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New criteria for exponential stability of a class of nonlinear continuous-time difference systems with delays

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ABSTRACT
In this paper, we present some new explicit criteria for exponential stability of positive monotone homogeneous continuous-time difference systems. Then, we apply the comparison principle to prove some novel criteria for exponential stability of general nonlinear continuous-time difference systems with delays, not necessarily monotone and homogeneous. The obtained criteria include many results existing in the literature as particular cases. Some examples are given to illustrate the obtained results.

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1. Introduction

This paper is devoted to the study of asymptotic properties of nonlinear monotone dynamical systems described by continuous-time difference equations.

Applications of continuous-time difference equations are well explained in several books, see, e.g. Hale and Lunel (1993) and Niculescu (2001), where mathematical models in economics and gas dynamics described by this type of equations are presented. Continuous-time difference equations with delays appear in problems of delay approximation of the partial differential equations describing the propagation phenomena in excitable media (Courtemanche et al., 1996), in stability analysis of time-delay differential equations involving system transformations (Kharitonov & Melchor-Aguilar, 2002; Melchor-Aguilar et al., 2010) and difference operators in neutral functional differential equations (Hale & Lunel, 1993; Ngoc et al., 2007) and in coupled differential-difference equations (Pepe, 2005).

Stability of systems is always an important research topic in the theory of control of dynamical systems. Given a widespread interest in applications of continuous-time difference systems, stability analysis for this class of systems has recently attracted a good deal of attention, see, e.g. Carvalho (1996), Chitour et al. (2016), Damak et al. (2015, 2016), Di Loreto et al. (2016), Di Loreto & Loiseau (2012), Iuliis et al. (2017), Kharitonov (1996), Melchor-Aguilar (2013, 2019) and references therein. So far, most of works in this field have been concentrated on exponential stability analysis for linear time-invariant systems with discrete multiple delays or distributed delays of the form

$$\boldsymbol{x}(t) = \sum_{i=1}^{m} A_i \boldsymbol{x}(t-h_i) + \int_{-h}^{0} C(s)\boldsymbol{x}(t+s)\,\mathrm{d}s, \quad t \ge t_0, \quad (1)$$

or general linear time-invariant functional difference equations, by using either the systems spectral properties or the Lyapunov functions based approach. The more general case of time-varying linear systems (for instance, when matrix functions $A_i(t), C(t,s), t \ge t_0$, are considered in the system model (1)) has been studied in Ngoc and Huy (2015), by using the comparison principle and spectral properties and non-negative matrices. Nonlinear difference equations with delays have been considered in Gil and Cheng (2007), Melchor-Aguilar (2016) and Shaikhet (2004), mainly by Lyapunov functions methods. In particular, in Gil and Cheng (2007), some sufficient conditions for asymptotic stability of the zero solution of the semi-linear continuous-time difference systems with discrete delays

$$\boldsymbol{x}(t) = \sum_{i=1}^{m} A_i(t)\boldsymbol{x}(t-h_i) + F(t, \boldsymbol{x}(t-h_1), \boldsymbol{x}(t-h_2), \ldots, \boldsymbol{x}(t-h_m)), \quad t \ge 0, \quad (2)$$

have been obtained, by using characteristic matrix-valued function and Laplace transform, under the assumption that the nonlinear perturbation F is sufficiently small. These results have been generalised recently in Melchor-Aguilar (2016) to the case when F may include both discrete and integral delay terms.

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In this paper, we will study exponential stability of the class of *nonlinear monotone* continuous-time difference systems with both discrete delays and distributed delays of the general form

$$x(t) = F\left(t, x(t-h_1(t)), x(t-h_2(t)), \ldots, x(t-h_m(t)), \int_{-h(t)}^{0} G(t,s,x(t+s))\,\mathrm{d}s\right), \tag{3}$$

$t \geq t_0$, which clearly includes linear models (1) and (2) as special cases. It is worth noticing that monotone homogeneous systems comprise an important class of dynamical systems which preserve an order in their state space. This types of systems are of significant interest both for their practical applicability (Sontag, 2007) and theoretical properties (Smith, 2008) and, in recent years, problems of their stability analysis have attracted a good deal of attention, see, e.g. Dashkovskiy et al. (2006), Dirr et al. (2015) and Feyzmahdavian et al. (2014a, 2014b). Note that, up to now, most efforts have been dedicated to time-invariant monotone systems only. Exponential stability of time-varying monotone systems, with both discrete delays and distributed delays, is studied for the first time in this paper.

As a primary purpose, we will first establish some characterisations of exponential stability for a class of nonlinear monotone and homogeneous continuous-time difference systems of the form (3). These results are shown to include many known results in particular cases. Furthermore, the obtained results are applied to get sufficient conditions of exponential stability for a general class of nonlinear systems which can be upper-bounded by monotone and homogeneous systems (in particular, by positive linear systems). To the best of our knowledge, such general results are novel in the existing literature. In contrast to the traditional Lyapunov functional-based method, our approach relies on the comparison principle, with using monotonicity and homogeneity of functions and the spectral properties of non-negative matrices. Note additionally that a similar approach has been used to study exponential stability for other classes of nonlinear systems, see, e.g. Ngoc and Hieu (2013) (for discrete-time model) and Tian and Sun (2020) (for differential model). Differently, in this paper, we are dealing with exponential stability for the continuous-time difference systems, containing both multiple discrete delays and distributed delays.

The paper is organised as follows: In Section 1, we present the scientific significance and motivation that lead us to the problem of stability of general nonlinear continuous-time difference system of the form (3). In Section 2, we prove some novel criteria for exponential stability of the zero solution of (3) when F and G are monotone and positive homogeneous functions for all non-negative initial conditions, satisfying some matching assumption. In Section 3, we generalise the results to more general cases when F and G are not necessarily positive but are upper-bounded by some monotone and positive homogeneous functions. In Section 4, we summarise the main contributions of this paper.

Some notations. We now introduce some notations and present some preliminary results which will be of use in the paper. Throughout the paper, vectors are written in bold lower case letters and matrices in capital letters, except for the zero vector and the zero matrix, being denoted both as **0**. Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the set of all natural numbers, real numbers and complex numbers, respectively. Set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For given $n, m \in \mathbb{N}, \mathbb{R}^n$ denotes the vector space of all n-tuples of real numbers and $\mathbb{R}^{n\times m}$ denotes the space of $n \times m$ matrices with real components. For a given $k \in \mathbb{N}$, denote $\underline{k} := \{1, 2, \ldots, k\}$ and $\mathbb{R}^{nk} = \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{k \text{ times}}$. The set of all vectors (resp., matrices) in $\mathbb{R}^n$ (resp., $\mathbb{R}^{n\times m}$) with non-negative components is denoted by $\mathbb{R}^n_+$ (resp., $\mathbb{R}^{n\times m}_+$). The identity $n \times n$ matrix is denoted by I_n. Inequalities between real vectors and matrices will be understood componentwise. More precisely, for $x, y \in \mathbb{R}^n$, we write: $x \geq y$ if $x_i \geq y_i$ for $i \in \underline{n} := \{1, 2, \ldots, n\}$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i$ for $i \in \underline{n}$. Similar notations are adopted for matrices. If $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n$ and $P = (p_{ij}) \in \mathbb{R}^{l\times q}$, we denote $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^\top \in \mathbb{R}^n_+$, $|P| = (|p_{ij}|) \in \mathbb{R}^{l\times q}_+$. For any matrix $A \in \mathbb{R}^{n\times n}$, the *spectral radius* of A is denoted by $\rho(A) = \max\{|z| : z \in \mathbb{C},\ \det(zI_n - A) = 0\}$. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Schur stable* if $\rho(A) < 1$. Throughout the paper, the norm of vectors is assumed to be monotonic, that is $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in \mathbb{R}^n$, or equivalently, $\|x\| = \||x|\|$, $\forall x \in \mathbb{R}^n$, (see, e.g.) Note that every p-norm on $\mathbb{R}^n$, $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$, $1 \leq p < \infty$ and $\|x\|_\infty = \max_{i\in\underline{n}} |x_i|)$, are monotonic (Horn & Johnson, 1986). The norm $\|M\|$ of a matrix $M \in \mathbb{R}^{l\times q}$ is defined by $\|M\| = \max_{\|y\|=1} \|My\|$, where $\mathbb{R}^l$ and $\mathbb{R}^q$ are provided with some monotonic vector norms. Then, the following monotonicity property holds $\|P\| \leq \||P|\| \leq \|Q\|$, whenever $P \in \mathbb{R}^{l\times q}, Q \in \mathbb{R}^{l\times q}_+$, $|P| \leq Q$, see, e.g. Hinrichsen and Son (1998). For any $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n_+$ and any $\alpha \in \mathbb{R}$, we define the vector $x^\alpha = (x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha)^\top$. The *Hadamard product* of two arbitrary vectors $x, y \in \mathbb{R}^n$ is defined as $x \circ y = (x_1y_1, x_2y_2, \ldots, x_ny_n)^\top \in \mathbb{R}^n$. By induction, we can define the Hadamard product $\mathbf{H}_{j=1}^q z_j = z_1 \circ z_2 \circ \cdots \circ z_q$, of any q vectors in $\mathbb{R}^n$. It is easy to verify that if $\mathbb{R}^n$ is equipped with p-norm, $1 \leq p \leq \infty$, then

$$\| \mathop{\mathbf{H}}_{j=1}^{q} z_j^{\alpha_j} \| \leq \prod_{j=1}^{q} \|z_j\|^{\alpha_j} \quad \forall z_j \in \mathbb{R}^n_+,\ \forall \alpha_j \in \mathbb{R}_+.$$

For $k \in \mathbb{N}$ and $\delta > 0, \mathcal{B}^k(\delta) := \{x \in \mathbb{R}^k : \|x\| < \delta\}$ is the open ball of radius δ centred at the origin in $\mathbb{R}^k$. Finally, for any $\tau > 0$, $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of all continuous functions $\varphi : [-\tau, 0] \to \mathbb{R}^n$ equipped with the norm $\|\varphi\|_{[-\tau,0]} = \max_{s\in[-\tau,0]} \|\varphi(s)\|$.

