Optimal numerical integration and approximation of functions on $\mathbb{R}^d$ equipped with Gaussian measure

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Abstract

We investigate the numerical approximation of integrals over $\mathbb{R}^d$ equipped with the standard Gaussian measure γ for integrands belonging to the Gaussian-weighted Sobolev spaces $W^\alpha_p(\mathbb{R}^d, \gamma)$ of mixed smoothness $\alpha \in \mathbb{N}$ for $1 < p < \infty$. We prove the asymptotic order of the convergence of optimal quadratures based on n integration nodes and propose a novel method for constructing asymptotically optimal quadratures. As for related problems, we establish by a similar technique the asymptotic order of the linear, Kolmogorov and sampling n-widths in the Gaussian-weighted space $L_q(\mathbb{R}^d, \gamma)$ of the unit ball of $W^\alpha_p(\mathbb{R}^d, \gamma)$ for $1 \le q < p < \infty$ and $q = p = 2$.

Keywords and Phrases: Multivariate numerical integration; Quadrature; Multivariate approximation; Gaussian-weighted Sobolev space of mixed smoothness; n-Widths; Asymptotic order of convergence.

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1 Introduction

We investigate numerical approximation of integrals

$$I(f) := \int_{\mathbb{R}^d} f(\boldsymbol{x}) \, \gamma(\mathrm{d}\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{x}) g(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \tag{1.1}$$

for functions f belonging to the Gaussian-weighted Sobolev spaces $W^\alpha_p(\mathbb{R}^d, \gamma)$ of mixed smoothness $\alpha \in \mathbb{N}$ for $1 < p < \infty$ (see Section 2 for the definition), where $\gamma(\mathrm{d}\boldsymbol{x}) = g(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ is the d-dimensional standard Gaussian measure on $\mathbb{R}^d$ with the density

$$g(\boldsymbol{x}) := (2\pi)^{-d/2} \exp\left(-|\boldsymbol{x}|^2/2\right), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

To approximate this integral we use a (linear) quadrature defined by

$$I_n(f) := \sum_{i=1}^{n} \lambda_i f(\boldsymbol{x}_i) \tag{1.2}$$

with the convention $I_0(f) = 0$, where $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset \mathbb{R}^d$ are given integration nodes and $(\lambda_1, \ldots, \lambda_n)$ the integration weights. For convenience, we assume that some of the integration nodes may coincide. Let $\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)$ be the unit ball of $W_p^\alpha(\mathbb{R}^d, \gamma)$. The optimality of quadratures for $\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)$ is measured by the quantity

$$\mathrm{Int}_n(\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)) := \inf_{I_n} \sup_{f \in \boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)} |I(f) - I_n(f)|. \tag{1.3}$$

We are interested in the asymptotic order of this quantity when $n \to \infty$, as well as in constructing asymptotically optimal quadratures. We do not investigate the dependence on the dimension and the problem of tractability. The problem of multivariate numerical integration (1.1)–(1.2) has been studied in [12, 13, 7] for functions in certain Hermite spaces, in particular, the space $\mathcal{H}_{d,\alpha}$ in [7] which coincides with $W_2^\alpha(\mathbb{R}^d, \gamma)$ in terms of norm equivalence. So far the best result on this problem is

$$n^{-\alpha} (\log n)^{\frac{d-1}{2}} \ll \mathrm{Int}_n\big(\boldsymbol{W}_2^\alpha(\mathbb{R}^d, \gamma)\big) \ll n^{-\alpha} (\log n)^{\frac{d(2\alpha+3)}{4} - \frac{1}{2}},$$

which has been proven in [7]. Moreover, the upper bound is achieved by a translated and scaled quasi-Monte Carlo (QMC) quadrature based on Dick's higher order digital nets. We note the related work [14] which studied weighted integration via a change of variables for functions on $\mathbb{R}^d$ from non-weighted spaces of mixed smoothness.

The aim of this paper is to prove the asymptotic order of $\mathrm{Int}_n\big(\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)\big)$. Let us briefly describe the main results.

For $\alpha \in \mathbb{N}$ and $1 < p < \infty$, we construct an asymptotically optimal quadrature I_n^γ of the form (1.2) which gives the asymptotic order of the convergence

$$\sup_{f \in \boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) \gamma(\mathrm{d}\boldsymbol{x}) - I_n^\gamma(f) \right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)\big) \asymp n^{-\alpha} (\log n)^{\frac{d-1}{2}}. \tag{1.4}$$

In constructing I_n^γ, we propose a novel method assembling an asymptotically optimal quadrature for the related Sobolev spaces on the unit d-cube to the integer-shifted d-cubes which cover $\mathbb{R}^d$. The asymptotically optimal quadrature I_n^γ is based on very sparse integration nodes contained in a d-ball of radius $\sqrt{\log n}$.

As for related problems with a similar approach, we establish the asymptotic orders of linear n-widths λ_n, Kolmogorov n-widths d_n, and sampling n-widths ϱ_n of the set $\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)$ in the Gaussian-weighted space $L_q(\mathbb{R}^d, \gamma)$ (see Section 3 for definitions). For $\alpha \in \mathbb{N}$ and $1 \le q < p < \infty$ we prove that

$$\lambda_n \asymp d_n \asymp n^{-\alpha} (\log n)^{(d-1)\alpha}, \tag{1.5}$$

and with the additional condition $q = 2$,

$$\varrho_n \asymp n^{-\alpha} (\log n)^{(d-1)\alpha}. \tag{1.6}$$

For $\alpha \in \mathbb{N}$ and $q = p = 2$, we prove that

$$\lambda_n = d_n \asymp n^{-\frac{\alpha}{2}} (\log n)^{\frac{(d-1)\alpha}{2}}, \tag{1.7}$$

and with the additional condition $\alpha \geq 2$,

$$\varrho_n \asymp n^{-\frac{\alpha}{2}} (\log n)^{\frac{(d-1)\alpha}{2}}. \tag{1.8}$$

The asymptotic orders (1.5)–(1.8) show very different approximation results between the cases $q < p$ and $q = p = 2$. We conjecture that the asymptotic orders (1.7) and (1.8) still hold true for p, q with the restrictions $p = q \neq 2$ and $1 < p < \infty$. The case $1 \leq p < q < \infty$ of these n-widths is excluded from the consideration caused by the natural reason that in this case we do not have a continuous embedding of $W_p^\alpha(\mathbb{R}^d, \gamma)$ into $L_q(\mathbb{R}^d, \gamma)$. For example, the function $f(\boldsymbol{x}) = \prod_{i=1}^d \left(1 + x_i^2\right)^{-m} \exp\left(|\boldsymbol{x}|^2/(2p)\right)$ belongs to $W_p^\alpha(\mathbb{R}^d, \gamma)$ if $m > 1/2 + \alpha$. However, this function does not belong to $L_q(\mathbb{R}^d, \gamma)$ when $q > p$.

The paper is organized as follows. In Section 2, we prove the asymptotic order of $\mathrm{Int}_n\big(\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)\big)$ and construct asymptotically optimal quadratures. Section 3 is devoted to the proof of the asymptotic order of linear n-widths λ_n and Kolmogorov n-widths d_n for the cases $q < p$ and $q = p = 2$ and the construction of asymptotically optimal linear approximations. In this section we also give asymptotic order of sampling n-widths for the cases $q = 2 < p$ and $q = p = 2$. In Section 4, we illustrate our integration nodes in comparison with those used in [7] and give a numerical test for the results obtained in Section 2.

Notation. We write $\mathbb{R}_1 := \{x \in \mathbb{R} : x \geq 1\}$. For a Banach space E, denote by the bold symbol $\boldsymbol{E}$ the unit ball in E. The letter d is always reserved for the underlying dimension of $\mathbb{R}^d$, $\mathbb{N}^d$, etc. Vectors in $\mathbb{R}^d$ are denoted by boldface letters. For $\boldsymbol{x} \in \mathbb{R}^d$, x_i denotes the ith coordinate, i.e., $\boldsymbol{x} := (x_1, \ldots, x_d)$. If $1 \leq p \leq \infty$, we write $|\boldsymbol{x}|_p := \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}$ with the usual modification when $p = \infty$. When $p = 2$ we simply write $|\boldsymbol{x}|$. For the quantities A_n and B_n depending on n in an index set J we write $A_n \ll B_n$ if there exists some constant $C > 0$ independent of n such that $A_n \leq C B_n$ for all $n \in J$, and $A_n \asymp B_n$ if $A_n \ll B_n$ and $B_n \ll A_n$. General positive constants or positive constants depending on parameters $\alpha, d, \ldots$ are denoted by C or $C_{\alpha, d, \ldots}$, respectively. Values of constants C and $C_{\alpha,d}$ in general, are not specified except in the cases when they are precisely given, and may be different in various places. Denote by $|G|$ the cardinality of the finite set G.

2 Numerical integration

In this section, based on a quadrature on the d-cube $\mathbb{I}^d := \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ for numerical integration of functions from classical Sobolev spaces of mixed smoothness on $\mathbb{I}^d$, by assembling we construct a quadrature on $\mathbb{R}^d$ for numerical integration of functions from γ-weighted Sobolev spaces $W_p^\alpha(\mathbb{R}^d, \gamma)$ which preserves the convergence rate. As a consequence, we prove the asymptotic order of $\mathrm{Int}_n\big(\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)\big)$.

2.1 Assembling quadratures

We first introduce γ-weighted Sobolev spaces of mixed smoothness. Let $1 \leq p < \infty$ and Ω be a Lebesgue measurable set on $\mathbb{R}^d$. We define the γ-weighted space $L_p(\Omega, \gamma)$ to be the set of all functions f on Ω such that the norm

$$\|f\|_{L_p(\Omega,\gamma)} := \left(\int_\Omega |f(\boldsymbol{x})|^p \gamma(\mathrm{d}\boldsymbol{x}) \right)^{1/p} = \left(\int_\Omega |f(\boldsymbol{x})|^p g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right)^{1/p} < \infty.$$

For $\alpha \in \mathbb{N}$, we define the γ-weighted space $W_p^\alpha(\Omega, \gamma)$ to be the normed space of all functions $f \in L_p(\Omega, \gamma)$ such that the weak (generalized) partial derivative $D^{\boldsymbol{r}} f$ of order $\boldsymbol{r}$ belongs to $L_p(\Omega, \gamma)$ for all $\boldsymbol{r} \in \mathbb{N}_0^d$ satisfying $|\boldsymbol{r}|_\infty \le \alpha$. The norm of a function f in this space is defined by

$$\|f\|_{W_p^\alpha(\Omega,\gamma)} := \left(\sum_{|\boldsymbol{r}|_\infty \le \alpha} \|D^{\boldsymbol{r}} f\|_{L_p(\Omega,\gamma)}^p \right)^{1/p}. \tag{2.1}$$

The space $W_p^\alpha(\Omega)$ is defined as the classical Sobolev space by replacing $L_p(\Omega, \gamma)$ with $L_p(\Omega)$ in (2.1), where as usual, $L_p(\Omega)$ denotes the Lebesgue space of functions on Ω equipped with the usual p-integral norm. For technical convenience we use the conventions $\mathrm{Int}_n := \mathrm{Int}_{\lfloor n \rfloor}$ and $I_n := I_{\lfloor n \rfloor}$ for $n \in \mathbb{R}_1$.

