PROLONGATION OF REGULAR SINGULAR CONNECTIONS ON PUNCTURED AFFINE LINE OVER A HENSELIAN RING

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Abstract. For a strict henselian local ring R of equal characteristic 0, we establish an equivalence between the categories of regular singular connections on the formal punctured disk over R and on the puncture affine line over R. This generalizes the well-known equivalence established be Deligne when R is a field.

1. Introduction

Let C be an algebraically closed field of characteristic 0 and x be a variable. The formal punctured disk is the spectrum $\operatorname{Spec} C(\!(x)\!)$. It is equipped with the logarithmic derivation $\vartheta := x\dfrac{d}{dx}$. In [Del87, Proposition 13.35], Deligne establishes an interesting equivalence between regular singular connections on the formal punctured disk and the punctured affine line $\mathbb{P}^1_C \smallsetminus \{0, \infty\}$. This equivalence is seen by Deligne as a prolongation of a regular singular connection on the formal punctured disk to the affine line in oder to define the tangential fiber functor.

Deligne's equivalence was also considered by Katz in a more general settings [Kat87]. The analogues in characteristic p was essentially established by Gieseker in [Gie75] and treated in more detail by Kindler in [Kin15]. There is also generalization to the p-adic settings by Matsuda [Mat02], see also [And02].

Deligne's equivalence has been established in [HdST22, Theorem 10.1] for the case that the base field C is replaced by a complete local ring R. Their main idea is based on the completeness of R to show that two mentioned categories are equivalent to two other categories of limits. More precisely, they first identify each connection (morphism between connections) with a projective limit of connections (morphisms) over C, here limit is over Artin rings $R_k := R/\mathfrak{r}^k R$, where $\mathfrak{r}$ is the maximal ideal and $k \in \mathbb{N}$. In fact [HdST22] shows more, namely that the two categories of interest are equivalent to a third category, the category of linear representations of the additive group $\mathbb{Z}$ in finite R–modules. Although the completeness assumption seems crucial for the last equivalence, it is expected that for the original equivalence of Deligne one might require a milder assumption than completeness.

In this manuscript, we will deal with the case R is a noetherian henselian local C–algebra. Our main observation is the following theorem which is Theorem 4.1 and Corollary 4.2 in this text:

Date: July 31, 2023.

Key words and phrases. Regular-singular connection, henselian discrete valuation ring, deligne's equivalence (MSC 2020: 12H05, 13N15, 18M05).

The research of Phùng Hô Hai and Phạm Thanh Tâm is funded by the International Center for Research and Postgraduate Training in Mathematics (Institute of Mathematics, VAST, Vietnam) under grant number `ICRTM01_2020.06` and funded by Vingroup Joint Stock Company and supported by Vingroup Innovation Foundation (VinIF) under the project code VINIF.2021.DA00030.

The research of Đào Văn Thịnh was supported by Postdoctoral program of Vietnam Institute for Advanced study in Mathematics.

Theorem 1.1. *The restriction functor*

$$\mathbf{r} : \mathbf{MC}_{\mathrm{rs}}^{\mathrm{free}}(R[x^{\pm}]/R) \longrightarrow \mathbf{MC}_{\mathrm{rs}}^{\mathrm{free}}(R(\!(x)\!)/R)$$

is an equivalence. If R is moreover a discrete valuation C–algebra then we have an equivalence of the full categories

$$\mathbf{r} : \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R) \longrightarrow \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R).$$

Here $\mathbf{MC}_{\mathrm{rs}}(*/R)$ denotes the category of regular-singular connections on $*$ relatively over R, and $\mathbf{MC}_{\mathrm{rs}}^{\mathrm{free}}(*/R)$ denotes the full subcategory of objects with underlying modules being flat modules over $*$.

Our approach is based on the results of [HdST22], i.e., we first base change to the completion of R and then try to descend. The descend on the side of $R(\!(x)\!)$ is done by using the faithful flatness of the map $R[\![x]\!] \longrightarrow \widehat{R}[\![x]\!]$, in particular we show that any R-flat connection on $R(\!(x)\!)/R$ is in fact free as an $R(\!(x)\!)$-module, Proposition 2.9. The descend on the side of $R[x^{\pm 1}]$ is carried out using Popescu's theorem on presentation $\widehat{R}$ as limit of smooth R–algebras.

Since the equivalence with $\mathbf{Repr}_R(\mathbb{Z})$ is not available in our settings, we need a replacement, which is the category $\mathbf{End}_R$ of finite R-modules with endomorphism. The latter category can be seen as a Lie-algebra counterpart to $\mathbf{Repr}_R(\mathbb{Z})$. Proposition 2.12 shows that the Euler functor

$$\gamma\mathrm{eul}_{R[\![x]\!]} : \mathbf{End}_R^{\mathrm{free}} \longrightarrow \mathbf{MC}_{\mathrm{rs}}^{\mathrm{free}}(R(\!(x)\!)/R)$$

is faithful and essentially surjective. The similar statement for $R[x^{\pm 1}]$ is given in Proposition 3.5.

The paper is organized as follows. Section 2 is devoted to the category of regular singular connection on the formal relative punctured disk $\mathrm{Spec}R(\!(x)\!)$. After reviewing basic definition of relative connections with regular singularities we show the existence of We show that each $R(\!(x)\!)$–free connection admits an Euler form 2.11.

Section 3 is devoted to the category of regular singular connections on the punctured relative affine line. We show that a connection admits an Euler form, Proposition 3.5. The results obtained in these two sections are then used to proved the main theorem in Section 4.

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1.1. Notation and conventions.

- C is a fixed an algebraically closed field of characteristic 0.
- R is given noetherian henselian local C-algebra with maximal ideal $\mathfrak{r}$ and residue field isomorphic to C.
- $R(\!(x)\!)$ denotes the ring of formal Laurent series with coeffients from R and $R[\![x]\!]$ be the ring of formal power series. We have $R(\!(x)\!) = R[\![x]\!][x^{-1}]$.
- ϑ denotes the logarithmic derivative on $R(\!(x)\!)$:

$$\vartheta \sum a_n x^n = x\frac{d}{dx} \sum a_n x^n = \sum n a_n x^n.$$

- $\widehat{R}$ denotes the completion of R along the maximal ideal $\mathfrak{r}$.
- Sp_φ denotes the spectrum of the endomorphism $\varphi : V \to V$ of vector space over C.
- τ denotes fixed a subset τ of C such that the natural map $\tau \to C/\mathbb{Z}$ is bijective.
- $\mathbf{End}_R$ denotes the category of couples (V, A), consisting of a finite R-module V and a R-linear endomorphism $A : V \to V$, and *arrows* from (V, A) to (V', A') are R-linear morphisms $\varphi : V \to V'$ such that $A'\varphi = \varphi A$.

