Stability for discrete-time impulsive positive singular system with time delays

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Abstract

This paper investigates the impulsive stability analysis issues of discrete-time positive singular systems with time delay. First, the paper addresses the positivity problem of the system by providing sufficient conditions. Next, a new method based on state transformations is presented to derive a new delay-dependent criterion for the exponential stability of impulsive positive singular systems. Finally, the effectiveness of the proposed conditions is validated through a numerical example.

Keywords: Impulsive system, positive systems, discrete-time singular system, time delays.

1. Introduction

There exists a special class of dynamic systems called positive systems whose states and outputs are always nonnegative for any nonnegative inputs and nonnegative initial states [1]. Positive systems are widely applied in many fields like chemistry [2], ecology [3], and biomedicine [4]. Numerous important results on positive systems have been achieved, such as stability analysis and controller synthesis [1, 5–9], input-output property analysis [10, 11], and filter design [12, 13].

Impulsive systems have received considerable attention in the past few decades [14] due to their wide applications in many practical areas such as economics, mechanics, population dynamics, biological phenomena, etc. Many important results on various impulsive systems have been explored. By employing Lyapunov functions with discontinuity at the impulse times, the authors in [15] considered the problem of exponential stability of impulsive systems. Liu [16] established comparison principles of existence and uniqueness and stability of solutions for stochastic impulsive systems by combining Lyapunov-like function method and Itô's formula. The problem of input-to-state stability of impulsive systems with nonlinear perturbation was addressed in [17]. Feng and Cao [1] discussed stability analysis for impulsive switched singular

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systems. Li et al. [18] provided several criteria for uniform stability and globally asymptotical stability for impulsive systems via event-triggered impulsive control. The problem of noise-to-state stability and globally asymptotic stability was studied in [19] for a class of random nonlinear impulsive systems.

An impulsive positive system is positive system which is modeled with impulsive effect. Impulsive positive system can be used to represent certain classes of epidemiology [20], population models [21], and ecosystems [22], which having deterministic jumps in their dynamics. Recently, impulsive positive systems have received great attention from researchers. By using linear copositive Lyapunov functions, Zhang et al. [23] have proposed new conditions for impulsive positive linear systems without time delays for the first time. However, as we know, time delays are frequently encountered in many fields of science and engineering, and they are often a source of degradation in system performance or instability. Therefore, it is necessary to consider the problem of stability analysis for impulsive positive systems with time delays. Following these ideas, many researchers focus on stability analysis and control design problems for impulsive positive systems through various techniques[24–28]. By combining a copositive LKF and the average impulsive interval method, the authors in [24] established a sufficient criterion of global exponential stability via linear programming problems for impulsive positive systems with mixed time-varying delays. Hu et al. [25] derived some sufficient delay-independent conditions to guarantee stability and stabilization for impulsive positive delay systems based on the Lyapunov-Krasovskiis functional (LKF) method. Dwell-time stability and stabilization for linear positive impulsive and switched systems were investigated in [26]. By constructing a multiple linear copositive Lyapunov function and using the average dwell time method, the guaranteed cost finite-time boundedness problem for positive discrete-time impulsive switched systems was investigated in [29].

In the past three decades, singular systems theory has been widely investigated due to its important applications in many fields from the engineering point of view. Many interesting results have been explored for various singular systems such as linear delayed singular systems [30], TS fuzzy singular systems [31], stochastic polynomial fuzzy singular systems [32], switched linear singular systems [33], and fractional-order singular systems [34]. For singular systems with impulsive effects, few works on the problem of stability analysis can be found in the literature [14, 35–38].

It should be noted that almost all of the current results on stability analysis problems are focused on impulsive systems and singular systems, and few works are devoted to positive impulsive systems, not to mention impulsive positive singular systems. To the best of the authors'knowledge, the exponential stability analysis for discrete-time impulsive positive singular system with time delays have not extensively investigated yet. One of the primary difficulties is the presence of singularities, making applying standard linear analysis methods challenging. The singularity of the matrix combined with the non-negativity of variables of impulsive positive singular systems makes the problem more difficult. Therefore, the exponential stability analysis problem for the concerned systems is not trivial and still remains a technically challenging issue.

Motivated by the discussions mentioned above, this paper considers the impulsive stability

analysis issues for the discrete-time positive singular system with time delays. The main contributions of our paper are summarized as follows.
(i) For the first time, the problem of impulsive stability of the discrete-time positive singular system with time delay has been studied;
(ii) New characterizations of positivity of such systems are proposed;
(iii) New sufficient conditions on the exponential stability of impulsive singular systems are presented.

The rest of this paper is organized as follows. We present problem formulation, notations and some auxiliary results that will be used in next sections in Section 2. Furthermore, in Section 3, a new exponential stability criterion for discrete-time positive singular system with time delay is established. In Section 4, numerical example is studied. The conclusions are given in Section 5.

Notation: The sets of integers, positive integers, and nonnegative integers are represented by $\mathbb{Z}, \mathbb{Z}^+$, and $\mathbb{N}$, respectively. The space of all nonnegative (positive) vectors in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_{0,+}$ ($\mathbb{R}^n_+$), and the space of all real $(s \times q)$ matrices is denoted by $\mathbb{R}^{s\times q}$. I_n is the identity matrix with $n-$ dimensions. For $x \in \mathbb{R}^n$, $x \succeq 0$ if all of the entries in a vector x are nonnegative. For $K \in \mathbb{R}^{h\times h}$, if all of the off-diagonal entries in matrix K are nonnegative, the matrix is Metzler. The notation $||\cdot||$ refers to the vector 1-norm. For a matrix $K = (k_{ij})_{p\times q} \in \mathbb{R}^{p\times q}$, k_{ij} denotes the entry in row i and column j. $K \succ 0 (\succeq 0)$ indicates that all elements of the matrix K are positive (nonnegative). Given a vector λ, the weighted ℓ_∞ norm is defined as $\|x\|^\lambda_\infty = \max_{1\le i\le n}\left\{\dfrac{|x_i|}{\lambda_i}\right\}$. For $p < q, p, q \in \mathbb{N}, \overline{p,q} := p, p+1, \ldots, q$.