For numerical approximation of integrals $I^\Omega(f) := \int_\Omega f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$ over the set Ω, we need natural modifications $I_n^\Omega(f)$ for functions f on Ω, and $\mathrm{Int}_n^\Omega(F)$ for a set F of functions on Ω, of the definitions (1.2) and (1.3). For simplicity we will drop Ω from these notations if there is no misunderstanding.

Let $\alpha \in \mathbb{N}$, $1 < p < \infty$ and $a > 0$, $b \ge 0$. Assume that for the quadrature

$$I_m(f) := \sum_{i=1}^m \lambda_i f(\boldsymbol{x}_i), \quad \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\} \subset \mathbb{I}^d, \tag{2.2}$$

holds the convergence rate

$$\left| \int_{\mathbb{I}^d} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - I_m(f) \right| \le C m^{-a} (\log m)^b \|f\|_{W_p^\alpha(\mathbb{I}^d)}, \quad f \in W_p^\alpha(\mathbb{I}^d). \tag{2.3}$$

Then based on I_m, we will construct a quadrature on $\mathbb{R}^d$ which approximates the integral $I(f)$ with the same convergence rate for $f \in W_p^\alpha(\mathbb{R}^d, \gamma)$.

Our strategy is as follows. The integral $I(f)$ can be represented as the sum of component integrals over the integer-shifted d-cubes $\mathbb{I}_{\boldsymbol{k}}^d$ by

$$I(f) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \int_{\mathbb{I}_{\boldsymbol{k}}^d} f_{\boldsymbol{k}}(\boldsymbol{x}) g_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{2.4}$$

where for $\boldsymbol{k} \in \mathbb{Z}^d$, $\mathbb{I}_{\boldsymbol{k}}^d := \boldsymbol{k} + \mathbb{I}^d$ and for a function f on $\mathbb{R}^d$, $f_{\boldsymbol{k}}$ denotes the restriction of f to $\mathbb{I}_{\boldsymbol{k}}^d$. For a given $n \in \mathbb{R}_1$, we take "shifted" quadratures $I_{n_{\boldsymbol{k}}}$ of the form (2.2) for approximating the component integrals in the sum in (2.4). The integration nodes in $I_{n_{\boldsymbol{k}}}$, $\boldsymbol{k} \in \mathbb{Z}^d$, are taken so that they become sparser as $|\boldsymbol{k}|$ gets larger and

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^d} \lfloor n_{\boldsymbol{k}} \rfloor \le n.$$

In the next step, we "assemble" these shifted integration nodes to form a quadrature I_n^γ for approximating $I(f)$. Let us describe this construction in detail.

It is clear that if $f \in W^\alpha_p(\mathbb{R}^d,\gamma)$, then $f_{\boldsymbol{k}}(\cdot+\boldsymbol{k}) \in W^\alpha_p(\mathbb{I}^d)$, and

$$\begin{aligned}\|f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|_{W^\alpha_p(\mathbb{I}^d)} &= \Bigg(\sum_{|\boldsymbol{r}|_\infty\le\alpha}\|D^{\boldsymbol{r}}f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|^p_{L_p(\mathbb{I}^d)}\Bigg)^{1/p}\\ &= \Bigg(\sum_{|\boldsymbol{r}|_\infty\le\alpha}\|D^{\boldsymbol{r}}f_{\boldsymbol{k}}\|^p_{L_p(\mathbb{I}^d_{\boldsymbol{k}})}\Bigg)^{1/p}\\ &= \Bigg(\sum_{|\boldsymbol{r}|_\infty\le\alpha}(2\pi)^{d/2}\int_{\mathbb{I}^d_{\boldsymbol{k}}}e^{\frac{|\boldsymbol{x}|^2}{2}}|D^{\boldsymbol{r}}f_{\boldsymbol{k}}(\boldsymbol{x})|^p g(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Bigg)^{1/p}.\end{aligned} \tag{2.5}$$

When $\boldsymbol{x}\in\mathbb{I}^d_{\boldsymbol{k}}$ we have $e^{\frac{|\boldsymbol{x}|^2}{2}} \le e^{\frac{|\boldsymbol{k}+(\operatorname{sign}\boldsymbol{k})/2|^2}{2}}$, where $\operatorname{sign}\boldsymbol{k} := (\operatorname{sign}k_1,\dots,\operatorname{sign}k_d)$ and $\operatorname{sign}x := 1$ if $x\ge 0$, and $\operatorname{sign}x := -1$ otherwise for $x\in\mathbb{R}$. Therefore,

$$\|f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|_{W^\alpha_p(\mathbb{I}^d)} \le (2\pi)^{\frac{d}{2p}}e^{\frac{|\boldsymbol{k}+(\operatorname{sign}\boldsymbol{k})/2|^2}{2p}}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)}. \tag{2.6}$$

We have

$$\|g_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|_{W^\alpha_p(\mathbb{I}^d)} = \Bigg(\sum_{|\boldsymbol{r}|_\infty\le\alpha}\|D^{\boldsymbol{r}}g\|^p_{L_p(\mathbb{I}^d_{\boldsymbol{k}})}\Bigg)^{1/p}.$$

A direct computation shows that for $\boldsymbol{r}\in\mathbb{N}^d_0$ we have $D^{\boldsymbol{r}}g(\boldsymbol{x}) = P_{\boldsymbol{r}}(\boldsymbol{x})g(\boldsymbol{x})$ where $P_{\boldsymbol{r}}(\boldsymbol{x})$ is a polynomial of order $|\boldsymbol{r}|_1$ of $\boldsymbol{x}$. Moreover, we have $-|x|^2 \le \frac{1}{2} - |k-(\operatorname{sign}k)/2|^2$ for $x\in[-\frac12,\frac12]+k,\ k\in\mathbb{Z}$. Therefore for $\boldsymbol{x}\in\mathbb{I}^d_{\boldsymbol{k}}$ we get

$$|D^{\boldsymbol{r}}g(\boldsymbol{x})| = \left|(2\pi)^{-d/2}P_{\boldsymbol{r}}(\boldsymbol{x})e^{-\frac{|\boldsymbol{x}|^2}{2}}\right| \le Ce^{-\frac{|\boldsymbol{x}|^2}{2\tau'}} \le Ce^{-\frac{|\boldsymbol{k}-(\operatorname{sign}\boldsymbol{k})/2|^2}{2\tau'}} \le Ce^{-\frac{|\boldsymbol{k}|^2}{2\tau}}$$

for some τ' and τ such that $1<\tau'<\tau<p<\infty$. This implies that

$$\|g_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|_{W^\alpha_p(\mathbb{I}^d)} \le Ce^{-\frac{|\boldsymbol{k}|^2}{2\tau}} \tag{2.7}$$

with C independent of $\boldsymbol{k}\in\mathbb{Z}^d$. Since $W^\alpha_p(\mathbb{I}^d)$ is a multiplication algebra (see [15, Theorem 3.16]), from (2.6) and (2.7) we have that

$$f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})g_{\boldsymbol{k}}(\cdot+\boldsymbol{k}) \in W^\alpha_p(\mathbb{I}^d), \tag{2.8}$$

and

$$\begin{aligned}\|f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})g_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|_{W^\alpha_p(\mathbb{I}^d)} &\le C\|f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|_{W^\alpha_p(\mathbb{I}^d)}\cdot\|g_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|_{W^\alpha_p(\mathbb{I}^d)}\\ &\le Ce^{\frac{|\boldsymbol{k}+(\operatorname{sign}\boldsymbol{k})/2|^2}{2p}-\frac{|\boldsymbol{k}|^2}{2\tau}}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)}.\end{aligned} \tag{2.9}$$

For $1<\tau<p<\infty$, we choose $\delta>0$ so that

$$\max\left\{e^{-\frac{|\boldsymbol{k}-(\operatorname{sign}\boldsymbol{k})/2|^2}{2}\left(1-\frac1p\right)}, e^{\frac{|\boldsymbol{k}+(\operatorname{sign}\boldsymbol{k})/2|^2}{2p}-\frac{|\boldsymbol{k}|^2}{2\tau}}\right\} \le Ce^{-\delta|\boldsymbol{k}|^2} \tag{2.10}$$

for $\boldsymbol{k}\in\mathbb{Z}^d$, and therefore,

$$\|f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})g_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|_{W^\alpha_p(\mathbb{I}^d)} \le Ce^{-\delta|\boldsymbol{k}|^2}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)}, \qquad \boldsymbol{k}\in\mathbb{Z}^d. \tag{2.11}$$

We define for $n \in \mathbb{R}_1$,

$$\xi_n = \sqrt{\delta^{-1} 2a (\log n)}\,, \tag{2.12}$$

and for $\boldsymbol{k} \in \mathbb{Z}^d$,

$$n_{\boldsymbol{k}} = \begin{cases} \varrho n e^{-\frac{\delta}{2a}|\boldsymbol{k}|^2} & \text{if } |\boldsymbol{k}| < \xi_n, \\ 0 & \text{if } |\boldsymbol{k}| \geq \xi_n, \end{cases} \tag{2.13}$$

where $\varrho := 2^{-d}\left(1 - e^{-\frac{\delta}{2a}}\right)^d$. We have

$$\sum_{|\boldsymbol{k}|<\xi_n} n_{\boldsymbol{k}} \leq n. \tag{2.14}$$

Indeed,

$$\begin{aligned} \sum_{|\boldsymbol{k}|<\xi_n} n_{\boldsymbol{k}} = \sum_{|\boldsymbol{k}|<\xi_n} \varrho n e^{-\frac{\delta}{2\alpha}|\boldsymbol{k}|^2} &\leq 2^d \varrho n \sum_{s=0}^{\lfloor \xi_n \rfloor} \binom{s+d-1}{d-1} e^{-\frac{\delta}{2a}s^2} \\ &\leq 2^d \varrho n \sum_{s=0}^{\infty} \binom{s+d-1}{d-1} e^{-\frac{\delta}{2a}s} \leq n, \end{aligned}$$

where in the last estimate we used the well-known formula

$$\sum_{j=0}^{\infty} x^j \binom{j+k}{k} = (1-x)^{-k-1}, \ k \in \mathbb{N}_0, \ x \in (0,1). \tag{2.15}$$

We define

$$I_n(f) := \sum_{|\boldsymbol{k}|<\xi_n} I_{n_{\boldsymbol{k}}}(f_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) g_{\boldsymbol{k}}(\cdot + \boldsymbol{k})) = \sum_{|\boldsymbol{k}|<\xi_n} \sum_{j=1}^{\lfloor n_{\boldsymbol{k}} \rfloor} \lambda_j f_{\boldsymbol{k}}(\boldsymbol{x}_j + \boldsymbol{k}) g_{\boldsymbol{k}}(\boldsymbol{x}_j + \boldsymbol{k}), \tag{2.16}$$

or equivalently,

$$I_n(f) := \sum_{|\boldsymbol{k}|<\xi_n} \sum_{j=1}^{\lfloor n_{\boldsymbol{k}} \rfloor} \lambda_{\boldsymbol{k},j} f(\boldsymbol{x}_{\boldsymbol{k},j}) \tag{2.17}$$

as a quadrature for the approximate integration of γ-weighted functions f on $\mathbb{R}^d$, where $\boldsymbol{x}_{\boldsymbol{k},j} := \boldsymbol{x}_j + \boldsymbol{k}$ and $\lambda_{\boldsymbol{k},j} := \lambda_j g_{\boldsymbol{k}}(\boldsymbol{x}_j + \boldsymbol{k})$ (here for simplicity, with an abuse of notation the dependence of integration nodes and weights on the quadratures $I_{n_{\boldsymbol{k}}}$ is omitted). The integration nodes of the quadrature I_n are

$$\{\boldsymbol{x}_{\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_n,\, j = 1, \ldots, \lfloor n_{\boldsymbol{k}} \rfloor\} \subset \mathbb{R}^d, \tag{2.18}$$

and the integration weights

$$(\lambda_{\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_n,\, j = 1, \ldots, \lfloor n_{\boldsymbol{k}} \rfloor).$$

Due to (2.14), the number of integration nodes is not greater than n. From the definition we can see that the integration nodes are contained in the ball of radius $\xi_n^* := \sqrt{d}/2 + \xi_n$, i.e., $\{\boldsymbol{x}_{\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_n,\, j = 1, \ldots, \lfloor n_{\boldsymbol{k}} \rfloor\} \subset B(\xi_n^*) := \left\{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| \leq \xi_n^*\right\}$. The density of the integration nodes is exponentially decreasing in $|\boldsymbol{k}|$ to zero from the origin of $\mathbb{R}^d$ to the boundary of the ball $B(\xi_n^*)$, and the set of integration nodes is very sparse because of the choice of $n_{\boldsymbol{k}}$ as in (2.13).