- $\mathbf{End}_R^{\mathrm{free}}$ denotes the full subcategory of $\mathbf{End}_R$ whose objects are free $R-$modules.

sect.3

2. Regular singular connections on a punctured formal disk

The punctured formal disk over a ring R is defined to be $\mathrm{Spec}(R(\!(x)\!))$. The aim of this section is to describe those connections which are flat over R (where R is a henselian local ring). In fact, it is known that, for a connection, being flat over R is equivalent to being flat over $R(\!(x)\!)$ (cf. [HdST22, Theorem 8.18]). Our first main result, Proposition 2.9, shows that an R-flat connection is in fact free over $R(\!(x)\!)$. This is done following the idea of Drinfeld's theory of infite vector bundles [Dr06]. The second main result is to show that such an $R(\!(x)\!)$-free connection has the Euler form, see Proposition 2.12.

2.1. **Definitions and properties.** Let C be an algebraically closed field of characteristic zero. We review in this subsection the definitions and main properties of regular singular connections on a relative formal punctured disk, that is $\mathrm{Spec}(R(\!(x)\!))$, where $(R, \mathfrak{r})$ is a noetherian henselian local C-algebra of residue field C. Our reference is [HdST22, Section 8].

fn.20220607-04 **Definition 2.1** (Connections on the punctured formal disk)**.** The *category of connections* on the punctured formal disk over R or on $R(\!(x)\!)/R$, denoted $\mathbf{MC}(R(\!(x)\!)/R)$, has for

objects those couples (M, ∇) consisting of a finite $R(\!(x)\!)$-module M and a R-linear endomorphism $\nabla : M \to M$, called *the derivation*, satisfying Leibniz's rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$, and the

arrows from (M, ∇) to (M', ∇') are $R(\!(x)\!)$-linear morphisms $\varphi : M \to M'$ such that $\nabla'\varphi = \varphi\nabla$.

The R–flat connections on $R(\!(x)\!)/R$ enjoy the following remarkable property.

Definition 2.2 (Logarithmic connections)**.** The category of *logarithmic connections*, denoted $\mathbf{MC}_{\log}(R[\![x]\!]/R)$, has for

objects those couples $(\mathcal{M}, \nabla)$ consisting of a finite $R[\![x]\!]$-module and a R-linear endomorphism, called *the derivation*, $\nabla : \mathcal{M} \to \mathcal{M}$ satisfying Leibniz's rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$, and

arrows from $(\mathcal{M}, \nabla)$ to $(\mathcal{M}', \nabla')$ are $R[\![x]\!]$-linear morphisms $\varphi : \mathcal{M} \to \mathcal{M}'$ such that $\nabla'\varphi = \varphi\nabla$.

The two categories $\mathbf{MC}(R(\!(x)\!)/R)$ and $\mathbf{MC}_{\log}(R[\![x]\!]/R)$ are abelian categories and there is an evident R-linear functor

$$\gamma : \mathbf{MC}_{\log}(R[\![x]\!]/R) \longrightarrow \mathbf{MC}(R(\!(x)\!)/R).$$

fn.20220607-05 **Definition 2.3** (Regular singular connection)**.**

(1) An object $M \in \mathbf{MC}(R(\!(x)\!)/R)$ is said to be *regular-singular* if it is isomorphic to a certain $\gamma(\mathcal{M})$ for some $\mathcal{M} \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$. The full subcategory of regular-singular connections will be denoted by $\mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$.

(2) Given $M \in \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$, any object $\mathcal{M} \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$ such that $\gamma(\mathcal{M}) \simeq M$ is called a logarithmic model of M. In case the model $\mathcal{M}$ is $R[\![x]\!]$*–free*, it is called a *logarithmic lattice of* M.

ex.20220607-01 **Example 2.4** (Euler connections)**.** Let $(V, A) \in \mathbf{End}_R$ be given. The logarithmic connection associated to the couple (V, A) is defined by the couple $(R[\![x]\!] \otimes_R V, D_A)$, where

$$D_A(f \otimes v) = \vartheta(f) \otimes v + f \otimes Av.$$

This logarithmic connection is called an Euler connection associated to (V, A). Notation: $\mathrm{eul}(V, A)$.

The Euler connections yield a functor, denoted eul :

$$\mathrm{eul} : \mathbf{End}_R \longrightarrow \mathbf{MC}_{\log}(R[\![x]\!]/R).$$

It is straightforward to check that this is an R-linear, exact and faithful tensor functor. Combining eul with γ we have a functor

$$\gamma\mathrm{eul} : \mathbf{End}_R \longrightarrow \mathbf{MC}_{\mathrm{rs}}(R[\![x]\!]/R).$$

The ultimate aim of this section is to show that this functor yields an equivalence when restricted to objects with *exponents* lying in a fixed $\tau \subset C$ (Proposition 2.12). A crucial role plays the exponents.

For a regular singular (M, ∇), we consider a model $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$. The Leibniz rule implies that $\nabla(x\mathcal{M}) \subset x\mathcal{M}$. Hence, we obtain an R-linear endormorphism

16.05.2020-2 (1)
$$\mathrm{res}_\nabla : \mathcal{M}/(x) \longrightarrow \mathcal{M}/(x),$$

given by

16.05.2020-3 (2)
$$\mathrm{res}_\nabla(m + (x)) = \nabla(m) + (x).$$

Further, taking residue modulo $\mathfrak{r}$ we have the map

14.07.2022-1 (3)
$$\overline{\mathrm{res}}_\nabla : \mathcal{M}/(\mathfrak{r}, x) \longrightarrow \mathcal{M}/(\mathfrak{r}, x).$$

28.06.2021-6 **Definition 2.5** (Residue and exponents)**.** Let (M, ∇) be a regular singular connection.

(1) The R-linear map (1) is called the *residue* of ∇.
(2) The eigenvalues of $\overline{\mathrm{res}}_\nabla$ are called the (reduced) *exponents* of ∇. The set of exponents will be denoted by $\mathrm{Exp}(\mathcal{M}, \nabla)$, $\mathrm{Exp}(\nabla)$ or $\mathrm{Exp}(\mathcal{M})$ if no confusion may appear.
(3) If the exponents of Δ do not differ from each other by non-zero integers, the model is called Deligne-Manin.

We are going to show that any regular-singular connection over R has a Deligne-Manin logarithmic model. We first need a lemma:

lem.202307-01 **Lemma 2.6.** *Let* $(V, \varphi) \in \mathbf{End}_R$ *and* $\{\varrho_1, \ldots, \varrho_r\}$ *be the spectrum of* $\overline{\varphi}$ *the reduction modulo* $\mathfrak{r}$ *of* φ. *Let*

$$\overline{V} = \bigoplus_{i=1}^{r} \ker\left(\overline{\varphi} - \varrho_i\right)^{r_i}$$

be the decomposition of V *into the direct sum of generalized eigenspaces of* $\overline{\varphi}$. *Then there exists a direct sum*

$$V = V_1 \oplus \ldots \oplus V_r,$$

where V_i *is a* $\varphi-$*invariant* $R-$*submodules of* V, *such that its reduction modulo* $\mathfrak{r}$ *is* $\ker\left(\overline{\varphi} - \varrho_i\right)^{r_i}$ *for each* $1 \le i \le r$.

