Phi^{z}_{\alpha_0}(t) \in U$ for all $z \in \mathcal{V}_{\delta_\phi}(\phi)$ and $t > T^{\phi}$. Since $\overline{\mathbf{B}}_R \setminus S$ is compact, there is $\phi_1, \phi_2, ..., \phi_n$ such that $\overline{\mathbf{B}}_R \setminus S \subset \cup_{k=1}^{n} \mathcal{V}_{\delta_{\phi_k}}(\phi_k)$. Let $T = \max\{T^{\phi_1}, T^{\phi_2}, ..., T^{\phi_n}\}$. It is seen that

$$\Phi^{\phi}_{\alpha_0}(T) \in U \subset \mathcal{V}_d(\Gamma_{\alpha_0}) \text{ for all } \phi \in \overline{\mathbf{B}}_R \setminus S.$$

Similar as above, we have

$$\begin{aligned}\mu_{\alpha,\sigma}(B) &= \int_{\mathbf{B}_R\setminus S} P_{\alpha,\sigma}(T,\phi,B)\mu_{\alpha,\sigma}(d\phi) + \int_S P_{\alpha,\sigma}(T,\phi,B)\mu_{\alpha,\sigma}(d\phi) + \int_{\mathbf{B}_R^c} P_{\alpha,\sigma}(T,\phi,B)\mu_{\alpha,\sigma}(d\phi),\\ &\le \int_{\mathbf{B}_R\setminus S} P_{\alpha,\sigma}(T,\phi,B)\mu_{\alpha,\sigma}(d\phi) + \mu_{\alpha,\sigma}(S) + \mu_{\alpha,\sigma}(\mathbf{B}_R^c)\\ &\le \int_{\mathbf{B}_R\setminus S} P_{\alpha,\sigma}(T,\phi,B)\mu_{\alpha,\sigma}(d\phi) + \mu_{\alpha,\sigma}(S) + \varepsilon.\end{aligned} \tag{2.46}$$

e2.30

Using the Lemma 2.5 again with $\varsigma = d$ we can find $\delta_6 < \delta_5$ and κ_1 such that

$$\mathbb{P}\left\{\|\Phi^{\phi}_{\alpha,\sigma}(T) - \Phi^{\phi}_{\alpha}(T)\| \ge \varsigma\right\} \le \exp\left\{-\frac{\kappa_1}{\|\sigma\|^2}\right\},\ \alpha \in \mathcal{U}_{\delta_6}(\alpha_0), \phi \in \mathbf{B}_R \setminus S,\ 0 < \|\sigma\| < \kappa_1.$$

This inequality implies that

$$\mathbb{P}\left\{\|\Phi^{\phi}_{\alpha,\sigma}(T) \in \mathcal{V}\right\} \ge 1 - \exp\left\{-\frac{\kappa_1}{\|\sigma\|^2}\right\},\ \alpha \in \mathcal{U}_{\delta_6}(\alpha_0), \phi \in \mathbf{B}_R \setminus S,\ 0 < \|\sigma\| < \kappa_1.$$

Since $B \cap \mathcal{V} = \emptyset$,

$$P_{\alpha,\sigma}(T,\phi,B) = \mathbb{P}\left\{\|\Phi^{\phi}_{\alpha,\sigma}(T) \in B\right\} \le \exp\left\{-\frac{\kappa_1}{\|\sigma\|^2}\right\},\ \alpha \in \mathcal{U}_{\delta_6}(\alpha_0), \phi \in \mathbf{B}_R\setminus S,\ 0 < \|\sigma\| < \kappa_1.$$

Hence,

$$\mu_{\alpha,\sigma}(B) \le \exp\left\{-\frac{\kappa_1}{\|\sigma\|^2}\right\} + \mu_{\alpha,\sigma}(S) + \varepsilon.$$

Thus,

$$\limsup_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(B) \le \lim_{(\alpha,\sigma)\to(\alpha_0,0)} (\exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\} + \mu_{\alpha,\sigma}(S) + \varepsilon) = \varepsilon.$$