2. Exponential stability of positively monotone homogeneous systems

In this section, we shall prove some criteria for exponential stability of the zero solution of the continuous-time difference equations of the form (3) where $F(t, \cdot, \ldots, \cdot)$ and $G(t, s, \cdot)$ are assumed to be monotone homogeneous and non-negative functions.

First, we recall some definitions and known results necessary for establishing the main results of this paper.

Definition 2.1: Let $k, n \in \mathbb{N}$ and a closed convex cone $C \subset \mathbb{R}^k$ be given. A function $H(\cdot) : \mathbb{R}^k \longmapsto \mathbb{R}^n$ is said to be

(i) *monotone on* C if $H(\boldsymbol{u}) \geq H(\boldsymbol{v})$, for any $\boldsymbol{u}, \boldsymbol{v} \in C$ satisfying $\boldsymbol{u} \geq \boldsymbol{v}$;
(ii) *homogeneous on* C if $H(\lambda \boldsymbol{u}) = \lambda H(\boldsymbol{u})$, for all $\boldsymbol{u} \in C$ and all real numbers $\lambda > 0$.

If $C = \mathbb{R}^k$ or $C = \mathbb{R}^k_+$ in (i) and (ii), then H is simply said to be monotone homogeneous or, respectively, positively monotone homogeneous, if no confusion can arise.

Clearly, if H is monotone on $\mathbb{R}^k_+$ and, moreover, $H(\mathbf{0}) = \mathbf{0}$, then $H(\boldsymbol{x}) \in \mathbb{R}^n_+$ for all $\boldsymbol{x} \in \mathbb{R}^k_+$. Therefore, if H is a linear function, then H is monotone if and only if its matrix (with respect to the standard bases) is non-negative. These facts make the class of dynamical systems described by monotone and homogeneous functions having many properties similar to those of positive linear systems, in particular, those resulting from the famous Perron–Frobenius Theorem for non-negative matrices (see, e.g. Berman & Plemmons, 1979). Many efforts have been observed during the last two decades to extend known results on stability analysis for positive linear systems to this particular class of nonlinear systems (see, e.g. Feyzmahdavian et al., 2014a and the references given therein). For instance, it has been proved, based on Perron–Frobenius Theorem, that the discrete-time LTI positive system with time-varying delay

$$\boldsymbol{x}(t+1) = A\boldsymbol{x}(t) + B\boldsymbol{x}(t - \tau(t)), \quad t \in \mathbb{Z}_+,$$

where $A, B \in \mathbb{R}^{n\times n}_+$ and $0 \leq \tau(t) \leq \tau_{\max}$ is exponentially stable iff there exists some vector $\boldsymbol{v} \gg 0$ such that $(A + B)\boldsymbol{v} \ll \boldsymbol{v}$ (see, e.g. Haddad & Chellaboina, 2004; Liu et al., 2010). The extension of this result to monotone homogeneous systems has been obtained in Feyzmahdavian et al. (2014b) as follows: The discrete-time system

$$\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{x}(k)) + \boldsymbol{g}(\boldsymbol{x}(k - \tau(k))), \quad k \in \mathbb{Z}_+,$$

where $\boldsymbol{f}, \boldsymbol{g} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuous on $\mathbb{R}^n$, monotone and homogeneous on $\mathbb{R}^n_+$ and $\boldsymbol{f}(\mathbf{0}) = \boldsymbol{g}(\mathbf{0}) = \mathbf{0}$. Then the zero solution of this system is asymptotically stable if and only if there exists a vector $\boldsymbol{v} \gg \mathbf{0}$ such that $\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) \ll \boldsymbol{v}$. See also Mason and Verwoerd (2009) for the case of continuous-time differential equations.

The main objective of this section is to establish some new delay-independent, as well as delay-dependent, criteria of exponential stability for nonlinear continuous-time difference systems of the general form (3) under some monotonicity and homogeneity assumptions. It is important to note that (3) is a time-varying nonlinear system containing both multiple discrete delays and distributed delays. This general situation renders the stability analysis for (3) much more technology involved in comparison with the above time-invariant system. The cost to be paid is that only sufficient conditions for exponential stability of (3) will be established.

Consider the general nonlinear continuous-time difference system with discrete and distributed delays of the form (3), where $h(\cdot), h_i(\cdot) : \mathbb{R} \to \mathbb{R}_+, i \in \underline{m}$, are given continuous functions satisfying $0 < h(t) \leq \hat{h},\ 0 < h_i(t) \leq \hat{h}_i,\ i \in \underline{m}$; $F : \mathbb{R} \times \mathbb{R}^{n(m+1)} \longmapsto \mathbb{R}^n$ and $G : \mathbb{R} \times [-\hat{h}, 0] \times \mathbb{R}^n \longmapsto \mathbb{R}^n$ are given continuous functions.

Denote $\tau := \max\{\hat{h}, \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_m\}$. Fix $t_0 \geq 0$ and $\boldsymbol{\varphi} \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, consider (3) with the initial condition of the form

$$\boldsymbol{x}(s + t_0) = \boldsymbol{\varphi}(s), \quad s \in [-\tau, 0]. \tag{4}$$

By a solution of the initial value problem (3) and (4), we mean a *continuous* vector-valued function $\boldsymbol{x}(\cdot) : [-\tau + t_0, \infty) \longmapsto \mathbb{R}^n$ such that (4) holds and $\boldsymbol{x}(\cdot)$ satisfies (3) for all $t \geq t_0$.

Since functions $F, G, h_i, i \in \underline{m}$ and h are continuous, it follows that for any fixed $t_0 \geq 0$ and any given $\boldsymbol{\varphi} \in \mathcal{C} := \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, the initial value problem (3) and (4) has a unique continuous solution $\boldsymbol{x}(\cdot; t_0, \boldsymbol{\varphi})$, whenever φ satisfies the compatibility condition $\boldsymbol{\varphi} \in \mathcal{C}_{t_0}$, where

$$\mathcal{C}_{t_0} := \Big\{ \boldsymbol{\varphi} \in \mathcal{C} : \boldsymbol{\varphi}(0) = F(t_0, \boldsymbol{\varphi}(-h_1(t_0)), \ldots, \boldsymbol{\varphi}(-h_m(t_0)), \int_{-h(t_0)}^{0} G(t_0, s, \boldsymbol{\varphi}(s))\,\mathrm{d}s) \Big\}. \tag{5}$$

In what follows, the system (3) with the initial condition (4) satisfying the matching condition $\boldsymbol{\varphi} \in \mathcal{C}_{t_0}$ will be referred to as the system (3)–(5).

Remark 2.2: If the initial function $\boldsymbol{\varphi} \in \mathcal{C}$ is taken arbitrarily, then the initial value problem (3) and (4) still well admits, by an obvious iterative scheme, a unique *piecewise-continuous* solution $\boldsymbol{x}(t, t_0, \boldsymbol{\varphi}), t \geq t_0$, having possibly discontinuity at $t = t_0$ which will be propagated in time and hence discontinuity may occur again at $t_k = t_0 + k\tau$, $k = 1, 2, \ldots$ In general, instead of $\mathcal{C}$ one may choose a more general space (e.g. piecewise right-continuous functions space $\mathcal{PC}([-\tau, 0), \mathbb{R}^n)$ or even quadratic-integrable function space $\mathcal{L}_2([-\tau, 0], \mathbb{R}^n)$) but then the concept of solutions of the problem (3), (4) must be changed accordingly, see, e.g. Carvalho (1996), Melchor-Aguilar (2013, 2016) and Pepe (2014). In this paper, we will restrict ourselves to the notion of continuous solutions, so the space $\mathcal{C} := \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is chosen and the assumption that the initial function $\boldsymbol{\varphi} \in \mathcal{C}$ satisfies the matching condition (5) is needed and, consequently, the solution $\boldsymbol{x}(\cdot; t_0, \boldsymbol{\varphi})$ of (3) and (4) is continuous.

Definition 2.3: The dynamical system (3)–(5) is said to be *positive* if $\boldsymbol{x}(t; t_0, \boldsymbol{\varphi}) \in \mathbb{R}^n_+$ for all $t \geq t_0$ whenever $\boldsymbol{\varphi} \in \mathcal{C}^+_{t_0}$, where

$$\mathcal{C}^+_{t_0} := \{\boldsymbol{\varphi} \in \mathcal{C}_{t_0} :\ \boldsymbol{\varphi}(s) \geq 0 \text{ for all } s \in [-\tau, 0]\}. \tag{6}$$

It is easy to show that the system (3)–(5) is positive if $F(t, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{m+1}) \geq 0$ and $G(t, s, \boldsymbol{x}) \geq 0$, for all $t \geq 0, s \in [-\hat{h}, 0], \boldsymbol{x}_i \in \mathbb{R}^n_+, i \in \underline{m+1}$ and $\boldsymbol{x} \in \mathbb{R}^n_+$. This implies, in particular, that the system (3)–(5) is positive if, for each $t \geq 0, s \in [-\hat{h}, 0]$, the functions $F(t, \cdot)$ and $G(t, s, \cdot)$ are monotone on $\mathbb{R}^{n(m+1)}_+$ and $\mathbb{R}^n_+$, respectively.