2. Problem statement and preliminaries

Consider the following impulsive singular systems

$$\begin{cases} Ex(t+1) = Ax(t) + A_d x(t-h),\ t \neq t_m - 1, \\ \mathrm{x}_1(t_m) = H\mathrm{x}_1(t_m - 1), m \in \mathbb{Z}^+, \\ x(s) = \nu(s), s \in \{-h, -h+1, \cdots, 0\}, \end{cases} \tag{1}$$

where $x(\cdot) := (\mathrm{x}_1(\cdot),\ \mathrm{x}_2(\cdot))$, $\mathrm{x}_1(t) \in \mathbb{R}^r$ and $\mathrm{x}_2(t) \in \mathbb{R}^{n-r}$ is the state vector. The matrix $E \in \mathbb{R}^{n\times n}$ is singular and $\mathrm{rank}(E) = r < n$. $A,\ A_d$ are known constant matrices with appropriate dimensions. In this paper, we suppose that the matrices E, A, A_d have the following expression:

$$E := \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, A := \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},\ A_d := \begin{pmatrix} A_{d_1} & A_{d_2} \\ A_{d_3} & A_{d_4} \end{pmatrix},$$

$A_1, A_{d_1} \in \mathbb{R}^{r\times r}, A_2, A_{d_2} \in \mathbb{R}^{r\times(n-r)}, A_3, A_{d_3} \in \mathbb{R}^{(n-r)\times r}, A_4,\ A_{d_4} \in \mathbb{R}^{(n-r)\times(n-r)}, H \in \mathbb{R}^{r\times r}$. The delay $h \in \mathbb{Z}^+$. $\nu : [-h, 0] \to \mathbb{R}^n_+$ is a vector-valued initial function with the norm defined by $\|\nu\|_h = \sup_{-h\le s\le 0} \|\nu(s)\|$. The impulse sequence $\{t_m\}^\infty_{m=1}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_m < \cdots, t_m \to \infty$ for $m \to \infty$. Let us denote the state trajectory with the initial value (ν_1, ν_2) of the system (1) by $\mathrm{x}_1(t, \nu_1, \nu_2)$ and $\mathrm{x}_2(t, \nu_1, \nu_2)$.

Lemma 1. *Assume that the matrix A_4 in the system (1) satisfies the condition* $\det(A_4) \neq 0$. *Then, the solution of system (1) exists and is unique in $\mathbb{Z}^+$.*

Proof. From $\det(A_4) \neq 0$, for $t \in [0,h]$, the system (1) can be reduced to the system:

$$\begin{aligned} \mathrm{x}_1(t+1) =& A_1\mathrm{x}_1(t) + A_2\mathrm{x}_2(t) + A_{d_1}\nu_1(t-h) + A_{d_2}\nu_2(t-h),\ t \neq t_m - 1, \\ \mathrm{x}_2(t) =& -A_4^{-1}\left(A_3\mathrm{x}_1(t) + A_{d_3}\nu_1(t-h) + A_{d_4}\nu_2(t-h)\right), \\ \mathrm{x}_1(t_m) =& H\mathrm{x}_1(t_m - 1), m \in \mathbb{Z}^+, \end{aligned} \tag{2}$$

For $t = 0$, from the first equation of (2) we have

$$\mathrm{x}_1(1) = \left(A_1 - A_2A_4^{-1}A_3\right)\nu_1(0) + \left(A_{d_1} - A_2A_4^{-1}A_{d_3}\right)\nu_1(-h) + \left(A_{d_2} - A_2A_4^{-1}A_{d_4}\right)\nu_2(-h), \tag{3}$$

Substitute (3) into the second equation of (2) we get

$$\mathrm{x}_2(1) = -A_4^{-1}\left(A_3\mathrm{x}_1(t) + A_{d_3}\nu_1(1-h) + A_{d_4}\nu_2(1-h)\right),$$

Using the method of steps , we can find solution $\mathrm{x}_1(t)$ on $[0,h]$ of the first equation of (2). Once $\mathrm{x}_1(t)$ is known on $[0,h]$, from the second equation of (2) we can find solution $\mathrm{x}_2(t)$ on $[0,h]$. Repeating the process on $[nh,(n+1)h]$, $n \in \mathbb{N}$, we can find solution $x(t)$, $t \geq 0$, which completes the proof. □

Definition 1. [39] System (1) is said to be an impulsive positive system if for all $\nu(t) \succeq 0$, then $\mathrm{x}(t,\nu) \succeq 0$ for all $t \geq 0$.

Definition 2. The system (1) is said to be exponentially stable if $\exists\ \xi > 0, \alpha \in (0,1)$ such that for any $\nu(\cdot) \succeq 0$

$$\|x(t)\| \leq \xi\alpha^t\|\nu\|_h, t \geq 0.$$

Lemma 2. *[40] Let $K \in \mathbb{R}^{l\times l}$ be a Metzler matrix. Then the following statements are equivalent:*
i) $\exists\ \xi \in \mathbb{R}^l$ satisfies $\xi \succ 0$ and $K\xi \prec 0$.
ii) K is Hurwitz.
iii) K is nonsingular and $K^{-1} \preceq 0$.

