2-SELMER GROUP OF ODD HYPERELLIPTIC CURVES OVER FUNCTION FIELDS-II

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Abstract. In this paper, we show that the average size of 2-Selmer groups of hyperelliptic curves with a marked Weierstrass point and a marked non-Weierstrass point over function fields is 6.

Introduction

This paper and [9] are of a series of the study on 2-Selmer groups of odd hyperelliptic curves over function fields. More precisely, in this paper, we will compute the average size of 2-Selmer groups of Jacobians of hyperelliptic curves with two marked points: one Weierstrass point and one non-Weierstrass point. Recall that in [9], we consider the family of hyperelliptic curves with a marked Weierstrass point.

One of the reasons that we consider this family comes from Vinberg theory.

Before introducing the main result, we talk a little about the contribution of this work:

First, this build up a machinery method that could potentially apply for other cases such as. In this machine, we deal with higher genus case which is more chalengent due to numerious equations and relations we need to consider. Moreover, in this paper, we also handle the case with two Kostant sections (see ??), and highlight the contributions of Kostants sections to the average size. The author hope that

Secondly, from the geometric point of view, we may see that some results in function fields setting is stronger than the one in number fields case. Hence, people may look back the number fields case and strengthen the result.

Lastly, from the result of average size of 2-Selmer groups, we can deduce the information of the rank of the Modell-Weil group, which may be thought as the motivation of this work.

Over $\mathbb{Q}$, this problem was studied vastly by Bhargava and coauthors in series of papers (??). To the author best knowledge, their method can not be applied directly here. So instead, we use the approach of [4], which is more geometric. In order to go from the genus 1 case to the general case (higher genus), we need to go over some technical difficulities. Moreover, a new idea is needed in the counting sections problem (see section) in order to "cut off the points at infinity" and see the contribution of the Kostant sections to the average size.

0.1. **Notations.** Let C be a geometrically irreducible projective curve of genus g over the finite field $\mathbb{F}_q$. We also denote $K(C)$ to be the function field of C. We also assume that the characteristic of $\mathbb{F}_q = p > 2n$.

1. Weierstrass equation and height

In this section, we will consider the family of Hyperelliptic curves with a marked Weierstrass point and a marked non-Weierstrass point over a function field. By introducing the integral model of these hyperelliptic curves, we will able to define the height of hyperelliptic curves.

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The height will help us to order hyperelliptic curves and then we can state our main results where we take the average over this family of hyperelliptic curve. Furthermore, we can use the integral model to interpret each hyperelliptic curve as a $C-$point of a quotient stack.

Given a smooth hyperelliptic curve H of genus $m \geq 2$ over the function field K of the smooth curve C, assume that H has a marked rational Weierstrass point P_1 and a marked rational non-Weierstrass point P_2. Without loss of generality, we may assume that under the natural map $H \to \mathbb{P}^1$, P_1 maps to $\infty \in \mathbb{P}^1(K)$, and P_2 maps to $0 \in \mathbb{P}^1(K)$. Therefore, we have an affine Weierstrass equation of H:

$$y^2 = f(x) = x^{2m+1} + a_1x^{2m} + \cdots + a_{2m}x + e^2, \tag{1.1}$$

where $a_i \in K$ and $e \in K^\times$ such that the discriminant $\Delta(a_1, \ldots, a_{2m}, e)$ of the polynomial $f(x)$ is non-zero. Denote the multi-set $(a_1, \ldots, a_{2m}, e)$ by $\underline{a}$. Then $\underline{a}$ is unique up to the following identification:

$$(a_1, \cdots, a_{2m}, e) \equiv (\lambda^2 a_1, \cdots, \lambda^{4m} a_{2m}, \lambda^{2m+1} e) \qquad \lambda \in K^\times.$$

Now we define the minimal integral model of a given hyperelliptic curve H as follows (c.f. [1]). First of all, we choose an affine Weierstrass equation of H with $a_1, a_2, \ldots, a_{2m}, e \in K$ as above. Then for each point $v \in |C|$, we can choose an integer n_v which is the smallest integer satisfying that: the tuple

$$(\varpi_v^{2n_v} a_1, \varpi_v^{4n_v} a_2, \cdots, \varpi_v^{4mn_v} a_{2m}, \varpi_v^{(2m+1)n_v} e)$$

has coordinates in $\mathscr{O}_{K_v}$. Given $(n_v)_{v\in|C|}$, we define the invertible sheaf $\mathscr{L}_H \subset K$ whose sections over a Zariski open $U \subset C$ are given by

$$\mathscr{L}_H(U) = K \cap \big(\prod_{v\in U} \varpi_v^{-n_v} \mathscr{O}_{K_v}\big).$$

Then it is easy to see that $a_i \in H^0(C, \mathscr{L}_H^{\otimes 2i})$ and $e \in H^0(C, \mathscr{L}_H^{\otimes 2m+1})$. Furthermore, the stratum $(\mathscr{L}_H, \underline{a})$ is minimal in the sense that there is no proper subsheaf $\mathscr{M}$ of $\mathscr{L}_H$ such that $a_i \in H^0(C, \mathscr{M}^{\otimes 2i})$ and $e \in H^0(C, \mathscr{M}^{\otimes 2m+1})$. Conversely, given a minimal strata $(\mathscr{L}, \underline{a})$ satisfying that $\Delta(\underline{a}) \neq 0$, we consider a subscheme of $\mathbb{P}^2(\mathscr{L}^{2m+1} \oplus \mathscr{L}^2 \oplus \mathscr{O}_C)$ that is defined by

$$Z^{2m-1}Y^2 = X^{2m+1} + a_1ZX^{2m} + \cdots + a_{2m}Z^{2m}X + e^2Z^{2m+1}.$$

This is a flat family of curves $\mathscr{H} \to C$, and the generic fiber H is a hyperelliptic curve over $K(C)$ with a marked rational Weierstrass point $P_1 = [0 : 1 : 0]$ and a marked rational non-Weierstrass point $P_2 = [0 : e : 1]$. Furthermore, the associated minimal data of H is exactly $(\mathscr{L}, \underline{a})$. Hence, we have just shown the surjectivity of the following map $\phi_{\mathscr{L}}$ with a given line bundle $\mathscr{L}$ over C:

$$\phi_{\mathscr{L}} : \{\text{minimal tuples } (\mathscr{L}, \underline{a})\} \to \{\text{Hyperelliptic curves } (H, P_1, P_2) | \mathscr{L}_H \cong \mathscr{L}\}/\sim.$$

Moreover, the sizes of fibers of $\phi_{\mathscr{L}}$ can be calculated as follows

Proposition 1.1. *Given a line bundle $\mathscr{L}$ over C, the map $\phi_{\mathscr{L}}$ defined as above is surjective, and the preimage of any curve (H, P_1, P_2) is of size $\frac{|\mathbb{F}_q^\times|}{|Aut(H,P_1,P_2)|}$, where $Aut(H, P_1, P_2)$ denotes the subset of all elements in $Aut(H)$ which preserve the marked points P_1 and P_2.*

Proof. Suppose that (H, P_1, P_2) is a hyperelliptic curve with the associated minimal data $(\mathscr{L}, \underline{a})$. The tuple of sections $\underline{a}$ is well-defined upto the following identification:

$$\underline{a} \equiv \lambda.\underline{a} = (\lambda^2 a_1, \ldots, \lambda^{4m} a_{2m}, \lambda^{2m+1} e), \qquad \lambda \in \mathbb{F}_q^\times.$$

In the other words, there is a transitive action of $\mathbb{F}_q^\times$ on the fiber $\phi_{\mathscr{L}}^{-1}(H)$. Furthermore, the stabilizer of any element in $\phi_{\mathscr{L}}^{-1}(H)$ is exactly $\mathrm{Aut}(H, P_1, P_2)$. Hence, the size of $\phi_{\mathscr{L}}^{-1}(H; P_1, P_2)$ is $\frac{|\mathbb{F}_q^\times|}{|\mathrm{Aut}(H,P_1,P_2)|}$. □

Definition 1.2. (Height of hyperelliptic curve) The height of the hyperelliptic curve (H, P_1, P_2) is defined to be the degree of the associated line bundle $\mathscr{L}_H$.

Remark 1.3. Given $d \in \mathbb{Z}$, there are finitely many isomorphism classes of hyperelliptic curves over K whose height are less than d.

Now we are able to state the main theorem of this section. Recall that the 2-Selmer group of a given hyperelliptic curve H over $K(C)$ is the 2-Selmer group of the Jacobian E of H, and by definition it is the kernel of $\beta \circ \alpha : H^1(K, E[2]) \to \prod_{v \in |C|} H^1(K_v, E)$, where α, and β are natural maps in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(K)/2E(K) & \longrightarrow & H^1(K,E[2]) & \longrightarrow & H^1(K,E)[2] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\ 0 & \longrightarrow & \prod_v E(K_v)/2E(K_v) & \longrightarrow & \prod_v H^1(K_v,E[2]) & \xrightarrow{\beta} & \prod_v H^1(K_v,E)[2] & \longrightarrow & 0 \end{array}$$

Theorem 1.4. *Assume that* $q > 16^{\frac{m^2(2m+1)}{2m-1}}$*, then*

$$\limsup_{deg(\mathscr{L}) \to \infty} \frac{\sum\limits_{\substack{Hyperelliptic\ H \\ \mathscr{L}(H) \cong \mathscr{L}}} \frac{|Sel_2(H)|}{|Aut(H)|}}{\sum\limits_{\substack{Hyperelliptic\ H \\ \mathscr{L}(H) \cong \mathscr{L}}} \frac{1}{|Aut(H)|}}$$

$$\leq 4 \prod_{v \in |C|} \left(1 + c_{2m-1}|k(v)|^{-2} + \cdots + c_1|k(v)|^{-2m} - 2|k(v)|^{(2m+1)^2}\right) + 2 + f(q),$$

where $\lim_{q \to \infty} f(q) = 0$*, and* c_i *are constants which only depend on* m *and* $p = char(\mathbb{F}_q)$*.*

By Proposition 1.1, the above theorem is equivalent to

Theorem 1.5. *With the same hypothesis as in the previous theorem, we have that*

$$\limsup_{deg(\mathscr{L}) \to \infty} \frac{\sum\limits_{\substack{minimal\ (\mathscr{L}, \underline{a}) \\ \Delta(\underline{a}) \neq 0}} |Sel_2(H_{\underline{a}})|}{\sum\limits_{\substack{minimal\ (\mathscr{L}, \underline{a}) \\ \Delta(\underline{a}) \neq 0}} 1} \leq$$

$$4. \prod_{v \in |C|} \left(1 + c_{2m-1}|k(v)|^{-2} + \cdots + c_1|k(v)|^{-2m} - 2|k(v)|^{(2m+1)^2}\right) + 2 + f(q),$$

where $H_{\underline{a}}$ *is the hyperelliptic curve that is corresponding to the tuple of sections* $\underline{a}$, $\lim_{q \to \infty} f(q) = 0$*, and* $c_1, \ldots, c_{2m-1}$ *are constants which only depend on* m *and* p*.*

If we order the set of hyperelliptic curves over K by height, the following corollary of the above theorem give an upper bound for the average size of 2-Selmer groups:

Corollary 1.6. *Assume that* $q > 16^{\frac{m^2(2m+1)}{2m-1}}$*, then*

$$\limsup_{d \to \infty} \frac{\sum\limits_{\substack{(\mathscr{L}, \underline{a})\ is\ minimal \\ \Delta(\underline{a}) \neq 0; deg(\mathscr{L}) \leq d}} |Sel_2(H_{\underline{a}})|}{\sum\limits_{\substack{(\mathscr{L}, \underline{a})\ is\ minimal \\ \Delta(\underline{a}) \neq 0; deg(\mathscr{L}) \leq d}} 1}$$

$$\leq 4. \prod_{v \in |C|} \left(1 + c_{2m-1}|k(v)|^{-2} + \cdots + c_1|k(v)|^{-2m} - 2|k(v)|^{(2m+1)^2}\right) + 2 + f(q),$$

where $\lim_{q \to \infty} f(q) = 0$*, and* $c_1, \ldots, c_{2m-1}$ *are constants which are only depended on* m *and* p*.*

The above error term $f(q)$ can be removed and the limsup becomes the normal limit if we take the average over the set of transversal hyperelliptic curves. The transversality can be defined as follows:

Definition 1.7. Let H be a hyperelliptic curve over K with an associated minimal data $(\mathscr{L}, \underline{a})$. Then H is said to be transversal if the discriminant $\Delta(\underline{a}) \in H^0(C, \mathscr{L}^{4m(2m+1)})$ is square-free.

If we compute the average on the family of transversal hyperelliptic curves then we could remove the restriction $q > 16^{\frac{m^2(2m+1)}{2m-1}}$ and actually obtain the limit:

Theorem 1.8.

$$\lim_{d\to\infty} \frac{\sum\limits_{\substack{(\mathscr{L},\underline{a}) \text{ is transversal} \\ deg(\mathscr{L} \leq d}} |Sel_2(H_{\underline{a}})|}{\sum\limits_{\substack{(\mathscr{L},\underline{a}) \text{ is transversal} \\ deg(\mathscr{L} \leq d}} 1} = 6.$$

Remark 1.9. If we set $S = \mathrm{Spec}(k[a_1, \dots, a_{2m}, e]) \cong \mathbb{A}^{2m+1}$, then any tuple $(\mathscr{L}, \underline{a})$ can be seen as a $C-$point of the quotient stack $[S/\mathbb{G}_m]$, where the action of $\mathbb{G}_m$ on S is given by $\lambda.(a_1, \dots, a_{2m}, e) = (\lambda^2 a_1, \dots, \lambda^{4m} a_{2m}, \lambda^{2m+1} e)$.

Over $S = \mathbb{A}^{2m+1}$, the universal curve H_S is defined to be the subscheme of $\mathbb{P}^3(S)$:

$$Z^{2m-1}Y^2 = X^{2m+1} + a_1 Z X^{2m} + \cdots + a_{2m} Z^{2m} X + e^2 Z^{2m+1},$$

where $\underline{a} = (a_i, e) \in S$. This is a flat family of integral projective curves over S, hence, there exists the relative Jacobian $J_S = \mathrm{Pic}^0_{H_S/S}$ which is a group scheme locally of finite type over S. The next section will provide a close relation between $BJ_S[2]$ and 2-Selmer groups. Consequently, we will be able to restate our main theorems in the stack language.

2. 2-torsion group and 2-Selmer group

This section is almost identical to Section **??** in section **??**. We will state the main results and then give sketchy proofs if required.

Given a hyperelliptic curve (H, P_1, P_2) over the function field $K(C)$, let denote $\mathscr{H} \to C$ be the minimal integral model of H. We also have the relative generalized Jacobian $\mathcal{J}$ of $\mathscr{H}$ whose generic fiber is the Jacobian J of H. Recall that the set of isomorphism classes of $\mathcal{J}[2]-$torsors over C can be identified with the étale cohomology group $H^1(C, \mathcal{J}[2])$. By restriction to the generic fiber of C, we obtain a homomorphism

$$H^1(C, \mathcal{J}[2]) \to H^1(K, J[2]). \tag{2.1}$$

We obtain the following results:

Proposition 2.1. *The homomorphism (2.1) factors through the 2-Selmer group $Sel_2(J)$.*

And now in the transversal case, $\mathrm{Sel}_2(J)$ can be identified with $H^1(C, \mathcal{J}[2])$ via the above map.