Consider the system (3)–(5) and assume that the functions F and G satisfy the following condition:

$$F(t, \mathbf{0}, \ldots, \mathbf{0}) = G(t, s, \mathbf{0}) = \mathbf{0} \quad \forall t \geq t_0, s \in [-\hat{h}, 0]. \tag{7}$$

Then clearly the zero initial function $\boldsymbol{\varphi}(\cdot) = \mathbf{0}$ belongs to $\mathcal{C}_{t_0}$ and $\boldsymbol{x}_0(t) := \boldsymbol{x}(t, t_0, \mathbf{0}) \equiv \mathbf{0}, t \geq -\tau + t_0$ is a solution (called the *zero solution*) of (3)–(5).

Definition 2.4: (i) The zero solution of (3)–(5) is said to be *locally exponentially stable* (shortly, LES) if there exist $r_0 > 0, M > 0$ and $\beta \in (0,1)$ such that for any $\boldsymbol{\varphi} \in \mathcal{C}_{t_0}$,

$$\|\boldsymbol{\varphi}\|_{[-\tau,0]} \leq r_0 \Rightarrow \|\boldsymbol{x}(t,t_0,\boldsymbol{\varphi})\| \leq M\beta^{t-t_0}\|\boldsymbol{\varphi}\|_{[-\tau,0]}, \quad t \geq t_0. \tag{8}$$

(ii) The zero solution of (3)–(5) is said to be *globally exponentially stable* (shortly, GES) if r_0 in (8) can be taken as large as desired or, equivalently, there exist $M > 0$ and $\beta \in (0,1)$ such that for any $\boldsymbol{\varphi} \in \mathcal{C}_{t_0}$,

$$\|\boldsymbol{x}(t,t_0,\boldsymbol{\varphi})\| \leq M\beta^{t-t_0}\|\boldsymbol{\varphi}\|_{[-\tau,0]}, \quad t \geq t_0. \tag{9}$$

(iii) If (8) and (9) hold for any $\boldsymbol{\varphi} \in \mathcal{C}^+_{t_0}$, then the zero solution of (3)–(5) is said to be LES and GES *with non-negative initial conditions*, respectively.

We are now in a position to prove the main result of this section that gives a delay-dependent criterion for exponential stability of the zero solution of (3)–(5).

Theorem 2.5: *Consider the nonlinear continuous-time difference system* (3)–(5) *with continuous functions F, G. Assume that, for each* $t \geq t_0, s \in [-\hat{h}, 0]$*, the functions* $F(t,\cdot)$ *and* $G(t,s,\cdot)$ *are monotone and homogeneous on* $\mathbb{R}^{n(m+1)}_+$ *and* $\mathbb{R}^n_+$*, respectively. If there exist a vector* $\boldsymbol{p} \in \mathbb{R}^n_+, \boldsymbol{p} \gg \boldsymbol{0}$ *and* $\lambda > 1$ *such that*

$$F\left(t, \lambda^{h_1(t)}\boldsymbol{p}, \lambda^{h_2(t)}\boldsymbol{p}, \dots, \lambda^{h_m(t)}\boldsymbol{p}, \int_{-h(t)}^{0} G(t,s,\lambda^{-s}\boldsymbol{p})\,\mathrm{d}s\right) \ll \boldsymbol{p} \quad \forall t \geq t_0, \tag{10}$$

then the zero solution of (3)–(5) *is GES with non-negative initial conditions.*

Proof: First, it follows from the assumption on F, G that (7) holds and hence $\boldsymbol{x}(t) \equiv \boldsymbol{0}$, $t \geq -\tau + t_0$ is the zero solution of (3)–(5). Further, it is obvious that the system (3) is positive, and thus for any $\boldsymbol{\varphi} \in \mathcal{C}^+_{t_0}$, the corresponding solution $\boldsymbol{x}(t) = \boldsymbol{x}(t;t_0,\boldsymbol{\varphi})$ satisfies

$$\boldsymbol{x}(t) \geq 0, \quad \boldsymbol{x}(t-h(t)) \geq 0, \quad \boldsymbol{x}(t-h_i(t)) \geq 0 \quad \forall t \geq t_0, \quad \forall i \in \underline{m}. \tag{11}$$

We prove that there exist $M > 0$ and $\beta \in (0,1)$ such that

$$\|\boldsymbol{x}(t)\| \leq M\beta^{t-t_0}\|\boldsymbol{\varphi}\|_{[-\tau,0]}, \quad t \geq t_0, \tag{12}$$

for any $\boldsymbol{\varphi} \in \mathcal{C}^+_{t_0}$.

Take an arbitrary $\boldsymbol{\varphi} \in \mathcal{C}^+_{t_0}$ and let $\boldsymbol{x}(\cdot) := \boldsymbol{x}(\cdot;t_0,\boldsymbol{\varphi})$ be the solution of (3) and (4). Choose $K := K_{\boldsymbol{\varphi}} := \lambda\|\boldsymbol{\varphi}\|_{[-\tau,0]} \frac{1}{\min\{p_i, i \in \underline{n}\}}$, then

$$\boldsymbol{\varphi}(s) \ll K\boldsymbol{p} \quad \forall s \in [-\tau, 0]. \tag{13}$$

Setting $\beta = \lambda^{-1} \in (0,1)$, consider the vector-valued function

$$\boldsymbol{u}(t) = K\beta^{t-t_0}\boldsymbol{p}, \quad t \in \mathbb{R}. \tag{14}$$

It follows from (13) and the initial condition (4) that $\boldsymbol{x}(s+t_0) = \boldsymbol{\varphi}(s) \ll K\boldsymbol{p} = \boldsymbol{u}(t_0) \leq \boldsymbol{u}(s+t_0)$, $\forall s \in [-\tau, 0]$, or equivalently,

$$\boldsymbol{x}(t) \ll \boldsymbol{u}(t) \quad \forall t \in [-\tau + t_0, t_0]. \tag{15}$$

We show that

$$\boldsymbol{x}(t) \leq \boldsymbol{u}(t) \quad \forall t \geq t_0. \tag{16}$$

Assume to the contrary that (16) does not hold. Then, by the continuity, it follows from (15) that there exist $t_1 > t_0$ and $i_0 \in \underline{n}$ such that

$$\boldsymbol{x}(t) \ll \boldsymbol{u}(t) \quad \forall t \in [t_0, t_1) \quad \text{and} \quad x_{i_0}(t_1) = u_{i_0}(t_1). \tag{17}$$

By the assumptions (i), (ii) on monotonicity and homogeneity of F and G, we can deduce from (3), (10), (11), (14)–(16) and the inequality in (17) that

$$\begin{aligned}
x_{i_0}(t_1) &= F_{i_0}\left(t_1, \boldsymbol{x}(t_1 - h_1(t_1)), \dots, \boldsymbol{x}(t_1 - h_m(t_1)), \int_{-h(t_1)}^{0} G(t_1, s, \boldsymbol{x}(t_1+s))\,\mathrm{d}s\right) \\
&\leq F_{i_0}\left(t_1, \boldsymbol{u}(t_1 - h_1(t_1)), \dots, \boldsymbol{u}(t_1 - h_m(t_1)), \int_{-h(t_1)}^{0} G(t_1, s, \boldsymbol{u}(t_1+s))\,\mathrm{d}s\right) \\
&= F_{i_0}\left(t_1, K\beta^{t_1-h_1(t_1)-t_0}\boldsymbol{p}, \dots, K\beta^{t_1-h_m(t_1)-t_0}\boldsymbol{p}, \int_{-h(t)}^{0} G(t_1, s, K\beta^{t_1+s-t_0}\boldsymbol{p})\,\mathrm{d}s\right) \\
&= K\beta^{t_1-t_0}F_{i_0}\left(t_1, \beta^{-h_1(t_1)}\boldsymbol{p}, \dots, \beta^{-h_m(t_1)}\boldsymbol{p}, \int_{-h(t_1)}^{0} G(t_1, s, \beta^{s}\boldsymbol{p})\,\mathrm{d}s\right) \\
&= K\beta^{t_1-t_0}F_{i_0}\left(t_1, \lambda^{h_1(t_1)}\boldsymbol{p}, \dots, \lambda^{h_m(t_1)}\boldsymbol{p}, \int_{-h(t_1)}^{0} G(t_1, s, \lambda^{-s}\boldsymbol{p})\,\mathrm{d}s\right) \\
&< K\beta^{t_1-t_0}p_{i_0} = u_{i_0}(t_1).
\end{aligned}$$

However, this conflicts with the equality in (17). Thus, (16) holds and implies, by the monotonicity of vector norms, that $\|\boldsymbol{x}(t)\| = \||\boldsymbol{x}(t)|\| \leq \|\boldsymbol{u}(t)\|$. Setting $M := \frac{\lambda}{\min\{p_i, i \in \underline{n}\}}\|\boldsymbol{p}\|$, we see that M is independent of $\boldsymbol{\varphi}$, $M > 1$ and

$$\|\boldsymbol{x}(t)\| \le \|\boldsymbol{u}(t)\| = K\beta^{t-t_0}\|\boldsymbol{p}\| = \lambda\|\boldsymbol{\varphi}\|_{[-\tau,0]} \times \frac{1}{\min\{p_i, i \in \underline{n}\}}\beta^{t-t_0}\|\boldsymbol{p}\| = M\beta^{t-t_0}\|\boldsymbol{\varphi}\|_{[-\tau,0]},$$

for all $t \ge t_0$, as was to be shown. This completes the proof. ■

In the case of time-invariant systems, Theorem 2.5 implies the following delay-independent characterisation of exponential stability which is easier to verify than (10).