Proof. Let $R^n \longrightarrow M \longrightarrow 0$ be the projective cover of M that induces an isomorphism $C^n \longrightarrow M/\mathfrak{r}$. Then φ lifts to $\tilde{\varphi} : R^n \longrightarrow R^n$ and the residue of the characteristic polynomial of $\tilde{\varphi}$ is equal to the characteristic of $\overline{\varphi}$:

$$\overline{P_{\tilde{\varphi}}(T)} = P_{\overline{\varphi}}(T).$$

As R is a hensenlian ring, the factorization

$$P_{\overline{\varphi}}(T) = \prod_{i=1}^{r} \left(\overline{\varphi} - \varrho_i\right)^{r_i} = 0$$

lifts to a factorization

$$P_{\tilde{\varphi}}(T) = \prod_{i=1}^{r} g_i(T),$$

so that the ideal generated by by $\{g_i(T)\}_{1 \le i \le r}$ is $R[T]$, cf [Lei11, Proposition 2.8.3].

Let $V_i = \ker g_i(\varphi)$ for $i = 1, 2, \ldots, r$, then V_i is an $\varphi-$invariant R-submodule of V. As $P_{\overline{\varphi}}(\varphi) = 0$ on M, we conclude that

$$V = V_1 \oplus \ldots \oplus V_r.$$

Taking reduction modulo $\mathfrak{r}$ we have $V_i/\mathfrak{r} = \ker\left(\overline{\varphi} - \varrho_i\right)^{r_i}$. □

We now are ready to show the existence of a Deligne-Manin logarithmic model.

hm.20220607-04 **Proposition 2.7** (Deligne-Manin model)**.** *Let (M, ∇) be the regular-singular connection over $R(\!(x)\!)$. Then, there exists a logarithmic model $\mathcal{M} \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$ of (M, ∇) whose reduced exponents do not differ from each other by non-zero integers.*

Proof. Let $\mathcal{M}'$ be an arbitrary logarithmic model of (M, ∇). It is easy to see that $\mathbf{res}_{\mathcal{M}'} : \mathcal{M}'/x\mathcal{M}' \longrightarrow \mathcal{M}'/x\mathcal{M}'$ is an $R-$linear homomorphism. Denote by $\{\varrho_1, \ldots, \varrho_r\}$ the set of all reduced exponents (see Definition 2.5) of $\mathcal{M}'$. Write

$$\mathcal{M}'/(\mathfrak{r}, x)\mathcal{M}' = \bigoplus_{i=1}^{r} \ker(\overline{\mathbf{res}}_{\mathcal{M}'} - \varrho_i)^{r_i}$$

the Jordan decomposition of $\overline{\mathbf{res}}_{\mathcal{M}'}$. By applying Lemma 2.6 for $V = \mathcal{M}'/x\mathcal{M}'$ and $\varphi = \mathbf{res}_{\mathcal{M}'}$, there exists a decomposition

$$\mathcal{M}'/x\mathcal{M}' = \bigoplus_{i=1}^{r} V_i$$

having the following properties for each $i = 1, 2, \ldots, r$:

(i) V_i is a $\varphi-$invariant $R-$submodule of $\mathcal{M}'/x\mathcal{M}'$;
(ii) the reduction modulo $\mathfrak{r}$ of V_i is $\ker(\overline{\mathbf{res}}_{\mathcal{M}'} - \varrho_i)^{r_i}$.

We now use the shearing method to transform the set $\{\varrho_1, ..., \varrho_r\}$ into a set in which the difference of any two elements is either zero or a non-integer. It is enough to deal with case $r = 2$ and $\varrho_2 = \varrho_1 + 1$.

For each $i = 1, 2$, let $\mathbf{v}_i^0 = \{v_{i1}^0, ..., v_{in_i}^0\}$ be a basis of $\ker(\overline{\mathbf{res}}_{\mathcal{M}'} - \varrho_i)^{r_i}$ over C such that

$$\mathrm{Mat}(\overline{\mathbf{res}}_{\mathcal{M}'}, \mathbf{v}_i^0) = \mathrm{J}_{\varrho_i}$$

is a Jordan matrix with respect to ϱ_i. Let $\mathbf{v}_i = \{v_{i1}, ..., v_{in_i}\} \subset V_i$ be an arbitrary lift of $\mathbf{v}_i^0$, they generate V_i for each $i = 1, 2$. Let $\{m_{i1}, ..., m_{in_i}\}$ be a lift of $\mathbf{v}_i = \{v_{i1}, ..., v_{in_i}\}$ to $\mathcal{M}'$, they generates $\mathcal{M}'$ as an $R[\![x]\!]-$module, since $R[\![x]\!]$ is a local ring with maximal ideal $(\mathfrak{r}, x)$.

Let now define $\mathcal{M} = \langle xm_{11}, ..., xm_{1n_1}, m_{21}, ..., m_{2n_2}\rangle_{R[\![x]\!]}$. We claim this is a logarithmic model of M. Indeed it is obvious that $\nabla(xm_{1j}) \in \mathcal{M}$ for each $1 \le j \le n_1$. Further we have:

$$\mathbf{res}_\nabla(m_{2j} + (x)) = \nabla(v_{2j}) = \sum_{k=1}^{n_2} a_{2k} v_{2k},$$

since V_2 is $\varphi-$invariant. Hence,

$$\nabla(m_{2j}) - \sum_{k=1}^{n_2} a_{2k} m_{2k} \in x\mathcal{M}'.$$

Therefore, $\nabla(m_{2j}) \in \mathcal{M}$ for all $1 \le j \le n_2$. Thus, $\mathcal{M}$ is a logarithmic model of (M, ∇). The general case is proved by induction. □

2.2. *R*-**flat connections.** In this subsection, we will show that a connection in $\mathbf{MC}_{\mathrm{rs}}(R((x))/R)$ is flat over R iff it is free as an $R((x))$-module. Recall that being flat over R is equivalent to being flat over $R((x))$ (cf. [HdST22, Theorem 8.16]). Consider the $\mathfrak{r}$-adic completion $R \longrightarrow \widehat{R}$. Our method is first to use the base change $R((x)) \longrightarrow \widehat{R}((x))$ and then to use faithfully flat descend. The following lemma will be useful:

faith-flatness **Lemma 2.8.** *Let R be a noetherian local C–algebra. Then the homomorphism $R[\![x]\!] \longrightarrow \widehat{R}[\![x]\!]$ is faithfully flat. Consequently, the map $R((x)) \longrightarrow \widehat{R}((x))$ obtained by inverting x is also faithfully flat.*

Proof. Indeed, $\widehat{R}[\![x]\!]$ is $(\mathfrak{r}, x)$-adically complete, see [Mat86, Exercise 8.6]. Thus we can consider $\widehat{R}[\![x]\!]$ as the $(\mathfrak{r}, x)$-adic completion of $R[\![x]\!]$. As $R[\![x]\!]$ a noetherian local domain, we conclude that $\widehat{R}[\![x]\!]$ is faithfully flat over $R[\![x]\!]$ (cf. [Mat86, Theorem 8.14]). $\square$

thm.230704 **Proposition 2.9.** *Let (M, ∇) be a connection over $R((x))$ such that M is flat as an R–module. Then, M is a free $R((x))$-module.*

Proof. Let $(M, \nabla) \in \mathbf{MC}_{\mathrm{rs}}(R((x))/R)$ such that M is flat of R–modules. Then M is flat $R((x))$-module by [HdST22, Theorem 8.18]. Since $R((x))$ is noetherian, M is a projective $R((x))$-module. We show that it is $R((x))$-free.