Proposition 2.2. *If the hyperelliptic curve H is transversal, then*

$$|Sel_2(J)| = |H^1(C, \mathcal{J}[2])|$$

Proof. C.f. Proposition **??**. □

In general case, the size of $\mathrm{Sel}_2(J)$ and $H^1(C, \mathcal{J}[2])$ can be compared as follows:

Proposition 2.3. *We have that*

$$\begin{aligned} |Sel_2(J)| &\leq |H^1(C, \mathcal{J}[2])|, \quad \text{when } J[2](K) = 0, \\ |Sel_2(J)| &\leq 2^{2m-1}|H^1(C, \mathcal{J}[2])|, \qquad \text{otherwise.} \end{aligned}$$

To summarize, in general, $|\mathrm{Sel}_2(J)|$ is bounded by $|H^1(C, \mathcal{J}[2]|$ except in the case our Jacobian J has a 2-torsion $K-$rational point. However, if we make an assumption that the size of our base field q is large enough, then the contribution to the average of $\mathrm{Sel}_2(J)$ in this case is zero. More precisely, we have

Lemma 2.4. *If $q > 16^{\frac{m^2(2m+1)}{2m-1}}$, then the contribution of the case $J[2](K) \neq 0$ to the average is zero. In the other words, we have the following limit:*

$$\limsup_{deg(\mathscr{L})\to\infty} \frac{\displaystyle\sum_{\substack{\underline{a}\in H^0(C,\mathscr{L}^2\oplus\cdots\oplus\mathscr{L}^{4m}\oplus\mathscr{L}^{2m+1}) \\ J_{\underline{a}}[2](K)\neq\{0\}}} |H^1(C, \mathcal{J}_{\underline{a}}[2])|}{\displaystyle\sum_{\underline{a}\in H^0(C,\mathscr{L}^2\oplus\cdots\oplus\mathscr{L}^{4m}\oplus\mathscr{L}^{2m+1})} 1} = 0$$

Proof. Let H be the hyperelliptic curve over C defined by $(\mathscr{L}, \underline{a})$, then the smooth locus C' of the map $H \to C$ is determined by the condition $\Delta(\underline{a}) \neq 0$, where $\Delta \in H^0(C, \mathscr{L}^{4n(2n+1)})$. Denote by $\mathcal{J}$ the corresponding Jacobian of H. Then by the smoothness of H over C', any K_v-points of $\mathcal{J}$ can be extended as C'_v-points. By using the Selmer condition, we deduce that any element in the 2-Selmer group of J can be lifted to $\mathcal{J}[2]-$torsors over C'. Consequently, we get

$$|\mathrm{Sel}_2(J)| \leq |H^1(C', \mathcal{J}[2])|.$$

When $\mathcal{J}[2](C) \neq 0$, there exists a section $c \in H^0(C, \mathscr{L}^{\otimes 2})$ such that the $(x, z)-$ polynomial defining H has a factorization:

$$\begin{aligned} &x^{2m+1} + a_1x^{2m}z + \cdots + a_{2m}xz^{2m} + e^2z^{2m+1} \\ &= (x - cz)(x^{2m} + b_1x^{2m-1}z + b_2x^{2m-2}z^2 + \cdots + b_{2m}z^{2m}). \end{aligned}$$

It means that $\underline{a}$ can be expressed in terms of c and $\{b_j\}_{1\leq j\leq 2m}$, where $b_j \in H^0(C, \mathscr{L}^{2j})$, for all j, and $-c.b_{2m}$ is the square of a section in $H^0(C, \mathscr{L}^{2m+1})$. If $d = \deg(\mathscr{L})$ is large enough, then by using the Riemann-Roch theorem, the number of multiple sets $\underline{a}$ in this case is bounded above by

$$q^{2d+(2+4+\cdots+4m-2+2m+1)d+(2m+1)(1-g)} = q^{(4m^2+3)d+(2m+1)(1-g)}.$$

Now we consider the following interpretation for $\mathcal{J}[2]-$torsors: any $\mathcal{J}[2]-$torsors over C' can be considered as tame étale covers of C' of degree 2^{2m}. Hence, there is a natural map:

$$\phi : H^1(C', \mathcal{J}[2]) \to \{\text{tame étale covers of } C' \text{ of degree } 4^m\}.$$

The number $|C - C'|$ of points where H_α is singular is bounded by the degree of $\Delta(H_\alpha)$, so $|C - C'| \leq 4m(2m+1)d$. As a consequence, the number of topological generators of $\pi_1^{tame}(C')$ is less than $2g + 4m(2m+1)d$. The size of $H^1(C', \mathcal{J}[2])$ can be estimated if we understand the fiber of ϕ. Let M is a degree 4^m étale cover of C', then giving M the structure of $\mathcal{J}[2]-$torsor is equivalent to giving an action map:

$$\psi : \mathcal{J}[2] \times_{C'} M \longrightarrow M$$

which is compatible with the structure maps to C' and which satisfies the condition that the following natural map

$$\begin{aligned} \mathcal{J}[2] \times_{C'} M &\longrightarrow M \times_{C'} M \\ (g, m) &\mapsto (\psi(g, m), m) \end{aligned}$$

is an isomorphism.
Since $\mathcal{J}[2]$ and M are proper and flat over C', the map ψ is totally defined by the map

$$\psi_{K(C')} : (\mathcal{J}[2] \times_{C'} M)_{K(C')} \to M_{K(C')}$$

on generic fibers. As $K(C')-$vector spaces, $\dim(M_{K(C')}) = 2^{2m}$ and

$$\dim(\mathcal{J}[2] \times_{C'} M)_{K(C')} = 2^{4m}.$$

Hence, the number of maps giving M the structure of a $\mathcal{J}[2]-$torsors is bounded by $q^{2^{6m}}$. Thus, the size of any fiber of ϕ is also bounded by $q^{2^{6m}}$.

The above discussion deduces that the average in case $\mathcal{J}[2](C) \neq 0$ is bounded by:

$$\frac{q^{2^{6m}}.4^{m(2g+4m(2m+1)d)}.q^{(4m^2+3)d+(2m+1)(1-g)}}{q^{(2m+1)^2d+(2m+1)(1-g)}} = \frac{a.4^{m(4m(2m+1)d)}}{q^{(4m-2)d}},$$

where a is a constant independent of d. This goes to zero as d goes to infinity if $q^{4m-2} > 4^{4m^2(2m+1)}$, or equivalently $q > 16^{\frac{m^2(2m+1)}{2m-1}}$. The lemma is proved. □

From now on, we will assume that $q > 16^{\frac{m^2(2m+1)}{2m-1}}$ if we work in the general case, and there are no assumptions for the transversal case. Hence, we may assume that $|\mathrm{Sel}_2(J_{\underline{a}})| \leq |H^1(C, \mathcal{J}[2]|$ for any tuples $\underline{a}$, and $|\mathrm{Sel}_2(J_{\underline{a}})| = |H^1(C, \mathcal{J}[2]|$ if $\underline{a}$ is transversal. We now restate our main theorem in stack language as follows.

1. Recall that $S = \mathrm{Spec}(k[a_1, \ldots, a_{2m}, e]) \cong \mathbb{A}^{2m+1}$. Then any tuple $(\mathscr{L}, \underline{a})$ can be seen as a $C-$point of the quotient stack $[S/\mathbb{G}_m]$, where the action of $\mathbb{G}_m$ on S is given by $\lambda.(a_1, \ldots, a_{2m}, e) = (\lambda^2 a_1, \ldots, \lambda^{4m} a_{2m}, \lambda^{2m+1} e)$. We set $\mathscr{A} = \mathrm{Hom}(C, [S/\mathscr{G}_m])$. Then $\mathscr{A}(k)$ classifies the tuples $(\mathscr{L}, \underline{a})$.
2. Since the universal Jacobian J_S is a group scheme over S, similar to the argument in Section **??**, there is a natural induced action of $\mathbb{G}_m$ on J_S. Observe that we have a natural map of quotient stacks

$$[BJ_S[2]/\mathbb{G}_m] \xrightarrow{\psi} [S/\mathbb{G}_m].$$

Given a morphism $\alpha : C \to [S/\mathbb{G}_m]$, as in the step 1, we obtain a family of curve $H_\alpha \to C$. Denote $J_\alpha = \alpha^* J_S$, then J_α is exactly the relative Jacobian of H_α over C. A $J_\alpha[2]-$torsor over C can be seen as a morphism $\beta : C \to [BJ_S[2]/\mathbb{G}_m]$ that fits in the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\beta} & [BJ_S[2]/\mathbb{G}_m] \\ & \searrow^{\alpha} & \downarrow \psi \\ & & [S/\mathbb{G}_m] \end{array}$$

Hence, if we set

$$\mathscr{M} = \mathrm{Hom}(C, [BJ_S[2]/\mathbb{G}_m]),$$

then we have a natural map induced by ψ

$$b : \mathscr{M} \to \mathscr{A}$$

whose fiber $\mathscr{M}_\alpha$ over $\alpha \in \mathscr{A}(k)$ classifies isomorphism classes of $J_\alpha[2]-$torsors over C.

3. Notice that the natural map $\mathscr{M} \to \mathscr{A}$ is compatible with base maps to $\mathrm{Hom}(C, B\mathbb{G}_m)$, i.e., we have a commutative diagram:

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\quad b \quad} & \mathscr{A} \\ & \searrow^{\pi_{\mathscr{M}}} \quad {}^{\pi_{\mathscr{A}}}\swarrow & \\ & \mathrm{Hom}(C, B\mathbb{G}_m) & \end{array}$$

This implies that for any line bundle $\mathscr{L}$ over C,

$$|\mathscr{M}_{\mathscr{L}}(k)| = \sum_{\alpha \in \mathscr{A}_{\mathscr{L}}(k)} |H^1(C, J_\alpha[2])|.$$

For each line bundle $\mathscr{L}$, we denote $\mathscr{A}_{\mathscr{L}}^{min}(k)$ to be the set of $\underline{a} = (a_1, \dots, a_{2m}, e)$in $H^0(S \times^{\mathbb{G}_m} \mathscr{L})$ satisfying that $(\mathscr{L}, \underline{a})$ is minimal, $e \neq 0$, and $\Delta(\underline{a}) \neq 0$. If we set $\mathscr{M}_{\mathscr{L}}^{min}(k)$ to be the preimage $b^{-1}(\mathscr{A}_{\mathscr{L}}^{min}(k))$, then our main theorem 1.4 can be rewritten as

Theorem 2.5. *Suppose that $q > 16^{\frac{m^2(2m+1)}{2m-1}}$. Then we have that*

$$\limsup_{deg(\mathscr{L}) \to \infty} \frac{|\mathscr{M}_{\mathscr{L}}^{min}(k)|}{|\mathscr{A}_{\mathscr{L}}^{min}(k)|} \leq 4. \prod_{v \in |C|} \left(1 + c_{2m-1}|k(v)|^{-2} + \cdots + c_1 |k(v)|^{-2m}\right) + 2 + f(q),$$

where $lim_{q\to\infty} f(q) = 0$, and c_i are constants which depend only on m and p. If $p > 2m+1$, then c_i depends only on m.

Similarly, let $\mathscr{A}^{\mathrm{trans}}(k)$ be the subset of transversal elements $(\mathscr{L}, \underline{a})$ in $\mathscr{A}(k)$ such that $e \neq 0$, and $\mathscr{M}^{\mathrm{trans}}(k)$ be the preimage of $\mathscr{A}^{\mathrm{trans}}(k)$ under the natural map $\mathscr{M} \to \mathscr{A}$. Then in transversal case, we have the following limit:

Theorem 2.6.

$$\lim_{deg(\mathscr{L}) \to \infty} \frac{|\mathscr{M}_{\mathscr{L}}^{trans}(k)|}{|\mathscr{A}_{\mathscr{L}}^{trans}(k)|} = 6.$$

One of the main ingredients in the proof of the above theorems is the close relationship between the 2-torsion subgroups of the Jacobians of our hyperelliptic curves and the stabilizer group schemes of a representation of $\mathrm{SO}(2m+1) \times \mathrm{SO}(2m+1)$ that appears in the Vinberg theory of $\theta-$groups. In the next sections, we are going to introduce the relevant representation and then explain the above mentioned connection to 2-torsion subgroups of Jacobians.

3. Vinberg representation of $\mathrm{SO}(V_1) \times \mathrm{SO}(V_2)$

Let $(V_1, <\,|\,>_1)$ and $(V_2, <\,|\,>_2)$ be split $(2m+1)-$dimensional orthogonal spaces over k of discriminant 1 and -1 respectively. Then we can find a basis $\{f_1, f_2, \dots, f_{2m+1}\}$ of V_1 such that the Gram matrix of $<\,|\,>_1$ is

$$B = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

Similarly, there exists a basis $\{f'_1, f'_2, \dots, f'_{2m+1}\}$ of V_2 such that the Gram matrix of $<\,|\,>_2$ is $-B$. Now we can define the special orthogonal groups G_i that corresponds to V_1 and V_2 :

$$G_i := \mathrm{SO}(V_i) = \left\{ T \in \mathrm{GL}(V_i) \;\middle|\; T^*.T = I; \det(T) = 1 \right\},$$

where $T^* \in \mathrm{GL}(V)$ denotes the adjoint transformation of T which is uniquely determined by the formula

$$\langle Tv, w\rangle_i = \langle v, T^* w\rangle_i.$$

Notice that the matrix of T^* with respect to our standard basis (for both V_1 and V_2) can be obtained by taking the reflection about anti-diagonal of the matrix of T. Set $G = G_1 \times G_2$, $V = V_1 \oplus V_2$ and consider the following representation of G

$$\begin{aligned} W &= \Big\{\text{self-adjoint operators } T : V \to V \text{ with block diagonal zero}\Big\} \\ &= \Big\{T = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}; \quad A : V_2 \to V_1; \quad -A^* : V_1 \to V_2\Big\} \\ &\equiv V_1 \otimes V_2, \end{aligned}$$

where G acts on W by conjugation. For each element $T \in W$, the corresponding characteristic polynomial is of the form

$$g_T(x) = f_T(x^2) = x^{2n} + a_1 x^{2n-2} + \cdots + a_{n-1}x^2 + e^2,$$

where $n = 2m+1$, $e = \det(A)$, and $f_T(x)$ is the characteristic polynomial of $-A.A^*$. The functions $a_1, a_2, \ldots, a_{n-1}, e$ are homogenous $G-$invariant functions on W of degree $2, 4, \ldots, 2n-2$ and n respectively. We have a $G-$equivariant map

$$\pi : W \longrightarrow \mathrm{S} := \mathrm{Spec}\big(k[a_1, a_2, \ldots, a_{n-1}, e]\big),$$

where the action of G on S is trivial.

3.0.1. *Regular locus and two Kostant sections.*

Definition 3.1. (Kostant section) A Kostant section of (W, G) is a linear subvariety κ of W for which the restriction of function $k[W]^G \to k[\kappa]$ is an isomorphism.

From Vinberg theory in characteristic 0, there are exactly two Kostant sections (up to conjugation) in our case. For positive characteristic, Paul Levi [5] made it available with the assumption that $\mathrm{char}(\mathbb{F}_q)$ is good ($p > 3$ in our case). Notice that the number of Kostant sections (up to conjugation), by construction, equals to the number of $G(\overline{k})-$orbits of the nilpotent regular locus. In our case, we give the precise description of Kostant sections as follows: for each point $c = (a_1, a_2, \ldots, a_{n-1}, e) \in S$, we define an associated element T_c in W:

$$T_c = \begin{pmatrix} 0 & A_c \\ -A_c^* & 0 \end{pmatrix} \tag{3.1}$$

where

$$A_c = \begin{pmatrix} b_{2m} & \cdots & b_{m+1} & e & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & 0 & 0 & 0 & 1 & \vdots & 0 \\ b_m & \cdots & b_1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & 0 & 0 & 0 & \vdots & 0 \\ 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3.2}$$

where $b_i = \dfrac{(-1)^{i-1}a_i}{2}$. It could be checked that $\pi(T_c) = c$, so that we have defined a section of the invariant map π:

$$\begin{aligned} \kappa_1 : \mathrm{S} &\longrightarrow W \\ c &\longrightarrow T_c \end{aligned}$$

Similarly, we define

$$T'_c = \begin{pmatrix} 0 & A'_c \\ -A'^*_c & 0 \end{pmatrix} \tag{3.3}$$

where

$$A'_c = \begin{pmatrix} b_{2m} & 0 & \cdots & b_m & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m+2} & 0 & \cdots & b_2 & 0 & 1 & \cdots & 0 \\ b_{m+1} & 0 & \cdots & b_1 & 1 & 0 & \cdots & 0 \\ e & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.4}$$

Then we will obtain the second section of π:

$$\kappa_2 : \mathrm{S} \longrightarrow W$$
$$c \longrightarrow T'_c$$

Now recall the definition of regularity:

Definition 3.2. An element T in $W(\overline{k})$ is called to be regular if its stabilizer $\mathrm{Stab}_{G_{\overline{k}}}(T)$ is finite. The condition of being regular is open, and we write W^{reg} for the open subscheme of regular elements of W.

Proposition 3.3. *Over an algebraic closed field $\overline{k}$, any element in $W^{reg}(\overline{K})$ is conjugate by $G(\overline{k})$ to an element in one of two Kostant sections κ_i.*

Proof. This is a consequence of Theorem 0.14 in [5]. ☐

The following proposition gives us a necessary condition of being regular, it will be very helpful in the counting section.