Corollary 2.6: *Let $F(t, \boldsymbol{u}_1, \dots, \boldsymbol{u}_{m+1}) \equiv F(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{m+1})$, $G(t, s, \boldsymbol{u}) \equiv G(s, \boldsymbol{u})$ for all $t \ge t_0 \ge 0$. Suppose that for each $s \in [-\hat{h}, 0]$, the functions $F(\cdot)$ and $G(s, \cdot)$ are monotone and homogeneous on $\mathbb{R}_+^{n(m+1)}$ and $\mathbb{R}_+^n$, respectively. If there exists a vector $\boldsymbol{p} \in \mathbb{R}_+^n, \boldsymbol{p} \gg \boldsymbol{0}$, such that*

$$F\left(\boldsymbol{p}, \dots, \boldsymbol{p}, \int_{-\hat{h}}^{0} G(s, \boldsymbol{p})\,\mathrm{d}s\right) \ll \boldsymbol{p}, \tag{18}$$

then the zero solution of time-invariant system (3)–(5) is GES with non-negative initial conditions.

Proof: Since F and G are time-invariant, (18) implies that (10) holds for $\lambda = 1$. Therefore, by homogeneity and continuity of the functions F and G, one can choose a $\lambda > 1$ such that

$$F\left(\boldsymbol{p}, \dots, \boldsymbol{p}, \int_{-\hat{h}}^{0} G(s, \boldsymbol{p})\,\mathrm{d}s\right) \le F\left(\lambda^{\hat{h}_1}\boldsymbol{p}, \lambda^{\hat{h}_2}\boldsymbol{p}, \dots, \lambda^{\hat{h}_m}\boldsymbol{p}, \int_{-\hat{h}}^{0} G(s, \lambda^{-s}\boldsymbol{p})\,\mathrm{d}s\right) \ll \boldsymbol{p}.$$

Since $0 < h(t) \le \hat{h}$, $0 < h_i(t) \le \hat{h}_i$, $t \ge t_0$, $i \in \underline{m}$, it follows that

$$F\left(\lambda^{h_1(t)}\boldsymbol{p}, \lambda^{h_2(t)}\boldsymbol{p}, \dots, \lambda^{h_m(t)}\boldsymbol{p}, \int_{-h(t)}^{0} G(s, \lambda^{-s}\boldsymbol{p})\,\mathrm{d}s\right) \le F\left(\lambda^{\hat{h}_1}\boldsymbol{p}, \lambda^{\hat{h}_2}\boldsymbol{p}, \dots, \lambda^{\hat{h}_m}\boldsymbol{p}, \int_{-\hat{h}}^{0} G(s, \lambda^{-s}\boldsymbol{p})\,\mathrm{d}s\right) \ll \boldsymbol{p}, \quad \forall t \ge t_0,$$

implying that the system is GES with non-negative initial conditions, by Theorem 2.5. ■

Remark 2.7: The above result can be extended even for unbounded delays $h_i(t)$, $i \in \underline{m}$ satisfying the condition $\lim_{t\to+\infty}(t - h_i(t)) = +\infty$ (see, e.g. Assumption 4.1 in Feyzmahdavian et al., 2014a for a detailed similar reasoning).

Theorem 2.5 is a general result that includes or generalises many known results related to exponential stability of positive or monotone continuous-time difference systems. First, let F be of the form

$$F(t, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m, \boldsymbol{x}_{m+1}) = \sum_{i=1}^{m+1} f_i(t, \boldsymbol{x}_i). \tag{19}$$

Then it is clear that F is monotone and homogeneous on $\mathbb{R}_+^{n(m+1)}$, for each $t \ge t_0$, if all f_i are so on $\mathbb{R}_+^n$, $i \in \underline{m+1}$. Therefore, as an immediate consequence of Theorem 2.5, we have:

Theorem 2.8: *Let $f_i : \mathbb{R} \times \mathbb{R}^n \longmapsto \mathbb{R}^n, i \in \underline{m+1}$ and $g : \mathbb{R} \times [-\hat{h}, 0] \times \mathbb{R}^n \longmapsto \mathbb{R}^n$ be given continuous functions such that, for each $t \ge t_0, s \in [-\hat{h}, 0], f_i(t, \cdot)$ and $g(t, s, \cdot)$ are monotone and homogeneous on $\mathbb{R}_+^n$. Assume, moreover, that there exist a vector $\boldsymbol{p} \in \mathbb{R}_+^n, \boldsymbol{p} \gg \boldsymbol{0}$ and $\lambda > 1$, such that*

$$\sum_{i=1}^{m} f_i(t, \boldsymbol{p})\lambda^{h_i(t)} + f_{m+1}\left(t, \int_{-h(t)}^{0} g(t, s, \boldsymbol{p})\lambda^{-s}\,\mathrm{d}s\right) \ll \boldsymbol{p} \quad \forall t \ge t_0. \tag{20}$$

Then, the zero solution of the nonlinear equation with delays

$$\boldsymbol{x}(t) = \sum_{i=1}^{m} f_i(t, \boldsymbol{x}(t - h_i(t))) + f_{m+1}\left(t, \int_{-h(t)}^{0} g(t, s, \boldsymbol{x}(t+s))\,\mathrm{d}s\right), \quad t \ge t_0, \tag{21}$$

is GES with non-negative initial conditions.

In particular, if $h(t) \equiv \hat{h}, h_i(t) \equiv \hat{h}_i, t \ge t_0, i \in \underline{m}$ and $f_i, i \in \underline{m+1}$, g are linear non-negative functions, that is, $f_i(t, \boldsymbol{x}) = A_i(t)\boldsymbol{x}$, $g(t, s, \boldsymbol{x}) = C(t, s)\boldsymbol{x}$, with $A_i(t) \in \mathbb{R}_+^{n\times n}$ and $C(t, s) \in \mathbb{R}_+^{n\times n}$ for all $t \ge 0, s \in [-\hat{h}, 0]$, then the above Theorem 2.8 implies immediately the following result given in Ngoc and Huy (2015, Theorem 3(i)), noticing that for positive linear systems, GES with non-negative initial conditions implies GES.

Corollary 2.9: *The time-varying positive linear system with time-invariant delays*

$$\boldsymbol{x}(t) = \sum_{i=1}^{m} A_i(t)\boldsymbol{x}(t - \hat{h}_i) + A_{m+1}(t)\int_{-\hat{h}}^{0} C(t, s)\boldsymbol{x}(t+s)\,\mathrm{d}s, \quad t \ge t_0, \tag{22}$$

is GES if there exists a vector $\boldsymbol{p} \in \mathbb{R}_+^n$, $\boldsymbol{p} \gg 0$ and $\lambda > 1$ such that

$$\left(\sum_{i=1}^{m} A_i(t)\lambda^{\hat{h}_i} + A_{m+1}(t)\int_{-\hat{h}}^{0} C(t, s)\lambda^{-s}\,\mathrm{d}s\right)\boldsymbol{p} \ll \boldsymbol{p} \quad \forall t \ge t_0. \tag{23}$$

Further, let us consider a more general situation than (19) when F is of the form

$$F(t, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{m+1}) := \sum_{i=1}^{r} f_i(t, q_i(\boldsymbol{x}_1, \dots, \boldsymbol{x}_m)) + f_{r+1}(t, \boldsymbol{x}_{m+1}). \tag{24}$$

Here, a non-zero $r \in \mathbb{Z}_+$ is given and, for each $i \in \underline{r}$, $q_i : \mathbb{R}_+^{nm} \longmapsto \mathbb{R}^n$ is a continuous function given by

$$q_i(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m) = \mathop{\mathbf{H}}_{j=1}^{m} \boldsymbol{x}_j^{\alpha_{ij}} := \boldsymbol{x}_1^{\alpha_{i1}} \circ \boldsymbol{x}_2^{\alpha_{i2}} \circ \cdots \boldsymbol{x}_m^{\alpha_{im}}, \tag{25}$$

with $\alpha_{ij} \ge 0, j \in \underline{m}$ and $\circ$ denoting the Hadamard vector product. Then, it follows readily from the definition of the Hadamard vector product that q_i is monotone on $\mathbb{R}_+^{nm}$ and homogeneous on $\mathbb{R}_+^n$ if $\sum_{j=1}^m \alpha_{ij} = 1$. Moreover, it is obvious that $q_i(\boldsymbol{p}, \ldots, \boldsymbol{p}) = \boldsymbol{p}$, for any $\boldsymbol{p} \in \mathbb{R}_+^n$. Therefore, as another consequence of Theorem 2.5, we have:

Corollary 2.10: *Consider the nonlinear continuous-time difference system*

$$\boldsymbol{x}(t) = \sum_{i=1}^r f_i\big(t, q_i(\boldsymbol{x}(t-h_1(t)), \ldots, \boldsymbol{x}(t-h_m(t)))\big) + f_{r+1}\left(t, \int_{-h(t)}^0 g(t,s,\boldsymbol{x}(t+s))\,\mathrm{d}s\right), \quad t \geq t_0, \tag{26}$$

where the functions $f_i(t,\cdot), i \in \underline{r+1}$ and $g(t,s,\cdot)$ are monotone and homogeneous functions on $\mathbb{R}_+^n$ and $q_i, i \in \underline{r}$, are defined by (25) with $\sum_{j=1}^m \alpha_{ij} = 1$. If there exist a vector $\boldsymbol{p} \in \mathbb{R}_+^n$, $\boldsymbol{p} \gg \boldsymbol{0}$ and $\lambda > 1$ such that

$$\sum_{i=1}^r f_i(t,\boldsymbol{p})\lambda^{h_1(t)\alpha_{i1}+\cdots+h_m(t)\alpha_{im}} + f_{r+1}\left(t, \int_{-h(t)}^0 g(t,s,\boldsymbol{p})\lambda^{-s}\,\mathrm{d}s\right) \ll \boldsymbol{p} \quad \forall t \geq t_0, \tag{27}$$

then the zero solution $\boldsymbol{x}(t) \equiv \boldsymbol{0}$ of the system (26) is GES with non-negative initial conditions.

