# YOSIDA DISTANCE AND EXISTENCE OF INVARIANT MANIFOLDS IN THE INFINITE-DIMENSIONAL DYNAMICAL SYSTEMS

XUAN-QUANG BUI AND NGUYEN VAN MINH

ABSTRACT. Using a new concept of "Yosida distance" between two (unbounded) linear operators $A$ and $B$ in a Banach space $\mathbb{X}$ defined as $\limsup_{\mu\to+\infty} \|A_\mu - B_\mu\|$, where $A_\mu$ and $B_\mu$ are the Yosida approximations of $A$ and $B$, respectively, we study the persistence under small linear perturbation of exponential dichotomy in the linear evolution equations. This new concept of distance is also used to define the continuity of the proto-derivative of the operator $F$ in the equation $u'(t) = Fu(t)$, where $F\colon D(F) \subset \mathbb{X} \to \mathbb{X}$ is a nonlinear operator. We show that the above-mentioned equation has local stable and unstable invariant manifolds near an exponentially dichotomous equilibrium if the proto-derivative of $F$ is continuous. The Yosida distance approach and the obtained results seem to be new.

## 1. Introduction

It is well known in the qualitative theory of ordinary differential equations that the asymptotic behavior of linear equations of the form

$$u'(t) = Mu(t), \qquad u(t) \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{1.1}$$

where $M$ is a $n \times n$-matrix, persists under "small" perturbation that is either linear or nonlinear if the system has an exponential dichotomy, that is, all eigenvalues of $M$ are off the imaginary axis (see e.g. [7, 11, 12]), or equivalently, the unit circle does not intersect with the spectrum of $e^M$. For many decades, extensions of these classical results have been ones of the central topics in the theory of infinite dimensional dynamical systems with applications to partial differential equations and other types of evolution equations. Namely, for linear perturbation of a linear hyperbolic system in a Banach space

$$u'(t) = Au(t), \quad u(t) \in \mathbb{X}, \tag{1.2}$$

where $\mathbb{X}$ is a (complex) Banach space, $A\colon D(A) \subset \mathbb{X} \to \mathbb{X}$ is an unbounded linear operator, one often considers the equation

$$u'(t) = (A + B)u(t), \quad u(t) \in \mathbb{X}, \tag{1.3}$$

where $B\colon D(B) \subset \mathbb{X} \to \mathbb{X}$ is a linear operator. To justify for the "smallness" of the perturbation $B$ one often assumes that $B$ is bounded and its norm $\|B\|$ is small. This assumption on the "smallness" of perturbation $B$ actually limits its applicability of the obtained results to partial differential equations and other types of evolution equations. For this reason, when $A$ has some further properties like the generator of an analytic semigroup one can consider the perturbation $B$ among a more general class of linear operator that are $A$-bounded, that is, $\|Bx\| \le a\|Ax\| + c\|x\|$, $x \in D(A)$, for some fixed positive constants $a$ and $c$. Then, the smallness of $B$ is measured by the sizes of $\max\{a, c\}$. These approaches are discussed in [11, 12, 21, 22] for the generation of semigroups by $A + B$ that can be easily used to show that the exponential dichotomy of Eq. (1.3) persists under small perturbation.

---

2010 *Mathematics Subject Classification.* 34G10, 37D10, 34D20, 34C45, 34D09.

*Key words and phrases.* Yosida distance, proto-derivative, exponential dichotomy, invariant manifolds.

The work of the first author (X.-Q. Bui) is partly supported by the Vietnam Institute for Advanced Study in Mathematics (VIASM)..

For nonlinear perturbation of Eq. (1.2)

$$u'(t) = Au + Fu, \quad u(t) \in \mathbb{X}, \tag{1.4}$$

where $F : D(F) \subset \mathbb{X} \to \mathbb{X}$ is a nonlinear operator, the asymptotic behavior of Eq. (1.4) near the equilibrium is often described by the (local) invariant manifolds near the equilibrium, see e.g. [3, 4, 6, 10, 14, 16, 17, 18, 19, 24, 26]. Apart from the assumption that $F$ be differentiable in some sense, say, in the sense of Fréchet, the continuity of the derivatives is often determined by that of $F'(x)$ in certain spaces of bounded linear operators (see e.g. [10, 14, 16, 17]). This allows to include more classes of partial differential equations into the consideration.

In this paper we will take an attempt to propose a new approach to the perturbation theory for Eq. (1.2) in which the size of perturbation will be measured by the so-called "Yosida distance". This is a new concept of a pseudo metric defined on the set of all generators of $C_0$-semigroups (see Definition 3.1 below). This allows us to measure to the distance between two unbounded operators in many important classes as seen in our Lemma 3.2. Note that in our Example 5.4 distances between another class of operators than those in Lemma 3.2 could be determined as well. Remarkably, in the case of evolution equations under unbounded perturbations we can see the work done by Chow-Leiva [5]. Unlike that approach, our method by the Yosida approximations and Yosida distance is different and that can be easily used to study nonlinear perturbations as shown in the paper. We will use this Yosida distance to define the continuity of the proto-derivatives of $F$ in Eq. (1.4). For linear equations (1.3) we will show the persistence of the exponential dichotomy under small perturbation measured by Yosida distance between $A$ and $A + B$, see Theorem 3.4. The Yosida distance approach will be extended to nonlinear perturbation, namely, to study Eq. (1.4) where we allow $F$ to be proto-differentiable and its proto-derivative is continuous in the topology defined by the Yosida distance. We prove the existence of local stable and unstable invariant manifolds near an equilibrium of (1.4) if the linearized equation (1.3) has an exponential dichotomy, see Theorems 4.19 and 4.20. We note that in our Assumptions 2 and 3 conditions on the m-accretiveness of the operators only to guarantee that they generate semiflows, or in other words, the corresponding equations are well posed. At the end of the paper we provide some examples to show that our approach allows a generalization of known results as shown in Examples 5.1, 5.2 and 5.4. To the best of our knowledge, the Yosida distance approach to the perturbation of exponential dichotomy and the obtained results discussed in this paper are new. Though the concept of proto-differentiability is popularly used in Applied Nonlinear Analysis to deal with set valued operators, we use it in the paper in the context of single-valued operators, so some statements are adjusted to this restriction.

This paper is organized as follows: In Section 2, we first list some notations used in the paper. Then, we recall some background materials on accretive operators and generation of strongly continuous nonlinear semigroups. Section 3 contains the definition of Yosida distance between two linear operators and results on the roughness of exponential dichotomy in linear dynamical systems. Nonlinear perturbation of exponential dichotomy is presented in Section 4. In this section, using the concept of Yosida distance, we define the continuity of the proto-derivative of a nonlinear operator and prove the existence of invariant manifolds for nonlinear dynamical systems. Finally, in Section 5, we give some examples to show that the Yosida distance is a consistent tool in studying the existence of invariant manifolds.

## 2. Preliminaries

2.1. **Notations.** In this paper we will denote by $\mathbb{X}$, $\mathbb{Y}$ Banach spaces with corresponding norms. The symbols $\mathbb{R}$ and $\mathbb{C}$ stand for the fields of real and complex numbers, respectively. Denote by $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ the Banach space of all bounded linear operators from a Banach space $\mathbb{X}$ to a Banach space $\mathbb{Y}$. We will denote the domain of an operator $T$ by $D(T)$ and its range by $R(T)$. The resolvent set of a linear operator $T$ in a Banach space will be denoted by $\rho(T)$ while its spectrum is denoted by $\sigma(T)$. Let $P$ be a linear operator in a Banach space $\mathbb{X}$, as usual, $\mathrm{Ker}(P)$ and $\mathrm{Im}(P)$ are the notations of kernel and image of $P$. We shall denote by $B_r(0,\mathbb{Y})$ the ball of radius $r$ centered at 0 of $\mathbb{Y}$ and write $B_r(0,\mathbb{X}) = B_r(0)$ if this does not cause any confusion, when $\mathbb{X}$ is a given fixed Banach space. Given function $F$, the notations $F'(u)$, $dF(u)$, and $\partial F(u)$ will stand for the Fréchet derivative, Gâteaux derivative, and proto-derivative, respectively, of $F$ at $u$.

### 2.2. **Accretive operators and generation of strongly continuous nonlinear semigroups.**

**Definition 2.1** (accretive, m-accretive (see [2, 8]))**.** Let $B$ be a (possibly nonlinear multi-valued) operator in $\mathbb{X}$, then $B$ is called *accretive* if $(I+\lambda B)^{-1}$ exists as a single-valued function and

$$\left\|(I+\lambda B)^{-1}x - (I+\lambda B)^{-1}y\right\| \le \left\|x-y\right\|,$$

for all $x,\ y \in D((I+\lambda B)^{-1})$. An accretive operator $B$ is called *m-accretive* if $R(I+\lambda B) = \mathbb{X}$ for all (equivalently for some) $\lambda > 0$.

Below is the well known Crandall-Liggett Theorem on the generation of strongly continuous nonlinear semigroups.