Proposition 3.4. *Let $T = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix} \in W$ be a regular element, then at least one of two matrices $A.A^*$ and $A^*.A$ is regular (here, regularity of a matrix in GL_n means that its minimal polynomial and its characteristic polynomial coincide).*

Proof. This result can be shown by a direct calculation. In fact, without loss of generality, we may assume that $T \in \kappa_1$. Then it is easy to see that the product $A.A^*$ belongs to the Kostant section κ of the representation we considered in the last section (see Section **??**). Hence, $A.A^*$ is regular. ☐

Let F/k be a field extension. Given a regular operator $T \in W(F)$, we consider two quotient rings $L = F[x]/(f(x))$, and $M = F[x]/(g(x)) \cong F[T]$, where $g(x) = f(x^2)$ is the characteristic polynomial of T. We have an embedding of $F-$algebras: $L \hookrightarrow M$ by $x \mapsto x^2$. We can describe the stabilizer of T under the action of G as follow:

Proposition 3.5. *The stabilizer $Stab_G(T)(F)$ of a regular operator $T \in W$ whose characteristic polynomial is $g(x) = f(x^2)$ is isomorphic to the kernel of the norm map $Res_{L/F}(\mu_2) \to \mu_2$, where $L = F[x]/(f(x))$. In particular, the finite group scheme $Stab_G(T)$ has order 2^r over F, where $r+1$ is the number of distinct roots of $f(x)$ in the separable closure F^s.*

Proof. Any elements in the stabilizer of $T = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}$ is of the form $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where $B, C \in \mathrm{SO}(n)$ satisfying that $BAC^* = A$. By squaring T, we deduce that the submatrices B and C commute with the matrix AA^* and A^*A respectively. Without loss of generality, we may assume that our matrix T lies in the first Kostant section. Thus the matrix $A.A^*$ is regular in $\mathrm{GL}(V_1)$, and if we denote its characteristic is $f(x)$, we can identify L with $F[A.A^*]$. As in section 1, we know that the stabilizer of AA^* in $\mathrm{SO}(V_1)$ can be identified with

$$\{h \in L = F[x]/(f(x)) | h^2 = 1,\, Nm_{L/F}(h) = 1\} \cong \mathrm{Ker}\{Res_{L/F}(\mu_2) \xrightarrow{Nm} \mu_2\}.$$

Hence, given $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in \mathrm{Stab}_G(T)$, there exist uniquely an element $h(x) \in L$ such that $B = h(AA^*)$. For any polynomial $P(x) \in F[x]$ and two square matrices D and E, we can prove that $\det(P(D.E)) = \det(P(E.D))$. In fact, if we express $P(D.E)$ as $D.H + aI$, for some square matrix H, then $P(E.D) = H.D + aI$. Combining with the well-known equality $\det(I + B.C) = \det(I + C.B)$, we have completed the proof of $\det(P(D.E)) = \det(P(E.D))$. Applying this observation to the case $D = A^*$ and $E = A$, we observe that $\det(h(A^*A)) = 1$. On the other hand, since $f(x)$ is also the characteristic polynomial of A^*A, we deduce that $h(A^*A)^2 = I_n$ (the identity matrix). We have just seen that $h(A^*A) \in \mathrm{SO}(V_2)$. Since

$$B.A.h(A^*A)^* = h(AA^*).A.h(A^*A) = h(AA^*).h(AA^*).A = A,$$

the matrix $\begin{pmatrix} h(AA^*) & 0 \\ 0 & h(A^*A) \end{pmatrix}$ stabilizes T. We will now prove that $C = h(A^*A)$. First of all, by setting $C = h(A^*A) + C_1$, the matrix C_1 needs to satisfy that $C_1.A^* = 0$. If A^* is invertible then $C_1 = 0$. Otherwise, by elementary computation, we can see that the entries of C_1: $C_{ij} = 0$ for all $(i, j) : i \neq m$. Since the determinant of A is 0, we have that:

$$A = \begin{pmatrix} b_{2m} & \cdots & b_{m+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & 0 & 0 & 0 & 1 & \vdots & 0 \\ b_m & \cdots & b_1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & 0 & 0 & 0 & \vdots & 0 \\ 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

thus all entries in the middle row and column of the product A^*A are zeros. This implies that the central entry of C_1 is uniquely determined by the condition $\det(C_1 + h(A^*A)) = 1$, and hence it must be equal to zero. By the above discussion, the central entry of $(C_1 + C_1^*)h(A^*A)$ is 0, and every entries except the central one of $C_1C_1^*$ are zeros. On the other hand, by decomposing the product $C.C^*$, the condition $C \in \mathrm{SO}(V_2)$ is equivalent to $(C_1 + C_1^*)h(A^*A) + C_1.C_1^* = 0$. This equality implies that $C_1.C_1^* = 0$, thus $C_1 + C_1^* = 0$, and hence $C_1 = 0$. We have just proven the claim that $C = h(A^*A)$.

For the case $T \in \kappa_2$, similarly, by using the fact that $A^*.A$ is regular in $\mathrm{GL}(V_2)$, we also can identify $Stab_G(T)$ with a subset of L as above. And if T belongs to both sections, we can just need to choose one of them to start with.

The proof of the Proposition is completed. □

We can also compute the infinitesimal stabilizer as follows: the induced action of $\mathfrak{g} = Lie(G)$ on W is: for any element $X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{g} = so(V_1) \times so(V_2)$ and $T \in W(F)$, then $X * T = [X, T] = XT - TX$. If T is regular, let assume that $T \in \kappa_1$, then $X_1 = 0$ since $X_1 = X_1^*$ (this is a consequence of the fact that any matrix that commutes with $A.A^*$ is of the form $h(AA^*)$ for some polynomial $h(x) \in F[x]$). We deduce that $X_2.A^*$ must be the zero matrix. Using the trivial computation and the fact that $X_2 = -X_2^*$, we can show that $X_2 = 0$.

Thus, $\mathrm{Stab}_{\mathfrak{g}}(T)$ is trivial. This implies that the action map

$$\begin{array}{ccc} G \times_S W^{\mathrm{reg}} & \longrightarrow & W^{\mathrm{reg}} \times_S W^{\mathrm{reg}} \\ (g, v) & \mapsto & (g.v, v) \end{array}$$

is étale, and the universal stabilizer I of the action of G on W^{reg}

$$I = (G \times_S W^{\mathrm{reg}}) \times_{W^{\mathrm{reg}} \times_S W^{\mathrm{reg}}} W^{\mathrm{reg}},$$

where $W^{\mathrm{reg}} \to W^{\mathrm{reg}} \times W^{\mathrm{reg}}$ is the diagonal map, is a quasi-finite étale group scheme over W^{reg} (base change of an étale map is étale).

Proposition 3.6. *There exists a unique group scheme I_S over S equipped with a $G-$equivariant isomorphism $\pi^* I_S \to I$ over W^{reg}. Moreover, there is a $\mathbb{G}_m-$ equivariant isomorphism of stacks $[BI_S] \cong [W^{reg}/G]$, where BI_S is the relative classifying stack of I_S over S.*

Proof. By Proposition 3.5, we obtain an isomorphism of the following group scheme over S:

$$\phi : \kappa_2^* I \xrightarrow{\cong} \kappa_1^* I. \tag{3.5}$$

We will show that the group scheme $I_S := \kappa_1^* I$ satisfies our condition.

Firstly, by considering the action map

$$f_i : W_i := G \times \kappa_i(S) \to W^{\mathrm{reg}}, \ i = 1, 2,$$

we obtain an étale covering of W^{reg}. We will see that there is a canonical isomorphism from $\pi^* I_S$ to I over this covering. In fact, over W_1, for any $(g_1, \kappa_1(s_1)) \in W_1$, we have a natural morphism:

$$\begin{array}{ccc} (f_1^* I)_{(g_1, \kappa_1(s_1))} = Stab_G(g_1.\kappa_1(s_1).g_1^*) & \longrightarrow & (\pi^* \kappa_1^* I)_{(g_1, \kappa_1(s_1))} = Stab_G(\kappa_1(s_1)) \\ a & \longmapsto & g_1^*.a.g_1 \end{array}$$

Over W_2, the isomorphism from $\pi^* I_S$ to I is given by

$$\begin{array}{ccc} (f_1^* I)_{(g_2, \kappa_2(s_2))} = Stab_G(g_2.\kappa_2(s_2).g_2^*) & \longrightarrow & (\pi^* \kappa_1^* I)_{(g_2, \kappa_2(s_2))} = Stab_G(\kappa_1(s_2)) \\ a & \longmapsto & \phi(g_2^*.a.g_2), \end{array}$$

where ϕ is the isomorphism (3.5), for any $(g_2, \kappa_2(s_2)) \in W_2$.

The above isomorphism descends to an isomorphism over W^{reg} since it is a morphism of covering descent data, i.e., for each $(g_1, \kappa(s)) = (g_2, \kappa_2(s)) \in W_1 \times_{W^{\mathrm{reg}}} W_2$, the following diagram:

$$\begin{array}{ccc} Stab_G(g_1 \kappa_1(s) g_1^*) & \xrightarrow{ad(g_1^*)} & Stab_G(\kappa_1(s)) \\ \big\downarrow Id & & \big\downarrow Id \\ Stab_G(g_2 \kappa_2(s) g_2^*) & \xrightarrow{\phi \circ ad(g_2^*)} & Stab_G(\kappa_1(s)) \end{array} \tag{3.6}$$

is commutative. In fact, we can describe the morphism ϕ in this case as follows: let $g \in G$ be any element such that $g\kappa_2(s)g^* = \kappa_1(s)$. Then ϕ is given by

$$\begin{array}{ccc} Stab_G(\kappa_2(s)) & \longrightarrow & Stab_G(\kappa_1(s)) \\ a & \longmapsto & g.a.g^*. \end{array}$$

This description does not depend on the choice of g due to the fact that I is commutative. Hence, we may choose g to be $g_1^* g_2$. It is now easy to check that our diagram (3.6) is commutative. Moreover, we can see from the above construction that the morphism $\pi^* I_S \to I$ is $G-$equivariant.

The second statement can be shown by using the above construction and a similar argument as in the proof of Proposition **??**. $\square$

4. Stabilizer group scheme and Jacobian of hyperelliptic curve

For each element $T \in W^{\mathrm{reg}}$, let $f_T(x^2)$ be the characteristic polynomial of T, we consider the projective curve in $\mathbb{P}^3$ with the affine equation: $y^2 = f_T(x)$. As a result, we obtain a flat family $H_{W^{\mathrm{reg}}}$ of integral projective curves over W^{reg}. By the representability of the relative Picard functor, we obtain the scheme $\mathrm{Pic}_{H_{W^{\mathrm{reg}}}/W_{\mathrm{reg}}}$ locally of finite type over W^{reg}, and also the relative Jacobian $J_{W^{\mathrm{reg}}} = \mathrm{Pic}^0_{H_{W^{\mathrm{reg}}}/W_{\mathrm{reg}}}$ over W^{reg}. Over S, recall that the universal curve H_S is defined to be the subscheme of $\mathbb{P}^3(S)$:

$$Z^{2m-1}Y^2 = X^{2m+1} + a_1 Z X^{2m} + \cdots + a_{2m} Z^{2m} X + e^2 Z^{2m+1}.$$

This is a flat family of integral projective curves over S, hence we also can define the relative Jacobian $J_S = \mathrm{Pic}^0_{H_S/S}$. By definition, we obtain a canonical isomorphism

$$J_{W^{\mathrm{reg}}} \to J_S \times_S W^{\mathrm{reg}}.$$

Now we will see the connection between the 2-torsion subgroup $J_{W^{\mathrm{reg}}}[2]$ and the stabilizer $I_{W^{\mathrm{reg}}}$.

Proposition 4.1. *There is a canonical isomorphism of étale group schemes over W^{reg}:*

$$I_{W^{reg}} \cong J_{W^{reg}}[2]$$

Proof. Let $W^{\mathrm{reg}} = W_1 \cup W_2$, where W_i is the orbit of the Kostant section κ_i. Thus, it is enough to show that there is a canonical isomorphism $I_{W^{\mathrm{reg}}}|_{W_i} \cong J_{W^{\mathrm{reg}}}[2]|_{W_i}$. Let do it over W_1 and the case of W_2 will be similar.

Denote B_1 and B_2 the bilinear forms associated to the quadratic spaces V_1 and V_2. For each $T = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix} \in W_1$, the matrix $T_1 = -A.A^* \in \mathrm{GL}(V_1)$ is regular. We define $B_{1,T_1}(v_1, w_1) = B_1(v_1, T_1 w_1)$ for $v_1, w_1 \in V_1$. Then denote Q_1 and Q_{1,T_1} the corresponding quadratic forms on V_1. Define $\mathscr{P}$ to be the pencil of quadrics on the projective space $\mathbb{P}(V_1)$ spanned by Q_1 and Q_{1,T_1}, and set B to be the base locus of $\mathscr{P}$. In [10] Section 3, X. Wang showed that both I_T and $J_T[2]$ act simply transitively on the Fano variety of B whose points are projective $(n-1)-$planes contained in the smooth part of B. By varying T, we obtain that I_{W_1} and $J_{W_1}[2]$ share a common principal homogeneous space. Furthermore, by using the fact that these two actions commute, we obtain a canonical isomorphism of étale group schemes I_{W_1} and $J_{W_1}[2]$ (see Proposition **??**). □

Remark 4.2. The previous isomorphism $I_{W^{\mathrm{reg}}} \to J_{W^{\mathrm{reg}}}[2]$ is $G-$equivariant by construction. Hence, it descends to an isomorphism of group schemes over S: $I_S \to J_S[2]$. By Proposition 3.6, we have a $\mathbb{G}_m-$equivariant isomorphism of quotient stacks

$$BJ_S[2] \cong [W^{\mathrm{reg}}/G]$$

This provides another interpretation of $\mathscr{M}(k)$ as promised (for the definition of $\mathscr{M}$, see the discussion before Theorem 2.5): from the isomorphism

$$\mathscr{M} \cong \mathrm{Hom}(C, [W^{\mathrm{reg}}/G \times \mathbb{G}_m]),$$

we see that $\mathscr{M}_{\mathscr{L}}(k)$ classifies tuples $(\mathscr{E}, s)$ where $\mathscr{E}$ is a principal $G-$bundle and s is a global section of the vector bundle $(W^{\mathrm{reg}} \times^G \mathscr{E}) \otimes \mathscr{L}$. In the next section, we will try to estimate the size of $H^0(C, (W^{\mathrm{reg}} \times^G \mathscr{E}) \otimes \mathscr{L})$ for a given $G-$bundle $\mathscr{E}$.

5. Density of regular locus

To prove our theorem, we need to estimate the number of regular global sections of some vector bundles. It is not easy to calculate it directly. Instead, we will firstly estimate the total number of global sections. Then the results in ([4] section 5) tell us that we will be able to estimate the number of regular global sections if we know the density of the regular locus W^{reg} in W. The next subsection will help us to compute the local density.

5.0.1. *Orbits over finite fields via Galois cohomology.* The content of this section is based on the paper of Bhargava and Gross [2] where they described rational orbits with a fixed invariant via Galois cohomology. We adopt their arguments in our case to estimate the number of rational orbits and then the size of regular locus over finite fields.

Let k^s denote a separable closure of the finite field k. If M (respectively J) is a commutative finite étale group scheme (a smooth algebraic group) over k, we denote $H^1(k, M) = H^1(Gal(k^s/k), M(k^s))$ $(H^1(k, J) = H^1(Gal(k^s/k), J(k^s))$ respectively) to be the corresponding Galois cohomology group (pointed set of first cohomology classes). By Lang's theorem, we have that

$$H^1(k, G) = H^1(k, \mathrm{SO}(V_1) \times \mathrm{SO}(V_2)) = 0.$$

For the rest of this section, we assume that k is a finite field. Let $T \in W^{\mathrm{reg}}(k)$ be a regular self-adjoint operator with the invariant $a = (a_1, \ldots, a_{2m}, e) \in S(k)$, and let $G_T \subset G$ be the finite étale subgroup stabilizing T. For any self-adjoint operator L in $W(k)$ that is in the same orbit as T over k^s, we have $L = gTg^{-1}$ for some $g \in G(k^s)$. This defines an element in $H^1(k, G_T)$ as follows: for any $\sigma \in Gal(k^s/k)$, the element $c_\sigma = g^{-1}g^\sigma$ lies in $G_T(k^s)$, and the map $\sigma \to c_\sigma$ defines a $1-$cocycle on the Galois group with values in $G_S(k^s)$, and hence defines an element in $H^1(k, G_T)$. It can be checked that the cohomology class of that $1-$cocycle depends only on the $G(k)-$orbit of T. Conversely, given a $1-$cocycle c_σ, then it has the form $g^{-1}g^\sigma$ since $H^1(k, G) = 0$. So we obtain an associated operator $L = gTg^{-1}$ that is defined over k since $\sigma(L) = L$ for all $\sigma \in Gal(k^s/k)$ by the definition of the cocycle c_σ. We have just proved the statement $i)$ of the following proposition: (c.f. [2])

Proposition 5.1. i) *Given an operator $T \in W^{reg}(k)$, there is a bijection between the set of $G(k)-$orbits in $W^{reg}(k) \cap G(k^s).T$ and the set $H^1(k, G_T)$.*
ii) *For any $a = (a_1, \ldots, a_{2m}, e) \in S(k)$, the size of $W_a^{reg}(k)$ is bounded above by $2.|G(k)|$.*

Proof. For $ii)$, firstly recall that the action of $G(k^s)$ on $W_a^{\mathrm{reg}}(k^s)$ has at most two orbits. Hence, there exist T_1 and T_2 (they could be the same) in $W_a^{\mathrm{reg}}(k)$ such that $W_a^{\mathrm{reg}}(k) \subset (G(k^s).T_1 \cup G(k^s).T_2)$. We will finish the proof by proving that the size of $W_a^{\mathrm{reg}}(k) \cap G(k^s).T_1$ is equal to $|G(k)|$. In fact, by Proposition 3.19, if we set $f(x) = x^{2m+1} + a_1x^{2m} + \cdots + a_{2m}x + e^2$ and denote $L = k[x]/(f(x))$, then $G_T(k)$ is isomorphic to the kernel of the norm map: $Res_{L/k}(\mu_2) \to \mu_2$. By $i)$ and Kummer theory, the number of $G(k)-$orbits in $W^{\mathrm{reg}}(k)$ equals to $|(L^*/L^{*2})_{N\equiv 1}| = |L^*[2]_{N=1}| = |G_T(k)|$. Hence, we can finish the proof of $ii)$ by using the Orbit-Stabilizer theorem. □