Proposition 1. *The system (1) is an impulsive positive system if A_4 is a Metzler and Hurwitz matrix and A_1, A_2, A_3, H, A_d are nonnegative matrices.*

Proof. A_4 is a Metzler and Hurwitz matrix, then, using Lemma 2, we get $\det(A_4) \neq 0$, then system (1) has a unique solution, by Lemma 1. Furthermore, we get $-A_4^{-1} \succeq 0$. Note that $-A_4^{-1}A_3, -A_4^{-1}A_{d_3}, -A_4^{-1}A_{d_4}$ are nonnegative matrices because $-A_4^{-1}, A_3, A_{d_3}, A_{d_4}$, are nonnegative. From (1) we get

$$\begin{aligned} &\mathrm{x}_1(t+1) = A_1\mathrm{x}_1(t) + A_2\mathrm{x}_2(t) + A_{d_1}\mathrm{x}_1(t-h) + A_{d_2}\mathrm{x}_2(t-h), \\ &\mathrm{x}_2(t) = (-A_4)^{-1}\left(A_3\mathrm{x}_1(t) + A_{d_3}\mathrm{x}_1(t-h) + A_{d_4}\mathrm{x}_2(t-h)\right), \end{aligned} \tag{4}$$

for $t = t_m$, we get

$$\mathrm{x}_1(t_m) = H\mathrm{x}_1(t_m - 1), m \in \mathbb{Z}^+. \tag{5}$$

Firstly, we will prove that

$$x(t) \succeq 0,\ 0 \le t \le t_1 - 1. \tag{6}$$

For $t = 1$, from the first equation of (4), we have

$$\begin{aligned} \mathrm{x}_1(1) &= A_1\mathrm{x}_1(0) + A_2\mathrm{x}_2(0) + A_{d_1}\mathrm{x}_1(-h) + A_{d_2}\mathrm{x}_2(-h) \\ &= A_1\nu_1(0) + A_2\nu_2(0) + A_{d_1}\nu_1(-h) + A_{d_2}\nu_2(-h) \\ &\succeq 0, \end{aligned} \tag{7}$$

From (7) and the second equation of (4), we have

$$\begin{aligned} \mathrm{x}_2(1) &= (-A_4)^{-1}\big(A_3\mathrm{x}_1(1) + A_{d_3}\mathrm{x}_1(1-h) + A_{d_4}\mathrm{x}_2(1-h)\big) \\ &= (-A_4)^{-1}\big(A_3\mathrm{x}_1(1) + A_{d_3}\nu_1(1-h) + A_{d_4}\nu_2(1-h)\big) \\ &\succeq 0. \end{aligned} \tag{8}$$

The combination of inequalities (7) and (8) yields $x(1) \succeq 0$. Similarly, we have $x(t) \succeq 0, \forall t \in \{1, 2, \cdots, t_1 - 1\}$. Therefore, (6) holds.

For $t = t_1$ we obtain $\mathrm{x}_1(t_1) = H\mathrm{x}_1(t_1 - 1)$. We have $H \succeq 0$ and $\mathrm{x}_1(t_1 - 1) \succeq 0$ implies $\mathrm{x}_1(t_1) \succeq 0$. From this and the second equation of (4) we get $\mathrm{x}_2(t_1) \succeq 0$.
For $t_1 \le t \le t_2 - 1$, we show that

$$\mathrm{x}_i(t) \succeq 0, i = 1, 2. \tag{9}$$

Note that $x(t_1) \succeq 0$. Assume that (9) holds for all $t_k \le t_l,\ t_1 \le t_l < t_2 - 1$. We shall prove (9) holds for $t_k + 1$. From the first equation of (4), we have

$$\mathrm{x}_1(t_k+1) = A_1\mathrm{x}_1(t_k) + A_2\mathrm{x}_2(t_k) + A_{d_1}\mathrm{x}_1(t_k - h) + A_{d_2}\mathrm{x}_2(t_k - h) \succeq 0, \tag{10}$$

From (10) and the second equation of (4), we have

$$\mathrm{x}_2(t_k+1) = (-A_4)^{-1}\big(A_3\mathrm{x}_1(t_k+1) + A_{d_3}\mathrm{x}_1(t_k+1-h) + A_{d_4}\mathrm{x}_2(t_k+1-h)\big) \succeq 0. \tag{11}$$

Combining the inequalities (10) and (11) yields $x(t_k + 1) \succeq 0$. Therefore, (9) holds.

When $t = t_2$ we have $\mathrm{x}_1(t_2) = H\mathrm{x}_1(t_2 - 1)$. We have $H \succeq 0$ and $\mathrm{x}_1(t_2 - 1) \succeq 0$ implies $\mathrm{x}_1(t_2) \succeq 0$. From this and the second equation of (4) we get $\mathrm{x}_2(t_2) \succeq 0$. By repeating the same procedure, we obtain $\mathrm{x}_i(t) \succeq 0, i = 1, 2, t \ge 0$. This implies that system (1) is impulsive positive.
□