**Theorem 2.2** (see Crandall-Liggett [9])**.** *Let $A$ be a (possibly multivalued) nonlinear operator and $\omega$ be a real number such that $\omega I - A$ is accretive. If $R(I-\lambda A) \supset \overline{D(A)}$ for all sufficiently small positive $\lambda$, then*

$$\lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-1} x \tag{2.1}$$

*exists for all $x \in \overline{D(A)}$ and $t > 0$. Moreover, if $S(t)x$ is defined as the limit in* (2.1)*, then*

$$S(t+\tau) = S(t)S(\tau), \quad t,\ \tau \ge 0, \tag{2.2}$$

$$\lim_{t\downarrow 0} S(t)x = x, \quad x \in \overline{D(A)}, \tag{2.3}$$

$$\|S(t)\| \le e^{\omega t}, \quad t \ge 0. \tag{2.4}$$

In addition, if $A$ is a single-valued operator and $R(I-\lambda A) \supset \mathrm{clco}\ D(A)$ ("clco" means the closure of convex hull of $D(A)$), then $S(t)x$ is the solution of the Cauchy problem

$$\frac{du}{dt} = Au, \quad u(0) = x \in D(A). \tag{2.5}$$

**Definition 2.3** (Exponential dichotomy)**.** A linear semigroup $(T(t))_{t\ge 0}$ is said to have an *exponential dichotomy* or to be *hyperbolic* if there exist a bounded projection $P$ on $\mathbb{X}$ and positive constants $N$ and $\alpha$ satisfying

(1) $T(t)P = PT(t)$, for $t \ge 0$;
(2) $T(t)\big|_{\mathrm{Ker}(P)}$ is an isomorphism from $\mathrm{Ker}(P)$ onto $\mathrm{Ker}(P)$, for all $t \ge 0$, and its inverse on $\mathrm{Ker}(P)$ is defined by $T(-t) := \left(T(t)\big|_{\mathrm{Ker}(P)}\right)^{-1}$;
(3) the following estimates hold

$$\|T(t)x\| \le Ne^{-\beta t}\|x\|, \quad \text{for all } t \ge 0, \quad x \in \mathrm{Im}(P), \tag{2.6}$$

$$\|T(-t)x\| \le Ne^{-\beta t}\|x\|, \quad \text{for all } t \ge 0, \quad x \in \mathrm{Ker}(P). \tag{2.7}$$

The projection $P$ is called the *dichotomy projection* for the hyperbolic semigroup $(T(t))_{t\ge 0}$, and the constants $N$ and $\alpha$ are called *dichotomy constants.*

The following result is well known:

**Lemma 2.4.** *Let $(T(t))_{t\ge 0}$ be a $C_0$-semigroup. Then, it has an exponential dichotomy if and only if $\sigma(T(1)) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset$.*

*Proof.* See Engel-Nagel [12, 1.17 Theorem]. □

From Lemma 2.4, it is easy to prove the following result:

**Lemma 2.5.** *Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup that has an exponential dichotomy. Then, $(S(t))_{t\geq 0}$ has an exponential dichotomy provided that $S(1)$ is sufficiently close to $T(1)$.*

## 3. Yosida Distance and Roughness of Exponential Dichotomy

We begin this section with the concept of Yosida distance between two linear operators. To this end, we recall the concept of Yosida approximation of a linear operator in the definition below (see e.g. [21, 29]).

Given operator $A$ in a Banach space $\mathbb{X}$ with $\rho(A) \supset [\omega, \infty)$, where $\omega$ is a given number, the *Yosida approximation* $A_\lambda$ is defined as $A_\lambda := \lambda^2 R(\lambda, A) - \lambda I$ for sufficiently large $\lambda$.

**Definition 3.1** (Yosida distance)**.** The *Yosida distance* between two linear operators $A$ and $B$ satisfying $\rho(A) \supset [\omega, \infty)$ and $\rho(B) \supset [\omega, \infty)$, where $\omega$ is a given number, is defined to be

$$d_Y(A, B) := \limsup_{\mu\to+\infty} \|A_\mu - B_\mu\|. \tag{3.1}$$

**Lemma 3.2.** *The following assertions are valid:*

(i) *Let $A$, $B$ be the generators of contraction semigroups. Assume further that $D(A) = D(B)$. Then, $A = B$, provided that $d_Y(A, B) = 0$.*
(ii) *Let $A$, $B \in \mathcal{L}(\mathbb{X})$. Then*
$$d_Y(A, B) = \|A - B\|. \tag{3.2}$$
(iii) *If $A$ is the generator of a $C_0$-semigroup $T(t)$ such that $\|T(t)\| \leq Me^{\omega t}$, and $C$ is a bounded operator, then $d_Y(A, A + C)$ is finite. Moreover,*
$$d_Y(A, A + C) \leq M^2\|C\| \tag{3.3}$$
(iv) *Let $A$ be the generator of an analytic semigroup $(T(t))_{t\geq 0}$ such that $\|T(t)\| \leq Me^{\omega t}$, and $C$ be an $A$-bounded operator, that is, $D(C) \supset D(A)$, and there are positive constants $a$, $c$ such that*
$$\|Cx\| \leq a\|Ax\| + c\|x\|. \tag{3.4}$$
*Then, there exists a constant $\delta > 0$ such that if $0 \leq a \leq \delta$, the distance $d_Y(A, A + C)$ is finite. Moreover,*
$$d_Y(A, A + C) \leq aKM + cM^2, \tag{3.5}$$
*where $K$ and $M$ are positive constants that depend only on $A$.*

*Proof.*

(i) From the semigroup theory (see Pazy [21, Lemma 1.3.3]), if $A$ is the generator of a contraction $C_0$-semigroup, then
$$\lim_{\mu\to+\infty} A_\mu x = Ax, \quad x \in D(A). \tag{3.6}$$
Therefore, for $x \in D(A) = D(B)$, $Ax = Bx$.
(ii) Firstly, we have
$$\begin{aligned} &R(\mu, A) - R(\mu, B) \\ &= (\mu - B)R(\mu, B)R(\mu, A) - R(\mu, B)R(\mu, A)(\mu - A). \end{aligned}$$
Set $C_\mu := R(\mu, B)R(\mu, A)$. Then
$$\begin{aligned} R(\mu, A) - R(\mu, B) &= (\mu - B)C_\mu - C_\mu(\mu - A) \\ &= \mu C_\mu + BC_\mu - C_\mu\mu - C_\mu A \\ &= BC_\mu - C_\mu A. \end{aligned}$$

Thus,

$$\mu^2\|R(\mu,A)-R(\mu,B)\| = \mu^2 \left\|BC_\mu - C_\mu A\right\|.$$

We will show that for a bounded linear operator $A$ in $\mathbb{X}$ the following is valid

$$\lim_{\mu\to+\infty} R(\mu,A) = 0. \tag{3.7}$$

In fact, for sufficiently large $\mu$, say $\mu > \|A\|$, using the Neuman series, for large $\mu$, we have

$$\begin{aligned}\|R(\mu,A)\| = \frac{1}{\mu}\left\|R\left(1,\frac{1}{\mu}A\right)\right\| &\le \frac{1}{\mu}\left\|\sum_{n=0}^{\infty}\left(\frac{1}{\mu}A\right)^n\right\| \\ &\le \frac{1}{\mu}\frac{1}{1-\frac{1}{\mu}\|A\|}.\end{aligned}$$

This proves (3.7). Next, by (3.7) and the identity $(\mu - A)R(\mu,A) = I$ we have that

$$\lim_{\mu\to+\infty} \mu R(\mu,A) = I,$$

so

$$\lim_{\mu\to+\infty} \mu^2 C_\mu = I,$$

Finally, we have

$$d_Y(A,B) := \limsup_{\mu\to+\infty} \mu^2 \left\|BC_\mu - C_\mu A\right\| = \|B-A\|,$$

and this finishes the proof.

(iii) By a simple computation we have

$$R(\mu,A+C) - R(\mu,A) = R(\mu,A+C)CR(\mu,A). \tag{3.8}$$

It is known (see e.g. Pazy [21, Theorem 1.1, p. 76] that $A+C$ with $D(A+C) = D(A)$ generates a $C_0$-semigroup $(S(t))_{t\ge0}$ satisfying

$$\|S(t)\| \le Me^{(\omega+M\|C\|)t}, \tag{3.9}$$

so by the Hille-Yosida Theorem

$$\|R(\mu,A)\| \le \frac{M}{\mu-\omega}, \quad \|R(\mu,A+C)\| \le \frac{M}{\mu-(\omega+M\|C\|)},$$

for certain positive constants $M$ and $\omega$. Therefore,

$$\begin{aligned}&\limsup_{\mu\to\infty} \mu^2\|R(\mu,A) - R(\mu,A+C)\| \\ &\le \limsup_{\mu\to\infty} \frac{\mu^2 M^2\|C\|}{(\mu-\omega)(\mu-(\omega+M\|C\|)} \\ &= M^2\|C\| < \infty.\end{aligned} \tag{3.10}$$

(iv) By Pazy [21, Theorem 2.1, p. 80] and the remark that follows it, there exists a positive constant $\delta$ such that if $0 \le a \le \delta$, then, $A+C$ generates an analytic semigroup $(S(t))_{t\ge0}$ that satisfies

$$\|S(t)\| \le Me^{(\omega+\Lambda(c))t},$$

where $\lim_{c\to0}\Lambda(c) = 0$. Therefore, by (3.8), as $A$ generates an analytic semigroup there are positive constants $K$ and $N$ (see Pazy [21, Theorem 5.5, p. 65]) such that if $\mu > N$, then

$$\|AR(\mu,A)\| \le \frac{K}{\mu}.$$

Hence,

$$\begin{aligned}d_Y(A,A+C) &= \limsup_{\mu\to\infty} \mu^2\|R(\mu,A+C) - R(\mu,A)\| \\ &\le \limsup_{\mu\to\infty} \frac{\mu^2 M}{\mu-\omega-\Lambda(c)}\left(a\|AR(\mu,A)\| + c\|R(\mu,A)\|\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{\mu\to\infty} \frac{\mu^2 M}{\mu-\omega-\Lambda(c)}\left(\frac{aK}{\mu}+\frac{cM}{\mu-\omega}\right)\\
&\leq aKM + cM^2.
\end{aligned}
$$

The proof is completed. □

*Remark* 3.3. The reader is also referred to Example 5.4 for another example where the distance between two unbounded linear operators $A$ and $A+B$ that are not included in Parts (iii) and (iv) of the above lemma.