5.0.2. *Regular locus in the transversal case.* Recall that in Section **??** we have computed the density of regular sections and also transversal regular sections by using the results of Poonen (see [6]). By looking back to our method there, we can see that it is essentially based on the fact that any regular vectors with the same invariant are conjugate over algeraic closed field. It is no longer true in our current situation where we have two Kostant sections. But if we restrict to the transversal part, we still have:

Proposition 5.2. *Denote $k = \mathbb{F}_q$ a finite field and $\overline{k}$ its algeraic closure. Let $f(x) \in k[x]$ satisfy the condition that the order of its roots in $\overline{k}$ is at most 2, and if x divides $f(x)$ then $x^2 \nmid f(x)$. Then the action of $G(\overline{k})$ on $V_f^{reg}(\overline{k})$ is transitive.*

Proof. In [7], a similar result for a separable characteristic polynomial $f(x)$ (the regular semi-simple case) is given. Here we will try to generalize the result for $f(x)$ with some conditions which later on can be seen to be closely related to the transversal condition. Given two elements X and T in $V_f^{\mathrm{reg}}(k)$, without loss of generality, we may assume that $X \in \kappa_1$ and $T \in \kappa_2$, or precisely :

$$X = \kappa_1(f) = \begin{pmatrix} 0 & A_1 \\ -A_1^* & 0 \end{pmatrix}$$
$$T = \kappa_2(f) = \begin{pmatrix} 0 & A_2 \\ -A_2^* & 0 \end{pmatrix},$$

such that $A_1A_1^*$ and $A_2^*A_2$ are regular in GL_n. Since $-A_1A_1^*$ is regular, $f(x)$ is also the minimal polynomial of $-A_1A_1^*$. Equivalently, for each root λ_i of $f(x)$ of order n_i, the vector space of generalized eigenvectors of $-A_1A_1^*$ corresponding to λ_i has dimension n_i. By the hypothesis of $f(x)$, we have 4 cases of roots as follows:

Case 1: If $\lambda \neq 0$ is a simple root of $f(x)$, then $\pm\sqrt{\lambda}$ are single roots of $f(x^2)$ the characteristic polynomial of X. If $v_{\sqrt{\lambda}}$ is the unique (up to scalar) non-zero $\sqrt{\lambda}-$eigenvector of X, then it will have the form

$$v_{\sqrt{\lambda}} = \begin{pmatrix} v_\lambda \\ \frac{-1}{\sqrt{\lambda}}A_1^*v_\lambda \end{pmatrix},$$

where v_λ is the unique $\lambda-$eigenvector of $-A_1A_1^*$. Similarly, for $(-\sqrt{\lambda})$, we can choose an eigenvector as follows:

$$v_{-\sqrt{\lambda}} = \begin{pmatrix} v_\lambda \\ \frac{1}{\sqrt{\lambda}}A_1^*v_\lambda \end{pmatrix}.$$

Case 2: If $\lambda = 0$ is a simple root of $f(x)$, then (up to scalar) we denote v_0 and v_0^* to be the unique non-zero $0-$eigenvector of A_1 and A_1^* respectively. In that case, a basis of the $2-$dimensional vector space of $0-$eigenvectors of H is

$$\left\{ \begin{pmatrix} 0 \\ v_0 \end{pmatrix}; \begin{pmatrix} v_0^* \\ 0 \end{pmatrix} \right\}$$

Case 3: If $\lambda \neq 0$ is a double root of $f(x)$ and the eigenspace $V_{1,\lambda}$ of $-A_1A_1^*$ corresponding to λ has dimension 2. Then we can choose an orthogonal basis $\{v_1, v_2\}$ of $V_{1,\lambda}$ with respect to the quadratic form (V_1, Q_1). Then the product in (V, Q):

$$\langle (v_1, A^*v_1); (v_2, A^*v_2) \rangle = \langle v_1, v_2 \rangle + \langle A^*v_1, A^*v_2 \rangle = 0 + \langle AA^*v_1, v_2 \rangle = 0.$$

This helps us to define an orthogonal basis of $V_{\sqrt{\lambda}} \oplus V_{-\sqrt{\lambda}} \subset V$:

$$\left\{ \begin{pmatrix} v_1 \\ \frac{1}{\sqrt{\lambda}}A_1^*v_1 \end{pmatrix}; \begin{pmatrix} v_2 \\ \frac{1}{\sqrt{\lambda}}A_1^*v_2 \end{pmatrix}; \begin{pmatrix} v_1 \\ \frac{-1}{\sqrt{\lambda}}A_1^*v_1 \end{pmatrix}; \begin{pmatrix} v_2 \\ \frac{-1}{\sqrt{\lambda}}A_1^*v_2 \end{pmatrix} \right\}$$

Case 4: If $\lambda \neq 0$ is a double root of $f(x)$ and the eigenspace $V_{1,\lambda}$ of $-A_1A_1^*$ corresponding to λ has dimension 1. Then there are an eigenvector and an generalized eigenvector of X in V that is corresponding to $\sqrt{\lambda}$, we denote them by v_1 and v_2 respectively. Then firstly, as in case 1, we obtain two eigenvectors of X corresponding to the eigenvalues $\pm\sqrt{\lambda}$:

$$v_{\sqrt{\lambda}} = \begin{pmatrix} v_1 \\ \frac{-1}{\sqrt{\lambda}}A_1^*v_1 \end{pmatrix}, v_{-\sqrt{\lambda}} = \begin{pmatrix} v_1 \\ \frac{1}{\sqrt{\lambda}}A_1^*v_1 \end{pmatrix}.$$

Moreover, the following vector is a generalized eigenvector in $V_{\sqrt{\lambda}}$:

$$\begin{pmatrix} v_2 \\ \frac{-1}{\sqrt{\lambda}}A_1^*v_2 + \frac{1}{\lambda}A_1^*v_1 \end{pmatrix}.$$

By replacing v_2 by $v_2 + c.v_1$ in the above formula, we still get a generalized eigenvector. Therefore we can choose the constant c to get an orthogonal basis for $V_{\sqrt{\lambda}}$. From that we also obtain an orthogonal basis for $V_{-\sqrt{\lambda}}$:

$$\left\{ \begin{pmatrix} v_1 \\ \frac{1}{\sqrt{\lambda}} A_1^* v_1 \end{pmatrix}; \begin{pmatrix} v_2 \\ \frac{1}{\sqrt{\lambda}} A_1^* v_2 + \frac{-1}{\lambda} A_1^* v_1 \end{pmatrix} \right\}$$

The upshot is that we have just constructed an orthogonal basis of (V, Q) that consists of generalized eigenvectors of X such that: for each root $\lambda \neq 0$ of $f(x)$, we will have one (or two) pair (pairs) of eigenvectors $w_{\pm\sqrt{\lambda}}$ of X such that $w_{\sqrt{\lambda}} + w_{-\sqrt{\lambda}} \in V_1$ and $w_{\sqrt{\lambda}} - w_{-\sqrt{\lambda}} \in V_2$. If λ is as in case 4, we also have a pair of generalized eigenvectors ${}^g w_{\pm\sqrt{\lambda}}$ with the same properties. If $\lambda = 0$, then we will have two eigenvector $w_{1,0}$ and $w_{2,0}$ satisfying $w_{i,0} \in V_i$.

Similarly, we can also construct an orthogonal basis $\{w'_{\pm\sqrt{\lambda}}, {}^g w'_{\pm\sqrt{\lambda}}, w'_{1,0}, w'_{2,0}\}$ that consists of generalized eigenvectors of T having the same properties as above (here we write down all possible generalized eigenvectors; for a specific case they may not appear in that orthogonal basis). Since $\langle w_{\sqrt{\lambda}} + w_{-\sqrt{\lambda}}, w_{\sqrt{\lambda}} - w_{-\sqrt{\lambda}} \rangle = 0$, we have that $Q(w_{\sqrt{\lambda}}) = Q(w_{-\sqrt{\lambda}})$. Similarly, two vectors in all of these pairs ${}^g w_{\pm\sqrt{\lambda}}, w'_{\pm\sqrt{\lambda}}, {}^g w_{\pm\sqrt{\lambda}}$ have the same norm w.r.t Q. By scaling, we may assume that $Q(w_{\pm\sqrt{\lambda}}) = Q(w'_{\pm\sqrt{\lambda}})$; $Q({}^g w_{\pm\sqrt{\lambda}}) = Q({}^g w'_{\pm\sqrt{\lambda}})$ and if $f(0) = 0$ we also assume that $Q(w_{i,0}) = Q(w'_{i,0})$ for $i = 1, 2$.

From the above construction, the linear map $g : V \to V$ taking the $w_{\pm\sqrt{\lambda}}, {}^g w_{\pm\sqrt{\lambda}}$, and $w_{i,0}$ (if we have) to $w'_{\pm\sqrt{\lambda}}, {}^g w'_{\pm\sqrt{\lambda}}$, and $w'_{i,0}$ respectively, is orthogonal, and conjugation by g takes T to X. Using the properties that

$$\{w_{\sqrt{\lambda}} + w_{-\sqrt{\lambda}};\ {}^g w_{\sqrt{\lambda}} + {}^g w_{-\sqrt{\lambda}};\ w_{1,0}\}_\lambda$$

span V_1 and

$$\{w_{\sqrt{\lambda}} - w_{-\sqrt{\lambda}};\ {}^g w_{\sqrt{\lambda}} - {}^g w_{-\sqrt{\lambda}};\ w_{2,0}\}_\lambda$$

span V_2 (similar for the orthogonal basis related to T) , we see that g preserves V_1 and V_2. Hence, $g \in O(V_1) \times O(V_2)$. Conjugating a matrix by g multiplies the Pfaffian of that matrix by the determinant of g. Hence, if the Pfaffian of X (also of T) is non-zero, we implies that $\det(g) = 1$. It means that $g \in H^\theta := \{h \in O(V_1) \times O(V_2) | \det(h) = 1\}$. Since G is the connected component of H^θ containing the identity, $H^\theta / G \cong \{I_{2n}, -I_{2n}\}$, and $-I_{2n}$ acts trivially on $W \cong V_1 \otimes V_2$, we implies that $W//H^\theta = W//G$. As a result, H and T are conjugated by an element in G.

If $\det(X) = \det(T) = 0$ and $\det(g) = -1$, then by considering g' that is exactly the same as g except that g' map $w_{1,0}$ to $-w'_{1,0}$, we still have that X and T are conjugated by g', and note that $\det(g') = 1$. The same arguments as above will now finish the proof. □

We also need the following lemma:

Lemma 5.3. *Let $k = \mathbb{F}_q$ denote a finite field, and $f \in S(k[[t]])$ is a polynomial of degree n with coefficients in the complete local ring $k[[t]]$. Assume that $ord_t(\Delta(f)) < 2$, then any element of $W_f(k[[t]])$ is regular, i.e., for any $T \in W_f(k[[t]])$, the image $\overline{T} = x(mod\, t)$ is in $W^{reg}(k)$.*

Proof. Denote $\overline{f} \in S(k)$ to be f (mod t). If $\Delta(f)$ is a unit in $k[[t]]$ then $\Delta(\overline{f}) = \overline{\Delta(f)} = \Delta(f)\, mod\, t$ is non-zero in k. Hence, by [7] (in case $\overline{f}(0) \neq 0$)and the previous Proposition (in case $\overline{f}(0) = 0$), $G(\overline{k})$ acts transitively on $W_{\overline{f}}(\overline{k})$, and consequently $W_{\overline{f}}(k) \subset W^{\text{reg}}(k)$.

Suppose $\text{ord}_t(\Delta(f)) = 1$, and we assume that $\overline{T} = T(mod\, t) \in W(k)$ is not regular. In this case, by Definition 3.2 of regularity and the proof of Proposition 5.2, we can deduce that both $A.A^* \in \text{GL}(V_1)$ and $A^*.A \in \text{GL}(V_2)$ are not regular, where $T = \begin{pmatrix} 0 & A_T \\ A_T^* & 0 \end{pmatrix} \in W_f(k[[t]]$, and $A = A_T(mod\, t)$. Now $A.A^*$ is not regular as an element in $\text{GL}(V_1)(k)$ is equivalent to that the dimension of the centralizer of $A.A^*$ in $\mathfrak{g}_k := Lie(\text{GL}(V_1))_k$ is not equal to the rank of GL(V_1), and hence it is at least rank(GL(V_1)) + 2 (see [8] III. 3.25). By setting $c = Cent_{\mathfrak{g}_{k((t))}}(A_T A_T^*) \cap \mathfrak{g}_{k[[t]]}$, we define an adjoint map

$$g := ad(A_T.A_T^*) : \mathfrak{g}_{k[[t]]}/c \to \mathfrak{g}_{k[[t]]}/c.$$

Then we have that $\det(g) = \Delta(A_T.A_T^*) = \Delta(f)$, up to units in $k[[t]]$. Since $A.A^*$ is not regular, $\overline{g} = g\, mod\, (t)$ has kernel of dimension at least 2, hence $\mathrm{ord}_t(\det(g) \geq 2$, a contradiction. □

Now we can compute the density of regular locus in the transversal case:

Proposition 5.4. *For any $v \in |C|$, we define*

$$\alpha_v = \frac{|\{f \in S(\mathscr{O}_{K_v}/(\varpi_v^2))|\Delta(f) \equiv 0 \ mod(\varpi_v^2)\}|}{|k(v)^{2n}|}$$

and

$$\beta_v = \frac{|\{x \in W(\mathscr{O}_{K_v}/(\varpi_v^2))|\Delta(x) \equiv 0 \ mod(\varpi_v^2)\}|}{|k(v)^{2n^2}|},$$

Then we have the following equalities

1.
$$\lim_{deg(\mathscr{L})\to\infty} \frac{|\Gamma(C, \mathscr{L}^{\otimes 2} \oplus \mathscr{L}^{\otimes 4} \oplus \cdots \oplus \mathscr{L}^{\otimes 2n-2} \oplus \mathscr{L}^{\otimes n})^{sf}|}{|\Gamma(C, \mathscr{L}^{\otimes 2} \oplus \mathscr{L}^{\otimes 4} \oplus \cdots \oplus \mathscr{L}^{\otimes 2n-2} \oplus \mathscr{L}^{\otimes n})|} = \prod_{v\in|C|}(1-\alpha_v).$$

2.
$$\lim_{deg(\mathscr{L})\to\infty} \frac{|\Gamma(C, W^{reg}(\mathscr{E}, \mathscr{L}))^{sf}|}{|\Gamma(C, W(\mathscr{E}, \mathscr{L}))|} = \prod_{v\in|C|}(1-\beta_v)$$

3.
$$\frac{\prod_{v\in|C|}(1-\beta_v)}{\prod_{v\in|C|}(1-\alpha_v)} = \prod_{v\in|C|}\frac{|G(k(v))|}{|k(v)|^{n^2-n}}$$

Here the upper script "sf" stands for "square free", i.e. $\Gamma()^{sf}$ is the set of sections whose invariants are transversal to the discriminant locus.

Proof. The first two equalities can be shown by using the results in [4] Section 5 and notice that by the previous lemma, any element in $W_a(\mathscr{O}_{K_v})$, where $\Delta(a) \not\equiv 0 \ \mathrm{mod}(\varpi_v^2)$, is regular. Now we will prove the last equality by showing that locally:

$$\frac{1-\beta_v}{1-\alpha_v} = \frac{|G(k(v))|}{|k(v)|^{n^2-n}}. \tag{5.1}$$

To do that, for a given transversal element $a = (a_1, \ldots, a_{n-1}, e) \in S(R)$, we will count the size of $W_a^{\mathrm{reg}}(R)$, where $R = k(v)[\epsilon]/(\epsilon^2)$. Set $T = \overline{T} + \epsilon H \in W_a^{\mathrm{reg}}(R)$ and $a = \overline{a} + \epsilon b$, where $\overline{T}, H \in W(k(v))$ and $\overline{a}, b = (b_2 + \ldots, b_{2n+1}) \in S(k(v))$. By Proposition 5.2, we firstly observe that there are $|G(k(v))|$ choices of $\overline{T}$ such that $\pi(\overline{T}) = \overline{a}$. With a fixed $\overline{T}$, by considering H and b as elements in the tangent spaces of W^{reg} and S, respectively, we can see that the tangent map:

$$d\pi : T_{\overline{T}}V^{\mathrm{reg}} \to T_{\overline{a}}S$$

will map H to b. Since $\pi : V^{\mathrm{reg}} \to S$ is smooth, the number of choices of H will be the size of the fiber of $d\pi$ at b, and it is equal to

$$|k(v)|^{\dim_{k(v)}(T_{\overline{T}}W^{\mathrm{reg}})-\dim_{k(v)}(T_{\overline{a}}S)} = |k(v)|^{n^2-n}.$$

By taking the sum over $a \in S^{\mathrm{trans}}(R)$ we will obtain the equality (5.1). □

5.0.3. *Regular locus in the general case.* Now we consider the general case (without the transversal property).

Proposition 5.5. 1. *We have the following limit*

$$\lim_{deg(\mathscr{L})\to\infty} \frac{|\Gamma(C, W^{reg}(\mathscr{E},\mathscr{L}))|}{|\Gamma(C, W(\mathscr{E},\mathscr{L}))|} = \prod_{v\in|C|} \frac{c_v}{|k(v)|^{n^2}},$$

where $c_v = |W^{reg}(k(v)|$.