3. Stability analysis

This section investigates the exponential stability analysis problem for system (1). One exponential stability criterion for system (1) is given by the theorem below. From now, we always assume that A_E, A_d, H are nonnegative matrices.
Let us denote:

$$
\begin{aligned}
&A_E := A + I_n - E,\\
&\mathscr{A}_1 := A_1 - A_2 A_4^{-1} A_3 := \left(a_{ij}\right)_{r\times r},\\
&\mathscr{A}_3 := -A_4^{-1} A_3 := \left(d_{ij}\right)_{(n-r)\times r},\\
&\bar{A}_{d_1} := A_{d_1} - A_2 A_4^{-1} A_{d_3} := \left(b_{ij}\right)_{r\times r},\\
&\bar{A}_{d_2} := A_{d_2} - A_2 A_4^{-1} A_{d_4} := \left(c_{ij}\right)_{r\times (n-r)},\\
&\bar{A}_{d_3} := -A_4^{-1} A_{d_3} := \left(e_{ij}\right)_{(n-r)\times r},\\
&\bar{A}_{d_4} := -A_4^{-1} A_{d_4} := \left(f_{ij}\right)_{(n-r)\times (n-r)},\\
&H := (h_{ij})_{r\times r},\\
&\Lambda_1 := (\lambda_1, \dots, \lambda_r) \in \mathbb{R}_+^r,\ \Lambda_2 = (\lambda_{r+1}, \dots, \lambda_n) \in \mathbb{R}_+^{n-r},\\
&\lambda := (\Lambda_1, \Lambda_2) \in \mathbb{R}_+^n.
\end{aligned}
$$

Theorem 3. *If there exist constants $\alpha \in (0,1),\ \delta \in (0,1)$, and $\lambda \in \mathbb{R}_+^n$ such that the following conditions hold*

$$\left(-\alpha E + A + \alpha^{-h} A_d\right) \lambda \prec 0, \tag{12}$$

$$\underline{\mathfrak{T}} \geq -\frac{1}{\delta} \log_\alpha R_\lambda, \tag{13}$$

$$R_\lambda^i = \frac{1}{\alpha} \sum_{j=1}^{r} h_{ij} \frac{\lambda_j}{\lambda_i} > 1, i = \overline{1,r},\ \ R_\lambda = \max_{1\leq i\leq r} \left\{R_\lambda^i\right\} > 1. \tag{14}$$

Then, under the minimum dwell-time $\underline{\mathfrak{T}}$ (i.e., the impulse time sequence fulfills $\inf_m \{t_m - t_{m-1}\} \geq \underline{\mathfrak{T}}, m \in \mathbb{Z}^+$), system (1) is an impulse positive and exponentially stable. Moreover, we have

$$\|x(t)\| \leq \left(\sup_{-h\leq s\leq 0} \|\nu(s)\|_\infty^\lambda\right) \|\lambda\|\, \alpha^{(1-\delta)t},\ t \geq 0.$$

Proof. From $A_E := A + I_n - E \succeq 0, A_d$ we have $A_1, A_2, A_3, A_4 + I_{n-r}, A_{d_i}, i = \overline{1,4}$ are nonnegative matrices. This implies that A_4 is a Metzler matrix. Setting $\lambda = (\Lambda_1,\ \Lambda_2) \in \mathbb{R}_+^r \times \mathbb{R}_+^{n-r}$. Using (12), we get

$$\left(A_4 + \alpha^{-h} A_{d_4}\right) \Lambda_2 \prec 0,$$

which implies that $A_4\Lambda_2 \prec 0$ because of $\alpha^{-h}A_{d_4}\Lambda_2 \succeq 0$. Combining this with Lemma 2, we get $\det(A_4) \neq 0$ and A_4 is a Hurwitz matrix, $-A_4^{-1} \succeq 0$. As a result, system (1) has a unique solution, according to Lemma 1. Moreover, from Proposition 1, we obtain that system (1) is an impulse positive. Since $A_{d_i}, i=\overline{1,4}, A_1, A_2, A_3$ are nonnegative, and $-A_4^{-1} \succeq 0$, then we obtain $\mathscr{A}_1$, $\mathscr{A}_3$, $\bar{A}_{d_1}$, $\bar{A}_{d_2}$, $\bar{A}_{d_3}$, $\bar{A}_{d_4}$ are nonnegative matrices. Since $\begin{pmatrix} I_r & -A_2A_4^{-1} \\ 0 & -A_4^{-1} \end{pmatrix}$ is a nonnegative matrix and nonsingular, from (12) we get:

$$\begin{pmatrix} I_r & -A_2A_4^{-1} \\ 0 & -A_4^{-1} \end{pmatrix} \left(-\alpha E + A + \alpha^{-h}A_d\right)\lambda \prec 0. \tag{15}$$

From (15), we have

$$\sum_{j=1}^{r}\left(a_{ij}+\alpha^{-h}b_{ij}\right)\lambda_j + \sum_{j=r+1}^{n}\alpha^{-h}c_{ij}\lambda_j - \alpha\lambda_i \prec 0, i=\overline{1,r}, \tag{16}$$

$$\sum_{j=1}^{r}\left(d_{ij}+\alpha^{-h}e_{ij}\right)\lambda_j + \sum_{j=r+1}^{n}\alpha^{-h}f_{ij}\lambda_j - \lambda_i \prec 0, i=\overline{r+1,n}. \tag{17}$$