The following theorem is the main result of this section on the roughness of exponential dichotomy.

**Theorem 3.4.** *Let $A$ be the generator of a $C_0$-semigroup that has an exponential dichotomy. Then, the $C_0$-semigroup generated by an operator $B$ also has an exponential dichotomy, provided that $d_Y(A,B)$ is sufficiently small.*

*Proof.* First, we assume that both semigroups generated by $A$ and $B$ are contraction $C_0$-semigroups. Then, by Pazy [21, Lemma 3.4], $e^{tA_\lambda}$ is a $C_0$-semigroup of contractions.

Let $C$ and $D$ be two bounded linear operators in a Banach space $\mathbb{X}$. We will estimate the growth of $e^{tC}-e^{tD}$. By the Variation-of-Constants Formula that is applied to the equation $x'(t)=Cx(t)+(D-C)x(t)$ and by setting $x(t)=e^{tD}x$ we have

$$
x(t)=e^{tC}x+\int_0^t e^{(t-s)C}(D-C)x(s)ds.
$$

For each $t\geq 0$, we have

$$
\begin{aligned}
\left\|e^{tC}x-e^{tD}x\right\| &\leq \int_0^t \left\|e^{(t-s)C}(D-C)e^{sD}x\right\| ds\\
&\leq t\|C-D\|\|e^{tC}\|\|e^{tD}\|\|x\|.
\end{aligned}
$$

Therefore,

$$
\|e^{tA_\lambda}-e^{tB_\lambda}\|\leq t\|A_\lambda-B_\lambda\|\|e^{tA_\lambda}\|\|e^{tB_\lambda}\|. \tag{3.11}
$$

Now we assume that $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are the $C_0$-semigroups generated by $A$ and $B$ that satisfy

$$
\|T(t)\|\leq Me^{\omega t},\quad \|S(t)\|\leq Me^{\omega t}
$$

for certain positive numbers $M$ and $\omega$. As is well known, for the Yosida approximation $A_\lambda$ of the generator $A$ of a $C_0$-semigroup $T(t)$ that satistifies $\|T(t)\|\leq Me^{\omega t}$ the following estimate of the growth is valid (see e.g. Pazy [21, (5.25)])

$$
\|e^{tA_\lambda}\|\leq Me^{2\omega t},\quad \|e^{tB_\lambda}\|\leq Me^{2\omega t}.
$$

Hence, we have

$$
\|e^{tA_\lambda}-e^{tB_\lambda}\|\leq tM^2\|A_\lambda-B_\lambda\|e^{4t\omega}\|. \tag{3.12}
$$

By Pazy [21, Theorem 5.5], for each $x\in\mathbb{X}$, we have

$$
\begin{aligned}
\|T(t)x-S(t)x\| &= \lim_{\lambda\to\infty}\|e^{tA_\lambda}x-e^{tB_\lambda}x\|\\
&\leq tM^2e^{4\omega t}\limsup_{\lambda\to\infty}\|A_\lambda-B_\lambda\|\\
&= tM^2e^{4\omega t}d_Y(A,B).
\end{aligned} \tag{3.13}
$$

Finally, if $d_Y(A,B)$ is sufficiently small, $\|T(1)-S(1)\|$ is sufficiently small as well, and thus, $(S(t))_{t\geq 0}$ has an exponential dichotomy. □

**Corollary 3.5.** *Let $A$ be the generator of an exponentially dichotomous $C_0$-semigroup $(T(t))_{t\geq 0}$ in $\mathbb{X}$ and $C$ be a bounded linear operator in $\mathbb{X}$. Then, the operator $A+C$ with domain $D(A+C)=D(A)$ generates an exponentially dichotomous $C_0$-semigroup, provided that $\|C\|$ is sufficiently small.*

*Proof.* Let $A$ generate a $C_0$-semigroup $(T(t))_{t\geq 0}$ that satisfies $\|T(t)\| \leq Me^{\omega}$. By (3.10) if $\|C\|$ is sufficiently small, then $d_Y(A, A+C)$ is sufficiently small as well, so by Theorem 3.4, the semigroup generated by $A+C$ also has an exponential dichotomy. $\square$

**Corollary 3.6.** *Let $A$ be the generator of a hyperbolic analytic semigroup $(T(t))_{t\geq 0}$ in $\mathbb{X}$ satisfying $\|T(t)\| \leq Me^{\omega t}$. Then, if $C$ is a $A$-bounded linear operator in $\mathbb{X}$, that is, it satisfies for all $x \in D(A)$*

$$\|Cx\| \leq a\|Ax\| + c\|x\|$$

*for certain positive constants $a$ and $c$. Then, $A+C$ generates an exponentially dichotomous analytic semigroup, provided that $a$ and $c$ are sufficiently small.*

*Proof.* By assumption for each $x \in D(A)$ we have

$$\begin{aligned} \|Cx\| &\leq |||C||| \cdot |||x||| \\ &\leq |||C||| \cdot \|Ax\| + |||C||| \cdot |||x|||. \end{aligned} \tag{3.14}$$

As is well known (see e.g. Pazy [21, Theorem 2.1, p. 80]) for sufficiently small $|||C|||$, the operator $A+C$ generates an analytic semigroup. Moreover, by (3.5) if $|||C|||$ is sufficiently small, $d_Y(A, A+C)$ is sufficiently small, so by Theorem 3.4, the semigroup generated by $A+C$ has an exponential dichotomy. $\square$

## 4. Nonlinear Perturbation of Exponential Dichotomy

We consider evolution equations of the form

$$u'(t) = Au(t), \tag{4.1}$$

where $A$ is a nonlinear single-valued operator from $D(A) \subset \mathbb{X}$ to $\mathbb{X}$. We will assume that $A(0) = 0$, $A$ is proto-differentiable in a neighborhood of 0 (see the definition below) and the linearized evolution equation at 0 (i.e. $u' = \partial A(0)u$) has an exponential dichotomy. Roughly speaking, our next result in this section to show that if the proto-derivative of $A$ is continuous at 0 in the Yosida distance' sense, then there exist stable and unstable invariant manifolds in a neighborhood of 0.

4.1. **Proto-Differentiability.** The convergence of sets in a complete metric space $(X, d)$ in this section is adapted from the similar concept from Rockafellar [23]. A family of sets $\{S_t\}_{t>0}$ in a complete metric space $(X, d)$ is said to *converge* to a set $S \subset X$ as $t \downarrow 0$, written

$$S = \lim_{t\downarrow 0} S_t, \tag{4.2}$$

if $S$ is closed and

$$\lim_{t\downarrow 0} \operatorname{dist}(w, S_t) = \operatorname{dist}(w, S), \qquad \text{for all } w \in X, \tag{4.3}$$

where "dist" denotes the distance $\operatorname{dist}(w, S) := \inf_{y\in S}\{d(x, y)\}$. It is often convenient to view (4.3) as the equation

$$S = \liminf_{t\downarrow 0} S_t = \limsup_{t\downarrow 0} S_t, \tag{4.4}$$

where

$$\liminf_{t\downarrow 0} S_t = \left\{ w \in \mathbb{X} : \limsup_{t\downarrow 0} \operatorname{dist}(w, S_t) = 0 \right\}, \tag{4.5}$$

and

$$\limsup_{t\downarrow 0} S_t = \left\{ w \in \mathbb{X} : \liminf_{t\downarrow 0} \operatorname{dist}(w, S_t) = 0 \right\}. \tag{4.6}$$

As in this paper all operators and functions under consideration are assumed to be sing-valued we will adapt the definition of proto-differentiability from Rockafellar [23] accordingly.

**Assumption 1.** *Let $G : D(G) \subset V \subset \mathbb{X} \to \mathbb{X}$ be an operator. We assume that domain $D(G)$ is an open subset of a vector subspace $V \subset \mathbb{X}$.*

**Definition 4.1** (Proto-differentiability)**.** Under Assumption 1 let $x \in D(G)$ be a given vector. For each $t \in (0,1)$, let $T_t \colon V \subset \mathbb{X} \to \mathbb{X}$ be an operator defined as

$$T_t(w) = \frac{G(x+tw) - G(x)}{t} \tag{4.7}$$

for each $w \in B_\varepsilon(x) \cap V$, where $\varepsilon$ is a sufficient small positive constant such that $\bar{B}_\varepsilon(x) \cap V \subset D(G)$. Then, we say that $G$ is *proto-differentiable* at $u$ if there is a linear operator $T \colon D(T) \subset V \to \mathbb{X}$ such that in $\bar{B}_\varepsilon(x) \times \mathbb{X}$ the graph of $T_t$ converges to the graph of $T$ as $t \downarrow 0$. In this case we write $\partial G(x) = T$.