2. *The above limit is bounded above by*

$$\zeta_C(2)^{-2}\dots\zeta_C(2m)^{-2}.\prod_{v\in|C|}\left(1 + c_{2m-1}|k(v)|^{-2} + \dots + c_1|k(v)|^{-2m}\right),$$

where c_i, for all i, is a constant which depends only on m and p. If $p > 2m+1$ then c_i only depends only on m.

Proof. The first statement is the consequence of Proposition 5.1.1 in [4] and a note that the subset that contains all of non-regular elements of W, is of codimensional 2. To prove the second part, for each element $a \in S(k(v))$, we will bound the size of $W_a^{\mathrm{reg}}(k(v))$. We have two cases:

Case 1: If a satisfies the hypothesis in the Proposition 3.24, then by Proposition 3.24, $|W_a^{\mathrm{reg}}(k(v))| = |G(k(v))|$.

Case 2: If a does not satisfy the hypothesis in the Proposition 3.24, then by Proposition 3.23 *ii*), $|W_a^{\mathrm{reg}}(k(v))| \leq 2|G(k(v))|$.

Our job now is to calculate the number of invariants $a = (a_1, \dots, a_{2m}, e)$ in the second case above. We also have several cases as follows:

Case 1: If the corresponding polynomial $f_a(x)$ is divided by x^2, then $a_{2m} = 0$ and $e = 0$. Hence, the total number of a's in this case is $|q^{2m-1}|$, where $q = |k(v)|$.

Case 2: If $f_a(x)$ has a root α of order $e > 2$ in $\overline{k(v)}$, we denote $m_\alpha(x)$ the minimal polynomial of α over $k(v)$, then

$$f_a(x) = m_\alpha(x)^e.g(x) \text{ \textbf{if} } m_\alpha(x) \text{ \textbf{is separable,}}$$

$$f_a(x) = m_\alpha(x).g(x), \textbf{where } m_\alpha(x) = h(x^{p^t}) \textbf{ for some } t \in \mathbb{N}$$

In both cases, a is defined by the coefficients of $m_\alpha(x)$ and $g(x)$. In the former case, if we set the degree of m_α and g by m_1 and m_2 respectively, then a can be defined by $m_1 + m_2 = 2m+1-(e-1)m_1$ coordinates. Hence, the total number of a in this case is bounded by $q^{2m+1-(e-1)m_1}$. In the later case, we also easily deduce that a is defied by at most $2m+1-p-1$ coordinates, thus, the total number of a's is bounded by $q^{2m+1-p-1}$. Note that we only have finite "types" of m_α ("type" here means the choice of the degree of m_α in the former case and the choice of $h(x^{p^t})$ in the later case). So the total number of $a \in S(k(v))$ satisfying the corresponding $f_a(x)$ has a root in $\overline{k(v)}$ of order at least 3 is bounded by

$$\sum_{i=1}^{2m-1} c_i q^i,$$

where c_i are constants that are only depended on m and p.

The upper bound of the limit in 1) is the consequence of the above calculation. □

5.0.4. *Regular locus in minimal case.* To take the average over the set of hyperelliptic curves, we need to consider the minimal data $(\mathscr{L}, \underline{a})$. Note that the transversal condition implies the minimal condition. Furthermore, the results in [4] Section 5 also help us to see that the density of the minimal locus is the product of local densities. The local condition for a minimal data is that at a closed point $v \in |C|$, the tuple of sections $\underline{a}$ does not come from $\mathscr{L}(-v)$. Thus, when $\deg(\mathscr{L}) >> 0$, the density of tuples $\underline{a}$ that come from $\mathscr{L}(-v)$ is

$$\frac{|H^0\big(C,\mathscr{L}(-v))^{\otimes 2}\oplus\cdots\oplus(\mathscr{L}(-v))^{\otimes 4m}\oplus(\mathscr{L}(-v))^{\otimes 2m+1}\big)|}{|H^0\big(C,\mathscr{L}^{\otimes 2}\oplus\cdots\oplus\mathscr{L}^{\otimes 4m}\oplus\mathscr{L}^{\otimes 2m+1}\big)|} \tag{5.2}$$

$$= \frac{1}{|k(v)|^{(2m+1)^2}} \tag{5.3}$$

We have just proved the following result:

Proposition 5.6. *Given a line bundle of sufficiently large degree, the density of minimal tuples* $\underline{a} \in H^0\big(C, \mathscr{L}^{\otimes 2} \oplus \cdots \oplus \mathscr{L}^{\otimes 4m} \oplus \mathscr{L}^{\otimes 2m+1}\big)$ *is* $\zeta_C((2m+1)^2)^{-1}$.

By using similar argument as in the previous subsection, we obtain the following estimate:

Proposition 5.7. *Given principal* $G-$*bundle* $\mathscr{E}$, *we denote the set of sections of* $W^{reg}(\mathscr{E}, \mathscr{L})$ *whose associated data* $(\mathscr{L}, \underline{a})$ *is minimal by* $\Gamma(C, W^{reg}(\mathscr{E}, \mathscr{L}))^{min}$. *Similarly for the notation* $\Gamma(C, \mathscr{L}^2 \oplus \cdots \oplus \mathscr{L}^{4m} \oplus \mathscr{L}^{2m+1})^{min}$ *- the set of minimal tuples* $\underline{a}$. *Then*

$$\lim_{deg(\mathscr{L})\to\infty} \frac{\dfrac{|\Gamma(C, W^{reg}(\mathscr{E}, \mathscr{L}))^{min}|}{|\Gamma(C, W(\mathscr{E}, \mathscr{L}))|}}{\dfrac{|\Gamma(C, \mathscr{L}^{\otimes 2} \oplus \mathscr{L}^{\otimes 4} \oplus \cdots \oplus \mathscr{L}^{\otimes 2n-2} \oplus \mathscr{L}^{\otimes n})^{min}|}{|\Gamma(C, \mathscr{L}^{\otimes 2} \oplus \mathscr{L}^{\otimes 4} \oplus \cdots \oplus \mathscr{L}^{\otimes 2n-2} \oplus \mathscr{L}^{\otimes n})|}}$$

$$\leq \zeta_C(2)^{-2} \dots \zeta_C(2m)^{-2}. \prod_{v\in|C|} \big(1 + c_{2m-1}|k(v)|^{-2} + \cdots + c_1|k(v)|^{-2m}$$

$$-2|k(v)|^{(2m+1)^2}\big),$$

where c_i *is the same as the one in Proposition 5.5.*

6. Counting

Let's recall some notations: V_1 and V_2 are orthogonal spaces over $k = \mathbb{F}_q$ of dimension $n = 2m+1$, $G = \mathrm{SO}(V_1) \times \mathrm{SO}(V_2)$ is split, and $W = V_1 \otimes V_2$ a representation of G. We can see each element in W as a skew-self adjoint matrix whose diagonal blocks are 0:

$$W(k) = \left\{ \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix} | A \in M_n(k) \right\}$$

where A^* is a matrix obtained from A by taking the transpose via the anti diagonal. The goal of this section is to estimate the following limit:

$$\lim_{\deg(\mathscr{L})\to\infty} \sum_{\mathscr{E}\in \mathrm{Bun}_G(k)} \frac{|H^0(C, (\mathscr{E} \times^G W^{\mathrm{reg}}) \otimes \mathscr{L})|}{|\mathrm{Aut}(\mathscr{E})|.|\mathscr{A}_{\mathscr{L}}(k)|}. \tag{6.1}$$

The denominator in the above limit can be easily calculated (using the similar arguments as those in section 1). In fact, $\mathscr{A}_{\mathscr{L}}(k)$ classifies hyperelliptic curves over C whose coefficients in their affine Weierstrass equation as in Section 3.1 all come from $\mathscr{L}$. This implies that when $\deg(\mathscr{L})$ is sufficiently large, we have

$$|\mathscr{A}_{\mathscr{L}}(k)| = |H^0(C, \mathscr{L}^{\otimes 2} \oplus \cdots \oplus \mathscr{L}^{\otimes 2n-2} \oplus \mathscr{L}^{\otimes n})| = q^{n^2 d + n(1-g)}$$

where d is the degree of $\mathscr{L}$ and g denotes the genus of the curve C.

6.0.1. *Automorphism group of a G-bundle.* If $\mathscr{E}$ is a $G-$bundle, then $\mathscr{E}$ can be expressed as the product $\mathscr{E}_1 \times \mathscr{E}_2$, where $\mathscr{E}_i$ are $\mathrm{SO}(V_i)-$bundles. Hence, $\mathrm{Aut}_G(\mathscr{E}) = \mathrm{Aut}_{\mathrm{SO}(V_1)}(\mathscr{E}_1) \times \mathrm{Aut}_{\mathrm{SO}(V_2)}(\mathscr{E}_2)$, and then we could apply the results in Section **??** to estimate the size of automorphism groups. Suppose that the bundle $\mathscr{E}_1$ has the canonical reduction (P_1, σ_1), and the parabolic subgroup P_1 has the Levi quotient given by

$$L_1 \cong \mathrm{GL}(r_1) \times \mathrm{GL}(r_2) \times \cdots \times \mathrm{GL}(r_t) \times \mathrm{SO}(r_0),$$

where $r_0 + 2\sum_{i=1}^h r_i = n = 2m+1$. In the other words, there exists a flag of isotropic subspaces

$$0 = V_{1,0} \subset V_{1,1} \subset \cdots \subset V_{1,h} \subset V_{1,h}^{\perp} \subset \cdots \subset V_{1,1}^{\perp} \subset V_1,$$

where $\dim(V_{1,i}/V_{1,i-1}) = r_i$ for $1 \le i \le h$ and $\dim(V_{1,t}^{\perp}/V_{1,h}) = r_0$. From that we obtain a filtration of the vector bundle $\mathscr{E}_1 \times^{\mathrm{SO}(V_1)} V_1$:

$$0 = \mathscr{E}_{1P_1} \times^{P_1} V_{1,0} \subset \cdots \subset \mathscr{E}_{1P_1} \times^{P_1} V_{1,h} \subset \mathscr{E}_{1P_1} \times^{P_1} V_{1,h}^{\perp} \subset \cdots \subset \mathscr{E}_{1P_1} \times^{P_1} V_{1,1}^{\perp} \subset \mathscr{E}_1 \times^{\mathrm{SO}(V_1)} V_1$$

such that the quotient bundles

$$X_i = \mathscr{E}_{1P_1} \times^{P_1} V_{1,i}/\mathscr{E}_{1P_1} \times^{P_1} V_{1,i-1} \text{ for } 1 \le i \le h \text{ and}$$

$$X_0 = \mathscr{E}_{1P_1} \times^{P_1} V_{1,h}^{\perp}/\mathscr{E}_{1P_1} \times^{P_1} V_{1,h}$$

are semistable (the semistable property comes from [3] Proposition 6.9 and our assumption on the characteristic of $\mathbb{F}_q$, see Section **??**). If we denote the slope of the vector bundle X_i by x_i, then by definition of the canonical reduction, we deduce that $x_1 > x_2 > \cdots > x_h > x_0 = 0$.

Similarly, for the $\mathrm{SO}(V_2)-$bundle $\mathscr{E}_2$ we associate it with a unique parabolic subgroup P_2 of $\mathrm{SO}(V_2)$ and a set of semistable vector bundles Y_i for $0 \le i \le l$ satisfying

$$t_0 + 2\sum_{i=1}^{l} t_i = n,$$

$$y_1 > y_2 > \cdots > y_l > y_0 = 0,$$

where t_i and y_i denote the rank and the slope of vector bundle Y_i, respectively. With these notations, we can estimate the size of the automorphic group as follows:

Proposition 6.1. (i) *There exists a constant c that depends only on n and g such that for any $G-$bundles $\mathscr{E}$ with canonical reduction to P, we have*

$$-c \le dim(Aut_G(\mathscr{E})) - dim(Aut_L(\mathscr{E}_L)) - \sum_{i=1}^{t} \big(h^0(\wedge^2 X_i) + h^0(X_i \otimes X_0)\big) -$$
$$- \sum_{t \ge j > i > 0} \big(h^0(X_i \otimes X_j) + h^0(X_i \otimes X_j^*)\big) - \sum_{i=1}^{l} \big(h^0(\wedge^2 Y_i) + h^0(Y_i \otimes X_0)\big) -$$
$$- \sum_{l \ge j > i > 0} \big(h^0(Y_i \otimes Y_j) + h^0(Y_i \otimes Y_j^*)\big) \le c$$

(ii) *In particular, if $x_i - x_{i+1} > 2g-2$ for all i and $y_j - y_{j+1} > 2g-2$ for all j, then the constant c in (i) can be taken to be 0.*

Proof. See Proposition **??**. □

6.0.2. *General setting.* Given a $G-$bundle $\mathscr{E}$ as above (with the canonical reduction to P and associated vector bundles X_i for $0 \leq i \leq t$, and Y_j for $0 \leq j \leq l$), we firstly assume that $x_i - x_{i+1}$ for $0 \leq i \leq t$, x_t, $y_j - y_{j+1}$ for $0 \leq j \leq l$, and y_l are all bigger than $2g-2$. This condition makes sure that the filtration associated with the canonical reduction of $\mathscr{E}$ is split. Precisely, we can express the vector bundles $\mathscr{E}_i \times^{\mathrm{SO}(V_i)} V_i$ as direct sums:

$$\mathscr{E}_1 \times^{\mathrm{SO}(V_1)} V_1 = X_0 \oplus \bigoplus_{i=1}^{t} \left(X_i \oplus X_i^*\right), \tag{6.2}$$

$$\mathscr{E}_2 \times^{\mathrm{SO}(V_2)} V_2 = Y_0 \oplus \bigoplus_{j=1}^{l} \left(Y_j \oplus Y_j^*\right). \tag{6.3}$$

As a result, any global sections of the vector bundle $\mathscr{E} \times^G W$ is of the following matrix form:

$$\begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix},$$

where A is the section of

$$\begin{pmatrix} X_1 \otimes Y_1 & X_1 \otimes Y_2 & \cdots & X_1 \otimes Y_0 & X_1 \otimes Y_l^* & \cdots & X_1 \otimes Y_1^* \\ X_2 \otimes Y_1 & X_2 \otimes Y_2 & \cdots & X_2 \otimes Y_0 & X_2 \otimes Y_l^* & \cdots & X_2 \otimes Y_1^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_0 \otimes Y_1 & X_0 \otimes Y_2 & \cdots & X_0 \otimes Y_0 & X_0 \otimes Y_l^* & \cdots & X_0 \otimes Y_1^* \\ X_t^* \otimes Y_1 & X_t^* \otimes Y_2 & \cdots & X_t^* \otimes Y_0 & X_t^* \otimes Y_l^* & \cdots & X_t^* \otimes Y_1^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_1^* \otimes Y_1 & X_1^* \otimes Y_2 & \cdots & X_1^* \otimes Y_0 & X_1^* \otimes Y_l^* & \cdots & X_1^* \otimes Y_1^* \end{pmatrix} \tag{6.4}$$

In general, the above matrix defines the semi-stable filtration of our vector bundle $\mathscr{E} \times^G W$. As a consequence, we obtain an upper bound of $|H^0(\mathscr{E} \times^G W)|$ by taking the product of the numbers of global sections of semi-stable vector bundles that are entries of the matrix (6.4). Moreover, in order to define a hyperelliptic curve with two remarked points, the global section

$$\alpha = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}$$

of the vector bundle $(\mathscr{E} \times^G W) \otimes \mathscr{L}$ need to satisfy the conditions that $det(A) \neq 0$ and the discriminant $\Delta(H_\alpha)$ of the associated curve H_α which is, by definition, is the discriminant of the characteristic polynomial of $A.A^*$, is non-zero. The following Proposition will give some necessary conditions on A to satisfy the above properties:

Proposition 6.2. *Let* $\alpha = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}$ *be an element of* $W(K)$. *We denote* H_α *to be the associated curve as in Section 4. Suppose that* H_α *is a hyperelliptic curve with two marked points. Then*

i) *There is no zero bottom right* $i \times (2m+2-i)$ *blocks, for any* $1 \leq i < 2m+2$, *in* A_α.
ii) *For any* $1 < i < 2m+1$, *the bottom right* $i \times (2m+1-i)$ *and* $(2m+1-i) \times i$ *blocks in* A_α *are not both zero.*

Proof. The first statement is easy to see from the condition that $det(A_\alpha) \neq 0$. The observation is a direct consequence of [7] lemma 7.5, and it corresponds to the condition that $\Delta(H_\alpha) \neq 0$. □

Before going to the detail computation, we list all of cases and their contribution to the average in the table 1.

Case	Hypothesis	Subcases	Contr.
1	$max\{x_i + y_i\}_{i,j} > d$ and $t + l < 2m$	$t = 0$ or $l = 0$	0
		$t = l = 1$	0
		$t > 1$ or $l > 1$	0
2	$t = l = m$ and $x_1 + y_1 > d$	$x_i = y_i + d = (2m - 2i + 2)d \; \forall 1 \leq i \leq m$	1
		$y_i = x_i + d = (2m - 2i + 2)d \; \forall 1 \leq i \leq m$	1
		otherwise	f(q)
3	$max\{x_i + y_j\}_{i,j} \leq d$	$d - 2g + 2 \leq max\{x_i + y_j\}_{i,j} \leq d$	0
		$max\{x_i + y_j\}_{i,j} < d - 2g + 2$	4

TABLE 1. Contribution to the average.