According to Proposition 2.7, there exists a logarithmic model $\mathcal{M} \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$ of (M, ∇) such that all (reduced) exponents of $\overline{\mathrm{res}}_{\nabla}$ belongs to τ. Hence

$$\widehat{\mathcal{M}} := \mathcal{M} \otimes_{R[\![x]\!]} \widehat{R}[\![x]\!]$$

is a logarithmic model of

$$\widehat{M} := M \otimes_{R[\![x]\!]} \widehat{R}((x)).$$

As $\widehat{R}[\![x]\!]$ is the completion of $R[\![x]\!]$ by the $(\mathfrak{r}, x)$-adic topology (cf. Lemma 2.8), $\widehat{M}$ is the completion of M in this topology. Hence

$$\widehat{M}/(\mathfrak{r}, x) = M/(\mathfrak{r}, x).$$

This implies all (reduced) exponents of $\widehat{M}$ are in τ.

By [HdST22, Theorem 9.1], $\widehat{\mathcal{M}}$ is a Deligne-Manin model of $\widehat{M}$. As $\widehat{M}$ is flat over $\widehat{R}((x))$, [HdST22, Corollary 9.4] implies that $\widehat{\mathcal{M}}$ is a free $\widehat{R}[\![x]\!]$-module. Now Lemma 2.8 implies that the model $\mathcal{M}$ is flat over $R[\![x]\!]$ which is a free $R[\![x]\!]$ since $R[\![x]\!]$ is local. Therefore M is a free $R((x))$-module. $\square$

2.3. **Euler form for connections on free** $R((x))$**–modules.** We now show that any regular singular connection (M, ∇), where M is a free $R((x))$-module, is isomorphic to a connection of Euler form, in other words, the functor $\mathrm{eul} : \mathbf{End}_R^{\mathrm{free}} \to \mathbf{MC}_{\mathrm{rs}}^{\mathrm{free}}(R((x))/R)$ is essentially surjective.

We use the classical method to eliminate the power of x in the denominator, cf. [ABC20, Section 8]. The following lemma is essential.

lemma3.9 **Lemma 2.10.** *Let Q be a matrix in $\mathrm{GL}_n(K((x))) \cap \mathrm{M}_n(R((x)))$. Then one can write Q as a product $Q'.(Q'')^{-1}$, where*

$$Q' \in \mathrm{GL}_n(R((x))) \text{ and } Q'' \in \mathrm{GL}_n(K[\![x]\!]).$$

Proof. By multiplying Q by a power of x, we may assume that $Q \in \mathrm{M}_n(R[\![x]\!])$. Then $\det(Q)$ can be written uniquely as:

$$\det(Q) = x^d u, \text{ where } u \in K[\![x]\!]^{\times}.$$

If $d = 0$, then $Q \in \mathrm{GL}_n(K[\![x]\!])$, so our statement is trivial.

Assume that $d>0$, we now show that there exists a matrix $Q_1' \in \mathrm{GL}_n\big(R(\!(x)\!)\big)$ such that
$$Q_1'Q \in \mathrm{M}_n\big(R[\![x]\!]\big), \text{ and } \det\,(Q_1'Q) = x^{d-1}.u,$$
and then our lemma will follow by induction on d. Let $\lambda_1,\dots,\lambda_n \in R$ be the coefficients of a non-trivial dependence relation between the rows of $Q_{|x=0}$. Without loss of generality, we may assume that $\lambda_1 = 1$. Then we consider the following matrix:

matrix1
$$Q_1' = \begin{bmatrix} 1/x & \lambda_2/x & \dots & \lambda_n/x \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{4}$$

Then it is easy to check that $Q_1'.Q \in \mathrm{M}_n\big(R[\![x]\!]\big)$, and $\det(Q_1'.Q) = x^{d-1}.u$. Moreover, the inverse of Q_1' is

matrix1b
$$(Q_1')^{-1} = \begin{bmatrix} x & -\lambda_2 & \dots & -\lambda_n \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{5}$$

which is in $\mathrm{GL}_n\big(R(\!(x)\!)\big)$. This completes our proof. $\square$

prop 3.9 **Proposition 2.11.** *Let $(M,\nabla) \in \mathbf{MC}^{\mathrm{free}}_{\mathrm{rs}}(R(\!(x)\!)/R)$. Then there exists a logarithmic lattice of M, i.e. an $R[\![x]\!]$–free logarithmic connection $\mathcal{M}$ such that $\gamma(\mathcal{M}) \cong M$.*

Proof. As R is noetherian, so $\mathfrak{r} = \langle r_1,...,r_s\rangle_R$. Then, for each $f(x) = \sum\limits_{i\geq i_0} a_i x^i$ in $\mathfrak{r}(\!(x)\!)$, there exist $a_{ij} \in R$ such that $a_i = \sum\limits_{j=1}^{s} a_{ij} r_j$ for each $i \geq i_0$. This implies that
$$f(x) = \sum_{i\geq i_0}\Big(\sum_{j=1}^{s} a_{ij} r_j\Big)x^i = \sum_{j=1}^{s} r_j\Big[\sum_{i\geq i_0} a_{ij}x^i\Big].$$
We note that $r_j \sum\limits_{i\geq i_0} a_{ij}x^i \in \mathfrak{r}(\!(x)\!)$ is correspond to $r_j \otimes \sum\limits_{i\geq i_0} a_{ij}x^i \in \mathfrak{r}(\!(x)\!)$ in $\mathfrak{r}\otimes R(\!(x)\!)$, and hence $\mathfrak{r}\otimes_R R(\!(x)\!) \cong \mathfrak{r}(\!(x)\!)$. Therefore, we obtain that $\mathfrak{r}R(\!(x)\!) = \mathfrak{r}(\!(x)\!)$ is a maximal ideal in $R(\!(x)\!)$ as the quotient $R(\!(x)\!)/\mathfrak{r}(\!(x)\!) \cong C(\!(x)\!)$ is a field.