Theorem 2.1 *Let $\alpha \in \mathbb{N}$, $1 < p < \infty$ and $a > 0$, $b \ge 0$. Assume that for any $m \in \mathbb{R}_1$, there is a quadrature I_m of the form* (2.2) *satisfying* (2.3)*. Then for the quadrature I_n defined as in* (2.17) *we have*

$$\left| \int_{\mathbb{R}^d} f(\boldsymbol{x})\gamma(\mathrm{d}\boldsymbol{x}) - I_n(f) \right| \ll n^{-a}(\log n)^b \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)}, \quad f \in W_p^\alpha(\mathbb{R}^d,\gamma). \tag{2.19}$$

Proof. Let $f \in W_p^\alpha(\mathbb{R}^d,\gamma)$ and $m \in \mathbb{R}_1$. For the quadrature I_m for functions on $\mathbb{I}^d$ in the assumption, from (2.3) and (2.11) we have

$$\left| \int_{\mathbb{I}^d} f_{\boldsymbol{k}}(\boldsymbol{x}+\boldsymbol{k}) g_{\boldsymbol{k}}(\boldsymbol{x}+\boldsymbol{k}) \mathrm{d}\boldsymbol{x} - I_m(f_{\boldsymbol{k}}(\cdot+\boldsymbol{k}) g_{\boldsymbol{k}}(\cdot+\boldsymbol{k})) \right| \ll m^{-a}(\log m)^b e^{-\delta|\boldsymbol{k}|^2} \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)}. \tag{2.20}$$

From (2.4) and (2.16) it follows that

$$\begin{aligned}\left| \int_{\mathbb{R}^d} f(\boldsymbol{x})\gamma(\mathrm{d}\boldsymbol{x}) - I_n(f) \right| \le & \sum_{|\boldsymbol{k}|<\xi_n} \left| \int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x}) g_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - I_{n_{\boldsymbol{k}}}(f_{\boldsymbol{k}}(\cdot+\boldsymbol{k}) g_{\boldsymbol{k}}(\cdot+\boldsymbol{k})) \right| \\ & + \sum_{|\boldsymbol{k}|\ge\xi_n} \left| \int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x}) g_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right|.\end{aligned}$$

For each term in the first sum by (2.20) we derive the estimates

$$\begin{aligned}&\left| \int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x}) g_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - I_{n_{\boldsymbol{k}}}(f_{\boldsymbol{k}}(\cdot+\boldsymbol{k}) g_{\boldsymbol{k}}(\cdot+\boldsymbol{k})) \right| \\ &\qquad = \left| \int_{\mathbb{I}^d} f_{\boldsymbol{k}}(\boldsymbol{x}+\boldsymbol{k}) g_{\boldsymbol{k}}(\boldsymbol{x}+\boldsymbol{k}) \mathrm{d}\boldsymbol{x} - I_{n_{\boldsymbol{k}}}(f_{\boldsymbol{k}}(\cdot+\boldsymbol{k}) g_{\boldsymbol{k}}(\cdot+\boldsymbol{k})) \right| \\ &\qquad \ll n_{\boldsymbol{k}}^{-a}(\log n_{\boldsymbol{k}})^b e^{-\delta|\boldsymbol{k}|^2} \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)} \\ &\qquad \ll (n e^{-\frac{\delta}{2a}|\boldsymbol{k}|^2})^{-a}(\log n)^b e^{-\delta|\boldsymbol{k}|^2} \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)} \\ &\qquad = n^{-a}(\log n)^b e^{-\frac{|\boldsymbol{k}|^2\delta}{2}} \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{|\boldsymbol{k}|<\xi_n} \left| \int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x}) g_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - I_{n_{\boldsymbol{k}}}(f_{\boldsymbol{k}}(\cdot+\boldsymbol{k}) g_{\boldsymbol{k}}(\cdot+\boldsymbol{k})) \right| &\ll \sum_{|\boldsymbol{k}|<\xi_n} n^{-a}(\log n)^b e^{-\frac{|\boldsymbol{k}|^2\delta}{2}} \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)} \\ &\ll n^{-a}(\log n)^b \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)}.\end{aligned}$$

For each term in the second sum we get by Hölder's inequality and (2.10),

$$\begin{aligned}\left| \int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x}) g_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right| &\le \left(\int_{\mathbb{I}^d_{\boldsymbol{k}}} |f_{\boldsymbol{k}}(\boldsymbol{x})|^p g_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}} \left(\int_{\mathbb{I}^d_{\boldsymbol{k}}} g_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right)^{1-\frac{1}{p}} \\ &\ll e^{-\frac{|\boldsymbol{k}-(\operatorname{sign}\boldsymbol{k})/2|^2}{2}(1-\frac{1}{p})} \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)} \\ &\ll e^{-\delta|\boldsymbol{k}|^2} \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)},\end{aligned}$$

which implies

$$
\begin{aligned}
\sum_{|\boldsymbol{k}|\geq\xi_n}\left|\int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x})g_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right| &\ll \sum_{|\boldsymbol{k}|\geq\xi_n} e^{-\delta|\boldsymbol{k}|^2}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)} \\
&\leq 2^d\sum_{s=\lceil\xi_n\rceil}^{\infty} e^{-s^2\delta}\binom{s+d-1}{d-1}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)} \\
&\leq 2^d e^{-\xi_n^2\delta(1-\varepsilon)}\sum_{s=\lceil\xi_n\rceil}^{\infty} e^{-s^2\varepsilon\delta}\binom{s+d-1}{d-1}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)} \\
&\ll e^{-\xi_n^2\delta(1-\varepsilon)}\sum_{s=0}^{\infty} e^{-s\varepsilon\delta}\binom{s+d-1}{d-1}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)}
\end{aligned}
\tag{2.21}
$$

with $\varepsilon\in(0,1/2)$. Using (2.15) we get

$$
\sum_{|\boldsymbol{k}|\geq\xi_n}\left|\int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x})g_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right| \ll e^{-2a(1-\varepsilon)\log n}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)} \ll n^{-a}(\log n)^b\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)}. \tag{2.22}
$$

Summing up, we have proven (2.19). $\square$

Some important quadratures such as the Frolov quadrature and the QMC quadrature based on Fibonacci lattice rules ($d=2$) are constructively designed for functions on $\mathbb{R}^d$ with support contained in the unit d-cube or for 1-periodic functions. To employ them for constructing a quadrature for functions on $\mathbb{R}^d$ we need to modify those constructions.

Assume that there is a quadrature I_m of the form (2.2) with the integration nodes $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_m\}\subset\left(-\frac12,\frac12\right)^d$ and weights $(\lambda_1,\dots,\lambda_m)$ such that the convergence rate

$$
\left|\int_{\mathbb{I}^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - I_m(f)\right| \leq Cm^{-a}(\log m)^b\|f\|_{W^\alpha_p(\mathbb{I}^d)},\quad f\in \mathring{W}^\alpha_p(\mathbb{I}^d) \tag{2.23}
$$

holds, where $\mathring{W}^\alpha_p(\mathbb{I}^d)$ denotes the space of functions in $W^\alpha_p(\mathbb{R}^d)$ with support contained in $\mathbb{I}^d$. Then based on the quadrature I_m, we propose two constructions of quadratures which approximate the integral $\int_{\mathbb{R}^d} f(\boldsymbol{x})\gamma(\mathrm{d}\boldsymbol{x})$ with the same convergence rate for $f\in W^\alpha_p(\mathbb{R}^d,\gamma)$.

The first method is a preliminary change of variables to transform the quadrature I_m into a quadrature for functions in $W^\alpha_p(\mathbb{I}^d)$ which gives the same convergence rate, and then apply the construction as in (2.17). Let us describe it. Let $k\in\mathbb{N}$ and ψ_k be the function defined by

$$
\psi_k(t) = \begin{cases} C_k\int_0^t(\frac14-\xi^2)^k\,\mathrm{d}\xi, & t\in[-\frac12,\frac12], \\ \frac12, & t>\frac12, \\ -\frac12, & t<-\frac12, \end{cases} \tag{2.24}
$$

where $C_k=\left(\int_{-1/2}^{1/2}(\frac14-\xi^2)^k\,\mathrm{d}\xi\right)^{-1}$. Observe that ψ_k is a one-to-one mapping on $[-\frac12,\frac12]$ and ψ_k' has compact support on $[-\frac12,\frac12]$. If $f\in W^\alpha_p(\mathbb{I}^d)$, a change of variable yields that

$$
\int_{\mathbb{I}^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\mathbb{I}^d}\big(T_{\psi_k}f\big)(\boldsymbol{x})\mathrm{d}\boldsymbol{x},
$$

where

$$
\big(T_{\psi_k}f\big)(\boldsymbol{x}) := \psi_k'(x_1)\cdot\ldots\cdot\psi_k'(x_d)f\big(\psi_k(x_1),\dots,\psi_k(x_d)\big),\ \boldsymbol{x}\in\mathbb{I}^d.
$$

Observe that the function $T_{\psi_k} f$ has support contained in $\mathbb{I}^d$. If $T_{\psi_k} f$ belongs to $\mathring{W}_p^\alpha(\mathbb{I}^d)$, then a quadrature with the integration nodes $\{\tilde{\boldsymbol{x}}_1, \ldots, \tilde{\boldsymbol{x}}_m\} \subset \mathbb{I}^d$ and weights $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m)$ for the function f can be defined as

$$\tilde{I}_m(f) := I_m(T_{\psi_k} f) = \sum_{j=1}^m \tilde{\lambda}_j f(\tilde{\boldsymbol{x}}_j),$$

where $\tilde{\boldsymbol{x}}_j = (\psi_k(x_{j,1}), \ldots, \psi_k(x_{j,d}))$ and $\tilde{\lambda}_j = \lambda_j \psi_k'(x_{j,1}) \cdot \ldots \cdot \psi_k'(x_{j,d})$. Hence, our task is finding a condition on k so that the mapping

$$f \mapsto T_{\psi_k} f$$

is a bounded operator from $W_p^\alpha(\mathbb{I}^d)$ to $\mathring{W}_p^\alpha(\mathbb{I}^d)$. A first result was proved by Bykovskii [2] where he showed that T_{ψ_k} is bounded in $W_2^\alpha(\mathbb{I}^d)$ if $k \geq 2\alpha + 1$. This result has been extended by Temlyakov, see [22, Theorem IV.4.1], to $W_p^\alpha(\mathbb{I}^d)$ under the condition $k \geq \left\lfloor \frac{\alpha p}{p-1} \right\rfloor + 1$. A recent improvement $k > \alpha + 1$ was obtained in [16].