*Remark* 4.2. As all functions considered in this paper are assumed to be single-valued, the proto-differentiablity of a function $G$ mentioned in Definition 4.1 can be stated equivalently as follows (see [1, 15]): $G$ is proto-differentiable at $x \in D(G)$ if and only if $\partial_i G(x) = \partial_s G(x)$, where $\partial_i G(x)$ and $\partial_s G(x)$ are defined as:

(1) The linear operator $\partial_i G(x)$ is defined at all $u \in \mathbb{X}$ and its value at $u$ is $v =: \partial_i G(x)u$ if for each sequence $\{t_n\} \downarrow 0$, there exists a sequence $\{(u_n, v_n)\} \subset \mathbb{X} \times \mathbb{X}$ such that $(u_n, v_n) \to (u, v)$ in $\mathbb{X} \times \mathbb{X}$, $x + t_n u_n \in D(G)$ and

$$\frac{G(x+t_n u_n) - G(x)}{t_n} = v_n. \tag{4.8}$$

(2) The linear operator $\partial_s G(x)$ is defined at all $u \in \mathbb{X}$ and its value at $u$ is $v =: \partial_s G(x)u$ if there exists a sequence $\{t_n\} \downarrow 0$, and a sequence $\{(u_n, v_n)\} \subset \mathbb{X} \times \mathbb{X}$ such that $(u_n, v_n) \to (u, v)$ in $\mathbb{X} \times \mathbb{X}$, $x + t_n u_n \in D(G)$ and

$$\frac{G(x+t_n u_n) - G(x)}{t_n} = v_n. \tag{4.9}$$

By definition it is apparent that $\partial_i G(x) \subset \partial_s G(x)$.

Before we proceed to studying the nonlinear perturbation of exponential dichotomy we consider some special cases.

**Example 4.3.** Consider $G(x) = Ux$, where $U \colon D(U) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator. Then, $G$ is proto-differentiable at every $x \in D(U)$ and $\partial G(x) = U$.

*Proof.* Indeed, we have

$$D_t(w) = \frac{G(u+tw) - G(u)}{t} = \frac{U(u+tw) - Uu}{t} = \frac{U(tw)}{t} = Uw,$$

so, the operator $D_t$ and $U$ are identical in any neighborhood of $u$, and their graphs must be the same. $\square$

**Definition 4.4** (Gâteaux differentiability)**.** Suppose $\mathbb{X}$ and $\mathbb{X}$ are Banach spaces, $U \subseteq X$ is open, and $F : U \subset \mathbb{X} \to \mathbb{Y}$. Operator $F$ is *Gâteaux differentiable* at $u \in U$ if there exists an operator $L \in \mathcal{L}(\mathbb{X}, \mathbb{X})$ such that

$$\lim_{\tau \to 0} \frac{F(u+\tau w) - F(u)}{\tau} = Lw,$$

for all $w \in \mathbb{X}$. In this case we denote the Gâteaux derivative of $F$ at $u$ by $dF(u)$.

**Proposition 4.5.** *Let $G \colon D(G) = V \subset \mathbb{X} \to \mathbb{X}$ and let $V$ be equipped with the graph norm $|||x||| := \|x\| + \|Sx\|$, where $S : D(S) = V \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator. Then, the following assertions are true:*

*(i) If $G$ is a Gâteaux differentiable operator from $(V, |||\cdot|||)$ to $(\mathbb{X}, \|\cdot\|)$ and $T$ is its derivative at $x$. Then*

$$T \subset \partial_i G(x). \tag{4.10}$$

*(ii) If $G$ is a Fréchet differentiable operator from $(V, |||\cdot|||)$ to $(\mathbb{X}, \|\cdot\|)$ and $T$ is its derivative at $x \in V$ that is a closed operator in $\mathbb{X}$. Then, $G$ is proto-differentiable at $x \in V$ and*

$$\partial G(x) = T. \tag{4.11}$$

*Moreover, if $H = S + G$, then $\partial H(x) = S + G'(x)$, for all $x \in \mathbb{X}$.*

*Proof.* Part (i): By definition, as $G$ is a Gâteaux differentiable operator from $(V, |||\cdot|||)$ to $(\mathbb{X}, \|\cdot\|)$ and $T$ is its Gâteaux derivative at $x$, we have

$$\lim_{t\to 0} \frac{\|G(x+tw) - G(x) - T(tw)\|}{|||tw|||} = 0, \tag{4.12}$$

where $w \in B_\varepsilon(x) \cap V$ for a sufficiently small positive $\varepsilon$. Therefore,

$$0 = \lim_{t\to 0} \frac{\|G(x+tw) - G(x) - T(tw)\|}{\|tw\| + \|S(tw)\|}. \tag{4.13}$$

We will prove that $T \subset \partial_i G(x) \subset \partial_s G(x) \subset T$. Let $0 \neq w \in D(T)$ and $\{t_n\} \downarrow 0$ be any sequence. By (4.13),

$$0 = \lim_{t\to 0} \frac{\|G(x+t_n w) - G(x) - T(t_n w)\|}{\|t_n w\| + \|S(t_n w)\|}. \tag{4.14}$$

Set

$$u_n = w, \quad v_n := \frac{G(x+t_n w) - G(x)}{t_n}.$$

Then, for $n$ large enough $x + t_n w \in B_\varepsilon(x) \cap V \subset D(G)$, and setting $y := G(x)$ we have

$$y + t_n v_n = G(x) + t_n \frac{G(x+t_n w) - G(x)}{t_n} = G(x+t_n w),$$

so, $(x + t_n u_n, y + t_n v_n) \in \text{graph}(G)$. Obviously, $u_n \to w$. We will show that

$$v_n = \frac{G(x+t_n w) - G(x)}{t_n} \to Tw, \quad \text{as } n \to \infty. \tag{4.15}$$

By (4.14)

$$\lim_{n\to\infty} \frac{\left\| \frac{G(x+t_n w) - G(x)}{t_n} - T(w) \right\|}{\|w\| + \|S(w)\|}. \tag{4.16}$$

This shows that (4.15) is valid. Finally, we have that $(w, Tw) \in \text{graph}(\partial_i G(x))$. By the arbitrary nature of $w$, this yields that

$$T \subset \partial_i(G)(x). \tag{4.17}$$

Part (ii): Suppose that $(u, v) \in \text{graph}(\partial_s(G))$. This means that there exists a sequence $\{t_n\} \downarrow 0$ and sequence $(u_n, v_n) \to (u, v) \in \mathbb{X} \times \mathbb{X}$ such that $G(x + t_n u_n) = G(x) + t_n v_n$ for each $n$. As $G$ is Fréchet differentiable at $x$ we have

$$\lim_{n\to\infty} \frac{\|G(x+t_n u_n) - G(x) - T(t_n u_n)\|}{|||t_n u_n|||} = 0, \tag{4.18}$$

so, as $u_n \to u \neq 0$,

$$0 = \lim_{n\to\infty} \frac{\left\| \frac{G(x+t_n u_n) - G(x)}{t_n} - T(u_n) \right\|}{\|u_n\| + \|S(u_n)\|} \tag{4.19}$$

$$= \lim_{n\to\infty} \left\| \frac{G(x+t_n u_n) - G(x)}{t_n} - T(u_n) \right\| \tag{4.20}$$

$$= \lim_{n\to\infty} \|v_n - T(u_n)\|. \tag{4.21}$$

As $v_n \to v \in \mathbb{X}$, we have that $Tu_n \to v$ as $n \to \infty$. Since $T$ is a closed operator in $\mathbb{X}$ this yields that $u \in D(T)$ and $Tu = v$. In other words, $\partial_s G(x) \subset T$. Combined with Part (i) and the fact from the definition that $\partial_i G(x) \subset \partial_s G(x)$ we have

$$T \subset \partial_i G(x) \subset \partial_s G(x) \subset T.$$

This yields that $\partial G(x)$ exists and is equal to $T$, completing the proof. $\square$

**Proposition 4.6.** *Consider $x' = G(x) = Ux + F(x)$, where the linear operator $U$ is the generator of a $C_0$-semigroup in $\mathbb{X}$ and $F(\cdot)$ is Fréchet differentiable at $x \in \mathbb{X}$. Then, $G$ is proto-differentiable, and*

$$\partial G(x) = U + F'(x). \tag{4.22}$$

*Proof.* We have

$$\begin{aligned} D_t(w) = \frac{G(u+tw) - G(u)}{t} &= \frac{U(u+tw) - Uu + F(u+tw) - F(u)}{t} \\ &= Uw + \frac{F(u+tw) - F(u)}{t}. \end{aligned}$$

Since $F(x)$ is Fréchet differentiable, we have

$$\lim_{t\to 0} \left( \frac{F(u+tw) - F(u)}{t} - F'(u) \right) = 0.$$

Set $W := U + F'(x)$. We will show that

$$W \subset \partial_i G(x) \subset \partial_s G(x) \subset W. \tag{4.23}$$

First, we show that $W \subset \partial_i G(x)$. Let $0 \neq w \in D(W)$ and $\{t_n\} \downarrow 0$ be any sequence. By assumption,

$$\begin{aligned} 0 &= \lim_{t\to 0} \frac{\|F(x+t_n w) - F(x) - F'(x)(t_n w)\|}{\|t_n w\|} \\ &= \lim_{t\to 0} \frac{\|G(x+t_n w) - G(x) - W(t_n w)\|}{\|t_n w\|} \qquad (4.24) \\ &= \lim_{n\to\infty} \frac{\left\| \frac{G(x+t_n w) - G(x)}{t_n} - W(w) \right\|}{\|w\|}. \qquad (4.25) \end{aligned}$$