6.0.3. *Case 1: $max\{x_i + y_j\}_{i,j} > d$ and $t + l < 2m$.* The contribution to the average, in this case, is equal to zero. That is the corollary of the following proposition.

Proposition 6.3. *Given a $G-$bundle $\mathscr{E}$ with the associated data $\{P, x_i, y_j\}_{\substack{0\leq i\leq t \\ 0\leq j\leq l}}$ such that $max\{x_i + y_j\}_{i,j} > d$. If $t \neq l$ or $t = l \neq m$, then*

$$\frac{|\{\alpha \in H^0(C, (\mathscr{E} \times^G W) \otimes \mathscr{L}) | Det(A_\alpha) \neq 0, \Delta(H_\alpha) \neq 0\}|}{|Aut_G(\mathscr{E})|.|\mathscr{A}_{\mathscr{L}}(k)|} \leq \frac{c}{|Aut_L(\mathscr{E}_L)|.q^d},$$

where d is the degree of the line bundle $\mathscr{L}$, and c is a constant that depends only on g and n.

Proof. Let we consider some initial cases:

Case 1: If $l = 0$, or $t = 0$. Without loss of generality, we may assume that $t = 0$. Then the condition $max\{x_i + y_j\}_{i,j} > d$ becomes $y_1 > d$. This implies that the semi-stable vector bundle $X_0 \otimes Y_1^* \otimes \mathscr{L}$ has negative degree, thus, it has no non-trivial section. By Proposition 6.2 i), the set

$$\{\alpha \in H^0(C, (\mathscr{E} \times^G W) \otimes \mathscr{L}) | Det(A_\alpha) \neq 0, \Delta(H_\alpha) \neq 0\}$$

is empty, so obviously, our Proposition is true in this case.

Case 2: If $t = l = 1$, $x_1 > d$, and $y_1 > d$. In this case, the vector bundles $X_0 \otimes Y_1^* \otimes \mathscr{L}$, $X_1^* \otimes Y_1 \otimes \mathscr{L}$, and $X_1^* \otimes Y_1^* \otimes \mathscr{L}$ have negative degrees. By Proposition **??**, they have no global section, hence, any sections α in $H^0(C, (\mathscr{E} \times^G W) \otimes \mathscr{L})$ will have the following form:

$$A_\alpha = \begin{pmatrix} A & B & C \\ D & E & 0 \\ F & 0 & 0 \end{pmatrix},$$

where $A, C, F \in Mat(r_1 \times t_1); B \in Mat(r_1 \times t_0); D \in Mat(r_0 \times t_1)$; and $E \in Mat(r_0 \times t_0)$. This implies that

$$A_\alpha.A_\alpha^* = \begin{pmatrix} C.F^* & B.E^* + C.D^* & 2A.C^* + B.B^* \\ 0 & E.E^* & B.E^* + C.D^* \\ 0 & 0 & C.F^* \end{pmatrix}.$$

We deduce that $\Delta(H_\alpha) = 0$, thus, the generic fiber of H_α is not a hyperelliptic curve over $K(C)$.

Case 3: If $t = l = 1$, $x_1 > d$, $y_1 \leq d$, and $x_1 - y_1 \leq d$. In this case,

$$H^0\big((\mathscr{E} \times^G W) \otimes \mathscr{L}\big)$$

$$= \begin{pmatrix} H^0(X_1 \otimes Y_1 \otimes \mathscr{L}) & H^0(X_1 \otimes Y_0 \otimes \mathscr{L}) & H^0(X_1 \otimes Y_1^* \otimes \mathscr{L}) \\ H^0(X_0 \otimes Y_1 \otimes \mathscr{L}) & H^0(X_0 \otimes Y_0 \otimes \mathscr{L}) & H^0(X_0 \otimes Y_1^* \otimes \mathscr{L}) \\ H^0(X_1^* \otimes Y_1 \otimes \mathscr{L}) & 0 & 0 \end{pmatrix}.$$

So for any section α with

$$A_\alpha = \begin{pmatrix} A & B & C \\ D & E & F \\ G & 0 & 0 \end{pmatrix},$$

where $G \in H^0(C, X_1^* \otimes Y_1)$, to make sure that $\det(A_\alpha) \neq 0$, by Proposition 6.2, we need to put an extra condition that $r_1 \leq t_1$. Hence,

$$\begin{aligned} &\frac{|H^0(C, (\mathscr{E} \times^G W) \otimes \mathscr{L})|}{|\mathrm{Aut}(\mathscr{E})|.|\mathscr{A}_{\mathscr{L}}(k)|} \\ &= \frac{q^{r_1t_0x_1 + r_1t_1(x_1+y_1) + d(n^2 - r_1t_0 - r_1t_1)}}{|\mathrm{Aut}(X_1) \times \mathrm{Aut}(X_0) \times \mathrm{Aut}(Y_1) \times \mathrm{Aut}(Y_0)|.q^{r_1x_1(r_1-1+r_0)+t_1y_1(t_1-1+t_0)+dn^2}} \\ &= \frac{q^{-r_1x_1(t_1-r_1-1)-t_1y_1(t_1-r_1+t_0-1)-dr_1(t_0+t_1)}}{|\mathrm{Aut}(X_1) \times \mathrm{Aut}(X_0) \times \mathrm{Aut}(Y_1) \times \mathrm{Aut}(Y_0)|} \\ &\leq \frac{1}{q^{-dr_1+dr_1(t_0+t_1)}|\mathrm{Aut}(X_1) \times \mathrm{Aut}(X_0) \times \mathrm{Aut}(Y_1) \times \mathrm{Aut}(Y_0)|.} \end{aligned}$$

This implies the Proposition.

Case 4: If $t = l = 1$, $x_1 \leq d$, $y_1 \leq d$, and $x_1 + y_1 > d$. Then similar to the above calculations, we obtain that

$$\begin{aligned} &\frac{|H^0(C, (\mathscr{E} \times^G W) \otimes \mathscr{L})|}{|\mathrm{Aut}(\mathscr{E})|.|\mathscr{A}_{\mathscr{L}}(k)|} \\ &= \frac{q^{r_1t_1(x_1+y_1)+d(n^2-r_1t_1)}}{|\mathrm{Aut}(X_1) \times \mathrm{Aut}(X_0) \times \mathrm{Aut}(Y_1) \times \mathrm{Aut}(Y_0)|.q^{r_1x_1(r_1-1+r_0)+t_1y_1(t_1-1+t_0)+dn^2}} \\ &= \frac{1}{q^{r_1x_1(r_1+r_0-t_1-1)+t_1y_1(t_1-r_1+t_0-1)+dr_1t_1}|\mathrm{Aut}(X_1) \times \mathrm{Aut}(X_0) \times \mathrm{Aut}(Y_1) \times \mathrm{Aut}(Y_0)|} \\ &\leq \frac{1}{q^{dr_1t_1}|\mathrm{Aut}(X_1) \times \mathrm{Aut}(X_0) \times \mathrm{Aut}(Y_1) \times \mathrm{Aut}(Y_0)|.} \end{aligned}$$

To sum up, we have just proved the Proposition in the cases that $t = 0$ or $l = 0$ or $t = l = 1$.

Now we will prove this proposition by induction. Assume that the statement is true for all pair (t', l'), where $t' \leq t$, $l' \leq l$, and $t' + l' < t + l$. Now having fixed numbers of X_i and Y_j, we will find $(x_i, r_i, y_j, t_j)_{0 \leq i \leq t; 0 \leq j \leq l}$ such that the fractional expression:

$$A = \frac{|\{\alpha \in H^0(C, (\mathscr{E} \times^G W) \otimes \mathscr{L}) | Det(A_\alpha) \neq 0, \Delta(H_\alpha) \neq 0\}|}{|\mathrm{Aut}_G(\mathscr{E})|.|\mathscr{A}_{\mathscr{L}}(k)|}$$

is "maximal".

Note that to prove our inequality, firstly we can make use of the semi-stable filtration associated to the canonical reduction of $\mathscr{E}$. Then we approximate the dimensions of each components in that filtration by their degrees. More precisely, we can replace $H^0(C, (\mathscr{E} \times^G W) \otimes \mathscr{L})$ in the numerator of A by

$$H^0\big((X_1 \oplus \cdots \oplus X_t \oplus X_0 \oplus X_t^* \oplus \cdots \oplus X_1^*) \otimes (Y_1 \oplus \cdots \oplus Y_l \oplus Y_0 \oplus Y_l^* \oplus \cdots \oplus Y_1^*) \otimes \mathscr{L}\big).$$

And the denominator of A can be replaced by:

$$|\mathrm{Aut}_L(\mathscr{E}_L)|.q^{\sum_{i=1}^t \left(r_ix_i(r_i-1+2r_{i+1}+\cdots+2r_t+r_0)\right)+\sum_{j=1}^l \left(t_jy_j(t_j-1+2t_{j+1}+\cdots+2t_l+t_0)\right)+n^2}.$$

We now prove a stronger result: there exist a constant c that does not depend on d such that

$$\frac{|\{\alpha \in H^0\left((X_1 \oplus \cdots \oplus X_t \oplus X_0 \oplus X_t^* \oplus \cdots \oplus X_1^*) \otimes (Y_1 \oplus \cdots \oplus Y_l \oplus Y_0 \oplus Y_l^* \oplus \cdots \oplus Y_1^*) \otimes \mathscr{L}\right) | \alpha \text{ satisfies Proposition 6.2}\}|}{q^{\sum_{i=1}^t \left(r_ix_i(r_i-1+2r_{i+1}+\cdots+2r_t+r_0)\right)+\sum_{j=1}^l \left(t_jy_j(t_j-1+2t_{j+1}+\cdots+2t_l+t_0)\right)+n^2}} \leq \frac{c}{q^d}.$$

We denote the above left hand side fraction by A'. Given the value of the slopes x_i and y_j, we will find the rank r_i and t_j such that A' is as large as possible. The main idea is the following: we fix all of r_i and t_j except r_1 and r_2, and assume that the necessary conditions in Proposition 6.2 do not give any relations between r_i and t_j. Then $r_1 + r_2$ is a fixed number, so we could consider r_1 as the only variable in our fraction A'. The numerator of A' is a power of q with the power is a linear expression of r_1, and the denominator is a power of q with the power is a degree 2 polynomial of r_1. Moreover, since

$$\begin{aligned}&r_1x_1(r_1 + 2r_2 + \cdots + 2r_t + r_0) + r_2x_2(r_2 + 2r_3 + \cdots + 2r_t + r_0)\\&= x_1r_1(n - 1 - r_1) + x_2r_2^2 + ax_1 + b\\&= -(x_1 - x_2)r_1^2 + a'x_1 + b',\end{aligned}$$

where a', b' are some constants, we implies that A' will obtain the maximal value at the extreme values of r_1 which we can choose as follows:

- If $X_2 + Y_1 > d$, we can take the extreme values of r_1 to be 0 and $r_1 + r_2$. If r_1 equals to 0 or $r_1 + r_2$, we could apply the inductive hypothesis to prove our Proposition.
- If $X_1 + Y_2 > d$, then by applying a similar argument for t_1 and t_2, we may reduce the value of l by choosing the extreme values of t_1 to be 0 or $t_1 + t_2$.
- If $X_2 + Y_1 \leq d$ and $X_1 + Y_2 \leq d$, then we choose the extreme values of r_1 to be 1 and $r_1 + r_2$, the extreme values of t_1 to be 1 and $t_1 + t_2$. Thus, the induction could be applied to all of cases here except the case when $r_1 = t_1 = 1$. We leave it to the end of the proof where we will deal with the case $r_1 = t_1$.

Coming back to the general case: we will prove that if the values of x_i, y_j, for all i and j, are fixed, A' is maximal and it does not satisfy this Proposition, then $r_1 = t_1$. Firstly, we assume that $x_1 \geq y_1 > d$. Base on the necessary conditions in Proposition 6.2, we consider the following subcases:

Case 1: There exist e and f bigger than 1 such that $x_1 - y_e \leq d$, $x_1 - y_{e+1} > d$, $y_1 - x_f \leq d$, and $y_1 - x_{f+1} > d$. Then Proposition 6.2 implies that

$$t_1 + t_2 + \cdots + t_e \geq r_1 + \cdots + r_{e'},$$

and

$$r_1 + r_2 + \cdots + r_f \geq t_1 + \cdots + t_{f'},$$

where e' is the biggest number satisfying that $x_{e'} - y_{e+1} > d$, and similarly, f' is the biggest number satisfying that $y_{f'} - x_{f+1} > d$. If $e' > 1$ then by fixing everything except r_1 and r_2, we observe that A' is maximal when $r_1 = 0$ or $r_2 = 0$. By induction, A' will satisfy this Proposition. Similarly for the case $f' > 1$, hence we can assume that $e' = f' = 1$. If $r_1 + r_2 + \cdots + r_f \geq t_1 + t_2$ then by using the same argument as before, we conclude that A' is bigger if $t_1 = 0$ or $t_2 = 0$, thus A' will satisfy this Proposition. If $r_1 + r_2 + \cdots + r_f < t_1 + t_2$, then the condition $t_1 + t_2 + \cdots + t_e \geq r_1$ can be ignored. As a result, r_1 and r_2 will always go in pair in every inequalities that are implied by Proposition 6.2. So A' will satisfy this Proposition in this case.

Case 2 Without loss of generality, we assume that $e = 1$ and $f > 1$, then

$$t_1 \geq r_1 + \cdots + r_{e'}$$

$$r_1 + r_2 + \cdots + r_f \geq t_1 + \cdots + t_{f'},$$

where e', f' are defined in the same way as above. Similar to the case 1, if $e' > 1$ then A' will satisfy the Proposition.

If $f' > 1$, then A' is maximal only if $t_1 = r_1$ or $t_1 = t_1 + t_2$. So if A' is maximal and does not satisfy the Proposition then $t_1 = r_1$.

If $f' = 1$ and $r_1 + \cdots + r_f \geq t_1 + t_2$, then we will have the same conclusion as the case $f' > 1$ above.

If $f' = 1$ and $r_1 + \cdots + r_f < t_1 + t_2$, then by considering the pair (t_1, t_2), we imply that A' is maximal only if $t_1 = x_1$ or $t_1 = x_1 + \cdots x_f$. In the later case, we argue similarly as the case $e' > 1$ to conclude that A' satisfies the Proposition.

Case 3 If $e = f = 1$, we can deduce that $r_1 = t_1$ from Proposition 6.2.