Set $\overline{M} := M/\mathfrak{r}M$, the reduction modulo $\mathfrak{r}$ of M. $\overline{M}$ is an object of $\mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)$. It is well-known that there exists a cyclic vector $\overline{m}$ of $\overline{M}$, i.e., the set $\{\nabla^{(i)}(\overline{m})\}_{0\leq i\leq n-1}$ forms a basis of $\overline{M}$ over $C(\!(x)\!)$, see, e.g., [ABC20, Lemma 8.3.3 (1)]. We claim that any lift $m \in M$ of $\overline{m}$ is a cyclic vector of
$$M_{K(\!(x)\!)} = M \otimes_{R(\!(x)\!)} K(\!(x)\!).$$

It is to prove that the set $\{\nabla^{(i)}(m)\}_{0\leq i\leq n-1}$ forms a basis of $M_{K(\!(x)\!))}$. Let $A := R(\!(x)\!)_{\mathfrak{r}}$, the localization of $R(\!(x)\!)$ at the ideal $\mathfrak{r}$. Then A is a local ring with residue field being $C(\!(x)\!)$ and fraction field being a subfield of $K(\!(x)\!)$. By Nakayama lemma, the set $\{\nabla^{(i)}(m)\}_{0\leq i\leq n-1}$ forms a basis of the A–module $M\otimes_{R(\!(x)\!)} A$. We have inclusions
$$M \longrightarrow M\otimes_{R(\!(x)\!)} A \longrightarrow M\otimes_{R(\!(x)\!)} K(\!(x)\!) = M_{K(\!(x)\!)}.$$
Thus $\{\nabla^{(i)}(m)\}_{0\leq i\leq n-1}$ forms a basis of $M_{K(\!(x)\!)}$.

Set $\mathbf{e} = \{\nabla^{(i)}(m)\}_{0\leq i\leq n-1}$. Apply Theorem 8.3.3 (3) in [ABC20] to the regular singular connection $M_{K(\!(x)\!)}$, we have
$$\nabla(\mathbf{e}) = \mathbf{e}H,$$

with some matrix H in $\mathrm{M}_n(K[\![x]\!])$. Let $\mathbf{f}$ be a basis of M over $R(\!(x)\!)$. We have a presentation of $\mathbf{e}$ via $\mathbf{f}$:

$$\mathbf{e} = \mathbf{f}Q,$$

for some matrix Q in $\mathrm{M}_n\big(R(\!(x)\!)\big)$. As $\mathbf{e}$ and $\mathbf{f}$ are both bases for $M_{K(\!(x)\!)}$, we have $Q \in \mathrm{GL}_n(K(\!(x)\!))$. Lemma 2.10 is applied for Q to obtain a decomposition

$$Q = Q'.(Q'')^{-1},$$

where $Q' \in \mathrm{GL}_n\big(R(\!(x)\!)\big)$ and $Q'' \in \mathrm{GL}_n\big(K[\![x]\!]\big)$. Thus $\mathbf{f}' := \mathbf{f}Q' = \mathbf{e}Q''$ is also a basis of M.

As $\mathbf{f}' := \mathbf{f}Q'$, computation shows

$$\nabla(\mathbf{f}') = \mathbf{f}'H_1,$$

for some H_1 in $\mathrm{M}_n\big(R(\!(x)\!)\big)$. On the other hand, since $\mathbf{f}' = \mathbf{e}Q''$, we have

$$H_1 = (Q'')^{-1}HQ'' + (Q'')^{-1}\vartheta(Q) \in \mathrm{M}_n\big(K[\![x]\!]\big).$$

So $H_1 \in \mathrm{M}_n\big(R(\!(x)\!)\big) \cap \mathrm{M}_n\big(K[\![x]\!]\big) = \mathrm{M}_n\big(R[\![x]\!]\big)$. This shows that $\langle \mathbf{f}' \rangle_{R[\![x]\!]}$ is the sought logarithmic lattice. $\square$

We now arrive at the following theorem.

m.2022070-8-01 **Proposition 2.12.** *The functor*

$$\gamma\mathrm{eul} : \mathbf{End}_R^{\mathrm{free}} \longrightarrow \mathbf{MC}_{\mathrm{rs}}^{\mathrm{free}}(R(\!(x)\!)/R)$$

is faithful and essentially surjective. This functor is not full. Assume that τ is a set of representatives of $C/\mathbb{Z}$ in C, which contains 0. Then its restriction to the full subcategory of all objects (V, A) in $\mathbf{End}_R^{\mathrm{free}}$ such that the spectrum of $A : V/\mathfrak{r} \to V/\mathfrak{r}$ is contained in τ, is indeed full.

Proof. Essential surjectivity follows from Propositions 2.7 and 2.11. Faithfulness is obvious.

We now concentrate on the verification of the last claim. Let (V, A) and (W, B) be objects of $\mathbf{End}_R^{\mathrm{free}}$ and suppose that the eigenvalues of the C-linear endomorphisms of $V/\mathfrak{r}$ and $W/\mathfrak{r}$ associated respectively to A and B lie in τ. On $H = \mathrm{Hom}_R(V, W)$, consider the endomorphism $T : h \mapsto hA - Bh$; we then obtain an object (H, T) of $\mathbf{End}_R^{\mathrm{free}}$. Let us note in passing that the spectrum of the C-linear endomorphism $T_0 : H/\mathfrak{r} \to H/\mathfrak{r}$ is built up from the differences of eigenvalues of $A_0 : V/\mathfrak{r} \to V/\mathfrak{r}$ and $B_0 : W/\mathfrak{r} \to W/\mathfrak{r}$; in particular, $\mathrm{Sp}_{T_0} \cap \mathbb{Z} \subset \{0\}$. Consequently, for each $k \in \mathbb{N}$, the spectrum of the C-linear endomorphism $T_k : H/\mathfrak{r}^{k+1} \to H/\mathfrak{r}^{k+1}$ contains no integer except perhaps 0. This is because $\mathrm{Sp}_{T_k} = \mathrm{Sp}_{T_0}$ [HdST22, Prp. 8.11]. It is a simple matter to see that the $\mathrm{Hom}_{\mathbf{MC}}(\gamma\mathrm{eul}(V, A), \gamma\mathrm{eul}(W, B))$ corresponds to the horizontal elements of $\gamma\mathrm{eul}(H, T)$. After picking a basis of H, a horizontal section of $\gamma\mathrm{eul}(H, T)$ amounts to a vector $\boldsymbol{h} \in R(\!(x)\!)^r$ such that

$$\vartheta\boldsymbol{h} = -T\boldsymbol{h}.$$

Writing $\boldsymbol{h} = \sum_{i \geq i_0} \boldsymbol{h}_i x^i$, we see that

$$T\boldsymbol{h}_i = -i\boldsymbol{h}_i.$$

Now, let $i \neq 0$. Then the image of $\boldsymbol{h}_i$ in $R_k^{\oplus r}$ must be zero, since $i \notin \mathrm{Sp}_{T_k}$. Hence, $\boldsymbol{h}_i = 0$ except perhaps for $i = 0$. This proves that any arrow

$$h : \gamma\mathrm{eul}(V, A) \longrightarrow \gamma\mathrm{eul}(W, B)$$

comes from an arrow $V \to W$.

$\square$

2.4. **The case of a discrete valuation C-algebra.** Previously, we have described the structure of an $R(\!(x)\!)$–free connection. We still have no conclusions for arbitrary connections. So let us in this section suppose that

$$R \text{ is a DVR and } \mathfrak{r} = Rt.$$

em.20220612-01 **Lemma 2.13.** *(i) The ring $R(\!(x)\!)$ is a principal ideal domain.*

(ii) Let (E, ∇) be a connection over $R(\!(x)\!)$ such that E is free of R–torsion. Then, E is a free $R(\!(x)\!)$–module.