Set

$$u_n = w, \quad v_n := \frac{G(x+t_n w) - G(x)}{t_n}.$$

Then, for $n$ large enough $x + t_n w \in B_\varepsilon(x) \cap V \subset D(G)$, and setting $y := G(x)$ we have

$$y + t_n v_n = G(x) + t_n \frac{G(x+t_n w) - G(x)}{t_n} = G(x + t_n w),$$

so, $(x + t_n u_n, y + t_n v_n) \in \text{graph}(G)$. Obviously, $u_n \to w$. We will show that

$$v_n = \frac{G(x+t_n w) - G(x)}{t_n} \to Ww \tag{4.26}$$

as $n \to \infty$. By (4.25)

$$\lim_{n\to\infty} \frac{\left\| \frac{G(x+t_n w) - G(x)}{t_n} - W(w) \right\|}{\|w\|}. \tag{4.27}$$

This shows that (4.26) is valid. Finally, this shows that $(w, Ww) \in \text{graph}(\partial_i G(x))$. By the arbitrary nature of $w$, this yields that

$$W \subset \partial_i(G)(x). \tag{4.28}$$

Next, we will show that $\partial_s G(x) \subset W$. Suppose that $(u, v) \in \text{graph}(\partial_s(G))$. This means that there exists a sequence $\{t_n\} \downarrow 0$ and sequence $(u_n, v_n) \to (u, v) \in \mathbb{X} \times \mathbb{X}$ such that $G(x + t_n u_n) = G(x) + t_n v_n$ for each $n$. As $F$ is Fréchet differentiable at $x$ we have

$$\begin{aligned} 0 &= \lim_{t\to 0} \frac{\|F(x+t_n u_n) - F(x) - F'(x)(t_n u_n)\|}{\|t_n u_n\|} \\ &= \lim_{t\to 0} \frac{\|G(x+t_n u_n) - G(x) - W(t_n u_n)\|}{\|t_n u_n\|} \\ &= 0, \end{aligned}$$

so, as $u_n \to u \neq 0$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{\left\| \frac{G(x+t_n u_n)-G(x)}{t_n} - T(u_n) \right\|}{\|u_n\| + \|S(u_n)\|} \\ &= \lim_{n\to\infty} \left\| \frac{G(x+t_n u_n)-G(x)}{t_n} - W(u_n) \right\| \\ &= \lim_{n\to\infty} \|v_n - W(u_n)\|. \end{aligned}$$

As $v_n \to v \in \mathbb{X}$, $Wu_n \to v$ as $n \to \infty$. Since $U$ is a closed operator in $\mathbb{X}$ and $F'(x)$ is a bounded operator, the operator $W$ is closed, so this yields that $u \in D(W)$ and $Wu = v$. In other words, $\partial_s G(x) \subset W$. Finally, (4.23) holds true, completing the proof. ☐

4.2. **Families of linear operators depending continuously on a parameter.** Recall that $d_Y(U, V)$ stands for the Yosida distance between two closed operators $U$ and $V$ in $\mathbb{X}$.

**Definition 4.7.** Let $A_\alpha$ be a family of (possibly unbounded) linear operators in $\mathbb{X}$ with parameter $\alpha \in S$, where $S$ is a metric space satisfying $d_Y(A_{\alpha_0}, A_\beta) < \infty$ for every $\beta \in S$ and $\alpha_0$ is a certain element of $S$. The family of such operators $A_\alpha$ is said to be *continuous* at $\alpha_0 \in S$ if

$$\lim_{\alpha\to\alpha_0} d_Y(A_\alpha, A_{\alpha_0}) = 0. \tag{4.29}$$

**Example 4.8.** Let $A$ be the generator of a $C_0$-semigroup. Then, the family $A_C := \{A + C, C \in \mathcal{L}(\mathbb{X})\}$ is continuous at $\alpha_0 := C = 0$. In fact, as in Lemma 3.2, if the semigroup $T(t)$ generated by $A$ satisfies $\|T(t)\| \leq Me^{\omega t}$, then

$$d_Y(A, A+C) \leq M^2 \|C\|.$$

Therefore, when $C \to 0$ in $\mathcal{L}(\mathbb{X})$, $d_Y(A, A+C) \to 0$.

**Example 4.9.** Let $A$ be the generator of an analytic semigroup and $C$ be an $A$-bounded operator. Since $D(C) \supset D(A)$ and there are constants $a$, $c$ such that

$$\|Cx\| \leq a\|Ax\| + c\|x\|,$$

the restriction of $C$ on $D(A)$ is a bounded linear operator from $(D(A), |||\cdot|||)$ to $(\mathbb{X}, \|\cdot\|)$, where

$$|||x||| := \|x\| + \|Ax\|, \qquad x \in D(A).$$

If we define $\mathcal{F}$ to be the family of all operators of the form $A_C := A + C$ where $C$ is $A$-bounded, and the metric space $S$ as $\mathcal{L}((D(A), |||\cdot|||), \mathbb{X})$, then

$$\lim_{C\to 0} d_Y(A_C, A_0) = 0.$$

4.3. **Existence of invariant manifolds.** This subsection deals with the existence of invariant manifolds.

**Assumption 2.**

(A2.1) *There exists a real number $\omega$ such that $\omega I - A$ is m-accretive;*
(A2.2) *The proto-derivative $\partial A(x)$ exists for each $x \in D(A)$ as a single-valued linear operator in $\mathbb{X}$ such that $\omega I - A$ is m-accretive.*
(A2.3) *Yosida distance $d_Y(\partial A(x), \partial A(0))$ satisfies*

$$\sup_{x\in D(A)} d_Y(\partial A(x), \partial A(0)) = \varepsilon < \infty. \tag{4.30}$$

The assumption yields that the operator $A$ generates a nonlinear semigroup in $\overline{D(A)}$ by the Crandall-Liggett Theorem (Theorem 2.2).

Let us consider the following family of equations associated with Eq. (4.1)

$$\begin{cases} \dfrac{d}{dt}u_\lambda(t) = A^\lambda u_\lambda(t), \quad 0 \le t \le 1, \\ \quad u_\lambda(0) = x, \end{cases} \tag{$E_\lambda$}$$

where

$$J_\lambda^A := (I - \lambda A)^{-1}, \quad A^\lambda := \frac{1}{\lambda}\left(J_\lambda^A - I\right), \tag{4.31}$$

for $\lambda > 0$ and $\lambda\omega < 1$. Note that with our notations if $U$ is a linear operator, then

$$J_\lambda^U = (I - \lambda U)^{-1} = \frac{1}{\lambda} R\left(\frac{1}{\lambda}, U\right).$$

Therefore,

$$U^\lambda = \frac{1}{\lambda}\left(\frac{1}{\lambda} R\left(\frac{1}{\lambda}, U\right) - I\right) = U_{1/\lambda},$$

where $U_\xi$ is the Yosida approximation of a linear operator $U$ with parameter $\xi$. Therefore, the Yosida distance between two linear operators $U$ and $V$ can be written as

$$\begin{aligned} d_Y(U,V) &= \limsup_{\lambda\downarrow 0} \frac{1}{\lambda^2}\left\| R\left(\frac{1}{\lambda}, U\right) - R\left(\frac{1}{\lambda}, V\right)\right\| \\ &= \limsup_{\lambda\downarrow 0} \frac{1}{\lambda^2}\left\| \left(\frac{1}{\lambda} - U\right)^{-1} - \left(\frac{1}{\lambda} - V\right)^{-1}\right\| \\ &= \limsup_{\lambda\downarrow 0} \frac{1}{\lambda}\left\| J_\lambda^U - J_\lambda^V \right\| \qquad (4.32) \\ &= \limsup_{\lambda\downarrow 0} \frac{1}{\lambda}\left\| U_{1/\lambda} - V_{1/\lambda}\right\|. \qquad (4.33) \end{aligned}$$

Recall that if $Q$ is a nonlinear operator such that $\omega I - Q$ is m-accretive if and only if

$$\left\| J_\lambda^Q x - J_\lambda^Q y\right\| \le \frac{1}{1-\lambda\omega}\|x - y\|, \quad x,\, y \in \mathbb{X}. \tag{4.34}$$

By Kato [15, Lemmas 1.1, 1.2], $A^\lambda$ is proto-differentiable in a neighborhood of 0 and $\partial A^\lambda(x) = dA^\lambda(x)$ for $x$ in a small neighborhood of 0. Consider the linearized equation for equation $(E_\lambda)$ along a solution $S_\lambda(t)x_0$

$$\begin{cases} \dfrac{d}{dt}v_\lambda(t) + dA^\lambda(S_\lambda(t)x_0)v_\lambda(t) = 0, \quad 0 \le t \le 1, \\ \qquad\qquad v_\lambda(0) = w. \end{cases} \tag{$L_\lambda; w$}$$

Since for each sufficiently small (but positive) $\lambda$, $A^\lambda(x)$ is a Lipschitz operator with $A^\lambda(0) = 0$, it generates a a semigroup $(S_\lambda(t))_{t\ge 0}$. Moreover, $u_\lambda(t) := S_\lambda(t)x$ is the classical solution of $(E_\lambda)$. By Kato [15, Proposition 3.1] the solution of the Cauchy problem $(E_\lambda)$ satisfies

$$\|v_\lambda(t)\| \le e^{\mu t}\|w\|, \tag{4.35}$$

where $\mu = \omega/(1-\lambda\omega)$, $0 < \lambda < 1/\omega$.