The second method is to decompose functions in $W_p^\alpha(\mathbb{R}^d, \gamma)$ into a sum of functions on $\mathbb{R}^d$ having support contained in integer translations of the d-cube $\mathbb{I}_\theta^d := \left[-\frac{\theta}{2}, \frac{\theta}{2}\right]$ for a fixed $\theta > 1$. Then the quadrature for $W_p^\alpha(\mathbb{R}^d, \gamma)$ is the sum of integer-translated dilations of I_m. Details of this construction are presented below.

First observe that

$$\mathbb{R}^d = \bigcup_{\boldsymbol{k} \in \mathbb{Z}^d} \mathbb{I}_{\theta, \boldsymbol{k}}^d,$$

where $\mathbb{I}_{\theta,\boldsymbol{k}}^d := \mathbb{I}_\theta^d + \boldsymbol{k}$. It is well-known that one can constructively define a partition of unity $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ such that

(i) $\varphi_{\boldsymbol{k}} \in C_0^\infty(\mathbb{R}^d)$ and $0 \leq \varphi_{\boldsymbol{k}}(\boldsymbol{x}) \leq 1, \quad \boldsymbol{x} \in \mathbb{R}^d, \quad \boldsymbol{k} \in \mathbb{Z}^d$;

(ii) $\operatorname{supp} \varphi_{\boldsymbol{k}}$ are contained in the interior of $\mathbb{I}_{\theta,\boldsymbol{k}}^d$, $\boldsymbol{k} \in \mathbb{Z}^d$;

(iii) $\sum_{\boldsymbol{k} \in \mathbb{Z}^d} \varphi_{\boldsymbol{k}}(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \mathbb{R}^d$;

(iv) $\|\varphi_{\boldsymbol{k}}\|_{W_p^\alpha(\mathbb{I}_{\theta,\boldsymbol{k}}^d)} \leq C_{\alpha,d,\theta}, \quad \boldsymbol{k} \in \mathbb{Z}^d$,

(see, e.g., [18, Chapter VI, 1.3]). By the items (ii) and (iii) the integral $\int_{\mathbb{R}^d} f(\boldsymbol{x}) \gamma(\mathrm{d}\boldsymbol{x})$ can be represented as

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) \gamma(\mathrm{d}\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \int_{\mathbb{I}_{\theta,\boldsymbol{k}}^d} f_{\theta,\boldsymbol{k}}(\boldsymbol{x}) g_{\theta,\boldsymbol{k}}(\boldsymbol{x}) \varphi_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{2.25}$$

where $f_{\theta,\boldsymbol{k}}$ and $g_{\theta,\boldsymbol{k}}$ denote the restrictions of f and g on $\mathbb{I}_{\theta,\boldsymbol{k}}^d$, respectively. The quadrature (2.2) induces the quadrature

$$I_{\theta,m}(f) := \sum_{i=1}^m \lambda_{\theta,i} f(\boldsymbol{x}_{\theta,i}), \tag{2.26}$$

for functions f on $\mathbb{I}_\theta^d$, where $\boldsymbol{x}_{\theta,i} := \theta \boldsymbol{x}_i$ and $\lambda_{\theta,i} := \theta \lambda_i$.

Denote by $\mathring{W}_p^\alpha(\mathbb{I}_\theta^d)$ the subspace of functions in $W_p^\alpha(\mathbb{R}^d)$ with support contained in $\mathbb{I}_\theta^d$. From (2.23) the error bound

$$\left| \int_{\mathbb{I}_\theta^d} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - I_{\theta,m}(f) \right| \ll m^{-a} (\log m)^b \|f\|_{W_p^\alpha(\mathbb{I}_\theta^d)}$$

holds for every $f \in \mathring{W}_p^\alpha(\mathbb{I}_\theta^d)$. Let $f \in W_p^\alpha(\mathbb{R}^d, \gamma)$. It is clear that $f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in W_p^\alpha(\mathbb{I}_\theta^d)$ and similar to (2.5) and (2.6) we get

$$\|f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W_p^\alpha(\mathbb{I}_\theta^d)} \ll e^{\frac{|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2p}} \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)}, \quad f \in W_p^\alpha(\mathbb{R}^d, \gamma), \quad \boldsymbol{k} \in \mathbb{Z}^d.$$

Similarly to (2.8) and (2.9), by additionally using the items (ii) and (iv) we have that

$$f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) g_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) \varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in \mathring{W}_p^\alpha(\mathbb{I}_\theta^d),$$

and

$$\|f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) g_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) \varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W_p^\alpha(\mathbb{I}_\theta^d)} \ll e^{\frac{|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2p} - \frac{|\boldsymbol{k}|^2}{2\tau}} \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)},$$

where τ is a fixed number satisfying the inequalities $1 < \tau < p < \infty$. We choose $\delta > 0$ so that

$$\max\left\{ e^{-\frac{|\boldsymbol{k}-(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2}(1-\frac{1}{p})}, e^{\frac{|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2p} - \frac{|\boldsymbol{k}|^2}{2\tau}} \right\} \le C e^{-\delta |\boldsymbol{k}|^2}, \quad \boldsymbol{k} \in \mathbb{Z}^d.$$

For $n \in \mathbb{R}_1$, let ξ_n and $n_{\boldsymbol{k}}$ be given as in (2.12) and (2.13), respectively. Noting (2.25) and (2.26), we define

$$I_{\theta,n}(f) := \sum_{|\boldsymbol{k}| < \xi_n} I_{\theta,n_{\boldsymbol{k}}}\big(f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) g_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) \varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\big),$$

or equivalently,

$$I_{\theta,n}(f) := \sum_{|\boldsymbol{k}| < \xi_n} \sum_{j=1}^{\lfloor n_{\boldsymbol{k}} \rfloor} \lambda_{\theta,\boldsymbol{k},j} f(\boldsymbol{x}_{\theta,\boldsymbol{k},j}) \tag{2.27}$$

as a linear quadrature for the approximate integration of γ-weighted functions f on $\mathbb{R}^d$ where $\boldsymbol{x}_{\theta,\boldsymbol{k},j} := \boldsymbol{x}_{\theta,j} + \boldsymbol{k}$ and $\lambda_{\theta,\boldsymbol{k},j} := \lambda_{\theta,j} g_{\boldsymbol{k}}(\boldsymbol{x}_{\theta,\boldsymbol{k},j}) \varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\theta,\boldsymbol{k},j})$. The integration nodes of the quadrature $I_{\theta,n}$ are

$$\{\boldsymbol{x}_{\theta,\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_n,\ j = 1, \dots, \lfloor n_{\boldsymbol{k}} \rfloor\} \subset \mathbb{R}^d, \tag{2.28}$$

and the weights

$$(\lambda_{\theta,\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_n,\ j = 1, \dots, \lfloor n_{\boldsymbol{k}} \rfloor).$$

Due to (2.14), the number of integration nodes is not greater than n. Moreover, from the definition we can see that the integration nodes are contained in the ball of radius $\xi_{\theta,n}^* := \theta\sqrt{d}/2 + \xi_n$, i.e.,

$$\{\boldsymbol{x}_{\theta,\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_{\theta,n}^*,\ j = 1, \dots, \lfloor n_{\boldsymbol{k}} \rfloor\} \subset B(\xi_{\theta,n}^*) := \left\{ \boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| \le \xi_{\theta,n}^* \right\}.$$

Notice that the set of integration nodes (2.28) possesses similar sparsity properties as the set (2.18).

In a way similar to the proof of Theorem 2.1 we derive

Theorem 2.2 *Let $\alpha \in \mathbb{N}$, $1 < p < \infty$ and $a > 0$, $b \geq 0$, $\theta > 1$. Assume that for any $m \in \mathbb{R}_1$, there is a quadrature I_m of the form (2.2) with $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\} \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d$ satisfying (2.23). Then for the quadrature $I_{\theta,n}$ defined as in (2.27) we have*

$$\left| \int_{\mathbb{R}^d} f(\boldsymbol{x})\gamma(\mathrm{d}\boldsymbol{x}) - I_{\theta,n}(f) \right| \ll n^{-a}(\log n)^b \|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)}, \quad f \in W_p^\alpha(\mathbb{R}^d, \gamma). \tag{2.29}$$

As noticed in Introduction, we do not study the dimension dependence for error estimates of integration. Hence the hidden constant in the bound (2.29) may depend on the dimension d and may increase exponentially in d. Therefore, for very large d, the resulting algorithm may not be practical.

2.2 Asymptotic order of optimal numerical integration

In this subsection, we prove the asymptotic order of optimal numerical integration as formulated in (1.4) based on Theorem 2.2 and known results on numerical integration for functions from $W_p^\alpha(\mathbb{I}^d)$.

Theorem 2.3 *Let $\alpha \in \mathbb{N}$ and $1 < p < \infty$. Then one can construct an asymptotically optimal family of quadratures of the form (2.27) $\left(I_n^\gamma\right)_{n\in\mathbb{R}_1}$ such that*

$$\sup_{f\in \boldsymbol{W}_p^\alpha(\mathbb{R}^d,\gamma)} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x})\gamma(\mathrm{d}\boldsymbol{x}) - I_n^\gamma(f) \right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}_p^\alpha(\mathbb{R}^d,\gamma)\big) \asymp n^{-\alpha}(\log n)^{\frac{d-1}{2}}. \tag{2.30}$$

Proof. Let $I_{\mathrm{F},m}$ be the Frolov quadrature for functions in $\mathring{W}_p^\alpha(\mathbb{I}^d)$ (see, e.g., [4, Chapter 8] for the definition) in the form (2.2) with $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\} \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d$. It was proven in [10] for $p = 2$, and in [17] for $1 < p < \infty$ that

$$\left| \int_{\mathbb{I}^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - I_{\mathrm{F},m}(f) \right| \leq Cm^{-\alpha}(\log m)^{\frac{d-1}{2}} \|f\|_{\mathring{W}_p^\alpha(\mathbb{I}^d)}, \quad f \in \mathring{W}_p^\alpha(\mathbb{I}^d). \tag{2.31}$$

For a fixed $\theta > 1$, we define $I_n^\gamma := I_{\theta,n}$ as the quadrature described in Theorem 2.2 for $a = \alpha$ and $b = \frac{d-1}{2}$, based on $I_m = I_{\mathrm{F},m}$. By Theorem 2.2 and (2.31) we prove the upper bound in (2.30).

Since for $f \in \mathring{W}_p^\alpha(\mathbb{I}^d)$

$$\|f\|_{W_p^\alpha(\mathbb{R}^d,\gamma)} \leq (2\pi)^{-\frac{d}{2p}} \|f\|_{\mathring{W}_p^\alpha(\mathbb{I}^d)},$$

we get

$$\mathrm{Int}_n\big(\boldsymbol{W}_p^\alpha(\mathbb{R}^d,\gamma)\big) \gg \mathrm{Int}_n(\mathring{\boldsymbol{W}}_p^\alpha(\mathbb{I}^d)).$$

Hence the lower bound in (2.30) follows from the lower bound $\mathrm{Int}_n(\mathring{\boldsymbol{W}}_p^\alpha(\mathbb{I}^d)) \gg n^{-\alpha}(\log n)^{\frac{d-1}{2}}$ proven in [20]. □

Besides Frolov quadratures, there are many quadratures for efficient numerical integration for functions on $\mathbb{I}^d$ to list. We refer the reader to [4, Chapter 8] for bibliography and historical comments as well as related results, in particular, the asymptotic order

$$\mathrm{Int}_m\big(\boldsymbol{W}_p^\alpha(\mathbb{I}^d)\big) \asymp m^{-\alpha}(\log m)^{\frac{d-1}{2}}$$

for $1 < p < \infty$. We recall only some of them, especially those which give asymptotic order of optimal integration.