Proof. (i) Each ideal $\mathfrak{a} \subset R(\!(x)\!)$ is of the form $\mathfrak{A} \cdot R(\!(x)\!)$, where $\mathfrak{A} = \mathfrak{a} \cap R[\![x]\!]$, cf. [Mat86, Theorem 4.1, p.22]. Since $R[\![x]\!]$ is complete for the x-adic topology, any generator of the $R[\![x]\!]/(x)$–module $\mathfrak{A}/x\mathfrak{A} \simeq (R[\![x]\!]/(x)) \otimes_{R[\![x]\!]} \mathfrak{A}$ is a generator of $\mathfrak{A}$ [Mat86, Theorem 8.4, p.58]. Now, it is a simple matter to see that $x\mathfrak{A} = (x) \cap \mathfrak{A}$, and hence that $(R[\![x]\!]/(x)) \otimes_{R[\![x]\!]} \mathfrak{A}$ is an ideal of R, which is therefore generated by one element.

(ii) As E is flat over $R(\!(x)\!)$, the theorem of structure for finite modules over principal ideal domains then assures that E is free over $R(\!(x)\!)$. □

The easiest case is when the connection is $\mathfrak{r}$–torsion: $(M, \nabla) \in \mathbf{MC}_{\rm rs}(R(\!(x)\!)/R)$ is a connection such that $\mathfrak{r}M = 0$. Then (M, ∇) can be identified with $M/\mathfrak{r}M \in \mathbf{MC}_{\rm rs}(C(\!(x)\!)/C)$. Such a connection is Euler (cf. [HdST22]), i.e. has the form $\gamma\mathrm{eul}(V, A)$ where V is a C-vector space. This connection is certainly a quotient of the Euler connection $\gamma\mathrm{eul}(V \otimes_C R, A \otimes \mathrm{id})$. Once this property has been brought to light, the general case follows a technique of [DH18].

Proposition 2.14. *Each object of $\mathbf{MC}_{\rm rs}(R(\!(x)\!)/R)$ is a quotient of a certain (E, ∇) in $\mathbf{MC}_{\rm rs}(R(\!(x)\!)/R)$ such that E is a free $R(\!(x)\!)$-module.*

Proof. The proof is almost identical to the that of [DH18, Proposition 5.2.2], but some care has to be taken to assure that the connections constructed are regular singular. In view of Proposition **??** and Lemma 2.13, the requirement "E is a free $R(\!(x)\!)$–module" can be weakened to "E is free of R-torsion." Let us make this precise: given an R–module W, define

$$\begin{aligned} W_{\rm tors} &= \bigcup_k (0 : \mathfrak{r}^k)_W \\ &= \{w \in W \,:\, \text{some power of } t \text{ annihilates } w\}. \end{aligned}$$

Being free of R-torsion means that $W_{\rm tors} = 0$.

Given $M \in \mathbf{MC}_{\rm rs}(R(\!(x)\!)/R)$, we define

$$r(M) = \min\{s \in \mathbb{N} \,:\, \mathfrak{r}^s M_{\rm tors} = 0\}.$$

We shall proceed by induction on $r(E)$, the case $r(E) = 0$ being trivial. Assume $r(E) = 1$, so that $\mathfrak{r}E$ is free of R-torsion. Let $q : E \to Q$ be the quotient by $\mathfrak{r}E$; since Q is annihilated by $\mathfrak{r}$, it is an object of $\mathbf{MC}_{\rm rs}(C(\!(x)\!)/C)$ and as such has the form $\gamma\mathrm{eul}(V, A)$ where V is a C-vector space [HdST22, Cor. 4.3]. This connection is certainly a quotient of the Euler connection

$$\widetilde{Q} := \gamma\mathrm{eul}(V \otimes_C R, A \otimes \mathrm{id}).$$

We then have exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{r}E \arrow[r] & E \arrow[r, "q"] \arrow[dr, phantom, "\square"] & Q \arrow[r] & 0 \\ 0 \arrow[r] & \mathfrak{r}E \arrow[r] \arrow[u, "\sim"] & \widetilde{E} \arrow[r] \arrow[u] & \widetilde{Q} \arrow[r] \arrow[u, two heads] & 0, \end{tikzcd}$$

where the rightmost square is cartesian and $\widetilde{E} \to E$ is in fact surjective. Since $\mathfrak{t}E$ and $\widetilde{Q}$ are free of R-torsion, so is $\widetilde{E}$. Since $\widetilde{E}$ is a subobject of $\widetilde{Q} \oplus E$, we can appeal to [HdST22, Lemma 8.3] to assure that it is regular-singular.

Let us now assume that $r(E) > 1$. Let $N = \{e \in E : te = 0\}$ and denote by $q : E \to Q$ the quotient by N. It then follows that $t^{r(E)-1}Q_{\mathrm{tors}} = 0$, so that $r(C) \leq r(E) - 1$. By induction there exists $\widetilde{Q}$ free and a surjection $\widetilde{Q} \to Q$. We arrive at commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & E \arrow[r, "q"] & Q \arrow[r] & 0 \\ 0 \arrow[r] & N \arrow[r] \arrow[u, "\sim"] & \widetilde{E} \arrow[r] \arrow[u] \arrow[ur, phantom, "\square"] & \widetilde{Q} \arrow[r] \arrow[u, two heads] & 0, \end{tikzcd}$$

where the rightmost square is cartesian and $\widetilde{E} \to E$ is surjective. Since $\widetilde{Q}$ is free of R-torsion, we conclude that $\widetilde{E}_{\mathrm{tors}} = N$, so that $r(\widetilde{E}) \leq 1$. We can therefore find $\widetilde{E}_1$ and a surjection $\widetilde{E}_1 \to \widetilde{E}$ and consequently a surjection $\widetilde{E}_1 \to E$. $\square$

cor.12.07.22-1 **Corollary 2.15.** *The functor* $\gamma\mathrm{eul}_{R[\![x]\!]} : \mathbf{End}_R \longrightarrow \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$ *is essentially surjective.*

Proof. Let $M \in \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$ be given; because of the Proposition, we can find an exact sequence in $\mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$:

$$E \xrightarrow{\Phi} F \longrightarrow M \longrightarrow 0,$$

where E and F are free objects of $\mathbf{MC}^{\mathrm{free}}_{\mathrm{rs}}(R(\!(x)\!)/R)$. Because of Propsoition 2.12, we can assume that

$$E \simeq \gamma\mathrm{eul}(V, A) \quad \text{and} \quad F \simeq \gamma\mathrm{eul}(W, B)$$

where A and B. In this case, $\Phi = \gamma eul(\varphi)$, again by Proposition 2.12 and hence M is isomorphic to $\gamma\mathrm{eul}(\operatorname{Coker}\varphi)$. $\square$

3. Regular singular connections on $\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$

sect.4

Our aim in this section is to show the equivalence between $\mathbf{End}_R$ and $\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$. Our strategy for the proof is different from the previous section. Instead of using the shearing technique, we rely on the known result for complete discrete valuation ring and use Popescu's theorem to descend from $\widehat{R}$ to R.