By removing all parts related to X_1 and Y_1, and then apply the same argument as above, it can be seen that A' is maximal and it does not satisfy the Proposition only if $r_2 = t_2$. Continue this way we obtain that the only case we need to take care is the case $h = l$ and $r_i = t_i$ for all $0 \le i \le t$. In this case, we could also assume that $x_1 - y_1 \le d$, $x_1 - y_2 > d$, and let f is the number between 2 and h satisfying $y_1 - x_f \le d$, and $y_1 - x_{f+1} > d$ (here we denote $x_{t+1} := x_0 = 0$), then the power of q related to X_1 and Y_1 in A' can be approximated as follows:

$$\begin{aligned}
e = &\sum_{i=1}^{t} \big(h^0(X_i \otimes Y_1 \otimes \mathscr{L}) + h^0(X_1^* \otimes Y_1 \otimes \mathscr{L})\big) + h^0(X_0 \otimes Y_1 \otimes \mathscr{L}) + \\
&+ h^0(X_1 \otimes Y_0 \otimes \mathscr{L}) + \sum_{i=2}^{t} \Big(h^0(X_1 \otimes Y_i \otimes \mathscr{L}) + h^0(X_1 \otimes Y_i^* \otimes \mathscr{L})\Big) - \\
&- \sum_{i=2}^{t} \big(h^0(X_1 \otimes X_i) + h^0(X_1 \otimes X_i^*) + h^0(Y_1 \otimes Y_i) + h^0(Y_1 \otimes Y_i^*)\big) + \\
&+ \sum_{i=1}^{f} h^0(X_i \otimes Y_1^* \otimes \mathscr{L}) - h^0(\wedge^2 X_1) - h^0(\wedge^2 Y_1) - h^0(X_1 \otimes X_0) - \\
&\qquad - h^0(Y_1 \otimes Y_0) - 4(r_1 n - r_1^2) \\
\approx\ & r_1 x_1 (r_1 + 2r_2 + \cdots + 2r_t + r_0) + r_1 y_1 (r_1 + \cdots + r_f + 2r_{f+1} + \cdots + 2r_t + r_0) \\
&+ r_1(r_2 x_2 + \cdots + r_f x_f) - r_1 d(r_1 + 3r_2 + \cdots + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\qquad - r_1(x_1 + y_1)(r_1 - 1 + 2r_2 + \cdots + 2r_t + r_0) \\
=\ & r_1\big(x_1 - y_1(r_2 + \cdots + r_f - 1) + (r_2 x_2 + \cdots + r_f x_f) - d(r_1 + 3r_2 + \cdots + 3r_f \\
&\qquad + 4r_{f+1} + \cdots + 4r_t + 2r_0)\big)
\end{aligned}$$

Additionally, the condition $\Delta(H_\alpha) \neq 0$ implies that x_i and y_i need to satisfy the following conditions:

$$\begin{aligned}
|x_i - y_i| &\le d \\
|x_i - x_{i+1}| &\le 2d \\
|y_i - y_{i+1}| &\le 2d \\
x_h \le d \quad &\text{or} \quad y_h \le d.
\end{aligned}$$

We now can analyze the above power e further as follows:

If $y_1 \le x_f$, $r_i \neq 1$ for some i then

$$\begin{aligned}
e/r_1 &\approx x_1 + y_1 + \sum_{i=2}^{f} r_i(x_i - y_1) - d(r_1 + 3r_2 + \cdots + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le 2y_1 + d + \sum_{i=2}^{f} r_i d - d(r_1 + 3r_2 + \cdots + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le 2x_f + d - d(r_1 + 2r_2 + \cdots + 2r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le d(4t - 4f + 5) - d(4t - 2f + 2) \\
&\le -d
\end{aligned}$$

If $y_1 \le x_f$, $r_i = 1$ for all i then

$$\begin{aligned}
e/r_1 &\approx x_1 + y_1 + \sum_{i=2}^{f} r_i(x_i - y_1) - d(r_1 + 3r_2 + \cdots + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le 2x_f + d + \sum_{i=2}^{f-1} r_i d - d(r_1 + 3r_2 + \cdots + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le 2x_f + d - d(r_1 + 2r_2 + \cdots + 2r_{f-1} + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le d(4t - 4f + 5) - d(4t - 2f + 2) \\
&\le -d
\end{aligned}$$

If there exist $2 \le h \le f-1$ such that $x_h \ge y_1 > x_{h+1}$, then

$$\begin{aligned}
e/r_1 &\approx x_1 + y_1 + \sum_{i=2}^{f} r_i(x_i - y_1) - d(r_1 + 3r_2 + \cdots + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le 2x_h + d + d(r_2 + \cdots + r_h) - d(r_1 + 3r_2 + \cdots + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le d(4t - 4h + 5) - d(r_1 + 2r_2 + \cdots + 2r_h + 3r_{h+1} + \cdots + 3r_f + 4r_{f+1} + \cdots + \\
&\qquad + 4r_t + 2r_0) \\
&\le d(4t - 4h + 5) - d(4t - 2h + 2) \\
&< -d
\end{aligned}$$

If $y_1 > x_2$, and $f > 2$, then

$$\begin{aligned}
e/r_1 &\approx x_1 + y_1 + \sum_{i=2}^{f} r_i(x_i - y_1) - d(r_1 + 3r_2 + \cdots + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le 2x_f + 3d - d(r_1 + 3r_2 + \cdots + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le d(4t - 4f + 7) - d(4t - f) \\
&< -d
\end{aligned}$$

If $y_1 > x_2$, and $f = 2$, then

$$\begin{aligned}
e/r_1 &\approx x_1 + y_1 + r_2(x_2 - y_1) - d(r_1 + 3r_2 + \cdots + 3r_f + 4r_{f+1} + \cdots + 4r_t + 2r_0) \\
&\le x_1 + x_2 - d(4t - f + 1) \qquad \text{(since there exists } i \text{ such that } r_i \ne 1) \\
&\le d(4t - 4f + 6) - d(4t - f + 1) \\
&= -d
\end{aligned}$$

We have just finished the proof under the assumption $x_1 \ge y_1 > d$.

Now we consider the case $x_1 > d$ and $y_1 \le d$. Proposition 6.2 implies that

$$t_1 + \cdots + t_e \ge r_1 + \ldots r_{e'},$$

where e is the biggest number satisfying that $x_1 - y_e \le d$ (e exists since $x_1 - y_1 \le d$ and $x_1 > d$), and e' is the biggest number such that $x_{e'} - y_{e+1} > d$. Using exactly the same argument as in the case $y_1 > d$, we deduce that $e = e' = 1$ is the necessary condition to make the value of A' maximal. Additionally, if we assume that A' does not satisfy the Proposition, then by induction and the same argument as the first case, we deduce that $t = l$ and $r_i = t_i$ for all i. Now we will finish this case by considering the part related to X_1 and Y_1 in A': let denote f

be the biggest number such that $y_1 + x_f > d$, then the power of q related to x_1 and y_1 in A' is

$$\begin{aligned}
e = &\sum_{i=1}^{t} \big(h^0(X_i \otimes Y_1 \otimes \mathscr{L}) + h^0(X_1^* \otimes Y_1 \otimes \mathscr{L})\big) + h^0(X_0 \otimes Y_1 \otimes \mathscr{L}) + h^0(X_1 \otimes Y_0 \otimes \mathscr{L}) + \\
&+ \sum_{i=2}^{t} \big(h^0(X_1 \otimes Y_i \otimes \mathscr{L}) + h^0(X_1 \otimes Y_i^* \otimes \mathscr{L})\big) + \sum_{i=0}^{t} h^0(X_i \otimes Y_1^* \otimes \mathscr{L}) + \\
&+ \sum_{i=f+1}^{t} h^0(X_i^* \otimes Y_1^* \otimes \mathscr{L}) - h^0(\wedge^2 X_1) - h^0(\wedge^2 Y_1) - h^0(X_1 \otimes X_0) - h^0(Y_1 \otimes Y_0) - \\
&- \sum_{i=2}^{t} \big(h^0(X_1 \otimes X_i) + h^0(X_1 \otimes X_i^*) + h^0(Y_1 \otimes Y_i) + h^0(Y_1 \otimes Y_i^*)\big) - 4d(r_1 n - r_1^2) \\
\approx\ & r_1 x_1 (r_1 + 2r_2 + \cdots + 2r_t + r_0) + r_1 y_1 (r_1 + \cdots + r_f) + \\
&+ r_1 (r_2 x_2 + \cdots + r_f x_f) - r_1 d (r_1 + 3r_2 + \cdots + 3r_f + 2r_{f+1} + \cdots + 2r_t + r_0) \\
&- r_1 (x_1 + y_1)(r_1 - 1 + 2r_2 + \cdots + 2r_t + r_0) \\
=\ & r_1 \big(x_1 - y_1 (r_2 + \cdots + r_f + 2r_{f+1} + \cdots + 2r_t + r_0 - 1) + (r_2 x_2 + \cdots + r_f x_f) - \\
&\qquad - d(r_1 + 3r_2 + \cdots + 3r_f + 2r_{f+1} + \cdots + 2r_t + r_0)\big) \\
<\ & r_1 \big(2d + 2d(r_2 + \cdots + r_f) - d(r_1 + 3r_2 + \cdots + 3r_f + 2r_{f+1} + \cdots + 2r_t + r_0)\big) \\
\le\ & -d
\end{aligned}$$

By induction, we have just proved the Proposition in the case $x_1 > d$ and $y_1 \le d$.

The last case we need to consider is $x_1 < d$, $y_1 < d$, and $x_1 + y_1 > d$. In this case, the condition X becomes empty. Hence, if h is different than 1, by fixing the sum $r_1 + r_2$ and using the same argument as above, we deduce that A' satisfies the Proposition. Similar story for the case $l \neq 1$. Thus we only need to consider the case $t = l = 1$, but we have already treated this case at the beginning of this section.

The proof is completed. □

Remark 6.4. From the above discussion we can see that the case that could contribute a positive portion to the average, is the case where all of r_i and t_j equal to 1, and also the differences $x_i - x_{i+1}$ and $y_j - y_{j+1}$ are close to $2d$. We have two ideas cases as follows:

Kostant 1: $x_i = 2m - 2i + 2, y_i = 2m - 2i + 1$ for all $1 \le i \le m$;
Kostant 2: $y_i = 2m - 2i + 2, x_i = 2m - 2i + 1$ for all $1 \le i \le m$.

The reason we named them Kostant is that they reflex the role of two Kostant sections in our average. We will see those cases in the next subsection.

6.0.4. *Case 2: P is the Borel subgroup and $x_1 + y_1 > d$.* Based on the remark at the end of the previous subsection, we divide the case 2 into some subcases as follows:

Case 1: If $t = l = m$, i.e. X_i and Y_j are all line bundles, and $(4m-3)d < x_1 + x_2 < (4m-2)d$. In this case, the relating X_1, Y_1 part of A is $\dfrac{n_1}{d_1}$ where

$$\begin{aligned}
n_1 = &\prod_{i=2}^{m} \Big(|H^0(X_i \otimes Y_1 \otimes \mathscr{L})|.|H^0(X_i^* \otimes Y_1 \otimes \mathscr{L})|.|H^0(X_1 \otimes Y_i \otimes \mathscr{L})|.|H^0(X_1 \otimes Y_i^* \otimes \mathscr{L})| \Big) \\
&\times |H^0(X_1 \otimes Y_1 \otimes \mathscr{L})|.|H^0(X_0 \otimes Y_1 \otimes \mathscr{L})|.|H^0(X_1 \otimes Y_0 \otimes \mathscr{L})|.|H^0(X_1 \otimes Y_1^* \otimes \mathscr{L})| \\
&\times |H^0(X_2 \otimes Y_1^* \otimes \mathscr{L})| \\
= &|H^0(X_1^* \otimes Y_1 \otimes \mathscr{L})|.|H^0(X_2 \otimes Y_1^* \otimes \mathscr{L})|.q^{nx_1 + (n-2)y_1 + (2n-2)d + (2n-2)(1-g)}
\end{aligned}$$

and

$$\begin{aligned}
d_1 &= q^{(n^2 - (n-2)^2)d + 2(1-g) + x_1(n-2) + y_1(n-2) + (2n-4)(1-g)}.|\mathrm{Aut}(X_1)||\mathrm{Aut}(Y_1)| \\
&= (q-1)^2 q^{(4n-4)d + (2n-2)(1-g) + x_1(n-2) + y_1(n-2)}.
\end{aligned}$$

Hence, the contribution of this range to the average is bounded above by

$$\begin{aligned}
&\sum_{d/2+x_2<y_1<d+x_2} \quad \sum_{y_1\le x_1<y_1+d} \frac{n_1(X_1,Y_1)}{d_1(X_1,Y_1)} \\
=& \sum_{d/2+x_2<y_1<d+x_2} \quad \sum_{y_1\le x_1<y_1+d} \frac{|H^0(X_1^*\otimes Y_1\otimes \mathscr{L})|.|H^0(X_2\otimes Y_1^*\otimes \mathscr{L})|q^{2x_1}}{(q-1)^2q^{(2n-2)d}} \\
\le& \sum_{d/2+x_2<y_1<d+x_2} \quad \sum_{y_1\le x_1<y_1+d} \frac{T}{(q-1)^2q^{x_1+x_2-(2n-4)d}} \\
\le& \sum_{d/2+x_2<y_1<d+x_2} \frac{2T}{(q-1)^2q^{y_1+d-1+x_2-(2n-4)d}} \\
\le& \frac{2T}{(q-1)^2},
\end{aligned}$$

where T is a constant that depends only on C.

Case 2: If X_i and Y_j are all line bundles, $x_i = 2m-2i+2$, $y_i = 2m-2i$ for all $1\le i\le m$. Then $\deg(X_2\otimes Y_1^*\otimes\mathscr{L}) = 0$, and therefore it will has no non-trivial global sections if it is a non-trivial line bundle. And if $H^0(X_2\otimes Y_1^*\otimes\mathscr{L}) = 0$, we can see that $\Delta(H_\alpha) = 0$ for all $\alpha\in H^0\big((\mathscr{E}\times_G W)\otimes\mathscr{L}\big)$. Hence, we can assume that $X_2\otimes Y_1^* = \mathscr{L}^*$. On the other hand, to make sure that $\det(A_\alpha)\ne 0$, we need to have that $X_1^*\otimes Y_1 = \mathscr{L}^*$. Similarly, we will obtain the following necessary conditions: $X_i\cong\mathscr{L}^{2m-2i+2}, Y_i\cong\mathscr{L}^{2m-2i}$. We now will show that any regular sections will factor through the first Kostant section. Firstly, let recall the form of A_α for any $\alpha\in H^0((\mathscr{E}\times^G W)\otimes\mathscr{L})$ satisfying $\det(A_\alpha)\ne 0$ and $\Delta(H_\alpha)\ne 0$:

$$A_\alpha = \begin{pmatrix}
* & \cdots & * & * & * & * & \cdots & * \\
* & \cdots & * & * & * & * & \cdots & c_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots \\
* & \cdots & * & * & * & c_{m-1} & \cdots & 0 \\
* & \cdots & * & * & c_m & 0 & \cdots & 0 \\
* & \cdots & c'_m & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c'_1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix} \tag{6.5}$$

where $c_i$and c'_i are non-zero constants. Notice that the action of G on W is

$$\begin{pmatrix} B & 0_n \\ 0_n & C\end{pmatrix}\cdot\begin{pmatrix} 0_n & A_\alpha \\ -A_\alpha^* & 0_n\end{pmatrix}\cdot\begin{pmatrix} B^* & 0_n \\ 0_n & C^*\end{pmatrix} = \begin{pmatrix} 0_n & BA_\alpha C^* \\ -CA_\alpha^*B^* & 0_n\end{pmatrix}.$$

Since c_i and c'_i are non-zero, we can make them to be 1 as follows: take B to be a diagonal matrix with the diagonal entries $b_{ii} =$, then we will have

$$A'_\alpha = \begin{pmatrix} \prod_{i=1}^m (c_i c'_i) &&&&&& \\ & \ddots &&&&& \\ && c'_m c_m &&&& \\ &&& 1 &&& \\ &&&& (c'_m c_m)^{-1} && \\ &&&&& \ddots & \\ &&&&&& \left(\prod_{i=1}^m (c_i c'_i)\right)^{-1} \end{pmatrix} .A_\alpha$$

$$= \begin{pmatrix} * & \cdots & * & * & * & * & \cdots & * \\ * & \cdots & * & * & * & * & \cdots & b_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots \\ * & \cdots & * & * & * & b_{m-1} & \cdots & 0 \\ * & \cdots & * & * & b_m & 0 & \cdots & 0 \\ * & \cdots & b_m & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_1^{-1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

After that we multiply the right hand side of A_α with

$$C = diag(b_1, b_2, \cdots, b_m, 1, b_m^{-1}, \cdots, b_1^{-1}) \in \mathrm{SO}(V_2),$$

then the resulting matrix will have the property we mentioned before. Now we can assume that in our matrix A_α, the entries c_i and c'_i are all equal to 1. We continue to multiply the left and the right of A_α by some orthogonal matrices to transform it into Kostant form as follows:

Step 1: We firstly transfer the entries $a_{1,i}$ for $m+2 \leq i \leq n$ into zero by multiplying on the left of A_α by the following special orthogonal matrix (upper triangular matrix)

$$B = \begin{pmatrix} 1 & -a_{1,n} & \cdots & -a_{1,m+3} & -a_{1,m+2} & 0 & \cdots & 0 & -a_{1,m+2}^2/2 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & a_{1,m+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & a_{1,n} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{6.6}$$

Step 2: Transfer the entries $a_{j,1}$, where $m+2 \leq j \leq n-1$, into zero. In this step we need to multiply the right hand side of A_α by the following lower triangular matrix:

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ -a_{n-1,1} & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{m+2,1} & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_{m+2,1} & \cdots & a_{n-1,1} & 1 \end{pmatrix}. \tag{6.7}$$

Step 3: If $n = 3$ then we can skip this step and go directly to the step 4. Hence, by using an inductive argument, we can assume that our statement is true for n small. More precisely, if we consider the submatrix A'_α obtained by removing the first column, the first row, the last row, and the last column, then we can transfer it into the Kostant form by multiplying (canonically) on the left by some (upper triangular) special orthogonal matrices, and on the right by some (lower triangular) special orthogonal matrices. From this induction, by multiplying A_α by

$$B = \begin{pmatrix} 1 & & \\ & B' & \\ & & 1 \end{pmatrix} \text{ on the left, and}$$

$$C = \begin{pmatrix} 1 & & \\ & C' & \\ & & 1 \end{pmatrix} \text{ on the right,}$$

where B' and C' are appropriate upper and lower triangular matrices, respectively, A_α will have the following form:

$$A_\alpha = \begin{pmatrix} * & * & \cdots & * & * & 0 & 0 & \cdots & 0 & 0 \\ * & * & \cdots & * & * & 0 & 0 & \cdots & 0 & 1 \\ * & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ * & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ * & * & \cdots & * & * & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{6.8}$$

We need to emphasise the upper and lower triangular properties here because they help us to keep the entries of A_α considered in the previous steps to be equal to zero.

Step 4: Multiply on the right of A_α by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{2,m+1} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2,2} & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ -a_{2,m+1}^2/2 & -a_{2,2} & \cdots & -a_{2,m} & -a_{2,m+1} & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{6.9}$$

will make the entries $a_{2,i}$ for $2 \le i \le m+1$ to be zero.

Step 5: Finally, multiply on the left of A_α by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & a_{m,1} & \cdots & a_{2,1} & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & -a_{2,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & -a_{m,1} \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{6.10}$$

then the entries $a_{j,1}$ for $2 \le j \le m$ will be zero. Thus, the matrix A_α is belong to the first Kostant.