Since ε is arbitrary,

$$\limsup_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(B) = 0.$$

The proof is complete. □

Corollary 2.7. *Suppose that all of assumptions of the theorem 2.6 hold. If H is continuous and bounded function defined on $\mathbb{R}^2_+$, then*

$$\lim_{(\alpha,\sigma)\to(\alpha_0,0)} \int_{\mathbb{R}^2_+} H(x)d\mu_{\alpha,\sigma}(x) = \frac{1}{T^*}\int_0^{T^*} H(\Phi^{\overline{\phi}}_{\alpha_0}(t))dt \tag{2.47}$$

where $\overline{\phi}$ is any point on Γ_{α_0} and T^* is the period of the limit cycle, i.e., $\Phi^{\overline{\phi}}_{\alpha_0}(t+T^*) = \Phi^{\overline{\phi}}_{\alpha_0}(t)$.

Proof. Let $\widehat{\Phi}_{\alpha,\sigma}(\cdot)$ be the stationary solution of (1.5), whose distribution is $\mu_{\alpha,\sigma}$. We see that

$$\int_{\mathbb{R}^2_+} H(\phi)d\mu_{\alpha,\sigma}(\phi) = \mathbb{E}H(\widehat{\Phi}_{\alpha,\sigma}(t)), \quad \text{for all } \ t \geq 0.$$

In particular,

$$\int_{\mathbb{R}^2_+} H(\phi)d\mu_{\alpha,\sigma}(\phi) = \frac{1}{T^*}\int_0^{T^*} H(\widehat{\Phi}_{\alpha,\sigma}(t))dt$$

where T^* is the period of the cycle. Since the measure $\mu_{\alpha,\sigma}(\cdot)$ becomes concentrated on the cycle Γ_{α_0} as (α,σ) approaches to $(\alpha_0, 0)$ and H is bounded continuous function, we obtain

$$\lim_{(\alpha,\sigma)\to(\alpha_0,0)} \int_{\mathbb{R}^2_+} H(\phi)d\mu_{\alpha,\sigma}(\phi) = \frac{1}{T^*}\int_0^{T^*} H(\Phi^{\overline{\phi}}_{\alpha_0}(t))dt,$$

where $\overline{\phi}$ is any point on the limit cycle Γ_{α_0}. The proof is completed. □

thm2.2 **Theorem 2.8.** *Let $\alpha_0 = (r_0, m_0, \beta_0, \gamma_0, a_0, b_0, c_0)$ be a vector of parameters of the system (1.1) such that $\lambda_{\alpha_0} > 0$ and $b_0 \geq \min\left\{\frac{c_0}{\beta_0}, \frac{m_0^2\beta_0^2 - c_0^2\gamma_0^2}{\gamma_0\beta_0(m_0\beta_0 - c_0\gamma_0) + m_0 r_0 \beta_0^2}\right\}$.*

Then, there exist $\delta > 0$ and $\overline{\sigma} > 0$ such that for all $\alpha \in \mathcal{U}_\delta(\alpha_0)$ and $0 < \sigma \leq \overline{\sigma}$, the process $(x_{\alpha,\sigma}(t), y_{\alpha,\sigma}(t))$ has a stationary distribution $\mu_{\alpha,\sigma}$ concentrated on $\mathbb{R}^{2,\circ}_+$. Further for any open set $\mathcal{V}$ containing the positive equilibrium point $\phi^*_{\alpha_0} = (x^*_{\alpha_0}, y^*_{\alpha_0})$ of the system (1.1)

$$\lim_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(\mathcal{V}) = 1. \tag{2.48}$$ e2.31

Proof. The proof is quite similar as the one of Theorem 2.4. So, we omit it here. □