Specifically, when all functions f_i, g are linear in $\boldsymbol{x}$, the system (26) takes the form

$$\boldsymbol{x}(t) = \sum_{i=1}^r A_i(t) \mathop{\mathbf{H}}_{j=1}^{m} \boldsymbol{x}^{\alpha_{ij}}(t-h_j(t)) + A_{r+1}(t)\int_{-h(t)}^0 C(t,s)\boldsymbol{x}(t+s)\,\mathrm{d}s, \quad t \geq t_0. \tag{28}$$

In this case, Theorem 2.5 and Corollary 2.10 yield the following result which gives some delay-dependent criteria of global exponential stability for this class of nonlinear systems.

Theorem 2.11: *Consider the time-varying nonlinear positive system with delay* (28) *where matrix functions $A_i(t), i \in \underline{r+1}, C(t,s)$ are non-negative and $\alpha_{ij} \geq 0$, $\sum_{j=1}^m \alpha_{ij} = 1$, $i \in \underline{r}$. Then, the zero solution of* (28) *is GES with non-negative initial conditions if one of the following conditions holds:*

(i) *There exist a vector $\boldsymbol{p} \in \mathbb{R}_+^n, \boldsymbol{p} \gg \boldsymbol{0}$ and $\lambda > 1$, such that*

$$\left(\sum_{i=1}^r A_i(t)\lambda^{h_1(t)\alpha_{i1}+\cdots+h_m(t)\alpha_{im}} + A_{r+1}(t)\int_{-h(t)}^0 C(t,s)\lambda^{-s}\,\mathrm{d}s\right)\boldsymbol{p} \ll \boldsymbol{p} \quad \forall t \geq t_0. \tag{29}$$

(ii) *There exists a Schur stable matrix $D \in \mathbb{R}_+^{n\times n}$, such that*

$$\left(\sum_{i=1}^r A_i(t) + A_{r+1}(t)\int_{-h(t)}^0 C(t,s)\,\mathrm{d}s\right) \leq D \quad \forall t \geq t_0. \tag{30}$$

(iii) *There exists $\lambda > 1$, such that*

$$\sum_{i=1}^r \|A_i(t)\|\lambda^{h_1(t)\alpha_{i1}+\cdots+h_m(t)\alpha_{im}} + \|A_{r+1}(t)\|\int_{-h(t)}^0 \|C(t,s)\|\lambda^{-s}\,\mathrm{d}s < 1 \quad \forall t \geq t_0. \tag{31}$$

Proof: First, it follows straightforwardly from Corollary 2.10 that if (i) holds then the nonlinear system (28) is GES with non-negative initial conditions. We show that (ii) implies (i). Since $D \in \mathbb{R}_+^{n\times n}, \rho(D) < 1$, there is a vector $\boldsymbol{p} \in \mathbb{R}^n, \boldsymbol{p} \gg \boldsymbol{0}$ such that $D\boldsymbol{p} \ll \boldsymbol{p}$, see, e.g. Ngoc and Hieu (2013, Theorem 1.2). By continuity, one can choose $\lambda > 1$, with $\lambda - 1$ sufficiently small such that

$$\lambda^\tau D\boldsymbol{p} \ll \boldsymbol{p}. \tag{32}$$

(recall that, by definition, $\tau := \max\{\hat{h}, \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_m\} > 0$.) Since $\sum_{j=1}^m \alpha_{ij} = 1, i \in \underline{r}$, we deduce easily

$$\begin{aligned}
&\left(\sum_{i=1}^r A_i(t)\lambda^{h_1(t)\alpha_{i1}+\cdots+h_m(t)\alpha_{im}} + A_{r+1}(t)\int_{-h(t)}^0 C(t,s)\lambda^{-s}\,\mathrm{d}s\right)\boldsymbol{p} \\
&\leq \left(\sum_{i=1}^r A_i(t)\lambda^{\hat{h}_1\alpha_{i1}+\cdots+\hat{h}_m\alpha_{im}} + A_{r+1}(t)\int_{-h(t)}^0 C(t,s)\lambda^{-s}\,\mathrm{d}s\right)\boldsymbol{p} \\
&\leq \left(\sum_{i=1}^r A_i(t)\lambda^{\tau[\alpha_{i1}+\cdots+\alpha_{im}]} + A_{r+1}(t)\int_{-h(t)}^0 C(t,s)\lambda^{\tau}\,\mathrm{d}s\right)\boldsymbol{p} \\
&= \lambda^\tau\left(\sum_{i=1}^r A_i(t) + A_{r+1}(t)\int_{-h(t)}^0 C(t,s)\,\mathrm{d}s\right)\boldsymbol{p} \\
&\overset{(30)}{\leq} \lambda^\tau D\boldsymbol{p} \overset{(32)}{\ll} \boldsymbol{p} \quad \forall t \geq t_0.
\end{aligned}$$

Thus, (i) holds. It remains to show that the system (28) is GES with non-negative initial conditions, provided (iii) holds. To this end, assuming (iii) and letting $\boldsymbol{x}(\cdot) := \boldsymbol{x}(\cdot; t_0, \boldsymbol{\varphi})$ be the solution of (28) with an arbitrary initial condition $\boldsymbol{\varphi} \in \mathcal{C}_{t_0}^+, \boldsymbol{\varphi} \neq 0$ we show that there exist $M > 0$ and $\beta \in (0,1)$, such that

$$\|\boldsymbol{x}(t)\| \leq M\|\boldsymbol{\varphi}\|_{[-\tau,0]}\beta^{t-t_0} \quad \forall t \geq t_0. \tag{33}$$

Without loss of generality, we assume that $\mathbb{R}^n$ is equipped with the ∞-norm. Setting $K := \lambda\|\boldsymbol{\varphi}\|_{[-\tau,0]}$ and $\beta := \lambda^{-1} \in (0,1)$, it is straightforwardly verified that

$$\|\boldsymbol{x}(t)\| < K\beta^{t-t_0} \quad \forall t \in [-\tau + t_0, t_0]. \tag{34}$$

We will prove that

$$\|\boldsymbol{x}(t)\| \leq K\beta^{t-t_0} \quad \forall t \geq t_0, \tag{35}$$

which immediately implies (33). Assume to the contrary that there exists $s_0 > t_0$ such that $\|\boldsymbol{x}(s_0)\| > K\beta^{s_0-t_0}$. Define

$$t_1 := \inf\{s_0 |\ s_0 > t_0, \|\boldsymbol{x}(s_0)\| > K\beta^{s_0-t_0}\}.$$

Then, by continuity of $\boldsymbol{x}(t)$, we have $t_1 > t_0$ and

$$\|\boldsymbol{x}(t)\| < K\beta^{t-t_0}, \quad \forall t \in [-\tau + t_0, t_1) \quad \text{and} \quad \|\boldsymbol{x}(t_1)\| = K\beta^{t_1-t_0}. \tag{36}$$

Therefore, we can deduce, by using the property of vector Hadamard product and (28), (31) and (36),

$$
\begin{aligned}
\|\boldsymbol{x}(t_1)\| &\leq \Bigg\| \sum_{i=1}^{r} A_i(t_1) \mathop{\mathbf{H}}_{j=1}^{m} \boldsymbol{x}^{\alpha_{ij}}(t_1 - h_j(t_1)) \\
&\quad + A_{r+1}(t_1) \int_{-h(t_1)}^{0} C(t_1, s)\boldsymbol{x}(t_1 + s)\,\mathrm{d}s \Bigg\| \\
&\leq \sum_{i=1}^{r} \|A_i(t_1)\| \left(\prod_{j=1}^{m} \|\boldsymbol{x}(t_1 - h_j(t_1))\|^{\alpha_{ij}} \right) \\
&\quad + \|A_{r+1}(t_1)\| \int_{-h(t_1)}^{0} \|C(t_1, s)\| \|\boldsymbol{x}(t_1 + s)\|\,\mathrm{d}s \\
&\leq \sum_{i=1}^{r} \|A_i(t_1)\| \left(\prod_{j=1}^{m} (K\beta^{t_1 - h_j(t_1) - t_0})^{\alpha_{ij}} \right) \\
&\quad + \|A_{r+1}(t_1)\| \int_{-h(t_1)}^{0} \|C(t_1, s)\| K\beta^{t_1 + s - t_0}\,\mathrm{d}s \\
&= K\beta^{t_1 - t_0} \sum_{i=1}^{r} \|A_i(t_1)\| \left(\prod_{j=1}^{m} \beta^{-h_j(t_1)\alpha_{ij}} \right) \\
&\quad + K\beta^{t_1 - t_0} \|A_{r+1}(t_1)\| \int_{-h(t_1)}^{0} \|C(t_1, s)\| \beta^{s}\,\mathrm{d}s \\
&= K\beta^{t_1 - t_0} \Bigg(\sum_{i=1}^{r} \|A_i(t_1)\| \gamma^{h_1(t_1)\alpha_{i1} + \cdots + h_m(t_1)\alpha_{im}} \\
&\quad + \|A_{r+1}(t_1)\| \int_{-h(t_1)}^{0} \|C(t_1, s)\| \gamma^{-s}\,\mathrm{d}s \Bigg) \\
&< K\beta^{t_1 - t_0}.
\end{aligned}
$$

This, however, contradicts the equality in (36) and completes the proof. ■

Remark 2.12: In the particular case when $r = m, \alpha_{ii} = 1, \alpha_{ij} = 0\ (i \neq j), i, j \in \underline{m},\ h(t) \equiv \hat{h}, h_i(t) \equiv \hat{h}_i, t \geq t_0, i \in \underline{m}$, the system (28) is reduced to linear system of the form (22) that was considered in Ngoc and Huy (2015). Therefore, Theorem 2.11 can be considered as a generalisation of the main result (Ngoc & Huy, 2015, Theorem 3) to the class of nonlinear positive systems of the form (28).