Case 3: The same condition as in the previous case except that we switch from X_i to Y_i, and vice versa. Then we can prove that any regular sections of $(\mathscr{E}\times_G W)\otimes\mathscr{L}$ will factor through the second Kostant section. Hence, the contribution of this case to the average is 1.

6.0.5. *Case 3:* $x_1+y_1\leq d$. We are going to see that the contribution of this case to the average is equal to the Tamagawa number of G. The main reason is that all of the entries in the matrix filtration (6.4) have non-trivial global sections. Firstly, we may ignore a small case as follows:

Case 1: If $d-2g+2\leq x_1+y_1\leq d$. By Proposition **??** we are able to bound the dimension of $H^0((\mathscr{E}\times^G V)\otimes\mathscr{L})$ as follows:

$$\begin{aligned}
&h^0((\mathscr{E}\times^G V)\otimes\mathscr{L})\\
\leq&\sum_{i=1}^{t}\sum_{j=1}^{l}\big(h^0(X_i\otimes Y_j\otimes\mathscr{L})+h^0(X_i^*\otimes Y_j\otimes\mathscr{L})+h^0(X_i\otimes Y_j^*\otimes\mathscr{L})\\
&+h^0(X_i^*\otimes Y_j^*\otimes\mathscr{L})\big)+\sum_{j=1}^{l}\big(h^0(X_0\otimes Y_i\otimes\mathscr{L}+h^0(X_0\otimes Y_i^*\otimes\mathscr{L})\big)\\
+&\sum_{i=1}^{t}\big(h^0(X_i\otimes Y_0\otimes\mathscr{L})+h^0(X_i^*\otimes Y_0\otimes\mathscr{L})\big)+h^0(X_0\otimes Y_0\otimes\mathscr{L})\\
\leq&\, n^2+d.n^2.
\end{aligned}$$

Furthermore, if we fix the rank r_1 of the semi-stable vector bundle X_1, there exists a constant A_1 such that for any integer d_1 we have that $|\mathrm{Bun}_{r_1,d_1}^{semi-stable}(\mathbb{F}_q)|\leq A_1$. In fact, set $d_1=a.r_1+d_2$ for some $0\leq d_2<r_1$, then $|\mathrm{Bun}_{r_1,d_1}^{semi-stable}(\mathbb{F}_q)|=|\mathrm{Bun}_{r_1,d_2}^{semi-stable}(\mathbb{F}_q)|$ because of the assumption that our curve C has an $\mathbb{F}_q-$ rational point. Notice that $|\mathrm{Bun}_{r_1,d_2}^{semi-stable}(\mathbb{F}_q)|$ is finite for any d_2, hence we can choose A_1 to be the maximal number among $|\mathrm{Bun}_{r_1,d_2}^{semi-stable}(\mathbb{F}_q)|$ for $0\leq d_2<r_1$. We also can choose the common bound A_1 for $|\mathrm{Bun}_{r_1,d_1}^{semi-stable}(\mathbb{F}_q)|$ when r_1 varies in the period $[1,m]$.

Now we fix a parabolic subgroup P of G and denote the set of $G-$bundles whose canonical reductions are reductions to P by Bun^P. Then the contribution of Bun^P to the average in this case is:

$$\begin{aligned}
&\sum_{d-2g+2\leq x_1+y_1\leq d}\ \sum_{\mathscr{E}\in\mathrm{Bun}^P_{x_i,y_j}}\frac{|H^0\big((\mathscr{E}\times^G V)\otimes\mathscr{L}\big)|}{|\mathscr{A}_{\mathscr{L}}(\mathbb{F}_q)|}d\mathscr{E}\\
\leq\ & c.\sum_{d-2g+2\leq x_1+y_1\leq d}\frac{A_1.q^{n^2+d.n^2}}{q^{r_1x_1(r_1-1+\cdots+2r_t+r_0)+t_1y_1(t_1-1+\cdots+2t_l+t_0)}.q^{n^2d+n(1-g)}}\\
&\qquad(\text{c depends only on } P,n \text{ and the genus } g)\\
\leq\ & b.\sum_{d-2g+2\leq x_1+y_1\leq d}\frac{1}{q^{r_1x_1+t_1y_1}}\\
\leq\ & b.\sum_{t=d-2g+2}^{d}\frac{t-1}{q^t}.\\
&\qquad(\text{b depends only on } P,n \text{ and } g)
\end{aligned}$$

By taking limit $d\to\infty$, the above upper bound implies that the contribution of this case to the average equals zero.

Case 2: The last case we need consider is $x_1+y_1<d-2g+2$, i.e. the slope of any consecutive semistable quotient in the "filtration" 6.4 of the vector bundle $(\mathscr{E}\times^G V)\otimes\mathscr{L}$ is

strictly bigger than $2g-2$. Consequently, we obtain the following equality:

$$\begin{aligned}
&h^0((\mathscr{E}\times^G V)\otimes\mathscr{L})\\
&=\textstyle\sum_{i=1}^t\sum_{j=1}^l \big(h^0(X_i\otimes Y_j\otimes\mathscr{L})+h^0(X_i^*\otimes Y_j\otimes\mathscr{L})+h^0(X_i\otimes Y_j^*\otimes\mathscr{L})\\
&\quad+h^0(X_i^*\otimes Y_j^*\otimes\mathscr{L})\big)+\textstyle\sum_{j=1}^l\big(h^0(X_0\otimes Y_i\otimes\mathscr{L}+h^0(X_0\otimes Y_i^*\otimes\mathscr{L})\big)+\\
&\quad+\textstyle\sum_{i=1}^t\big(h^0(X_i\otimes Y_0\otimes\mathscr{L})+h^0(X_i^*\otimes Y_0\otimes\mathscr{L})\big)+h^0(X_0\otimes Y_0\otimes\mathscr{L})\\
&=n^2(1-g)+d.n^2.
\end{aligned}$$

Notice that in case the $G-$bundle $\mathscr{E}$ is semistable, we also have the above equality because $\deg(\mathscr{E}\times^G V)=0$. Since the Tamagawa number of G is 4, by considering the counting measure weighted by the size of automorphism groups on $\mathrm{Bun}_G(\mathbb{F}_q)$ and using a similar argument in the proof of equality (**??**) in section **??**, we have that

$$\begin{aligned}
|\mathrm{Bun}_G(\mathbb{F}_q)| &:= \sum_{\mathscr{E}\in\mathrm{Bun}_G(\mathbb{F}_q)}\frac{1}{|\mathrm{Aut}_{\mathrm{Bun}_G(\mathbb{F}_q)}(\mathscr{E})|}=4.q^{(4m^2+2m)(g-1)}.\prod_{x\in|C|}\frac{|\kappa(x)|^{\dim(G)}}{|G(\kappa(x))|}\\
&= 4.q^{(4m^2+2m)(g-1)}.\zeta_C(2).\zeta_C(4)\dots\zeta_C(2m). \qquad (6.11)
\end{aligned}$$

Denote $\mathrm{Bun}_G^{<d-2g+2}(\mathbb{F}_q)$ to be the set of $G-$bundles whose associated datum

$$\{x_i,y_j,r_i,t_j\}_{1\le i\le t+1;1\le j\le l+1}$$

satisfy the condition that $x_1+y_1<d-2g+2$. Then the contribution of the current case to the average is:

$$\begin{aligned}
&\lim_{d\to\infty}\int\limits_{\mathrm{Bun}_G^{<d-2g+2}(\mathbb{F}_q)}\frac{|\mathscr{M}_{L,\mathscr{E}}(k)|}{|\mathscr{A}_L(k)|}d\mathscr{E}\\
=\ &\lim_{d\to\infty}\frac{1}{q^{n^2d+n(1-g)}}\int\limits_{\mathrm{Bun}_G^{<d-2g+2}(\mathbb{F}_q)}|H^0(C,V(\mathscr{E},\mathscr{L})^{\mathrm{reg}}|d\mathscr{E}\\
=\ &\lim_{d\to\infty}\frac{|H^0(C,V(\mathscr{E},\mathscr{L}))|}{q^{n^2d+n(1-g)}}\int\limits_{\mathrm{Bun}_G^{<d-2g+2}(\mathbb{F}_q)}\frac{|H^0(C,V(\mathscr{E},\mathscr{L})^{\mathrm{reg}}|}{|H^0(C,V(\mathscr{E},\mathscr{L}))|}d\mathscr{E}\\
\le\ &\frac{q^{n^2d+n^2(1-g)}\int\limits_{\mathrm{Bun}_G(\mathbb{F}_q)}\prod_{i=1}^m\zeta_C(2i)^{-2}.\prod_{v\in|C|}\big(1+\sum_{j=1}^{2m-1}c_j|k(v)|^{-(2m+1-j)}\big)d\mathscr{E}}{q^{n^2d+n(1-g)}}\\
&\text{(By Proposition 5.5)}\\
=\ &q^{(4m^2+2m)(1-g)}.\prod_{v\in|C|}\big(1+\sum_{j=1}^{2m-1}c_j|k(v)|^{-(2m+1-j)}\big)|\mathrm{Bun}_G(\mathbb{F}_q)|.\prod_{i=1}^m\zeta_C(2i)^{-2}\\
=\ &4.\prod_{v\in|C|}\big(1+c_{2m-1}|k(v)|^{-2}+\dots+c_1|k(v)|^{-2m}\big),
\end{aligned}$$

where c_i, for $1\le i\le 2m-1$, is the constant defined in Proposition 5.5.

6.0.6. *The transversal case.* In this subsection, we consider a special family of hyperelliptic curves, the transversal family. The purpose of this subsection is to show that in the transversal case, we can exclude the case $(4m-3)d<x_1+x_2<(4m-2)d$ in Subsection 6.0.4 above. Then, the average in this case will not contain the rational function of q. In fact, it is just a consequence of Lemma **??** as we can see now:

Proposition 6.5. *Assume that $\mathscr{E}$ is a $G-$bundle satisfying the conditions in the case 6 above, i.e. X_i and Y_j are line bundles for all i,j, and also $(4m-3)d < x_1+x_2 < (4m-2)d$. Suppose that s is a global section of $(\mathscr{E} \times^G V) \otimes \mathscr{L}$. Then, for d sufficiently large, the discriminant section $\Delta(s)$ is not square-free.*

Proof. By definition, if s is of the form $\begin{pmatrix} 0_n & A \\ -A^* & 0_n \end{pmatrix}$, then $\Delta(s)$ is the discriminant of $A.A^*$. It is easy to see that $A.A^*$ is of the following matrix form:

$$\begin{pmatrix} * & * & \cdots & * & * & * \\ x_1 & * & \cdots & * & * & * \\ 0 & x_2 & \cdots & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x_2 & * & * \\ 0 & 0 & \cdots & 0 & x_1 & * \end{pmatrix},$$

where $x_i \in H^0(C, (X_i^* \otimes X_{i+1}) \otimes \mathscr{L}^{\otimes 2})$ and here $X_{n+1} := X_0$. The necessary conditions of $\det(s) \neq 0$, combining with the hypothesis, imply that $d < x_i - x_{i+1} \leq 2d$ for all i, and there is at least one index i such that $d < x_i - x_{i+1} < 2d$. Thus, Lemma **??** will help us to finish the proof. ☐

Now we will consider the transversal version of the case $x_1+y_1 < d$ in Subsection 6.0.5. The only difference here will come from the density of the regular locus as we can see as follows:

$$\begin{aligned}
&\lim_{d\to\infty} \frac{\int_{\mathrm{Bun}_G^{<d-2g+2}(k)} |\mathscr{M}^{\mathrm{trans}}_{\mathscr{L},\mathscr{E}}(k)| d\mathscr{E}}{|\mathscr{A}^{\mathrm{trans}}_{\mathscr{L}}(k)|} = \lim_{d\to\infty} \frac{\int_{\mathrm{Bun}_G^{<d-2g+2}(k)} |H^0(C, V^{\mathrm{reg}}(\mathscr{E},\mathscr{L}))^{sf}| d\mathscr{E}}{\frac{|\mathscr{A}^{\mathrm{trans}}_{\mathscr{L}}(k)|}{|\mathscr{A}_{\mathscr{L}}(k)|}.|\mathscr{A}_{\mathscr{L}}(k)|} \\
= &\lim_{d\to\infty} \frac{|H^0(C,V(\mathscr{E},\mathscr{L}))|}{q^{n^2d+n(1-g)}}.\frac{|\mathscr{A}_{\mathscr{L}}(k)|}{|\mathscr{A}^{\mathrm{trans}}_{\mathscr{L}}(k)|}.\int_{\mathrm{Bun}_G^{<d-2g+2}(k)} \frac{|H^0(C, V^{\mathrm{reg}}(\mathscr{E},\mathscr{L}))^{sf}}{|H^0(C,V(\mathscr{E},\mathscr{L}))|} |d\mathscr{E} \\
= &\lim_{d\to\infty} q^{(n^2-n)(1-g)}.\int_{\mathrm{Bun}_G^{<d-2g+2}(k)} \frac{\frac{|H^0(C, V^{\mathrm{reg}}(\mathscr{E},\mathscr{L}))^{sf}|}{|H^0(C,V(\mathscr{E},\mathscr{L}))|}}{\frac{|\mathscr{A}^{\mathrm{trans}}_{\mathscr{L}}(k)|}{|\mathscr{A}_{\mathscr{L}}(k)|}} d\mathscr{E} \\
= &\; q^{(n^2-n)(1-g)} \int_{\mathrm{Bun}_G(\mathbb{F}_q)} \zeta_C(2)^{-2}\ldots\zeta_C(2m)^{-2} d\mathscr{E} \qquad \text{(By Proposition 5.4)} \\
= &\; q^{(4m^2+2m)(1-g)}\zeta_C(2)^{-2}\ldots\zeta_C(2m)^{-2}.|\mathrm{Bun}_G(\mathbb{F}_q)| \\
= &\; 4 \qquad \text{(By 6.11)}.
\end{aligned}$$

7. Proof of main theorems 2.5 and 2.6

Firstly, it is easy to see that when $\deg(\mathscr{L})$ goes to infinity, the volume of the set of elements $\underline{a} = (a_1, \ldots, a_{2m}, 0) \in \mathscr{A}_{\mathscr{L}}(k)$ goes to zero. By [6] Lemma 4.1, we can also exclude all of elements $\underline{a}$ with zero discriminant.

In the transversal case, Theorem 2.6 is a direct consequence of the previous section and the above observations. Moreover, similar to section 2, we also obtain the lower bound for the average size of 2-Selmer group as follows:

Corollary 7.1.

$$\liminf_{deg(\mathscr{L})\to\infty} \frac{\sum\limits_{\substack{\alpha\in[S/\mathbb{G}_m](C)\\ \mathscr{L}(H_\alpha)\cong\mathscr{L}}} |Sel_2(E_\alpha)|}{\sum\limits_{\substack{\alpha\in[S/\mathbb{G}_m](C)\\ \mathscr{L}(H_\alpha)\cong\mathscr{L}}} 1} \geq 6\prod_{v\in|C|}(1-\alpha_v),$$

where $\alpha_v = \dfrac{|\{x\in S(\mathscr{O}_{K_v}/(\varpi_v^2))|\Delta(x)\equiv 0 \ mod(\varpi_v^2)\}|}{|k(v)^{2n}|}$.

Proof. See Corollary **??**. □

Theorem 2.5 for the general case also can be proved by the same manner as above. Notice that in this theorem we need to put an extra condition on q, that is $q > 16^{\frac{m^2(2m+1)}{2m-1}}$, because we only have the inequalities between $|\mathrm{Sel}_2(J)|$ and $|H^1(C,\mathcal{J}[2])|$ as in Proposition 2.3 but not the equality as in the transversal case. Finally, to obtain the average size of 2-Selmer groups of hyperelliptic curves, we need to take care of the minimal locus. By looking at the counting section, we can see that if we restrict to the minimal locus, then we only have some changes as follows: in the case 1 of Subsection 6.0.4, the fractional function of q will has an extra factor $\zeta_C((2m+1)^2)$. And in the case $x_1+y_1<d-2g+2$, we use Proposition 5.7 instead of 5.5.

To sum up, we have just proved the following theorem:

Theorem 7.2. *Suppose that* $q > 16^{\frac{m^2(2m+1)}{2m-1}}$. *Then we have that*

$$\limsup_{deg(\mathscr{L})\to\infty} \frac{\sum\limits_{\substack{H \text{ is hyperelliptic}\\ \mathscr{L}(H)\cong\mathscr{L}}} \frac{|Sel_2(H)|}{|Aut(H)|}}{\sum\limits_{\substack{H \text{ is hyperelliptic}\\ \mathscr{L}(H)\cong\mathscr{L}}} \frac{1}{|Aut(H)|}}$$

$$\leq 4.\prod_{v\in|C|}\left(1+c_{2m-1}|k(v)|^{-2}+\cdots+c_1|k(v)|^{-2m}-2|k(v)|^{(2m+1)^2}\right)$$

$$+2+f(q),$$

where $f(q)$ *is a rational function of* q *satisfying that* $\lim_{q\to\infty} f(q)=0$, *and* c_i, *for* $1\leq i\leq 2m-1$, *are constants which depend only on* m.

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