**Lemma 4.10.** *Define $\mathcal{S}_\lambda \colon \mathbb{X} \to C([0,1],\mathbb{X})$ by*

$$(\mathcal{S}_\lambda x)(t) = S_\lambda(t)x, \quad \text{for } t \in [0,1]. \tag{4.36}$$

*Then, we have*

1. *(1) $\mathcal{S}_\lambda$ is Gâteaux differentiable at each $x \in B_r(0)$ and the Gâteaux derivative $d\mathcal{S}_\lambda(x)$ represents a unique solution of $(L_\lambda; w)$,*
$$d\mathcal{S}_\lambda(x)w = v_\lambda(\cdot; 0, w) \tag{4.37}$$
*is solution of $(L_\lambda; w)$.*
2. *(2) Fix $w \in \mathbb{X}$. The mapping $z \mapsto d\mathcal{S}_\lambda(z)w$ is continuous from $B_r(0)$ into $C([0,1],\mathbb{X})$.*

*(3) For $x$, $y \in B_r(0)$,*

$$\mathcal{S}_\lambda y - \mathcal{S}_\lambda x = \int_0^1 d\mathcal{S}_\lambda(\theta y + (1-\theta)x)(y-x)d\theta \tag{4.38}$$

*in $C([0,1],\mathbb{X})$.*

*Proof.* See Kato [15, Proposition 3.2, Lemma 3.3, and Lemma 3.4]. □

**Lemma 4.11.** *Let $A(\cdot)$, $B(\cdot)$ be continuous functions from $[0,1]$ to $\mathcal{L}(\mathbb{X})$. Assume that $x(t)$ and $y(t)$ are the solutions of the Cauchy problems*

$$x'(t) = A(t)x(t), \quad x(0) = w,$$
$$y'(t) = B(t)y(t), \quad y(0) = w,$$

*respectively. Moreover, assume that the evolution operators of these two equations satisfy*

$$\|X(t,s)\| \le Me^{\omega(t-s)}, \quad \|Y(t,s)\| \le Me^{\omega(t-s)}, \qquad 0 \le s \le t \le 1.$$

*Then,*

$$\|x(t) - y(t)\| \le Me^{2\omega} \sup_{0\le\tau\le1} \|A(\tau) - B(\tau)\| \cdot \|w\|. \tag{4.39}$$

*Proof.* We have

$$y'(t) = A(t)y(t) + (B(t) - A(t))y(t),$$

so

$$y(t) = X(t,0)w + \int_0^t X(t,\tau)(B(\tau) - A(\tau))y(\tau)d\tau.$$

Next,

$$\begin{aligned} \|x(t) - y(t)\| &\le \int_0^t \|X(t,\tau)\| \cdot \|B(\tau) - A(\tau)\| \cdot \|y(\tau)\| d\tau \\ &\le Me^{2\omega} \sup_{0\le\tau\le1} \|A(\tau) - B(\tau)\| \cdot \|w\|. \end{aligned} \tag{4.40}$$

This completes the proof. □

**Corollary 4.12.** *Under Assumption 2, let $v_\lambda^x(t)$ and $v_\lambda^0(t)$ be solutions to $(L_\lambda; w)$ with $x_0 = x$ and $x_0 = 0$, respectively. There exists $0 < r_0 < r$ such that $\|S_\lambda(t)x\| \le e^{2\omega t}\|x\| < r/2$, for small enough $\lambda$, for all $x \in \mathbb{X}$ and $0 < t < 1$ the following estimate hold true*

$$\left\|v_\lambda^x(t) - v_\lambda^0(t)\right\| \le e^{2\mu}\|w\| \sup_{z\in\mathbb{X}} d_Y(\partial A(z), \partial A(0)), \tag{4.41}$$

*where $\mu = \omega/(1-\lambda\omega)$.*

*Proof.* Applying the Lemma 4.11 to $(L_\lambda; w)$ where

$$A(t) := dA^\lambda(S_\lambda(t)x), \quad B(t) := dA^\lambda(S_\lambda(t)0) = dA^\lambda(0)$$

and taking into account (4.35), we arrive at

$$\begin{aligned} \left\|v_\lambda^x(t) - v_\lambda^0(t)\right\| &\le e^{2\mu}\|w\| \sup_{0\le t\le1} \left\|dA^\lambda(S_\lambda(t)x) - dA^\lambda(S_\lambda(t)0)\right\| \\ &\le e^{2\mu}\|w\| \sup_{0\le t\le1} \limsup_{\lambda\downarrow0} \left\|dA^\lambda(S_\lambda(t)x) - dA^\lambda(S_\lambda(t)0)\right\| \\ &\le e^{2\mu}\|w\| \sup_{0\le t\le1} d_Y(\partial A(S(t)x), \partial A(S(t)0)) \\ &\le e^{2\mu}\|w\| \sup_{z\in\mathbb{X}} d_Y(\partial A(z), \partial A(0)). \end{aligned} \tag{4.42}$$

This finishes the proof. □

**Corollary 4.13.** *Let 0 be the stationary solution of* (4.1) *and let the Assumption 2 be made. Let us denote by* $(S(t))_{t\geq 0}$ *and* $(T(t))_{t\geq 0}$ *the* $C_0$*-semigroups generated by* $A$ *and* $\partial A(0)$*, respectively. Then the following statements are true*

$$\|\phi(x) - \phi(y)\| \leq e^{3\omega}\|x - y\| \sup_{z\in\mathbb{X}} d_Y(\partial A(z), \partial A(0)), \tag{4.43}$$

*for all* $x$, $y \in B_{r_0}(0)$ *and* $r_0$ *is sufficiently small positive real number.*

*Proof.* Set

$$\phi(t)x = S(t)x - T(t)x, \tag{4.44}$$

$$\phi_\lambda(x) = \mathcal{S}_\lambda(x) - \mathcal{T}_\lambda(x), \tag{4.45}$$

where

$$(\mathcal{S}_\lambda x)(t) = S_\lambda(t)x, \quad \text{for all } t \in [0,1],$$
$$(\mathcal{T}_\lambda x)(t) = T_\lambda(t)x, \quad \text{for all } t \in [0,1].$$

In order to prove that $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ are $\varepsilon_0$-close, it suffices to show that $\mathrm{Lip}(\phi_\lambda) \leq \varepsilon$, where $\varepsilon = \varepsilon(\varepsilon_0)$ is positive and independent of $x$, $y$, i.e. the following estimate

$$\|\phi_\lambda(x) - \phi_\lambda(y)\| \leq \varepsilon\|x - y\|, \quad \text{for all } x,\ y \in \mathbb{X} \tag{4.46}$$

holds. Then, by letting $\lambda \downarrow 0$, for fixed $t$, we have

$$\|\phi(t)x - \phi(t)(y)\| \leq \varepsilon\|x - y\|, \quad \text{for all } x,\ y \in \mathbb{X}, \quad t \in [0,1]. \tag{4.47}$$

Our task is now to prove (4.46). By Kato [15, Lemma 3.4], we have

$$\begin{aligned}
&\|\phi_\lambda(x) - \phi_\lambda(y)\| \\
&= \|\mathcal{S}_\lambda(x) - \mathcal{S}_\lambda(y) - \mathcal{T}_\lambda(x) + \mathcal{T}_\lambda(y)\| \\
&\leq \int_0^1 \|d\mathcal{S}_\lambda(\theta x + (1-\theta)y)(x - y) - d\mathcal{T}_\lambda(\theta x + (1-\theta)y)(x - y)\|\, d\theta.
\end{aligned} \tag{4.48}$$

Let $\eta = \theta x + (1-\theta)y$. Then

$$v_\lambda^\eta(t) = [d\mathcal{S}_\lambda(\eta)(x - y)](t) \tag{4.49}$$

and

$$v_\lambda^0(t) := v_\lambda(t) = [d\mathcal{T}_\lambda(\eta)(x - y)](t) \tag{4.50}$$

are solutions to $(E_\lambda)$ with the operators are $dA_\lambda(S_\lambda(t)\eta)$ and $dA_\lambda(0)$, respectively, and $w = x - y$. By Corollary 4.12, for $t \in [0,1]$, we have

$$\begin{aligned}
&\|d\mathcal{S}_\lambda(\theta x + (1-\theta)y)(x - y)(t) - d\mathcal{T}_\lambda(\theta x + (1-\theta)y)(x - y)(t)\| \\
&= \|v_\lambda^\eta(t) - v_\lambda(t)\| \\
&\leq e^{2\mu}\|x - y\| \sup_{z\in\mathbb{X}} d_Y(\partial A(z), \partial A(0)),
\end{aligned} \tag{4.51}$$

where $\mu = \omega/(1-\lambda\omega)$, $\omega$ is a fixed positive number that makes $\omega I - A$ m-accretive. Therefore,

$$\begin{aligned}
&\|d\mathcal{S}_\lambda(\theta x + (1-\theta)y)(x - y) - d\mathcal{T}_\lambda(\theta x + (1-\theta)y)(x - y)\| \\
&= \sup_{t\in[0,1]} \|d\mathcal{S}_\lambda(\theta x + (1-\theta)y)(x - y)(t) - d\mathcal{T}_\lambda(\theta x + (1-\theta)y)(x - y)(t)\| \\
&\leq e^{2\mu}\|x - y\| \sup_{z\in\mathbb{X}} d_Y(\partial A(z), \partial A(0)).
\end{aligned}$$