Remark 2.13: Note that a *discrete-time version* of the nonlinear equation of the form (28), but without distributed delays (i.e. when $A_{r+1}(\cdot) = \mathbf{0}$), has been considered in Nam et al. (2015) where, as the main result, a sufficient condition for global exponential stability similar to Corollary 2.10 has been established. Therefore, Corollary 2.10 can be considered as a generalisation of the result in Nam et al. (2015) to the case of continuous-time difference systems containing both discrete and distributed delays. Our proof is, moreover, less involved than Nam et al. (2015).

3. Exponential stability of general nonlinear systems

In this section, by using the comparison principle, we get the extension of the main results of the previous section to general nonlinear systems (3)–(5) without the assumption on positivity, monotonicity and homogeneity of F and G.

Theorem 3.1: *Consider the nonlinear continuous-time difference system* (3)–(5) *with continuous functions* F, G. *Let* $F_* \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}_+^{n(m+1)}, \mathbb{R}_+^n)$ *and* $G_* \in \mathcal{C}(\mathbb{R}_+ \times [-\hat{h}, 0] \times \mathbb{R}_+^n, \mathbb{R}_+^n)$ *be continuous functions such that for each* $t \geq 0, s \in [-\hat{h}, 0]$, $F_*(t, \cdot)$ *and* $G_*(t, s, \cdot)$ *are monotone and homogeneous on* $\mathbb{R}_+^{n(m+1)}$ *and* $\mathbb{R}_+^n$, *respectively. Assume, moreover, that there exists* $\delta > 0$ *such that, for each* $t \geq t_0, s \in [-\hat{h}, 0]$ *and each* $\boldsymbol{x}, \boldsymbol{x}_i \in \mathcal{B}^n(\delta), i \in \underline{m+1}$,

$$
|F(t, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{m+1})| \leq F_*(t, |\boldsymbol{x}_1|, |\boldsymbol{x}_2|, \ldots, |\boldsymbol{x}_{m+1}|), \tag{37}
$$

$$
|G(t, s, \boldsymbol{x})| \leq G_*(t, s, |\boldsymbol{x}|). \tag{38}
$$

Then, the zero solution of (3)–(5) *is LES if there exist* $\lambda > 1$ *and* $\boldsymbol{p} \in \mathbb{R}_+^n, \boldsymbol{p} \gg \mathbf{0}$, *satisfying*

$$
F_*\left(t, \lambda^{h_1(t)}\boldsymbol{p}, \ldots, \lambda^{h_m(t)}\boldsymbol{p}, \int_{-h(t)}^{0} G_*(t, s, \lambda^{-s}\boldsymbol{p})\,\mathrm{d}s\right) \ll \boldsymbol{p} \quad \forall t \geq t_0. \tag{39}
$$

If (37), (38) *hold for* $\delta = \infty$, *then the zero solution of* (3)–(5) *is GES.*

Proof: First, by the assumption on F_*, G_* and (37) and (38), it follows that the functions F, G satisfy (7) and hence the system (3)–(5) admits the zero solution. Further, the proof is divided into two steps.

Step 1: We show that there exist $r_0 > 0$ such that, for any solution $x(t) = x(t, t_0, \boldsymbol{\varphi})$ of (3)–(5),

$$
\|\boldsymbol{x}(t)\| \leq \delta \quad \text{and} \quad \left\| \int_{-h(t)}^{0} G(t, s, \boldsymbol{x}(t+s))\,\mathrm{d}s \right\| \leq \delta \quad \forall t \geq t_0, \tag{40}
$$

whenever $\|\boldsymbol{\varphi}\|_{[-\tau,0]} \leq r_0$.

By the positivity, monotonicity and homogeneity of F_* and G_*, and assumption (39), it follows that $\sup_{t \geq t_0} \| \int_{-h(t)}^{0} G_*(t, s, \boldsymbol{p})\,\mathrm{d}s\| < \infty$. Moreover, due to the homogeneity of F_* and G_*, it follows that (39) holds for any vector $\gamma\boldsymbol{p} \in \mathbb{R}_+^n$, $0 < \gamma < \varepsilon$, with a small enough $\varepsilon > 0$. Therefore, we can assume, without loss of generality, that $\boldsymbol{p}$ is chosen such that

$$
\|\boldsymbol{p}\| \leq \delta \quad \text{and} \quad \sup_{t \geq t_0} \left\| \int_{-h(t)}^{0} G_*(t, s, \boldsymbol{p})\,\mathrm{d}s \right\| \leq \delta. \tag{41}
$$

Denote $\boldsymbol{p} := (p_1, p_2, \ldots, p_n)^\top$. Since $p_i > 0, i \in \underline{n}$, we can choose $r_0 > 0$ so that $0 < r_0 < \min\{p_i, i \in \underline{n}\}$. Since the norm is monotonic, it follows that for any initial condition $\boldsymbol{\varphi} \in \mathcal{C}_{t_0}$ with $\|\boldsymbol{\varphi}\|_{[-\tau,0]} \leq r_0$ the corresponding solution satisfies $|\boldsymbol{x}(s + t_0)| = |\boldsymbol{\varphi}(s)| \ll \boldsymbol{p},\ s \in [-\tau, 0]$ and, therefore, $|\boldsymbol{x}(s)| \ll \boldsymbol{p},\ s \in [t_0 - \tau, t_0]$. We claim that $|\boldsymbol{x}(t)| \leq \boldsymbol{p},\ t \geq t_0$, or equivalently,

$$
|x_i(t)| \leq p_i, \quad t \geq t_0,\ i \in \underline{n}. \tag{42}
$$

Assume to the contrary that there exists $I \subset \underline{n}$ such that, for all $i \in I$, there exist $s_i > t_0$, $|x_i(s_i)| > p_i$. By continuity, it follows that

$$\exists s_i^* \in (t_0, s_i): \quad |x_i(s_i^*)| = p_i, \ i \in I. \tag{43}$$

Setting

$$t_1 := \min_{i\in I} \inf\{s^* : \ s_i^* > t_0, (43) \text{ holds}\}, \tag{44}$$

one has $t_1 > t_0$ and

$$|\boldsymbol{x}(t)| \le \boldsymbol{p}, \quad t \in [t_0 - \tau, t_1) \quad \text{and} \quad |x_{i_0}(t_1)| = p_{i_0}, \tag{45}$$

where i_0 is the index when the minimum (44) is attained. Then, (38), (41) and (45) imply that

$$\begin{aligned} &\left\| \int_{-h(t)}^{0} G(t, s, \boldsymbol{x}(t+s))\,\mathrm{d}s \right\| \le \left\| \int_{-h(t)}^{0} G_*(t, s, |\boldsymbol{x}(t+s)|)\,\mathrm{d}s \right\| \\ &\quad \le \left\| \int_{-h(t)}^{0} G_*(t, s, \boldsymbol{p})\,\mathrm{d}s \right\| \le \sup_{t\ge t_0} \left\| \int_{-h(t)}^{0} G_*(t, s, \boldsymbol{p})\,\mathrm{d}s \right\| \overset{(41)}{\le} \delta, \\ &\quad t \in [t_0, t_1). \end{aligned} \tag{46}$$

On the other hand, (3), (37)–(39), (45) and (46) imply that

$$\begin{aligned} |x_{i_0}(t_1)| &\le F_{*i_0}\Bigg(t_1, |\boldsymbol{x}(t_1 - h_1(t))|, \dots, |\boldsymbol{x}(t_1 - h_m(t))|, \\ &\qquad \int_{-h(t)}^{0} G_*(t_1, s, |\boldsymbol{x}(t_1+s)|)\,\mathrm{d}s\Bigg) \\ &\overset{(45),(46)}{\le} F_{*i_0}\left(t_1, \boldsymbol{p}, \dots, \boldsymbol{p}, \int_{-h(t)}^{0} G_*(t_1, s, \boldsymbol{p})\,\mathrm{d}s\right) \\ &\le F_{*i_0}\Bigg(t_1, \lambda^{h_1(t)}\boldsymbol{p}, \dots, \lambda^{h_m(t)}\boldsymbol{p}, \\ &\qquad \int_{-h(t)}^{0} G_*(t_1, s, \lambda^{-s}\boldsymbol{p})\,\mathrm{d}s\Bigg) p_{i_0}, \end{aligned}$$

conflicting with (45). Thus, we get $|\boldsymbol{x}(t)| \le \boldsymbol{p}$ and, by monotonicity of the vector norm and (41),

$$\|\boldsymbol{x}(t)\| \le \|\boldsymbol{p}\| \le \delta, \quad t \ge t_0, \tag{47}$$

as to be shown. The second inequality in (40) is obtained by the following reasoning:

$$\begin{aligned} &\left\| \int_{-h(t)}^{0} G(t, s, \boldsymbol{x}(t+s))\,\mathrm{d}s \right\| \overset{(47)}{\le} \left\| \int_{-h(t)}^{0} G_*(t, s, |\boldsymbol{x}(t+s)|)\,\mathrm{d}s \right\|, \\ &\overset{(40)}{\le} \left\| \int_{-h(t)}^{0} G_*(t, s, \boldsymbol{p})\,\mathrm{d}s \right\| \overset{(41)}{\le} \sup_{t\ge t_0} \left\| \int_{-h(t)}^{0} G_*(t, s, \boldsymbol{p})\,\mathrm{d}s \right\| \le \delta, \\ &\quad t \ge t_0. \end{aligned}$$