Let $x(t,\nu) = (\mathtt{x}_1(t), \mathtt{x}_2(t))$, where $\mathtt{x}_1(t) := (x_1(t),\dots,x_r(t))$, $\mathtt{x}_2(t) := (x_{r+1}(t),\dots,x_n(t))$, be the unique nontrivial solution of the system (1) with initial condition $\nu(\cdot)$. From Proposition 1, we have

$$x(t) \succeq 0, t \geq -h. \tag{18}$$

To prove that the system (1) is exponentially stable, consider the functions as follows: $\mathfrak{V}_i(x(t)) := \dfrac{x_i(t)}{\lambda_i}, i=\overline{1,n}$ and we choose the function $\mathfrak{V}(x(t)) = \max_{1\leq i\leq n}\{\mathfrak{V}_i(x(t))\} = \max_{1\leq i\leq n}\left\{\dfrac{x_i(t)}{\lambda_i}\right\}$, this implies that $x(t) \preceq \mathfrak{V}(x(t))\lambda$. From (16) and (17), we have

$$\sum_{j=1}^{r}\left(a_{ij}+\alpha^{-h}b_{ij}\right)\frac{\lambda_j}{\lambda_i} + \sum_{j=r+1}^{n}c_{ij}\alpha^{-h}\frac{\lambda_j}{\lambda_i} - \alpha < 0, i=\overline{1,r}, \tag{19}$$

and

$$\sum_{j=1}^{r}\left(d_{ij}+\alpha^{-h}e_{ij}\right)\frac{\lambda_j}{\lambda_i} + \sum_{j=r+1}^{n}\alpha^{-h}f_{ij}\frac{\lambda_j}{\lambda_i} - 1 < 0, i=\overline{r+1,n}. \tag{20}$$

It follows from (1) that

$$\begin{aligned} x_i(t+1) &= a_{ii}x_i(t) + \sum_{j=1,j\neq i}^{r}a_{ij}x_j(t) + \sum_{j=1}^{r}b_{ij}x_j(t-h) + \sum_{j=r+1}^{n}c_{ij}x_j(t-h), i=\overline{1,r}, \\ x_i(t) &= \sum_{j=1}^{r}d_{ij}x_j(t) + \sum_{j=1}^{r}e_{ij}x_j(t-h) + \sum_{j=r+1}^{n}f_{ij}x_j(t-h), i=\overline{r+1,n}. \end{aligned} \tag{21}$$

Setting $\|\nu\|_h^{\triangleright} := \sup_{-h\le s\le 0} \|\nu(s)\|_{\infty}^{\lambda}$, and

$$z_i(t) = \begin{cases} \dfrac{x_i(t)}{\lambda_i} - \alpha^t \|\nu\|_h^{\triangleright}, \ -h \le t \le 0, i = \overline{1,n}, \\ \dfrac{x_i(t)}{\lambda_i} - (R_\lambda)^{m-1} \alpha^t \|\nu\|_h^{\triangleright}, t_{m-1} \le t < t_m, m \in \mathbb{Z}^+. \end{cases} \tag{22}$$

We will show that

$$z_i(t) \le 0, \ \forall t \ge 0, \forall \ i = \overline{1,n}. \tag{23}$$

To prove (23), we divide into the following steps:
Step 1: We prove that (23) holds for $t \in [-h, 0]$. Indeed, from (18), $x(s) = \nu(s), s \in [-h, 0]$, $\alpha \in (0,1)$ and the definition of $\|\nu\|_h^{\triangleright}$, we obtain

$$\begin{aligned} z_i(t) = \frac{x_i(t)}{\lambda_i} - \alpha^t \|\nu\|_h^{\triangleright} &\le \|x(t)\|_{\infty}^{\lambda} - \alpha^t \|\nu\|_h^{\triangleright} \\ &\le \|\nu\|_h^{\triangleright} - \alpha^t \|\nu\|_h^{\triangleright} \le 0, i = \overline{1,n}, t \in [-h, 0]. \end{aligned} \tag{24}$$

Step 2: We prove that

$$z_i(t) \le 0, \forall \ i = \overline{1,n}, t \in (0, t_1 - 1]. \tag{25}$$

Assume that this is not true. Then $\exists \ m \in \{1, 2, \ldots, n\}$, and $\bar{t} \in (0, t_1 - 1]$ such that

$$z_i(t) \le 0, t \in (0, \bar{t} - 1], i = \overline{1,n}, \tag{26}$$

and

$$z_m(\bar{t}) > 0. \tag{27}$$

From (22), (24) and (26) we get

$$x(t) \preceq \alpha^t \|\nu\|_h^{\triangleright} \lambda, \ \forall \ t \in [-h, \bar{t} - 1]. \tag{28}$$

Now we consider two cases:

Case 1: If the index $m \in \{1,2,\ldots,r\}$, from (19), (21), (22) and (28) we have:

$$\begin{aligned}
z_m(\bar{t}) &= \frac{x_m(\bar{t})}{\lambda_m} - \alpha^{\bar{t}}\|\nu\|_h^{\triangleright} \\
&\leq \frac{1}{\lambda_m}\Big(a_{mm}x_m(\bar{t}-1) + \sum_{j=1,j\neq m}^{r} a_{mj}x_j(\bar{t}-1) + \sum_{j=1}^{r} b_{mj}x_j(\bar{t}-1-h) \\
&\qquad + \sum_{j=r+1}^{n} c_{mj}x_j(\bar{t}-1-h)\Big) - \alpha^{\bar{t}}\|\nu\|_h^{\triangleright} \\
&\leq \frac{1}{\lambda_m}\Big(a_{mm}\alpha^{\bar{t}-1}\|\nu\|_h^{\triangleright}\lambda_m + \sum_{j=1,j\neq m}^{r} a_{mj}\alpha^{\bar{t}-1}\|\nu\|_h^{\triangleright}\lambda_j + \sum_{j=1}^{r} b_{mj}\alpha^{-h}\alpha^{\bar{t}-1}\|\nu\|_h^{\triangleright}\lambda_j \\
&\qquad + \sum_{j=r+1}^{n} c_{mj}\alpha^{-h}\alpha^{\bar{t}-1}\|\nu\|_h^{\triangleright}\lambda_j\Big) - \alpha^{\bar{t}}\|\nu\|_h^{\triangleright} \\
&= \alpha^{\bar{t}-1}\|\nu\|_h^{\triangleright}\Big(\big(\sum_{j=1}^{r} a_{mj} + \alpha^{-h}b_{mj}\big)\frac{\lambda_j}{\lambda_m} + \sum_{j=r+1}^{n} \alpha^{-h}c_{mj}\frac{\lambda_j}{\lambda_m} - \alpha\Big) \\
&\underset{(19)}{<} 0,
\end{aligned} \tag{29}$$