Next, we have

$$\begin{aligned}
\|\phi_\lambda(x) - \phi_\lambda(y)\| &\leq \int_0^1 e^{2\mu}\|x - y\| \sup_{z\in\mathbb{X}} d_Y(\partial A(z), \partial A(0))d\theta \\
&= e^{2\mu}\|x - y\| \sup_{z\in\mathbb{X}} d_Y(\partial A(z), \partial A(0)).
\end{aligned}$$

Letting $\lambda \downarrow 0$, we arrive at

$$\|\phi(x) - \phi(y)\| \leq e^{3\omega}\|x - y\| \sup_{z\in\mathbb{X}} d_Y(\partial A(z), \partial A(0)). \tag{4.52}$$

This completes the proof of (4.43). □

**Definition 4.14** (Lipschitz invariant manifold)**.** Let $(S(t))_{t\geq 0}$ be a semigroup of (possibly nonlinear) operators on the Banach space $\mathbb{X}$. A set $\mathcal{M} \subset \mathbb{X}$ is said to be a *Lipschitz invariant manifold* for semigroup $(S(t))_{t\geq 0}$ if the phase space $\mathbb{X}$ is split into a direct sum $\mathbb{X} = \mathbb{X}^1 \oplus \mathbb{X}^2$, where $\mathbb{X}^1$ and $\mathbb{X}^2$ are closed subspaces of $\mathbb{X}$, and there exists a Lipschitz continuous mapping $\Phi\colon \mathbb{X}^1 \to \mathbb{X}^2$ so that $\mathcal{M} = \mathrm{graph}(\Phi)$ and $S(t)\mathcal{M} \subset \mathcal{M}$ for $t \geq 0$.

For brevity, a Lipschitz invariant manifold will be called simply invariant manifold if this does not cause any confusion.

**Definition 4.15** ($\varepsilon$-close (see Minh-Wu [19]))**.** Two semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$ are said to be *$\varepsilon$-close* if there exist a positive constant $\varepsilon$ such that

$$\|\phi(t)x - \phi(t)y\| \leq \varepsilon\|x - y\|, \quad \text{for all } t \in [0, 1], \quad x, y \in \mathbb{X}, \tag{4.53}$$

where

$$\phi(t)x := S(t)x - T(t)x, \quad \text{for all } x \in \mathbb{X}. \tag{4.54}$$

**Theorem 4.16** (Unstable invariant manifold)**.** *Under Assumption 2, let $0 \in \mathbb{X}$ be a stationary solution of Eq.* (4.1)*. Moreover, assume that the strongly continuous semigroup $(T(t))_{t\geq 0}$ has an exponential dichotomy with projection $P$. Then, there exists a positive constant $\varepsilon_0$, such that if $0 < \varepsilon < \varepsilon_0$, Eq.* (4.1) *has a unique invariant manifold $\mathcal{W}^{\mathrm{u}} \subset \mathbb{X}$, presented as graph of a Lipschitz continuous mapping $\Phi\colon \mathrm{Ker}(P) \to \mathrm{Im}(P)$. Moreover,* $\lim_{\varepsilon_0\to 0} \mathrm{Lip}(\Phi) = 0$.

*Proof.* The theorem is an immediate consequence of Minh-Wu [19, Lemmas 2.11, 2.12, 2.13] and Corollary 4.13. □

**Theorem 4.17** (Stable invariant manifold)**.** *With the assumptions in Theorem 4.16, the set*

$$\mathcal{W}^{\mathrm{s}} := \left\{ x \in \mathbb{X} : \lim_{t\to+\infty} S(t)x = 0 \right\} \tag{4.55}$$

*is a stable invariant manifold of Eq.* (4.1)*, represented by the graph of a Lipschitz continuous mapping $\Psi\colon \mathrm{Im}(P) \to \mathrm{Ker}(P)$, i.e. $\mathcal{W}^{\mathrm{s}} = \mathrm{graph}(\Psi)$ and $S(t)\mathcal{W}^{\mathrm{s}} \subset \mathcal{W}^{\mathrm{s}}$, for all $t \geq 0$.*

*Proof.* The theorem is an immediate consequence of Minh-Wu [19, Theorem 2.16] and Corollary 4.13. □

Below we present local versions of Theorems 4.16 and 4.17. Recall that $B_r(0, \mathbb{Y})$ stands for the ball of radius $r$ centered at 0 of $\mathbb{Y}$. Because we use $\mathbb{X}$ as the fixed phase space for Eq. (4.1), for brevity, we denote $B_r(0, \mathbb{X})$ by $B_r(0)$ if this does not cause any confusion.

**Assumption 3.**

- (A3.1) *There exists a real number $\omega$ such that $\omega I - A$ is m-accretive;*
- (A3.2) *There exists a positive $r > 0$ such that $A$ is proto-differentiable at every point $x \in B_r(0) \cap D(A)$ and the proto-derivative $\partial A(x)$ is a single-valued linear operator in $\mathbb{X}$ such that $\omega I - A$ is m-accretive.*
- (A3.3) *Yosida distance $d_Y(\partial A(x), \partial A(0))$ satisfies $d_Y(\partial A(x), \partial A(0)) < \infty$ for all $x \in B_r(0)$ and it satisfies*

$$\lim_{x\to 0} d_Y(\partial A(x), \partial A(0)) = 0. \tag{4.56}$$

This assumption yields that the operator $A$ is proto-differentiable in a neighborhood of 0 and the proto-derivative $\partial A(\cdot)$ is continuous at 0 in the Yosida distance's sense.

**Definition 4.18** (Local Lipschitz invariant manifold). Let $(S(t))_{t\geq 0}$ be a semigroup of (possibly nonlinear) operators on the Banach space $\mathbb{X}$. A set $\mathcal{N} \subset \mathbb{X}$ is said to be a *local Lipschitz invariant manifold* for semigroup $(S(t))_{t\geq 0}$ around an equilibrium 0 if $\mathbb{X}$ is split into a direct sum $\mathbb{X} = \mathbb{X}^1 \oplus \mathbb{X}^2$, where $\mathbb{X}^1$ and $\mathbb{X}^2$ are closed subspaces of $\mathbb{X}$, and there exists a Lipschitz continuous mapping $\Phi\colon B_r(0,\mathbb{X}^1) \to \mathbb{X}^2$ and an open neighborhood $U$ of 0 such that $\mathcal{N} \cap U = \mathrm{graph}(\Phi)$ and for each $t \geq 0$, $S(t)(\mathrm{graph}(\Phi)) \cap U \subset \mathrm{graph}(\Phi)$.

**Theorem 4.19** (Local unstable manifolds). *Under the Assumption 3, let* $0 \in \mathbb{X}$ *be a stationary solution of Eq.* (4.1). *Moreover, assume that the strongly continuous semigroup* $(T(t))_{t\geq 0}$ *has an exponential dichotomy with projection* $P$. *Then, there exists a neighborhood* $U$ *of* $0 \in \mathbb{X}$, *such that Eq.* (4.1) *has a unique local invariant manifold* $\mathcal{W}^{\mathrm{u}}_{\mathrm{loc}} \subset \mathbb{X}$, *presented as* $\mathrm{graph}(\Phi) \cap U$, *where* $\Phi\colon \mathrm{Ker}(P) \to \mathrm{Im}(P)$ *is a Lipschitz continuous mapping.*

*Proof.* For the functions $\phi(\cdot)$ defined as in (4.54), we consider the standard truncation procedure by defining

$$\phi_0(t)x := \begin{cases} \phi(t)x, & \text{if } \|x\| \leq r_0, \\ \phi(t)(r_0 x/\|x\|), & \text{if } \|x\| > r_0. \end{cases} \tag{4.57}$$

It can be shown that $\phi_0(t)$ is Lipschitz continuous with Lipschitz coefficient $\mathrm{Lip}(\phi_0(t)) = 2\mathrm{Lip}\left(\phi(t)\big|_{B_{r_0}(0)}\right)$ (see, for example, Webb [28, Proposition 3.10, p.95]). Hence, by modifying (4.42) and (4.43) we have

$$\|\phi_0(x) - \phi_0(y)\| \leq e^{3\omega}\|x-y\| \sup_{\|z\|\leq e^{\omega} r_0} d_Y(\partial A(z), \partial A(0)), \tag{4.58}$$

for all $x$, $y \in B_{r_0}(0)$ and $r_0$ is sufficiently small positive real number. Next, we have

$$\mathrm{Lip}\left(\phi(t)\big|_{B_{r_0}(0)}\right) \leq e^{3\omega} \sup_{\|z\|\leq e^{\omega} r_0} d_Y(\partial A(z), \partial A(0)). \tag{4.59}$$

Since $d_Y(\partial A(z), \partial A(0)) \to 0$ as $r_0 \to 0$, hence $\mathrm{Lip}\left(\phi(t)\big|_{B_{r_0}(0)}\right) \to 0$ as $r_0 \to 0$. Now, the assertions of Theorem 4.16 can be applied to $\phi_0(t)$. In fact, we choose $U = B_{r_0/2}(0,\mathbb{X}^2) \times B_{r_0/2}(0,\mathbb{X}^1) \subset B_{r_0}(0)$. By Theorem 4.16, $\phi_0(t)$ has an invariant manifold $\mathcal{M}$, $\phi_0(t)\mathcal{M} \subset \mathcal{M}$. Since $\mathcal{M} = \mathrm{graph}(\Phi)$, where $\Phi\colon \mathbb{X}^2 \to \mathbb{X}^1$, we have $\mathcal{N} := \mathrm{graph}\left(\Phi\big|_{B_{r_0/2}(0,\mathbb{X}^2)}\right) \subset \mathcal{M}$. This implies

$$\phi_0(t)\mathcal{N} \subset \phi_0(t)\mathcal{M},$$

that is,

$$\phi_0(t)\mathcal{N} \cap U \subset \phi_0(t)\mathcal{M} \cap U \subset \mathcal{M} \cap U.$$