Step 2: We show that there exist $M > 0$ and $\beta \in (0, 1)$ such that

$$\|\boldsymbol{x}(t)\| \le M\beta^{t-t_0} \|\boldsymbol{\varphi}\|_{[-\tau,0]}, \quad t \ge t_0, \tag{48}$$

for any $\boldsymbol{\varphi} \in \mathcal{C}_{t_0}$, $\|\boldsymbol{\varphi}\|_{[-\tau,0]} \le r_0$. The proof is similar to that of Theorem 2.5 (essentially based on (39) and (40), with some minor modifications) and hence is omitted. Therefore, the zero solution of (3)–(5) is LES. Finally, if (37)–(38) hold for $\delta = \infty$, then the proof of Step 2 also shows that

$$\|\boldsymbol{x}(t)\| \le M\beta^{t-t_0} \|\boldsymbol{\varphi}\|_{[-\tau,0]} \quad \forall t \ge t_0,$$

for any $\boldsymbol{\varphi} \in \mathcal{C}_{t_0}$. Therefore, the zero solution of (3)–(5) is GES. This completes the proof. ■

The functions F_* and G_* in Theorem 3.1 are called *upper bound* of F and G, respectively.

Remark 3.2: Roughly speaking, Theorem 3.1 means that if the nonlinear system (3) is upper bounded by a time-varying nonlinear positive monotone homogeneous system

$$\begin{aligned} \boldsymbol{x}(t) &= F_*(t, |\boldsymbol{x}(t - h_1(t))|, \dots, |\boldsymbol{x}(t - h_m(t))|, \\ &\qquad \int_{-h(t)}^{0} G_*(t, s, |\boldsymbol{x}(t+s)|)\,\mathrm{d}s), \quad t \ge t_0, \end{aligned} \tag{49}$$

whose the zero solution is LES (or, equivalently, GES, due to homogeneity of F_*, G_*) then so is the zero solution of (3)–(5). Note that in the recent work (Tian & Sun, 2020), a similar result has been proved for a class of time-varying differential equations with mixed delays which are assumed to be upper-bounded by a positive *time-invariant* monotone homogeneous system. Therefore, our result is more general, giving less restrictive delay-dependent criteria for exponential stability.

Combining Theorem 3.1 with the results obtained in the previous section for nonlinear positive monotone homogeneous systems will lead to different criteria for exponential stability of the zero solution of the system (3)–(5). In particular, the combination of Theorems 3.1 and 2.11 yields immediately the following:

Theorem 3.3: *Consider the nonlinear continuous-time difference system* (3)–(5) *with continuous functions F, G. Assume that there exist* $\delta > 0, r \in \mathbb{N}, A_i(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}_+^{n\times n}), i \in \underline{r+1}, C(\cdot,\cdot) \in \mathcal{C}(\mathbb{R} \times [-\hat{h}, 0], \mathbb{R}_+^{n\times n})$ *and* $\alpha_{ij} \in [0, 1]$, $\sum_{j=1}^{m} \alpha_{ij} = 1, i \in \underline{r}, j \in \underline{m}$ *such that*

$$|F(t, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{m+1})| \le \sum_{i=1}^{r} A_i(t) \mathop{\mathbf{H}}_{j=1}^{m} |\boldsymbol{x}_j|^{\alpha_{ij}} + A_{r+1}(t)|\boldsymbol{x}_{m+1}|, \tag{50}$$

$$\big|G(t, s, \boldsymbol{x})| \le C(t, s)|\boldsymbol{x}|, \tag{51}$$

for all $t \ge t_0, s \in [-\hat{h}, 0], \boldsymbol{x}, \boldsymbol{x}_j \in \mathcal{B}^n(\delta), j \in \underline{m+1}$. *Then the zero solution of* (3)–(5) *is LES provided that one of the conditions* (i), (ii), (iii) *of Theorem* 2.11 *holds for the non-negative matrix functions* $A_i(\cdot), i \in \underline{r+1}$ *and* $C(\cdot,\cdot)$.

In addition, if (50) *and* (51) *hold for* $\delta = \infty$, *then the zero solution of* (3)–(5) *is GES.*

Particularly, if $r = m, \alpha_{ii} = 1, \alpha_{ij} = 0$ $(i \ne j), i, j \in \underline{m}$, then Theorem 3.3 implies the following corollary.

Corollary 3.4: *Assume that the continuous functions F, G satisfy*

$$\left|F(t,\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_{m+1})\right| \le \sum_{i=1}^{m+1} A_i(t)|\boldsymbol{x}_i|, \quad |G(t,s,\boldsymbol{x})| \le C(t,s)|\boldsymbol{x}|, \tag{52}$$

for all $t \ge t_0$, $s \in [-\hat{h},0]$, $\boldsymbol{x},\boldsymbol{x}_i \in \mathbb{R}^n$, $i \in \underline{m+1}$. *Then the zero solution of* (3)–(5) *is GES provided that one of the conditions* (i), (ii), (iii) *of Theorem* 2.11 *holds for the non-negative matrix functions* $A_i(\cdot)$, $i \in \underline{m+1}$ *and* $C(\cdot,\cdot)$.

Remark 3.5: The above corollary is a generalisation of Theorem 2.2 in Ngoc and Hieu (2013) where a similar, but delay-independent, criterion has been proved for the class of nonlinear discrete-time systems.

In particular, from Corollary 3.4 and Theorem 2.11(ii), it follows immediately the following corollary.

Corollary 3.6: *Assume that there exist* $A_i \in \mathbb{R}_+^{n\times n}, i \in \underline{m+1}$ *and* $C(\cdot) \in \mathcal{C}([-\hat{h},0],\mathbb{R}_+^{n\times n})$ *such that*

$$\left|F(t,\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_{m+1})\right| \le \sum_{i=1}^{m+1} A_i|\boldsymbol{x}_i|, \quad |G(t,s,\boldsymbol{x})| \le C(s)|\boldsymbol{x}|, \tag{53}$$

for all $t \ge t_0, s \in [-\hat{h},0]$, $\boldsymbol{x},\boldsymbol{x}_i \in \mathbb{R}^n, i \in \underline{m+1}$. *Then the zero solution of* (3)–(5) *is GES provided that the non-negative matrix* $D := \sum_{i=1}^{m} A_i + A_{m+1}\int_{-h}^{0} C(s)\,\mathrm{d}s$ *is Schur stable, i.e.* $\rho(D) < 1$.

We provide now an application of the above result to deal with an existing example in the literature (Pepe, 2014). Consider the scalar nonlinear continuous-time difference equation

$$x(t) = \sum_{i=1}^{p} \lambda_i \tanh(x(t-\Delta_i)), \quad t \ge 0, \tag{54}$$

with the initial condition $x(s) = x_0(s)$, $s \in [-\Delta,0)$. Here, $\lambda_i \in \mathbb{R}$, $\Delta_i \in (0,\Delta], i \in \underline{p}$; Δ is a positive real number. In Pepe (2014, Example 1), under the assumption that x_0 is piece-wise continuous function, it has been shown, by using the Lyapunov approach, that the zero solution of (54) is globally asymptotically stable if $\sum_{i=1}^{p}|\lambda_i| \le 1$. The proof needs a lot of computations.

Let $x_0 \in C([-\Delta,0],\mathbb{R})$. Since $|\tanh(x)| \le |x|$, for any $x \in \mathbb{R}$, it follows immediately, by Corollary 3.6, that the zero solution of (54) is GES provided $\sum_{i=1}^{p}|\lambda_i| < 1$. Additionally, it is important to note that our above conclusion also holds when the delays in (54) are bounded time-varying delays $\Delta_i(t), i \in \underline{p}, t \in \mathbb{R}_+$.

Below, we present some simple nonlinear examples to illustrate the effectiveness of our results. Notice that the stability criteria obtained in Damak et al. (2015, 2016), Melchor-Aguilar (2016) and Ngoc and Huy (2015), for linear systems are not applicable to deal with these examples.