3.1. **Regular singular connections on the punctured line.** We fix a choice of local coordinates of $\mathbb{P}^1_R$ as follows: write $\mathbb{P}^1_R$ as the union of two affine lines $\mathbb{A}_0$ and $\mathbb{A}_\infty$, where $\mathbb{A}_0 = \mathrm{Spec}(R[x])$ and $\mathbb{A}_\infty = \mathrm{Spec}(R[y])$, with the transition function on their intersection $R[x^{\pm}] \cong R[y^{\pm}]$ is given by $x \mapsto y = x^{-1}$.

By the equality $y = x^{-1}$ we have

$$x\frac{d}{dx} = -y\frac{d}{dy},$$

therefore ϑ can be extended canonically to a global section, still denoted by ϑ, of the tangent sheaf of $\mathbb{P}^1_R$.

fn.20220607-08 **Definition 3.1** (Connection on punctured affine line)**.** The category of connections on on $R[x^{\pm}]$ or on the punctured affine line $\mathbb{P}^1_R \smallsetminus \{0, \infty\}$ over R, denoted $\mathbf{MC}(R[x^{\pm}]/R)$, has for *objects* those couples (M, ∇) consisting of a $R[x^{\pm}]$*–module of finite presentation* and a R–linear endomorphism $\nabla : M \to M$ satisfying Leibniz's rule

$$\nabla(fm) = \vartheta(f)m + f\nabla(m);$$

arrows between (M,∇) and (M',∇') are just $R[x^{\pm}]$–linear maps $\varphi : M \to M'$ satisfying $\nabla'\varphi = \varphi\nabla$.

It is well-known that for a connection (M,∇) on $R[x^{\pm}]/R$, M is $R[x^{\pm}]$–flat if and only if its is R–flat, cf., e.g., [dS09, p.82] or [DH18, Theorem 5.1.1]. Since $R[x^{\pm}]$ is a domain, this amounts to M being projective over $R[x^{\pm}]$.

n.20220607-08b **Definition 3.2** (Logarithmic connection on punctured affine line)**.** The category of *logarithmic connections* on the punctured affine line $\mathbb{P}^1_R$, denoted $\mathbf{MC}_{\log}(\mathbb{P}^1_R/R)$, has for

objects those couples $(\mathcal{M},\nabla)$ consisting of a coherent $\mathcal{O}_{\mathbb{P}^1_R}$–module and an R–linear endomorphism $\nabla : \mathcal{M} \to \mathcal{M}$ satisfying Leibniz's rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$ on all open subsets; and

arrows between $(\mathcal{M},\nabla)$ and $(\mathcal{M}',\nabla')$ are $\mathcal{O}_{\mathbb{P}^1_R}$–linear maps $\varphi : \mathcal{M} \to \mathcal{M}'$ satisfying $\nabla'\varphi = \varphi\nabla$.

We let

$$\gamma : \mathbf{MC}_{\log}(\mathbb{P}^1_R/R) \longrightarrow \mathbf{MC}(R[x^{\pm}]/R)$$

be the natural restriction functor.

Definition 3.3 (Regular singular connection on punctured affine line)**.**

(1) A connection (M,∇) in $\mathbf{MC}(R[x^{\pm}]/R)$ is *regular-singular* if $\gamma(\mathcal{M}) \simeq M$ for a certain $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}^1_R/R)$; in this case, any such $\mathcal{M}$ is a *logarithmic model* of M. In case $\mathcal{M}$ is a locally free $\mathcal{O}_{\mathbb{P}_R}$-module, we call $\mathcal{M}$ a logarithmic lattice of M (which is also free over $R(\!(x)\!)$).

(2) The full subcategory of $\mathbf{MC}(R[x^{\pm}]/R)$ having regular-singular connections as objects is denoted by $\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$.

ex.20220709-01 **Example 3.4** (Euler connections)**.** For an object $(V,A) \in \mathbf{End}_R$ we set

$$\mathrm{eul}_{\mathbb{P}^1}(V,A) := (\mathcal{O}_{\mathbb{P}^1} \otimes_R V, D_A),$$

where $D_A : \mathcal{O}_{\mathbb{P}^1_R} \otimes_R V \to \mathcal{O}_{\mathbb{P}^1_R} \otimes_R V$ is $R-$linear and defined by

$$D_A(f \otimes m) = \vartheta(f) \otimes v + f \otimes Av$$

on any open subsets of $\mathbb{P}^1_R$. Notation: $\mathrm{eul}_{\mathbb{P}^1}(V,A)$.

Thus we have functor $\mathrm{eul}_{\mathbb{P}^1} : \mathbf{End}_R \longrightarrow \mathbf{MC}_{\log}(\mathbb{P}^1_R/R)$ and, composing it with γ, the functor

$$\gamma\mathrm{eul}_{\mathbb{P}^1} : \mathbf{End}_R \longrightarrow \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R).$$

op.20220607-03 **Proposition 3.5.** *The functor*

$$\gamma\mathrm{eul}_{\mathbb{P}^1} : \mathbf{End}_R \longrightarrow \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$$

is faithful and essentially surjective.

Proof. The functor is obviously faithful. We proceed to show that it is essentially surjective. To see this we shall first base change to $\widehat{R}$ and use the known results from [HdST22], and then descent back to R by using Popescu's theorem.

Let $\widehat{R}$ be the $\mathfrak{r}$-adic completion of R. Let (M,∇) be an object in $\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$. By tensoring over R with $\widehat{R}$, we obtain an object in $\mathbf{MC}_{\mathrm{rs}}(\widehat{R}[x^{\pm}]/\widehat{R})$, denoted by $(M_{\widehat{R}},\nabla_{\widehat{R}})$.

Let us now write, according to Popescu [Pop86, Theorem 2.5],

$$\widehat{R} = \varinjlim_{\lambda\in L} S_\lambda$$

where each S_λ is a *smooth* R–algebra.

Let $(M,\nabla)\in\mathbf{MC}_{\rm rs}(R[x^{\pm}]/R)$. For each $\lambda\in L$, we let $(M_\lambda,\nabla_\lambda)$ stand for the object of $\mathbf{MC}(S_\lambda[x^{\pm}]/S_\lambda)$ defined, in the evident manner, by means of the functor $S_\lambda\otimes_R(-)$. We define $(\widehat{M},\widehat{\nabla})$ in the similar fashion.

Let $\mathfrak{A}:\mathfrak{V}\to\mathfrak{V}$ be an endomorphism of the finite $\widehat{R}$-module $\mathfrak{V}$ such that there exists an *isomorphism*

$$\mathfrak{f}:(\widehat{M},\widehat{\nabla})\ \longrightarrow\ (\widehat{R}[x^{\pm}]\otimes_{\widehat{R}}\mathfrak{V},D_{\mathfrak{A}})$$

in $\mathbf{MC}(\widehat{R}[x^{\pm}]/\widehat{R})$. The existence of this arrow is a consequence of [HdST22, Proposition 10.1] and Theorem 2.12.