which is in conflict with (27), $z_m(\bar{t}) > 0$, this implies that

$$z_i(t) \leq 0, \forall\, i = \overline{1,r}, t \in (0, t_1 - 1]. \tag{30}$$

Case 2: If the index $m \in \{r+1, r+2, \ldots, n\}$, from (20), (21), (22) and (28) we obtain:

$$\begin{aligned}
z_m(\bar{t}) &= \frac{x_m(\bar{t})}{\lambda_m} - \alpha^{\bar{t}}\|\nu\|_h^{\triangleright} \\
&= \frac{1}{\lambda_m}\Big(\sum_{j=1}^{r} d_{mj}x_j(\bar{t}) + \sum_{j=1}^{r} e_{mj}x_j(\bar{t}-h) + \sum_{j=r+1}^{n} f_{mj}x_j(\bar{t}-h)\Big) - \alpha^{\bar{t}}\|\nu\|_h^{\triangleright} \\
&\leq \frac{1}{\lambda_m}\Big(\sum_{j=1}^{r} d_{mj}\alpha^{\bar{t}}\|\nu\|_h^{\triangleright}\lambda_j + \sum_{j=1}^{r} e_{mj}\alpha^{-h}\alpha^{\bar{t}}\|\nu\|_h^{\triangleright}\lambda_j + \sum_{j=r+1}^{n} f_{mj}\alpha^{-h}\alpha^{\bar{t}}\|\nu\|_h^{\triangleright}\lambda_j\Big) - \alpha^{\bar{t}}\|\nu\|_h^{\triangleright} \\
&= \alpha^{\bar{t}}\|\nu\|_h^{\triangleright}\Big(\big(\sum_{j=1}^{r} d_{mj} + e_{mj}\alpha^{-h}\big)\frac{\lambda_j}{\lambda_m} + \sum_{j=r+1}^{n} f_{mj}\alpha^{-h}\frac{\lambda_j}{\lambda_m} - 1\Big) \\
&\underset{(20)}{<} 0,
\end{aligned} \tag{31}$$

which is in contradiction with $z_m(\bar{t}) > 0$, then we obtain

$$z_i(t) \leq 0, \forall\, i = \overline{r+1,n}, t \in (0, t_1 - 1]. \tag{32}$$

Combining together with (30) and (32) implies (25) hold.

Step 3: Suppose that $z_i(t) \leq 0, \forall\, i = \overline{1,n}, t \in [0, t_s), s \in \mathbb{Z}^+$, then

$$z_i(t) \leq 0, i = \overline{1,n}, t \in [t_{m-1}, t_m), m = 1,2,\ldots,s. \tag{33}$$

For $i=\overline{1,n}$, we prove that $z_i(t)\le 0$, $t\in[t_s,t_{s+1})$. First, we show that $z_i(t_s)\le 0, i=\overline{1,n}$. From (22) and (33), we get

$$\begin{aligned} &x_i(t) \underset{(22)}{\le} (R_\lambda)^{m-1}\|\nu\|_h^{\triangleright}\alpha^t\lambda_i, i=\overline{1,n}, t\in[t_{m-1},t_m),\ m=1,2,\dots,s,\\ &x(t_s-1)\preceq (R_\lambda)^{s-1}\alpha^{t_s-1}\|\nu\|_h^{\triangleright}\lambda. \end{aligned} \tag{34}$$

From (14), (22) and (34), for $i=1,\dots,r$, we get

$$\begin{aligned} z_i(t_s) &\underset{(22)}{=} \frac{x_i(t_s)}{\lambda_i}-(R_\lambda)^s\alpha^{t_s}\|\nu\|_h^{\triangleright}=\sum_{j=1}^r h_{ij}\frac{x_j(t_s-1)}{\lambda_i}-(R_\lambda)^s\alpha^{t_s}\|\nu\|_h^{\triangleright}\\ &\underset{(34)}{\le} (R_\lambda)^{s-1}\alpha^{t_s-1}\|\nu\|_h^{\triangleright}\sum_{j=1}^r h_{ij}\frac{\lambda_j}{\lambda_i}-(R_\lambda)^s\alpha^{t_s}\|\nu\|_h^{\triangleright}\\ &=(R_\lambda)^{s-1}\alpha^{t_s-1}\|\nu\|_h^{\triangleright}\left(\sum_{j=1}^r h_{ij}\frac{\lambda_j}{\lambda_i}-\alpha R_\lambda\right)\\ &\underset{(14)}{\le} 0, i=\overline{1,r}, \end{aligned} \tag{35}$$

then, we have

$$x_i(t_s)\le (R_\lambda)^s\alpha^{t_s}\|\nu\|_h^{\triangleright}\lambda_i, i=\overline{1,r}. \tag{36}$$

Note that $R_\lambda>1, h>0, t\ge 0$, then $\forall\, t\in[0,t_s)$, we get

$$x_i(t-h)\le (R_\lambda)^s\|\nu\|_h^{\triangleright}\alpha^{-h}\alpha^t\lambda_i, i=\overline{1,n}. \tag{37}$$