A quasi-Monte Carlo (QMC) quadrature based on a set of integration nodes $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\} \subset \mathbb{I}^d$ is of the form

$$I_m(f) = \frac{1}{m} \sum_{i=1}^{m} f(\boldsymbol{x}_i).$$

In [5, 6] for a prime number q the author introduced higher order digital nets over the finite field $\mathbb{F}_q := \{0, 1, \ldots, q-1\}$ equipped with the arithmetic operations modulo q. Such digital nets can achieve the convergence rate $m^{-\alpha}(\log m)^{d\alpha}$ with $m = q^s$ for functions from $W_2^\alpha(\mathbb{I}^d)$, see [8]. In the recent paper [11], the authors have shown that the asymptotic order of $\mathrm{Int}_m\big(\boldsymbol{W}_2^\alpha(\mathbb{I}^d)\big)$ can be achieved by Dick's digital nets $\{\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_{q^s}^*\}$ of order $(2\alpha + 1)$. Namely, they proved that

$$\left| \int_{\mathbb{I}^d} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - \frac{1}{m} \sum_{i=1}^{m} f(\boldsymbol{x}_i^*) \right| \le C m^{-\alpha} (\log m)^{\frac{d-1}{2}} \|f\|_{W_2^\alpha(\mathbb{I}^d)}, \quad f \in \ W_2^\alpha(\mathbb{I}^d), \quad m = q^s. \tag{2.32}$$

In the case $d = 2$ the QMC quadrature $I_m = I_{\Phi,m}$ based on Fibonacci lattice rules ($d = 2$) is also asymptotically optimal for numerical integration of periodic functions in $\tilde{W}_p^\alpha(\mathbb{I}^2)$, that is,

$$\left| \int_{\mathbb{I}^2} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - I_{\Phi,m}(f) \right| \le C m^{-\alpha} (\log m)^{\frac{1}{2}} \|f\|_{W_p^\alpha(\mathbb{I}^2)}, \quad f \in \tilde{W}_p^\alpha(\mathbb{I}^2), \tag{2.33}$$

where $\tilde{W}_p^\alpha(\mathbb{I}^2)$ denotes the subspace of $W_p^\alpha(\mathbb{I}^2)$ of all functions which can be extended to the whole $\mathbb{R}^2$ as 1-periodic functions in each variable. The estimate (2.33) was proven in [1] for $p = 2$ and in [21] for $1 < p < \infty$. The QMC quadrature $I_m = I_{\Phi,m}$ based on Fibonacci lattice rules ($d = 2$) is defined by

$$I_{\Phi,m}(f) := \frac{1}{b_m} \sum_{i=1}^{b_m} f\left(\left\{ \frac{i}{b_m} \right\} - \frac{1}{2}, \left\{ \frac{i b_{m-1}}{b_m} \right\} - \frac{1}{2} \right),$$

where $b_0 = b_1 = 1$, $b_m := b_{m-1} + b_{m-2}$ are the Fibonacci numbers and $\{x\}$ denotes the fractional part of the number x.

Therefore, from Theorems 2.1–2.3 and (2.32), (2.33) it follows that the QMC quadratures based on Dick's digital nets of order $(2\alpha + 1)$ and Fibonacci lattice rules ($d = 2$) can be used for assembling asymptotically optimal quadratures I_n^γ and $I_{\theta,n}^\gamma$ of the forms (2.17) and (2.27) for $\mathrm{Int}_n\big(\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma)\big)$, in the particular cases $p = 2$, $d \ge 2$, and $1 < p < \infty$, $d = 2$, respectively.

The sparse Smolyak grid $SG(\xi)$ in $\mathbb{I}^d$ is defined as the set of points:

$$SG(\xi) := \left\{ \boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}} := 2^{-\boldsymbol{k}} \boldsymbol{s} \in \mathbb{Z}^d : |\boldsymbol{k}|_1 \le \xi, \quad |s_i| \le 2^{k_i - 1}, \ i = 1, \ldots, d \right\}, \quad \xi \in \mathbb{R}_1.$$

For a given $m \in \mathbb{R}_1$, let ξ_m be the maximal number satisfying $|SG(\xi_m)| \le m$. Then we can constructively define a quadrature $I_m = I_{\mathrm{S},m}$ based on the integration nodes in $SG(\xi_m)$ so that

$$\left| \int_{\mathbb{I}^d} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - I_{\mathrm{S},m}(f) \right| \le C m^{-\alpha} (\log m)^{(d-1)(\alpha+1/2)} \|f\|_{W_p^\alpha(\mathbb{I}^d)}, \quad f \in W_p^\alpha(\mathbb{I}^d). \tag{2.34}$$

To understand this quadrature let us recall a detailed construction from [3, page 760]. Indeed, from the well-known embedding of $W_p^\alpha(\mathbb{I}^d)$ into the Besov space of mixed smoothness

$B^{\alpha}_{p,\max(p,2)}(\mathbb{I}^d)$ (see, e.g., [4, Lemma 3.4.1(iv)]), and the result on B-spline sampling recovery of functions from the last space it follows that one can constructively define a sampling recovery algorithm of the form

$$R_m(f) := \sum_{\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}} \in SG(\xi_m)} f(\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}}) \phi_{\boldsymbol{k},\boldsymbol{s}}$$

with certain B-splines $\phi_{\boldsymbol{k},\boldsymbol{s}}$, such that

$$\|f - R_m(f)\|_{L_1(\mathbb{I}^d)} \le C m^{-\alpha} (\log m)^{(d-1)(\alpha+1/2)} \|f\|_{W^{\alpha}_p(\mathbb{I}^d)}, \quad f \in W^{\alpha}_p(\mathbb{I}^d).$$

Then the quadrature $I_{\mathrm{S},m}$ can be defined as

$$I_{\mathrm{S},m}(f) := \sum_{\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}} \in SG(\xi_m)} \lambda_{\boldsymbol{k},\boldsymbol{s}} f(\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}}), \quad \lambda_{\boldsymbol{k},\boldsymbol{s}} := \int_{\mathbb{I}^d} \phi_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},$$

and (2.34) is implied by the obvious inequality $\left| \int_{\mathbb{I}^d} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - I_{\mathrm{S},m}(f) \right| \le \|f - R_m(f)\|_{L_1(\mathbb{I}^d)}$. Therefore, from Theorem 2.1 and (2.34) we can see that the Smolyak quadrature $I_{\mathrm{S},m}$ can be used for assembling a quadrature $I_{\mathrm{S},n}$ of the form (2.17) with “double” sparse integration nodes which gives the convergence rate

$$\left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) \gamma(\mathrm{d}\boldsymbol{x}) - I_{\mathrm{S},n}(f) \right| \ll n^{-\alpha} (\log n)^{(d-1)(\alpha+1/2)}, \quad f \in \boldsymbol{W}^{\alpha}_p(\mathbb{R}^d, \gamma).$$

3 Approximation

In this section we study the linear approximation and sampling recovery in $L_q(\mathbb{R}^d, \gamma)$ of functions from $W^{\alpha}_p(\mathbb{R}^d, \gamma)$, and the asymptotic optimality in terms of Kolmogorov n-widths and the linear n-widths and sampling n-widths for $1 \le q < p < \infty$ and $p = q = 2$.

Let $n \in \mathbb{N}$ and let X be a Banach space and F a central symmetric compact set in X. Then the Kolmogorov n-width of F is defined by

$$d_n(F, X) := \inf_{L_n} \sup_{f \in F} \inf_{g \in L_n} \|f - g\|_X,$$

where the left-most infimum is taken over all subspaces L_n of dimension $\le n$ in X. The linear n-width of the set F is defined by

$$\lambda_n(F, X) := \inf_{A_n} \sup_{f \in F} \|f - A_n(f)\|_X,$$

where the infimum is taken over all linear operators A_n in X with $\operatorname{rank} A_n \le n$. Notice that if X is a Hilbert space, then $\lambda_n(F, X) = d_n(F, X)$.

Let Ω be a domain in $\mathbb{R}^d$. Let $n \in \mathbb{N}$ and let X be a Banach space of functions on Ω and F a compact set in X. Given $\{\boldsymbol{x}_i\}_{i=1}^n \subset \Omega$, to approximately recover $f \in F$ from the sampled values $\{f(\boldsymbol{x}_i)\}_{i=1}^n$ we use a (linear) sampling algorithm defined by

$$R_n(f) := \sum_{i=1}^{n} f(\boldsymbol{x}_i) \varphi_i, \tag{3.1}$$

where $\{\varphi_i\}_{i=1}^n$ is a collection of n functions in X. For convenience, we assume that some points from $\{\boldsymbol{x}_i\}_{i=1}^n \subset \Omega$ and some functions from $\{\varphi_i\}_{i=1}^n$ may coincide. For $n \in \mathbb{N}$ we define the sampling n-width of the set F in X as

$$\varrho_n(F, X) := \inf_{\substack{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \Omega, \\ \varphi_1, \ldots, \varphi_n \in X}} \sup_{f \in F} \|f - R_n(f)\|_X,$$

where $R_n(f)$ is given by (3.1). Obviously, we have the inequalities

$$d_n(F, X) \leq \lambda_n(F, X) \leq \varrho_n(F, X). \tag{3.2}$$

There are other popular n-widths in approximation theory like the entropy n-widths, Gel'fand n-widths and Bernstein n-widths, etc. In particular, for optimality of numerical algorithms, the Gel'fand n-widths are very important, since optimal algorithms could be non-linear (for detail, see, e.g., [4, Section 6 and Section 9.6]). However, these n-widths are not in the scope of consideration of the present paper.

For technical convenience we use the conventions $A_n := A_{\lfloor n \rfloor}$, $R_n := R_{\lfloor n \rfloor}$, $d_n(F, X) := d_{\lfloor n \rfloor}(F, X)$, $\lambda_n(F, X) := \lambda_{\lfloor n \rfloor}(F, X)$ and $\varrho_n(F, X) := \varrho_{\lfloor n \rfloor}(F, X)$ for $n \in \mathbb{R}_1$.

For given α and p, q, we make use of the abbreviations:

$$\lambda_n := \lambda_n(\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma), L_q(\mathbb{R}^d, \gamma)), \quad d_n := d_n(\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma), L_q(\mathbb{R}^d, \gamma)),$$

$$\varrho_n := \varrho_n(\boldsymbol{W}_p^\alpha(\mathbb{R}^d, \gamma), L_q(\mathbb{R}^d, \gamma)).$$

We prove the asymptotic orders of λ_n, d_n and ϱ_n as well as constructively define asymptotically optimal linear approximation methods which are very different for the cases $1 \leq q < p < \infty$ and $q = p = 2$.