There exists α such that $\mathfrak{A}:\mathfrak{V}\to\mathfrak{V}$ is of the form

$$\mathrm{id}_{\widehat{R}}\underset{S_\alpha}{\otimes}A_\alpha:\widehat{R}\underset{S_\alpha}{\otimes}V_\alpha\longrightarrow\widehat{R}\underset{S_\alpha}{\otimes}V_\alpha$$

where A_α is an S_α-linear endomorphism of the finite S_α-module. See [EGA IV$_3$, 8.5.2(i)-(ii), p.20]. Given $\lambda\ge\alpha$, let $A_\lambda:V_\lambda\to V_\lambda$ be the base-change of A_α to $V_\lambda:=S_\lambda\otimes_{S_\alpha}V_\alpha$.

This allows us to define objects in

$$(S_\lambda[x^{\pm}]\otimes_{S_\alpha}V_\alpha,D_{A_\lambda})\in\mathbf{MC}(S_\lambda[x^{\pm}]/S_\lambda)$$

for all $\lambda\ge\alpha$ along the lines of 3.4.

There exists $\beta\ge\alpha$ such that $\mathfrak{f}$ is obtained from a certain

$$f_\beta\ :\ M_\beta\ \longrightarrow\ S_\beta[x^{\pm}]\underset{S_\alpha}{\otimes}V_\alpha$$

by base change $S_\beta\to\widehat{R}$. See [EGA IV$_3$, 8.5.2.1, p.21]. For convenience, let f_λ be the base-change of f_β for $\lambda\ge\beta$.

Let now $\{m_i\}\in M$ be a set of $R[x^{\pm}]$-module generators for M and write m_i^λ for the image of m_i in M_λ via the natural arrow $M\to M_\lambda$. Consider, for each $\lambda\ge\beta$, the elements

$$\delta_i^\lambda:=f_\lambda(\nabla_\lambda(m_i^\lambda))-D_{A_\lambda}(f_\lambda(m_i^\lambda))$$

of $S_\lambda[x^{\pm}]\otimes_{S_\alpha}V_\alpha$. We then conclude that for some $\lambda\ge\beta$, the elements δ_i^λ are all zero, and hence for a certain $\lambda\ge\beta$, the arrow

$$f_\lambda:M_\lambda\longrightarrow S_\lambda[x^{\pm}]\otimes_{S_\alpha}V_\alpha$$

is horizontal, which is verified without much effort.

Now, it is clear that $C\to C\otimes_R S_\lambda$ comes with a section to $C\to S_\lambda\otimes_R C$. Then, "Hensel's Lemma" [EGA IV$_4$, 18.5.11.(b)] shows that there exists a section $S_\lambda\to R$. Base changing the morphism f_λ via $S_\lambda\to R$ we obtain the desired isomorphism $M\to R[x^{\pm}]\otimes V$. □

4. Deligne's equivalence

sect.5

In this section we put things together to obtain the generalization of Deligne's equivalence to the case of strict henselian rings. Recall that Deligne proved in [Del87, Proposition 15.35] that for any field k of characteristic 0, the functor

$$\mathbf{r}:\mathbf{MC}_{\rm rs}(k[x^{\pm}])\longrightarrow\mathbf{MC}_{\rm rs}(k(\!(x)\!))$$

given by base change is indeed an equivalence. This has been generalized to an equivalence for k replace by a complete local noetherian C-algebra in [HdST22]. The aim of this section is the following theorem.

hm.20220623-01 **Theorem 4.1.** *Then the restriction functor*

$$\mathbf{r}:\mathbf{MC}_{\rm rs}^{\rm free}(R[x^{\pm}]/R)\longrightarrow\mathbf{MC}_{\rm rs}^{\rm free}(R(\!(x)\!)/R)$$

is an equivalence.

Proof. Essential surjectivity. By Proposition 2.12 and Proposition 3.5 the functor $\mathbf{r}$ is essentially surjective. Indeed, let (E,∇) be an arbitrary object in $\mathbf{MC}_{\mathrm{rs}}^{\mathrm{free}}(R(\!(x)\!)/R)$. According to Theorem 2.7, there exists $(V,A)\in \mathbf{End}_R^{\mathrm{free}}$ satisfying

$$\gamma\mathrm{eul}_{R[\![x]\!]}(V,A)\simeq (E,\nabla).$$

Then, $\gamma\mathrm{eul}_{\mathbb{P}^1_R}(V,A)=(R[x^{\pm}]\otimes_R V, D_A)$ is a regular singular connection in $\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$ which satisfies that $\mathbf{r}(R[x^{\pm}]\otimes_R V, D_A)\simeq (E,\nabla)$.

Faithfulness. This is obvious as the map $R[x^{\pm}]\longrightarrow R(\!(x)\!)$ is fully faithful as it is the localization at x of the completion map $R[x]\longrightarrow R[\![x]\!]$.

Fullness. Since $R[x^{\pm}]$ is noetherian, each flat module over $R[x^{\pm}]$ is projective. Hence, for each $(E,\nabla)\in \mathbf{MC}_{\mathrm{rs}}^{\mathrm{free}}(R[x^{\pm}]/R)$ then E is $R[x^{\pm}]$–projective. According to [Pop02, Theorem 1], the module E is $R[x^{\pm}]$–free.

Therefore, by using Proposition 3.5, we may work with Euler forms all the time. So given two objects $(V_1,A_1),(V_2,A_2)\in \mathbf{End}_R^{\mathrm{free}}$, we are going to show that the restriction map:

$$\mathbf{r}:\mathrm{Mor}_{R[x^{\pm}]}(\gamma\mathrm{eul}_{\mathbb{P}^1_R}(V_1,A_1),\gamma\mathrm{eul}_{\mathbb{P}^1_R}(V_2,A_2))\to \mathrm{Mor}_{R(\!(x)\!)}(\gamma\mathrm{eul}_{R[\![x]\!]}(V_1,A_1),\gamma\mathrm{eul}_{R[\![x]\!]}(V_2,A_2)),$$

is surjective. Indeed, fix bases $\mathbf{e}_i$ of V_i over R. Then any

$$f\in \mathrm{Mor}_{R(\!(x)\!)}(\gamma\mathrm{eul}_{R[\![x]\!]}(V_1,A_1),\gamma\mathrm{eul}_{R[\![x]\!]}(V_2,A_2))$$

is defined a matrix with coefficients from $R(\!(x)\!)$. On the other hand, after base changed to $\widehat{R}$, the above map is an isomorphism. This means the matrix elements of f also belong to $\widehat{R}[x^{\pm}]$. Now we have

$$R(\!(x)\!)\cap \widehat{R}[x^{\pm}]=R[x^{\pm}].$$

That is, f is defined over $R[x^{\pm}]$. $\square$

valence.hensel **Corollary 4.2.** *Let R be a henselian discrete valuation C-algebra with residue field isomorphic to C. Then the restriction functor*

$$\mathbf{r}:\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)\longrightarrow \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$$

is an equivalence.

Proof. This is a consequence of Corollary 2.15 and Proposition 3.5 and the theorem above. $\square$

Remark 4.3. As the original equivalence established by Deligne holds true over any field of characteristic 0, the above theorem also holds for any field C of characteristic 0. Indeed, the proof in [HdST22] of the corresponding claim (Theorem 10.1) holds in this more general setting.

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