For $i=r+1,\dots,n$, from (20), (21), (22), (34), (36) and (37) we achieve

$$\begin{aligned} z_i(t_s) &= \frac{x_i(t_s)}{\lambda_i}-\alpha^{t_s}\|\nu\|_h^{\triangleright}(R_\lambda)^s\\ &=\frac{1}{\lambda_i}\left(\sum_{j=1}^r d_{ij}x_j(t_s)+\sum_{j=1}^r e_{ij}x_j(t_s-h)+\sum_{j=r+1}^n f_{ij}x_j(t_s-h)\right)-\alpha^{t_s}\|\nu\|_h^{\triangleright}(R_\lambda)^s\\ &\le\frac{1}{\lambda_i}\left(\sum_{j=1}^r d_{ij}(R_\lambda)^s\alpha^{t_s}\|\nu\|_h^{\triangleright}\lambda_j+\sum_{j=1}^r (R_\lambda)^s e_{ij}\alpha^{-h}\alpha^{t_s}\|\nu\|_h^{\triangleright}\lambda_j\right.\\ &\qquad\left.+\sum_{j=r+1}^n (R_\lambda)^s f_{ij}\alpha^{-h}\alpha^{t_s}\|\nu\|_h^{\triangleright}\lambda_j\right)-(R_\lambda)^s\alpha^{t_s}\|\nu\|_h^{\triangleright}\\ &=(R_\lambda)^s\alpha^{t_s}\|\nu\|_h^{\triangleright}\left(\Big(\sum_{j=1}^r d_{ij}+e_{ij}\alpha^{-h}\Big)\frac{\lambda_j}{\lambda_i}+\sum_{j=r+1}^n f_{ij}\alpha^{-h}\frac{\lambda_j}{\lambda_i}-1\right)\\ &\underset{(20)}{<} 0. \end{aligned} \tag{38}$$

Combining inequalities (35) and (38), we get

$$z_i(t_s)\le 0, i=\overline{1,n}. \tag{39}$$

Therefore, we only need to show that $z_i(t) \le 0$ for all $i = \overline{1,n}, t \in (t_s, t_{s+1})$. Assume that this is not true, then $\exists\ p \in \{1,2,\ldots,n\}$, $\hat{t} \in (t_s, t_{s+1}-1]$ such that

$$z_i(t) \le 0, i = \overline{1,n}, t \in (t_s, \hat{t}-1], \tag{40}$$

$$z_p(\hat{t}) > 0. \tag{41}$$

Combining (33), (39) and (40) yields

$$\begin{aligned} &x(t) \preceq (R_\lambda)^{m-1} \|\nu\|_h^{\triangleright} \alpha^t \lambda,\ t \in [t_{m-1}, t_m), m = 1, \cdots, s. \\ &x(t) \preceq (R_\lambda)^{s} \|\nu\|_h^{\triangleright} \alpha^t \lambda,\ t \in [t_s, \hat{t}-1]. \end{aligned} \tag{42}$$

Now we consider two cases:
Case I: If the index $p \in \{1,2,\ldots,r\}$, from (19), (21), (22), (42) we have:

$$\begin{aligned} z_p(\hat{t}) &= \frac{x_p(\hat{t})}{\lambda_p} - (R_\lambda)^s \alpha^{\hat{t}} \|\nu\|_h^{\triangleright} \\ &\le \frac{1}{\lambda_p}\Big(a_{pp} x_p(\hat{t}-1) + \sum_{j=1, j\neq p}^{r} a_{pj} x_j(\hat{t}-1) + \sum_{j=1}^{r} b_{pj} x_j(\hat{t}-1-h) + \sum_{j=r+1}^{n} c_{pj} x_j(\hat{t}-1-h)\Big) \\ &\quad - (R_\lambda)^s \alpha^{\hat{t}} \|\nu\|_h^{\triangleright} \\ &\le \frac{1}{\lambda_p}\Big(a_{pp} (R_\lambda)^s \alpha^{\hat{t}-1} \|\nu\|_h^{\triangleright} \lambda_p + \sum_{j=1, j\neq p}^{r} a_{pj} (R_\lambda)^s \alpha^{\hat{t}-1} \|\nu\|_h^{\triangleright} \lambda_j + \sum_{j=1}^{r} b_{pj} (R_\lambda)^s \alpha^{-h} \alpha^{\hat{t}-1} \|\nu\|_h^{\triangleright} \lambda_j \\ &\qquad + \sum_{j=r+1}^{n} c_{pj} (R_\lambda)^s \alpha^{-h} \alpha^{\hat{t}-1} \|\nu\|_h^{\triangleright} \lambda_j\Big) - (R_\lambda)^s \alpha^{\hat{t}} \|\nu\|_h^{\triangleright} \\ &= (R_\lambda)^s \alpha^{\hat{t}-1} \|\nu\|_h^{\triangleright} \Big(\big(\sum_{j=1}^{r} a_{pj} + \alpha^{-h} b_{pj}\big) \frac{\lambda_j}{\lambda_p} + \sum_{j=r+1}^{n} \alpha^{-h} c_{pj} \frac{\lambda_j}{\lambda_p} - \alpha \Big) \\ &\underset{(19)}{<} 0, \end{aligned} \tag{43}$$

the opposite of (41), $z_p(\hat{t}) > 0$, therefore we obtain $z_i(t) \le 0,\ i = \overline{1,r}, t \in (t_s, t_{s+1})$.
Case II: If the index $p \in \{r+1, r+2, \ldots, n\}$, from (20), (21), (22), and (42) we obtain:

$$\begin{aligned} z_p(\hat{t}) &= \frac{x_p(\hat{t})}{\lambda_p} - (R_\lambda)^s \alpha^{\hat{t}} \|\nu\|_h^{\triangleright} \\ &= \frac{1}{\lambda_p}\Big(\sum_{j=1}^{r} d_{pj} x_j(\hat{t}) + \sum_{j=1}^{r} e_{pj} x_j(\hat{t}-h) + \sum_{j=r+1}^{n} f_{pj} x_j(\hat{t}-h)\Big) - (R_\lambda)^s \alpha^{\hat{t}} \|\nu\|_h^{\triangleright} \\ &\le \frac{1}{\lambda_p}\Big(\sum_{j=1}^{r} (R_\lambda)^s d_{pj} \alpha^{\hat{t}} \|\nu\|_h^{\triangleright} \lambda_j + \sum_{j=1}^{r} (R_\lambda)^s e_{pj} \alpha^{-h} \alpha^{\hat{t}} \|\nu\|_h^{\triangleright} \lambda_j + \sum_{j=r+1}^{n} (R_\lambda)^s f_{pj} \alpha^{-h} \alpha^{\hat{t}} \|\nu\|_h^{\triangleright} \lambda_j \Big) \\ &\qquad - (R_\lambda)^s \alpha^{\hat{t}} \|\nu\|_h^{\triangleright} \\ &= (R_\lambda)^s \alpha^{\hat{t}} \|\nu\|_h^{\triangleright} \Big(\big(\sum_{j=1}^{r} d_{pj} + e_{pj} \alpha^{-h}\big) \frac{\lambda_j}{\lambda_p} + \sum_{j=r+1}^{n} f_{pj} \alpha^{-h} \frac{\lambda_j}{\lambda_p} - 1 \Big) \underset{(20)}{<} 0, \end{aligned} \tag{44}$$

which contradicts with (41), $z_p(\hat{t}) > 0$, therefore we obtain $z_i(t) \le 0,\ i = \overline{r+1,n}, t \in (t_s, t_{s+1})$. As a result of the mathematical induction principle, we have established that $z_i(t) \le 0, t \ge 0, i = \overline{1,n}$. In this light, we get

$$\frac{x_i(t)}{\lambda_i} \le (R_\lambda)^{m-1} \alpha^t \|v\|_h^{\triangleright}, i = \overline{1,n}, t \in [t_{m-1}, t_m), m \in \mathbb{Z}^+,$$

then

$$\mathfrak{V}(x(t)) = \max_{1 \le i \le n} \left\{ \frac{x_i(t)}{\lambda_i} \right\} \le (R_\lambda)^{m-1} \alpha^t \|v\|_h^{\triangleright}, t \in [t_{m-1}, t_m), m \in \mathbb{Z}^+. \tag{45}$$

Using (13), we have

$$R_\lambda \le \alpha^{-\delta \mathfrak{T}} \le \alpha^{-\delta(t_m - t_{m-1})}, \quad m \in \mathbb{Z}^+$$

combine this with inequality (45) to get

$$\begin{aligned}
\|x(t)\| &\le \|\mathfrak{V}(x(t))\lambda\| = \mathfrak{V}(x(t))\|\lambda\| \le (R_\lambda)^{m-1} \alpha^t \|v\|_h^{\triangleright} \|\lambda\| \\
&\le \alpha^{-\delta t_1} \alpha^{-\delta(t_2 - t_1)} \dots \alpha^{-\delta(t_{m-1} - t_{m-2})} \alpha^t \|v\|_h^{\triangleright} \|\lambda\| \\
&= \alpha^{-\delta[t_1 + (t_2 - t_1) + \dots + (t_{m-1} - t_{m-2})]} \alpha^t \|v\|_h^{\triangleright} \|\lambda\| = \alpha^{-\delta t_{m-1}} \alpha^t \|v\|_h^{\triangleright} \|\lambda\| \\
&= \alpha^{-\delta(t_{m-1} - t)} \alpha^{-\delta t} \alpha^t \|v\|_h^{\triangleright} \|\lambda\| \le \alpha^{(1-\delta)t} \|v\|_h^{\triangleright} \|\lambda\|, t \in [t_{m-1}, t_m), m \in \mathbb{Z}^+,
\end{aligned}$$

thus

$$\|x(t)\| \le \alpha^{(1-\delta)t} \|v\|_h^{\triangleright} \|\lambda\|, t \ge 0.$$

This means that system (1) is exponentially stable. □

4. Numerical examples

Example 1 Consider system (1) where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0.15 & 0.2 & 0 \\ 0.1 & 0.2 & 0 \\ 0.15 & 0.2 & -0.7 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0 \\ 1.0 & 2.0 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0.5 \\ 2 & 0.2 \end{bmatrix},$$

and $h = 2$. We see that $A_E := A + I_3 - E, A_d, H$ are nonnegative matrices. For $\alpha = 0.8, \delta = 0.9$, we can quickly check that the conditions (12), (13), (14) are satisfied with $\lambda = [11\ 10\ 76]^T$. By some simple calculation, we have $R_\lambda^1 = 1.4545 > 1$, $R_\lambda^2 = 2.4 > 1$, and $R_\lambda = 2.4 > 1$. Thus, if $t_m - t_{m-1} \ge \mathfrak{T} \ge -\frac{1}{\delta}\log_\alpha R_\lambda, \approx 3.5310, m \in \mathbb{Z}^+$, then, from Theorem 3, the system is exponentially stable.

5. Conclusions

This research paper proposes a method for analyzing the exponential stability of discrete-time impulsive positive singular systems with time delay. It presents new results on impulsive exponential stability for discrete-time impulsive positive singular systems with time delay, which have not been previously reported in the literature. The proposed method is demonstrated to be effective through a numerical example.

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