3 Numerical Examples

sec:num

ex1 *Example* 3.1. Consider (1.1) having the parameter α_0 with $r_0 = 1, K_0 = 5, m_0 = 9, a_0 = 1.75, b_0 = 1, c_0 = 1, \gamma_0 = 0.6, \beta_0 = 0.5$. Direct calculation shows that $\lambda_\alpha = 2.7582 > 0$ and a positive equilibrium $(x^*, y^*) = (0.3, 0.233)$ with $Df(x^*, y^*)$ has two eigenvalues $0.0016 \pm 0.66\,i$. Further, $b_0 \le \min\left\{\frac{c_0}{\beta_0}, \frac{m_0^2\beta_0^2 - c_0^2\gamma_0^2}{\gamma_0\beta_0(m_0\beta_0 - c_0\gamma_0) + m_0 r_0 \beta_0^2}\right\}$. Thus, this system has a limit cycle Γ_0 simulated in Figure 1, starting from the point $(0.67, 0.2)$. Let $\mathcal{V}$ be an $\varepsilon-$neighbourhood of Γ_0 with $\varepsilon = 0.01$. For $\|\sigma\| \le 1$, we have $\lambda_{\alpha,\sigma} > 0$. This means that (1.5) has a unique stationary distribution $\mu_{\alpha,\sigma}$. We estimate the probability $\mu_{\alpha,\sigma}(\mathcal{V})$ as $\sigma \to 0$. To simplify the simulation, we fix all other parameters, excepted the variation of a and list results in the following table.

σ	(0.5,0.5)	(0.1,0.1)	(0.01,0.01)	(0.001,0.001)
a	1.8	1.75	1.72	1.71
$\lambda_{\alpha,\sigma}$	2.3975	2.72	2.7450	2.7500
$\mu_{\alpha,\sigma}(\mathcal{V})$	0.1575	0.3380	0.7001	0.8960

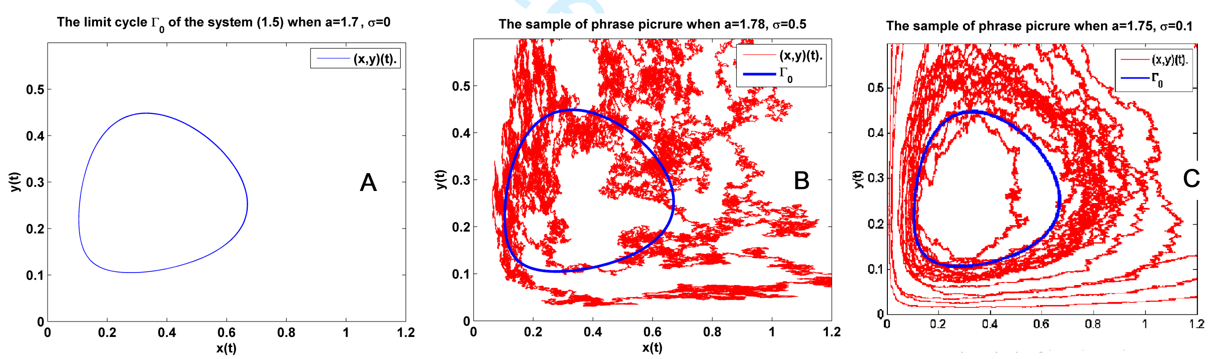

Figure 1: **A:** Limit cycle Γ_0 of (1.1); **B:** Sample path of (1.5) when $a = 1.78, \sigma = 0.5$; **C:** Sample path of (1.5) when $a = 1.75, \sigma = 0.1$. f1.1

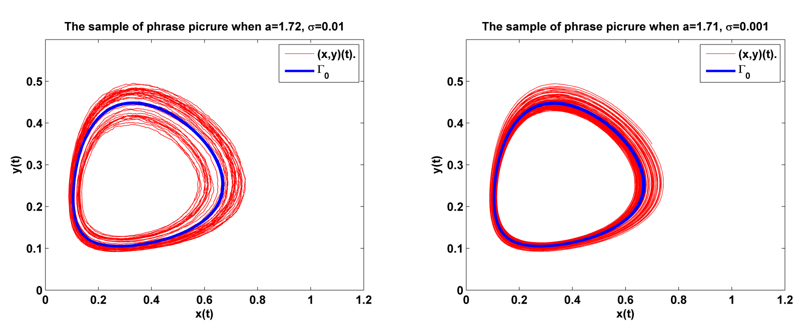

Figure 2: **D:** Sample path of (1.5) when $a = 1.2, \sigma = 0.01$; **E:** Sample path of (1.5) when $a = 1.71, \sigma = 0.001$. f1.2

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