3.1 The case $1 \leq q < p < \infty$

Let $\alpha \in \mathbb{N}$, $1 \leq q < p < \infty$ and $a > 0$, $b \geq 0$. Denote by $\tilde{L}_q(\mathbb{I}^d)$ and $\tilde{W}_p^\alpha(\mathbb{I}^d)$ the subspaces of $L_q(\mathbb{I}^d)$ and $W_p^\alpha(\mathbb{I}^d)$, respectively, of all functions f which can be extended to the whole $\mathbb{R}^d$ as 1-periodic functions in each variable (denoted again by f). Let A_m be a linear operator in $\tilde{L}_q(\mathbb{I}^d)$ of rank $\leq m$. Assume it holds that

$$\|f - A_m(f)\|_{\tilde{L}_q(\mathbb{I}^d)} \leq C m^{-a} (\log m)^b \|f\|_{\tilde{W}_p^\alpha(\mathbb{I}^d)}, \quad f \in \tilde{W}_p^\alpha(\mathbb{I}^d). \tag{3.3}$$

Then based on A_m, we will construct a linear operator A_m^γ in $L_q(\mathbb{R}^d, \gamma)$ which approximates $f \in W_p^\alpha(\mathbb{R}^d, \gamma)$ with the same convergence rate. Our strategy is similar to the problem of numerical integration considered in Subsection 2.1.

Fix a number θ with $\theta > 1$. Denote by $\tilde{L}_q(\mathbb{I}_\theta^d)$ and $\tilde{W}_p^\alpha(\mathbb{I}_\theta^d)$ the subspaces of $L_q(\mathbb{I}_\theta^d)$ and $W_p^\alpha(\mathbb{I}_\theta^d)$, respectively, of all functions f which can be extended to the whole $\mathbb{R}^d$ as θ-periodic functions in each variable (denoted again by f). A linear operator A_m induces the linear operator $A_{\theta,m}$ in $\tilde{L}_q(\mathbb{I}_\theta^d)$, defined for $f \in \tilde{L}_q(\mathbb{I}_\theta^d)$ by $A_{\theta,m}(f) := A_m(f(\cdot/\theta))$.

From (3.3) it follows that

$$\|f - A_{\theta,m}(f)\|_{\tilde{L}_q(\mathbb{I}_\theta^d)} \leq C m^{-a} (\log m)^b \|f\|_{\tilde{W}_p^\alpha(\mathbb{I}_\theta^d)}, \quad f \in \tilde{W}_p^\alpha(\mathbb{I}_\theta^d).$$

Since $q < p$, we can choose a fixed $\delta > 0$ such that

$$e^{\frac{|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2p} - \frac{|\boldsymbol{k}-(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2q}} \le C e^{-\delta |\boldsymbol{k}|^2}, \quad \boldsymbol{k} \in \mathbb{Z}^d. \tag{3.4}$$

For $n \in \mathbb{R}_1$, let ξ_n and $n_{\boldsymbol{k}}$ be given as in (2.12) and (2.13). Recall that we write $\mathbb{I}^d_{\theta,\boldsymbol{k}} := \boldsymbol{k} + \mathbb{I}^d_\theta$ for $\boldsymbol{k} \in \mathbb{Z}^d$, and $f_{\theta,\boldsymbol{k}}$ the restriction of f on $\mathbb{I}^d_{\theta,\boldsymbol{k}}$ for a function f on $\mathbb{R}^d$. Let $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ be the partition of unity satisfying items (i)–(iv), introduced in Subsection 2.1. Similarly to (2.8) and (2.9), by additionally using the items (ii) and (iv) we have that if $f \in W^\alpha_p(\mathbb{R}^d, \gamma)$, then

$$f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in \tilde{W}^\alpha_p(\mathbb{I}^d_\theta),$$

and it holds that

$$\|f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}^\alpha_p(\mathbb{I}^d_\theta)} \ll e^{\frac{|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2p}} \|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)}. \tag{3.5}$$

We define the linear operator $A^\gamma_{\theta,n}$ in $L_q(\mathbb{R}^d, \gamma)$ of rank $\le n$ by

$$\left(A^\gamma_{\theta,n} f\right)(\boldsymbol{x}) := \sum_{|\boldsymbol{k}|<\xi_n} \left(A_{\theta,n_{\boldsymbol{k}}} \tilde{f}_{\theta,\boldsymbol{k}}\right)(\boldsymbol{x} - \boldsymbol{k}), \tag{3.6}$$

where $\tilde{f}_{\theta,\boldsymbol{k}}(\boldsymbol{x}) = f_{\theta,\boldsymbol{k}}(\boldsymbol{x} + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\boldsymbol{x} + \boldsymbol{k})$. Indeed, by (2.14),

$$\operatorname{rank} A^\gamma_{\theta,n} \le \sum_{|\boldsymbol{k}|<\xi_n} \operatorname{rank} A_{\theta,n_{\boldsymbol{k}}} \le \sum_{|\boldsymbol{k}|<\xi_n} n_{\boldsymbol{k}} \le n.$$

Theorem 3.1 *Let $\alpha \in \mathbb{N}$, $1 \le q < p < \infty$ and $a > 0$, $b \ge 0$, $\theta > 1$. Assume that for any $m \in \mathbb{R}_1$, there is a linear operator A_m in $\tilde{L}_q(\mathbb{I}^d)$ of rank $\le m$ such that the convergence rate (3.3) holds. Then for any $n \in \mathbb{R}_1$, based on this linear operator one can construct the linear operator $A^\gamma_{\theta,n}$ in $L_q(\mathbb{R}^d,\gamma)$ of rank $\le n$ as in (3.6) so that*

$$\|f - A^\gamma_{\theta,n}(f)\|_{L_q(\mathbb{R}^d,\gamma)} \le C n^{-a} (\log n)^b \|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)}, \quad f \in W^\alpha_p(\mathbb{R}^d, \gamma). \tag{3.7}$$

Proof. The proof of this theorem is analogous to that of Theorem 2.2 with certain modifications. We give a short description of it. From the items (ii) and (iii) in Subsection 2.1 it is implied that

$$f = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}.$$

Hence we have

$$\|f - A^\gamma_{\theta,n}(f)\|_{L_q(\mathbb{R}^d,\gamma)} \le \sum_{|\boldsymbol{k}|<\xi_n} \left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - \left(A_{\theta,n_{\boldsymbol{k}}} \tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k}) \right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}},\gamma)} + \sum_{|\boldsymbol{k}|\ge\xi_n} \|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}},\gamma)}. \tag{3.8}$$

From (2.13), (3.3) and (3.5) we derive the estimates

$$\begin{aligned}
&\left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - \left(A_{\theta,n_{\boldsymbol{k}}} \tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k}) \right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}},\gamma)} \\
&\ll e^{-\frac{|\boldsymbol{k}-(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2q}} \left\| f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) - A_{\theta,n_{\boldsymbol{k}}} \tilde{f}_{\theta,\boldsymbol{k}} \right\|_{\tilde{L}_q(\mathbb{I}^d_\theta)} \\
&\ll e^{-\frac{|\boldsymbol{k}-(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2q}} n_{\boldsymbol{k}}^{-a} (\log n_{\boldsymbol{k}})^b \|f(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}^\alpha_p(\mathbb{I}^d_\theta)} \\
&\ll e^{\frac{|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2p} - \frac{|\boldsymbol{k}-(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2q}} \left(n e^{-\frac{\delta}{2a}|\boldsymbol{k}|^2}\right)^{-a} (\log n)^b \|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)}.
\end{aligned}$$

Using (3.4) we get

$$\left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - \left(A_{\theta,n_{\boldsymbol{k}}}\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot-\boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}},\gamma)} \ll e^{-\frac{\delta}{2}|\boldsymbol{k}|^2} n^{-a}(\log n)^b \|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)},$$

which implies

$$\begin{aligned}\sum_{|\boldsymbol{k}|<\xi_n} \left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - \left(A_{\theta,n_{\boldsymbol{k}}}\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot-\boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}},\gamma)} &\ll \sum_{|\boldsymbol{k}|<\xi_n} e^{-\frac{\delta}{2}|\boldsymbol{k}|^2} n^{-a}(\log n)^b \|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)} \\ &\ll n^{-a}(\log n)^b \|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)}.\end{aligned}$$

Similar to (2.21) and (2.22), we have for a fixed $\varepsilon \in (0,1/2)$,

$$\begin{aligned}\sum_{|\boldsymbol{k}|\ge\xi_n} \|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}},\gamma)} &\ll \sum_{|\boldsymbol{k}|\ge\xi_n} e^{-\frac{|\boldsymbol{k}-(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2q}+\frac{|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^2}{2p}} \|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)} \\ &\ll \sum_{|\boldsymbol{k}|\ge\xi_n} e^{-\delta|\boldsymbol{k}|^2}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)} \ll e^{-\delta(1-\varepsilon)\xi_n^2}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)} \\ &= e^{-2a(1-\varepsilon)\log n}\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)} \ll n^{-a}(\log n)^b\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)}.\end{aligned}$$

From the last two estimates and (3.8) we obtain (3.7). □

Lemma 3.2 *Let $\alpha \in \mathbb{N}$ and $1 \le q < p < \infty$. Then we have*

$$d_m(\tilde{\boldsymbol{W}}^\alpha_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)) \asymp m^{-\alpha}(\log m)^{(d-1)\alpha}.$$

Moreover, truncations on certain hyperbolic crosses of the Fourier series form an asymptotically optimal linear operator A_m in $\tilde{L}_q(\mathbb{I}^d)$ of rank $\le m$ such that

$$\|f - A_m(f)\|_{\tilde{L}_q(\mathbb{I}^d)} \ll m^{-\alpha}(\log m)^{(d-1)\alpha}\|f\|_{\tilde{W}^\alpha_p(\mathbb{I}^d)}, \quad f \in \tilde{W}^\alpha_p(\mathbb{I}^d). \tag{3.9}$$

For details on this lemma see, e.g., in [4, Theorems 4.2.5, 4.3.1 & 4.3.7] and related comments on the asymptotic optimality of the hyperbolic cross approximation.

We are now in the position to prove the main result in this section.

Theorem 3.3 *Let $\alpha \in \mathbb{N}$ and $1 \le q < p < \infty$. Then for any $n \in \mathbb{R}_1$, based on the linear operator A_m in Lemma 3.2 one can construct the linear operator A^γ_n in $L_q(\mathbb{R}^d,\gamma)$ of rank $\le n$ as in (3.6) so that*

$$\sup_{f\in \boldsymbol{W}^\alpha_p(\mathbb{R}^d,\gamma)} \|f - A^\gamma_n(f)\|_{L_q(\mathbb{R}^d,\gamma)} \asymp \lambda_n \asymp d_n \asymp n^{-\alpha}(\log n)^{(d-1)\alpha}. \tag{3.10}$$

Moreover, with the additional condition $q = 2$,

$$\varrho_n \asymp n^{-\alpha}(\log n)^{(d-1)\alpha}. \tag{3.11}$$

Proof. For a fixed $\theta > 1$, we define $A^\gamma_n := A^\gamma_{\theta,n}$ as the linear operator described in Theorem 3.1. The upper bounds in (3.10) follow from (3.9) and Theorem 3.1 with $a = \alpha$, $b = (d-1)\alpha$.