Example 3.7: Consider a scalar nonlinear continuous-time difference equation

$$x(t) = \sqrt{a^2(t)|x(t-h_1(t))x(t-h_2(t))| + [b(t)x(t-h_2(t))]^2} + \int_{-h(t)}^{0} \frac{c\,\mathrm{e}^{s-t^2}}{1+t^2} x(t+s)\,\mathrm{d}s, \quad t \ge 0. \tag{55}$$

Here, $h(\cdot), h_1(\cdot), h_2(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, $0 < h_1(t) \le \hat{h}_1, 0 < h_2(t) \le \hat{h}_2, 0 < h(t) \le \hat{h}$, for some $\hat{h}_1 > 0, \hat{h}_2 > 0, \hat{h} > 0$; $a(\cdot), b(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ are bounded continuous functions; c is a real number. Note that $x_e = 0$ is an equilibrium point of (55). Let

$$F(t,x_1,x_2,x_3) := \sqrt{a^2(t)|x_1x_2| + b^2(t)x_2^2} + x_3 \quad \text{and}$$

$$G(t,s,x) := \frac{c\,e^{s-t^2}x}{1+t^2},$$

with $t \in \mathbb{R}_+$, $s \in [-\hat{h},0]$, $x_1,x_2,x_3,x \in \mathbb{R}$. One has

$$|F(t,x_1,x_2,x_3)| \le |a(t)|\cdot|x_1|^{1/2}|x_2|^{1/2} + |b(t)|\cdot|x_2| + |x_3| \quad \forall t \in \mathbb{R}_+,\ x_1,x_2,x_3 \in \mathbb{R},$$

and

$$|G(t,s,x)| \le |c|\,e^s|x| \quad \forall t \in \mathbb{R}_+,\ s \in [-\hat{h},0],\ x \in \mathbb{R}.$$

By Theorem 3.3(iii), the zero solution of (55) is GES if

$$\sup_{t\in\mathbb{R}_+}|a(t)| + \sup_{t\in\mathbb{R}_+}|b(t)| + \int_{-\hat{h}}^{0}|c|\,e^s\,\mathrm{d}s < 1,$$

or equivalently,

$$\sup_{t\in\mathbb{R}_+}|a(t)| + \sup_{t\in\mathbb{R}_+}|b(t)| + |c|\,(1-e^{-\hat{h}}) < 1. \tag{56}$$

Figure 1 shows the simulated trajectories of the solutions $x(t)$ of (55), corresponding, respectively, to the initial functions

$$\phi_1(s) = 3s^2 + 0.5s + 0.5, \quad s \in [-1,0],$$

$$\phi_2(s) = \begin{cases} -8s-8, & s \in [-1,-0.5], \\ 10s+1, & s \in (-0.5,0], \end{cases}$$

when $a(t) = b(t) = \frac{1}{4}, c = 0, h_1(t) = 1, h_2(t) = 0.5, h(t) = \hat{h} = 1$, $t \in \mathbb{R}_+$. Then, $\sup_{t\in\mathbb{R}_+}|a(t)| + \sup_{t\in\mathbb{R}_+}|b(t)| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1$ and hence, (56) holds. The initial functions ϕ_1,ϕ_2 satisfy the matching condition

$$\phi_i(0) = \sqrt{\frac{1}{16}|\phi_i(-1)\phi_i(-0.5)| + \frac{1}{16}(\phi_i(-0.5))^2}, \quad i = 1,2.$$

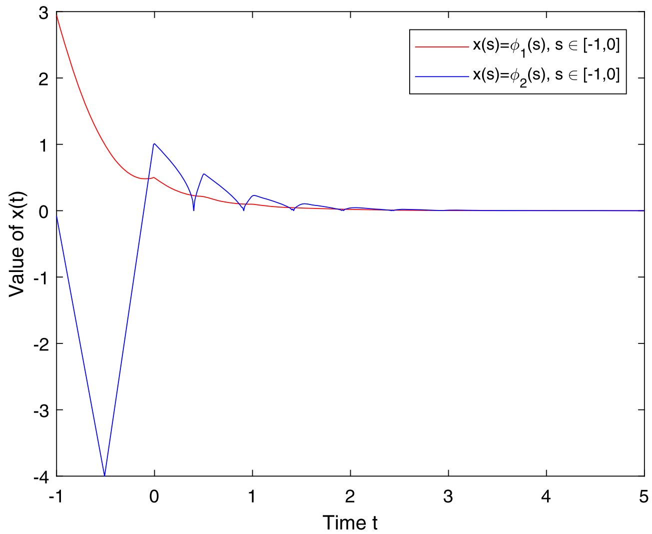

Figure 1. The zero solution of (55) is GES if (56) holds.

Example 3.8: Consider a nonlinear time-varying continuous-time difference system in $\mathbb{R}^2$ given by

$$\boldsymbol{x}(t) = H(t, \boldsymbol{x}(t-h_1(t)), \boldsymbol{x}(t-h_2(t))) + \int_{-2}^{0} G(t,s,\boldsymbol{x}(t+s))\,\mathrm{d}s, \quad t \geq 0. \tag{57}$$

Here, $h_1(t) = |\sin(t)| + 0.5,\ h_2(t) = \cos^2(t) + 1$; $H(\cdot,\cdot,\ldots,\cdot)$ and $G(\cdot,\cdot,\cdot)$ are defined by

$$H(t,\boldsymbol{x}_1,\boldsymbol{x}_2) := \begin{pmatrix} \frac{1}{8}x_{11}e^{-t^2x_{11}^2} + \frac{1}{8}x_{21} + \frac{1}{4(1+t^2)}x_{22} \\ x_{11}\sin t + \frac{1}{16}\ln(1+|x_{12}+x_{22}|) \end{pmatrix},$$

with $t \in \mathbb{R}_+$ and $\boldsymbol{x}_i := (x_{i1}, x_{i2})^\top \in \mathbb{R}^2,\ i = 1,2$ and

$$G(t,s,\boldsymbol{y}_1) := \begin{pmatrix} \frac{1}{8}e^{-t^2s^2}\sin y_{11} \\ \frac{3}{64}s^2 y_{12} \end{pmatrix}, \quad t \in \mathbb{R}_+,\ s \in [-2,0],\ \boldsymbol{y}_1 := (y_{11}, y_{12})^\top \in \mathbb{R}^2.$$

Let $F(t,\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) := H(t;\boldsymbol{x}_1,\boldsymbol{x}_2) + \boldsymbol{x}_3$. One has

$$\begin{cases} |F(t,\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)| \leq A_1|\boldsymbol{x}_1| + A_2|\boldsymbol{x}_2| + A_3|\boldsymbol{x}_3| \\ \qquad \forall t \in \mathbb{R}_+, \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^2, \\ |G(t,s,\boldsymbol{y})| \leq G(s)|\boldsymbol{y}| \quad \forall t \in \mathbb{R}_+,\ s \in [-2,0],\ \boldsymbol{y} \in \mathbb{R}^2, \end{cases}$$

with

$$A_1 := \begin{pmatrix} \frac{1}{8} & 0 \\ 1 & \frac{1}{16} \end{pmatrix}, \quad A_2 := \begin{pmatrix} \frac{1}{8} & \frac{1}{4} \\ 0 & \frac{1}{16} \end{pmatrix}, \quad A_3 = I_2 \quad \text{and}$$

$$G(s) := \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{3}{64}s^2 \end{pmatrix}, \quad s \in [-2,0].$$

Thus, the inequality (50)–(51) hold with the vector $\boldsymbol{x}_e = \boldsymbol{0}$. On the other hand, we have

$$\int_{-2}^{0} G(s)\,\mathrm{d}s = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}$$

and

$$\rho\left(A_1 + A_2 + A_3\int_{-2}^{0} G(s)\,\mathrm{d}s\right) = \rho\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & \frac{1}{4} \end{pmatrix} = \frac{3+\sqrt{17}}{8} < 1.$$

Therefore, the zero solution of (57) is GES, by Corollary 3.6.

Finally, as another application of our approach, we investigate stability of a simple linear continuous-time difference system of the form

$$\boldsymbol{x}(t+h) = A\boldsymbol{x}(t) + \int_{t-h}^{t} B\boldsymbol{x}(s)\,\mathrm{d}s, \quad t \geq t_0 - h, \tag{58}$$

where, $t_0 \geq 0, h > 0, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n}$ are given.

Let $u = t + h$, then (58) becomes $\boldsymbol{x}(u) = A\boldsymbol{x}(u-h) + \int_{u-2h}^{u-h} B\boldsymbol{x}(s)\,\mathrm{d}s,\ u \geq t_0$ or, equivalently,

$$\boldsymbol{x}(u) = A\boldsymbol{x}(u-h) + \int_{-h}^{0} B\boldsymbol{x}(u-h+w)\,\mathrm{d}w, \quad u \geq t_0. \tag{59}$$

By a similar proof as that of Theorems 2.5 and 3.1, it can be shown that (59) is GES if there is $\lambda > 1$ such that $|A|\lambda^h\boldsymbol{p} + \int_{-h}^{0}|B|\lambda^{h-s}\boldsymbol{p}\,\mathrm{d}s \ll \boldsymbol{p}$, for some $\boldsymbol{p} \in \mathbb{R}^n, p \gg 0$. The last inequality implies $(|A| + \int_{-h}^{0}|B|dw)\boldsymbol{p} \ll \boldsymbol{p}$, or equivalently, the nonnegative matrix $D := |A| + h|B| = |A| + \int_{-h}^{0}|B|\,d\,w$ is Schur stable, see, e.g. Ngoc and Hieu (2013, Theorem 1.2).

Our approach can be applied similarly for nonlinear systems more general than (58), for example, $\boldsymbol{x}(t+h) = F(t,\boldsymbol{x}(t),\int_{t-h}^{t}B\boldsymbol{x}(s)\,\mathrm{d}s)$, $t \geq t_0$, where $F(t,\cdot)$ is monotone and homogeneous on $\mathbb{R}_+^n \times \mathbb{R}_+^n$ or $F(t,\cdot)$ is 'bounded above' by a monotone homogeneous function $F_*(t,\cdot)$ on $\mathbb{R}_+^n \times \mathbb{R}_+^n$, for each $t \geq t_0$.

4. Concluding remark

By using the comparison principle, we present several explicit criteria for exponential stability of the zero solution for a general class of nonlinear delay continuous-time difference systems. The obtained results are shown to include or extend many known results proved previously either for linear time-invariant difference systems or nonlinear discrete-time systems.

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