If f is a 1-periodic function on $\mathbb{R}^d$ and $f \in \tilde{W}^\alpha_p(\mathbb{I}^d)$, then

$$\begin{aligned}
\|f\|_{W^\alpha_p(\mathbb{R}^d,\gamma)} &= \left((2\pi)^{-d/2} \sum_{|\boldsymbol{r}|_\infty \le \alpha} \int_{\mathbb{R}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x})|^p e^{-\frac{|\boldsymbol{x}|^2}{2}} \mathrm{d}\boldsymbol{x} \right)^{1/p} \\
&= (2\pi)^{-\frac{d}{2p}} \left(\sum_{|\boldsymbol{r}|_\infty \le \alpha} \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \int_{\mathbb{I}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x}+\boldsymbol{k})|^p e^{-\frac{|\boldsymbol{x}+\boldsymbol{k}|^2}{2}} \mathrm{d}\boldsymbol{x} \right)^{1/p} \\
&\ll \left(\sum_{|\boldsymbol{r}|_\infty \le \alpha} \int_{\mathbb{I}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x} \sum_{\boldsymbol{k} \in \mathbb{Z}^d} e^{-\frac{|\boldsymbol{k}-(\operatorname{sign} \boldsymbol{k})/2|^2}{2}} \right)^{1/p} \\
&\ll \|f\|_{\tilde{W}^\alpha_p(\mathbb{I}^d)},
\end{aligned}$$

and

$$\|f\|_{\tilde{L}_q(\mathbb{I}^d)} = \left((2\pi)^{\frac{d}{2}} \int_{\mathbb{I}^d} |f(\boldsymbol{x})|^q e^{\frac{|\boldsymbol{x}|^2}{2}} g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right)^{1/q} \le (2\pi)^{\frac{d}{2q}} e^{\frac{d}{8q}} \|f\|_{L_q(\mathbb{R}^d,\gamma)}.$$

Hence we get

$$\lambda_n \ge d_n \gg d_n(\tilde{\boldsymbol{W}}^\alpha_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)).$$

Now Lemma 3.2 implies the lower bounds in (3.10).

We now prove (3.11). Assume $q = 2$. The lower bound of (3.11) follows from (3.2) and (3.10). Let us verify the upper one. By (3.10) we have that

$$d_n \ll n^{-\alpha} (\log n)^{(d-1)\alpha}. \tag{3.12}$$

Notice that the separable normed space $W^\alpha_p(\mathbb{R}^d,\gamma)$ is continuously embedded into $L_2(\mathbb{R}^d,\gamma)$, and the evaluation functional $f \mapsto f(\boldsymbol{x})$ is continuous on the space $W^\alpha_p(\mathbb{R}^d,\gamma)$ for each $\boldsymbol{x} \in \mathbb{R}^d$. This means that $\boldsymbol{W}^\alpha_p(\mathbb{R}^d,\gamma)$ satisfies Assumption A in [9]. By [9, Corollary 4] and (3.12) we prove the upper bound:

$$\varrho_n \ll d_n \ll n^{-\alpha} (\log n)^{(d-1)\alpha}.$$

$\square$

3.2 The case $q = p = 2$

Our approach to this case, which is completely different from the one in the case $1 \le q < p < \infty$, is similar to the hyperbolic cross trigonometric approximation in the Hilbert space $\tilde{L}_2(\mathbb{I}^d)$ of periodic functions from the Sobolev space $\tilde{W}^\alpha_2(\mathbb{I}^d)$ (see, e.g., [4] for details). Here, in the approximation, the trigonometric polynomials are replaced by the Hermite polynomials.

For $k \in \mathbb{N}_0$, the normalized probabilistic Hermite polynomial H_k of degree k on $\mathbb{R}$ is defined by

$$H_k(x) := \frac{(-1)^k}{\sqrt{k!}} \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^k}{\mathrm{d}x^k} \exp\left(-\frac{x^2}{2}\right).$$

For every multi-degree $\boldsymbol{k} \in \mathbb{N}^d_0$, the d-variate Hermite polynomial $H_{\boldsymbol{k}}$ is defined by

$$H_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^d H_{k_j}(x_j), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

It is well-known that the Hermite polynomials $\{H_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$ constitute an orthonormal basis of the Hilbert space $L_2(\mathbb{R}^d,\gamma)$ (see, e.g., [19, Section 5.5]). In particular, every $f \in L_2(\mathbb{R}^d,\gamma)$ can be represented by the Hermite series

$$f = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \hat{f}(\boldsymbol{k}) H_{\boldsymbol{k}} \quad \text{with} \quad \hat{f}(\boldsymbol{k}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, H_{\boldsymbol{k}}(\boldsymbol{x}) \gamma(\mathrm{d}\boldsymbol{x}) \tag{3.13}$$

converging in the norm of $L_2(\mathbb{R}^d,\gamma)$, and in addition, there holds Parseval's identity

$$\|f\|^2_{L_2(\mathbb{R}^d,\gamma)} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} |\hat{f}(\boldsymbol{k})|^2. \tag{3.14}$$

For $\alpha \in \mathbb{N}_0$ and $\boldsymbol{k} \in \mathbb{N}_0^d$, we define

$$\rho_{\alpha,\boldsymbol{k}} := \prod_{j=1}^d (k_j + 1)^\alpha .$$

Lemma 3.4 *Let $\alpha \in \mathbb{N}_0$. Then we have that*

$$\|f\|^2_{W_2^\alpha(\mathbb{R}^d,\gamma)} \asymp \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \rho_{\alpha,\boldsymbol{k}} |\hat{f}(\boldsymbol{k})|^2, \quad f \in W_2^\alpha(\mathbb{R}^d,\gamma). \tag{3.15}$$

Proof. This lemma in an implicit form has been proven in [7, pages 687–688]. Let us prove it for completeness. From the formula for the rth derivative of the Hermite polynomial H_k

$$H_k^{(r)} = \begin{cases} \sqrt{\frac{k!}{(k-r)!}}\, H_{k-r}, & \text{if } k \ge r, \\ 0, & \text{otherwise,} \end{cases}$$

we deduce that for $f \in W_2^\alpha(\mathbb{R},\gamma)$ and $r \le \alpha$,

$$f^{(r)} = \sum_{k\ge r} \sqrt{\frac{k!}{(k-r)!}}\, \hat{f}(k) H_{k-r},$$

and hence,

$$\|f\|^2_{W_2^\alpha(\mathbb{R}^d,\gamma)} = \sum_{r_1=0}^{\alpha} \sum_{k_1\ge r_1} \frac{k_1!}{(k_1-r_1)!} \cdots \sum_{r_d=0}^{\alpha} \sum_{k_d\ge r_d} \frac{k_d!}{(k_d-r_d)!} |\hat{f}(k_1,\ldots,k_d)|^2. \tag{3.16}$$

From the last equality and the relation $\frac{k!}{(k-r)!} \asymp \rho_{r,k}$, $k \in \mathbb{N}_0$, it is easy to derive (3.15) for the case $d = 1$. In the case $d \ge 2$, (3.15) can be proven by induction on d with the help of the equality (3.16). ∎

We extend the space $W_2^\alpha(\mathbb{R}^d,\gamma)$ to any $\alpha > 0$. Denote by $\mathcal{H}^\alpha$ the space of all functions $f \in L_2(\mathbb{R}^d,\gamma)$ represented by the Hermite series (3.13) for which the norm

$$\|f\|_{\mathcal{H}^\alpha} := \left(\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \rho_{\alpha,\boldsymbol{k}} |\hat{f}(\boldsymbol{k})|^2 \right)^{1/2} \tag{3.17}$$

is finite. With this definition, we identify $W_2^\alpha(\mathbb{R}^d,\gamma)$ with $\mathcal{H}^\alpha$ for $\alpha \in \mathbb{N}$.

For functions $f \in \mathcal{H}^\alpha$, we construct a hyperbolic cross approximation based on truncations of the Hermite series (3.13). For the hyperbolic cross $G(\xi) := \left\{\boldsymbol{k} \in \mathbb{N}_0^d : \rho_{1,\boldsymbol{k}} \le \xi\right\}, \ \ \xi \in \mathbb{R}_1$, the truncation $S_\xi(f)$ of the Hermite series (3.13) on this set is defined by

$$S_\xi(f) := \sum_{\boldsymbol{k} \in G(\xi)} \hat{f}(\boldsymbol{k}) H_{\boldsymbol{k}}.$$

Notice that S_ξ is a linear projection from $L_2(\mathbb{R}^d,\gamma)$ onto the linear subspace $L(\xi)$ spanned by the Hermite polynomials $H_{\boldsymbol{k}}$, $\boldsymbol{k} \in G(\xi)$, and $\dim L(\xi) = |G(\xi)|$.

Recall that according to the section on notation in the introduction $\boldsymbol{\mathcal{H}}^\alpha$ denotes the unit ball in $\mathcal{H}^\alpha$.

Theorem 3.5 *Let $\alpha > 0$. Then we can construct a sequence $\{\xi_n\}_{n=2}^\infty$ with $|G(\xi_n)| \le n$ so that*

$$\sup_{f \in \boldsymbol{\mathcal{H}}^\alpha} \|f - S_{\xi_n}(f)\|_{L_2(\mathbb{R}^d,\gamma)} \asymp \lambda_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma)) = d_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma)) \asymp n^{-\frac{\alpha}{2}} (\log n)^{\frac{(d-1)\alpha}{2}}. \tag{3.18}$$

Moreover, with the additional condition $\alpha > 1$,

$$\varrho_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma)) \asymp n^{-\frac{\alpha}{2}} (\log n)^{\frac{(d-1)\alpha}{2}}. \tag{3.19}$$

Proof. Since $L_2(\mathbb{R}^d,\gamma)$ is a Hilbert space, we have the equality $\lambda_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma)) = d_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma))$ in (3.18). To prove the upper bounds in (3.18) it is sufficient to construct a sequence $\{\xi_n\}_{n=2}^\infty$ so that $|G(\xi_n)| \le n$ and

$$\sup_{f \in \boldsymbol{\mathcal{H}}^\alpha} \|f - S_{\xi_n}(f)\|_{L_2(\mathbb{R}^d,\gamma)} \ll n^{-\frac{\alpha}{2}} (\log n)^{\frac{(d-1)\alpha}{2}}. \tag{3.20}$$

From Parseval's identity (3.14) and Lemma 3.4 we have that for every $f \in \boldsymbol{W}_2^\alpha(\mathbb{R}^d,\gamma)$ and $\xi > 1$,

$$\|f - S_\xi(f)\|_{L_2(\mathbb{R}^d,\gamma)}^2 = \sum_{\boldsymbol{k} \notin G(\xi)} \hat{f}(\boldsymbol{k})^2 \ll \xi^{-\alpha} \sum_{\boldsymbol{k} \notin G(\xi)} \rho_{\alpha,\boldsymbol{k}} \hat{f}(\boldsymbol{k})^2 \ll \xi^{-\alpha} \|f\|_{W_2^\alpha(\mathbb{R}^d,\gamma)} \le \xi^{-\alpha}. \tag{3.21}$$

Let $\{\xi_n\}_{n=2}^\infty$ be the sequence of ξ_n defined as the largest number satisfying the condition $|G(\xi_n)| \le n$. From the relation $|G(\xi_n)| \asymp \xi_n (\log \xi_n)^{d-1}$, see, e.g., [22, page 130], we derive that $\xi_n^{-\alpha} \asymp n^{-\alpha} (\log n)^{(d-1)\alpha}$ which together with (3.21) yields (3.20).

To show the lower bounds of (3.18) we need Tikhomirov's theorem [23, Theorem 1] which states that if X is a Banach space and $U_{n+1}(\lambda)$ the ball of radius $\lambda > 0$ in a linear $n+1$-dimensional subspace of X, then $d_n(U_{n+1}(\lambda), X) = \lambda$. Further, if

$$U(\xi) := \left\{f \in L(\xi) : \|f\|_{L_2(\mathbb{R}^d,\gamma)} \le 1\right\}$$

and $f \in U(\xi)$, then by Parseval's identity (3.14) and the definition of $\boldsymbol{\mathcal{H}}^\alpha$, similarly to (3.21), we deduce that $\|f\|_{\mathcal{H}^\alpha} \ll \xi^{\alpha/2}$. This means that $C\xi^{\alpha/2} U(\xi) \subset \boldsymbol{\mathcal{H}}^\alpha$ for some $C > 0$. Let $\{\xi_n'\}_{n=2}^\infty$ be the sequence of ξ_n' defined as the smallest number satisfying the condition $|G(\xi_n')| \ge n+1$. Then $\dim L(\xi_n') = |G(\xi_n')| \ge n+1$, and similarly as in the upper estimation, $(\xi_n')^{-\alpha} \asymp n^{-\alpha} (\log n)^{(d-1)\alpha}$. By Tikhomirov's theorem for the smallest quantity d_n in (3.18) we have that

$$d_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma)) \ge d_n(C\xi^{\alpha/2} U(\xi_{n+1}'), L_2(\mathbb{R}^d,\gamma)) \gg (\xi_n')^{-\alpha} \asymp n^{-\frac{\alpha}{2}} (\log n)^{\frac{(d-1)\alpha}{2}}.$$

Let us prove (3.19). The lower bound of (3.19) follows from (3.18) and the inequality $\varrho_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma)) \ge \lambda_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma))$. We verify the upper one. By (3.18),

$$d_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma)) \ll n^{-\frac{\alpha}{2}} (\log n)^{\frac{(d-1)\alpha}{2}}. \tag{3.22}$$

Notice that for $\alpha > 1$, $\mathcal{H}^\alpha$ is a separable reproducing kernel Hilbert space with the reproducing kernel

$$K(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \rho_{\alpha,\boldsymbol{k}}^{-1} H_{\boldsymbol{k}}(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{y}). \tag{3.23}$$

From the orthonormality of the system $\{H_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$ it is easily seen that $K(\boldsymbol{x},\boldsymbol{y})$ satisfies the finite trace assumption

$$\int_{\mathbb{R}^d} K(\boldsymbol{x},\boldsymbol{x})\gamma(\mathrm{d}\boldsymbol{x}) \ < \ \infty. \tag{3.24}$$

Hence by [9, Corollary 2] we obtain $\varrho_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma)) \ll d_n(\boldsymbol{\mathcal{H}}^\alpha, L_2(\mathbb{R}^d,\gamma))$. This and (3.22) prove the upper bound of (3.19). ∎

In the case when $\alpha \in \mathbb{N}$, Theorem 3.5 yields the following result on sampling n-widths of the Sobolev class $\boldsymbol{W}_2^\alpha(\mathbb{R}^d,\gamma)$ of mixed smoothness α.

Corollary 3.6 *Let $\alpha \in \mathbb{N}$. Then we can construct a sequence $\{\xi_n\}_{n=2}^\infty$ with $|G(\xi_n)| \le n$ so that*

$$\sup_{f\in\boldsymbol{W}_2^\alpha(\mathbb{R}^d,\gamma)} \|f - S_{\xi_n}(f)\|_{L_2(\mathbb{R}^d,\gamma)} \asymp \lambda_n = d_n \asymp n^{-\frac{\alpha}{2}} (\log n)^{\frac{(d-1)\alpha}{2}}. \tag{3.25}$$

Moreover, with the additional condition $\alpha \ge 2$,

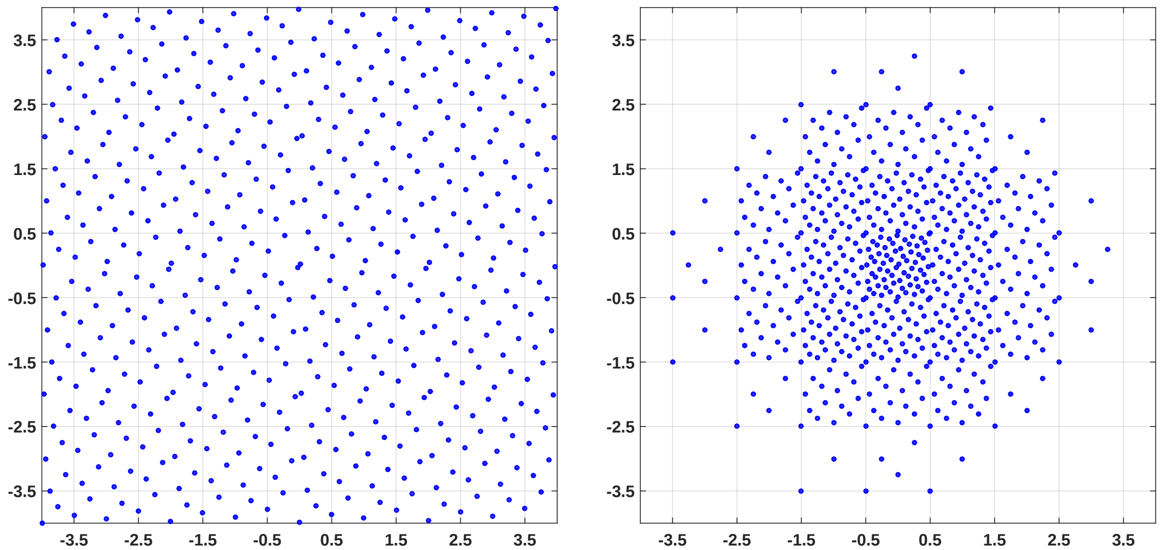
$$\varrho_n \asymp n^{-\frac{\alpha}{2}} (\log n)^{\frac{(d-1)\alpha}{2}}. \tag{3.26}$$

We stress that the assumption $\alpha > 1$ for (3.19) is vital since it is a necessary and sufficient condition for $\mathcal{H}^\alpha$ to be a separable reproducing kernel Hilbert space with the finite trace condition (3.24) and therefore, the result [9, Corollary 2] can be applied. We conjecture that the consequent asymptotic order (3.26) still holds true for $\alpha = 1$. Here it may require a different technique.

4 Numerical comparison with other quadratures

We illustrate the integration nodes of the quadratures constructed in the present paper, in comparison with the integration nodes used in [7]. Assume that $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\}$ are the integration nodes for an optimal quadrature I_n for functions in $W_p^\alpha(\mathbb{I}^2)$. Then the integration nodes in [7] are just a dilation of these nodes to the cube $[-C\sqrt{\log n}, C\sqrt{\log n}]^2$. Hence these nodes are distributed similarly on this cube. Differently, the integration nodes in our construction are formed from certain integer-shifted dilations of $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_m\}$ and contained in the ball of radius $C\sqrt{\log n}$. These nodes are dense when they are near the origin and getting sparser as they are farther from the origin. The illustration is given in Figure 1.

The following is a numerical test of our result for the cases $d = 1$ and $\alpha = 1, 2, 3$. We consider the algorithm for the space $W_2^\alpha(\mathbb{R})$. For numerical integration of functions in $\mathring{W}_2^\alpha(\mathbb{I})$ we use the Smolyak point set. Observe that these nodes give the optimal convergence rate since $d = 1$, see Section 2.2. In this test δ in (2.10) is chosen as $\delta = \frac{1}{6}$. We apply the method of change



Point set of Dick et al. [7] (512 points) Point set of our construction (560 points)

Figure 1: Distribution of integration nodes in [7] and in this paper.

of variable by ψ_3 to get asymptotically optimal integration nodes and weights for functions in $W_2^\alpha(\mathbb{I})$ where the function ψ_3 is defined as in (2.24). From these nodes and weights we get the optimal quadrature $\{x_1, \ldots, x_n\}$ and $\{\lambda_1, \ldots, \lambda_n\}$ for $W_2^\alpha(\mathbb{R}, \gamma)$ as described in Section 2.1. The error of this quadrature is given by

$$\mathrm{err} = \left(\left(1 - \sum_{i=1}^{n} \lambda_i \right)^2 + \sum_{k=1}^{\infty} \rho_{\alpha,k} \left(\sum_{i=1}^{n} \lambda_i H_k(x_i) \right)^2 \right)^{1/2},$$

see, e.g., [7, Section 4].

For the numerical computation this error is replaced by the truncated version

$$\mathrm{err}_m = \left(\left(1 - \sum_{i=1}^{n} \lambda_i \right)^2 + \sum_{k=1}^{m} \rho_{\alpha,k} \left(\sum_{i=1}^{n} \lambda_i H_k(x_i) \right)^2 \right)^{1/2}.$$

In our test we choose $m = 10^5$. Our result is given in Figure 2 which shows that the worst-case errors of the assembled quadratures for $\alpha \in \{1, 2, 3\}$ have convergence rate $\mathcal{O}(n^{-\alpha})$. It has been observed in [7] that the interlaced Sobol' sequence also gives the optimal convergence rates for numerical integration of $W_2^\alpha(\mathbb{R}, \gamma)$. The numerical result reaffirms the theory in this paper.

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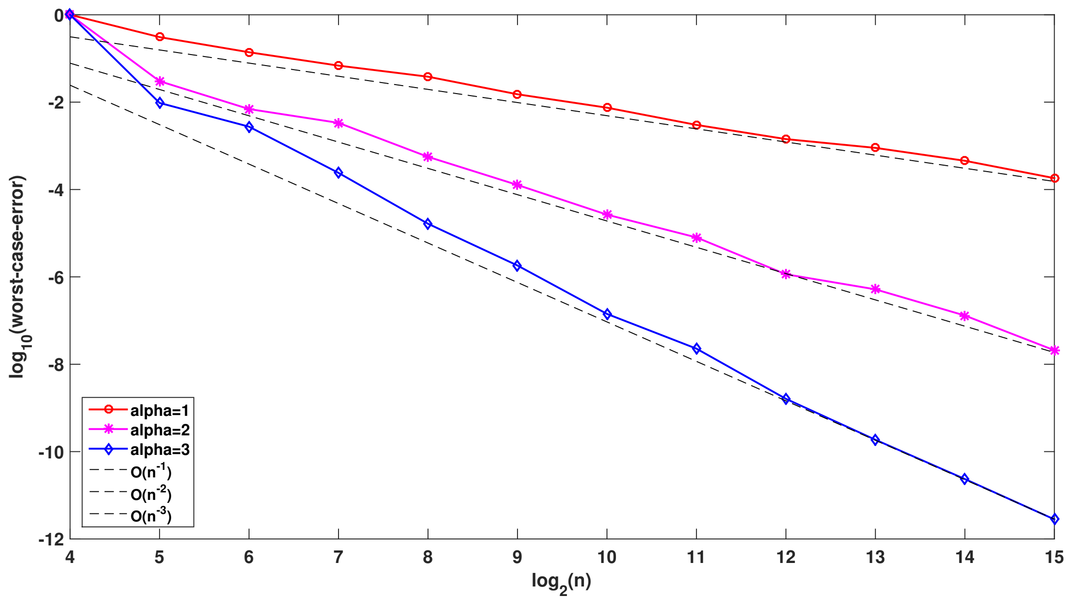

Figure 2: Errors of the assembling quadratures.

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