Note that $\mathcal{M} \cap U = \mathrm{graph}\left(\Phi\big|_{B_{r_0/2}(0,\mathbb{X}^2)}\right)$, so

$$\phi(t)\mathrm{graph}\left(\Phi\big|_{B_{r_0/2}(0,\mathbb{X}^2)}\right) \subset \mathrm{graph}\left(\Phi\big|_{B_{r_0/2}(0,\mathbb{X}^2)}\right).$$

The proof is complete. □

**Theorem 4.20** (Local stable manifolds). *Under the assumptions in Theorem 4.19, there exists a neighborhood* $U$ *of* $0 \in \mathbb{X}$ *such that the set*

$$\mathcal{W}^{\mathrm{s}}_{\mathrm{loc}} := \left\{ x \in U : \lim_{t\to+\infty} S(t)x = 0 \right\} \tag{4.60}$$

*is a local invariant manifold of Eq.* (4.1), *represented as* $\mathrm{graph}(\Psi) \cap U$, *where* $\Psi\colon \mathrm{Im}(P) \to \mathrm{Ker}(P)$ *is a Lipschitz continuous mapping.*

*Proof.* The proof is similar to that of Theorem 4.19 and so the details are omitted. □

## 5. Applications and Examples

**Example 5.1** (Semilinear equation)**.** Let $\mathbb{X}$ be a Banach space. Consider the semilinear equation

$$\frac{du(t)}{dt} = (L+F)u(t), \tag{5.1}$$

where $L\colon \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$, where $\omega$ is a certain positive number, $F\colon \mathbb{X} \to \mathbb{X}$ is a nonlinear Fréchet differentiable operator, and $\bar{u} = 0$ is the stationary solution of Eq. (5.1). Assume further that $F'$ is continuous in a neighborhood of 0 and $F'(0) = 0$.

Then, by Proposition 4.6, the operator $A := L+F$ is proto-differentable in a neighborhood of 0. Moreover, $\partial A(x) = L + F'(x)$ and then, by Lemma 3.2

$$\begin{aligned} d_Y(\partial A(x), \partial A(0)) &= d_Y(L+F'(x), L) \\ &\leq \|F'(x)\|. \end{aligned} \tag{5.2}$$

By a standard renorming $|x| = \sup_{t\geq 0} \|e^{-\omega t}T(t)x\|$ we can reduce the problem to the case where $L$ generates a contraction semigroup. Next, we can use the standard truncation procedure by defining

$$F_0(x) := \begin{cases} F(x), & \text{if } \|x\| \leq r_0, \\ F(r_0 x/\|x\|), & \text{if } \|x\| > r_0, \end{cases} \tag{5.3}$$

then, the function $F_0$ is globally Lipschitz. It is well known that in this case $-(L+F_0)$ is m-accretive (see e.g. [20, 27]). This process makes $A := L + F_0$ satisfies all assumptions of Assumption 2.

**Example 5.2** (Semilinear equation)**.** Consider Eq. (5.1) again with a different assumption that $L$ be the generator of an analytic $C_0$-semigroup $(T(t))_{t\geq 0}$ in $\mathbb{X}$ such that $\|T(t)\| \leq e^{\omega t}$, $t \geq 0$, and $|||x||| := \|x\| + \|Lx\|$ for all $x \in D(A)$, here we note that this norm $|||\cdot|||$ makes $(D(A), |||\cdot|||)$ a Banach space. Assume further that $F : (D(A), |||\cdot|||) \to \mathbb{X}$ be a Fréchet differentiable operator such that $F(0) = 0$, $F'(0) = 0$ and $F'(\cdot)$ is countinous in a neighborhood of 0. Then, by Proposition 4.5, $F$ is proto-differentiable as a function from $D(A) \subset \mathbb{X}$ to $\mathbb{X}$ and $\partial F(x) = F'(x)$. Therefore,

$$\partial(L+F)(x) = [L + F(x)]' = L + F'(x)$$

as $L$ is its Fréchet derivative, itself.

Next, by Lemma 3.2,

$$\begin{aligned} d_Y(\partial(L+F)(x), L) &= d_Y(L+F'(x), L) \\ &\leq K|||F'(x)|||, \end{aligned} \tag{5.4}$$

where $K$ is a constant depending only on $L$ and for $U \in \mathcal{L}(V, \mathbb{X})$, $|||U|||$ denotes the norm of $U$. If we denote $A := L + F$ in this case, then

$$\begin{aligned} d_Y(\partial A(x), \partial A(0)) &= d_Y(L+F'(x), L) \\ &\leq K|||F'(x)|||, \end{aligned} \tag{5.5}$$

where $K$ is a constant depending only on $L$. Therefore, if $F'(x)$ is continuous in $x$ around 0, then, Assumption 3 is made and Theorems 4.19 and 4.20 apply.

**Example 5.3.** As a concrete example from PDE of Example 5.2 we can take the following: Let us consider the initial value problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} - au(t,x) + \sin\left(\dfrac{\partial^2 u(t,x)}{\partial x^2}\right), & t \geq 0, \quad x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \geq 0, \\ u(0,x) = u_0(x), & x \in [0,\pi], \end{cases} \tag{5.6}$$

where $a$ is a constant and $u_0(\cdot) \in L^2[0,\pi]$. If we set $\mathbb{X} := L^2[0,\pi]$, then (see Pazy [21, Chapter 7]) or Lunardi [16]), the operator $Ay := y'' - ay$ with domain $D(A)$ consisting of all $y \in \mathbb{X}$ such that $y'$ is absolutely continuous such that $y'' \in \mathbb{X}$, $y(0) = y(\pi)$, will generate an analytic $C_0$-semigroup $(T(t))_{t\geq 0}$. Therefore, $(T(t))_{t\geq 0}$ has an exponential dichotomy if and only if the following equations

$$\lambda + a = -n^2, \quad n = 1, 2, \dots \tag{5.7}$$

have no zero root (see Travis-Webb [25, p. 414]). Since $F(y(\cdot)) := \sin(y(\cdot))$ is continuously differentiable in $y(\cdot) \in D(A)$ all our assumptions in Theorems 4.19 and 4.20 are made if $a \neq -n^2$ for any positive integer $n$.

**Example 5.4** (Age-dependent population dynamics)**.** Let $L^1 := L^1(0,\infty;\mathbb{R}^n)$ be a Bochner integrable function space, which norm is denoted by $\|\cdot\|_{L^1}$. Given two mappings $F\colon L^1 \to \mathbb{R}^n$ and $G\colon L^1 \to L^1$, we consider the following partial differential equation

$$\begin{cases} \dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = G(u(t,\cdot))(a), & t \geq 0, \quad a \geq 0, \\ \qquad\qquad u(t,0) = F(u(t,\cdot)) & t \geq 0. \end{cases} \tag{5.8}$$

For $i = 1, \dots, n$, define $K_i$, $J_i\colon L^1 \to [0,\infty)$ by

$$K_i\phi = \int_0^\infty k_i(a)\phi(a)da, \quad J_i\phi = \int_0^\infty j_i(a)\phi(a)da,$$

respectively, where $k_i$, $j_i\colon [0,\infty) \to \mathcal{L}(\mathbb{R}^n, [0,\infty))$ are given mappings. Then define $F\colon L^1 \to \mathbb{R}^n$ and $G\colon L^1 \to L^1$ by taking their $i$-th component as follows:

$$\begin{aligned} F(\phi)_i &= \int_0^\infty \beta_i(a, K_i(\phi)\phi_i(a)da \quad \text{ for } \phi = (\phi_i) \in L^1, \\ G(\phi)_i(a) &= -\mu(a, J_i\phi)\phi_i(a)da \quad \text{ a.e. } a > 0 \quad \text{ for } \phi = (\phi_i) \in L^1, \end{aligned}$$

where $\beta_1$, $\mu_i\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ are given functions (see Webb [28] for details).

In the following, we assume that $F\colon L^1 \to \mathbb{R}^n$ and $G\colon L^1 \to L^1$ are continuously Fréchet differentiable, i.e.,

(F) For any $\phi \in L^1$, there exists a $F'(\phi) \in \mathcal{L}(L^1, \mathbb{R}^n)$ such that
$$F(\phi + h) = F(\phi) + F'(\phi)h + o_F(h), \quad h \in L^1,$$
where $o_F\colon L^1 \to \mathbb{X}$, $\|o_F(h)\| \leq b_F(r)\|h\|_{L^1}$ for $\|h\|_{L^1} \leq r$, and $b_F\colon [0,\infty) \to [0,\infty)$ is a continuous increasing function satisfying $b_F(0) = 0$; and there exists a continuous increasing function $d_F\colon [0,\infty) \to [0,\infty)$ such that
$$\|F'(\phi) - F'(\psi)\|_{\mathcal{L}(L^1,\mathbb{X})} \leq d_F(r)\|\phi - \psi\|_{L^1},$$
for $\|\phi\|_{L^1} \leq r$, $\|\psi\|_{L^1} \leq r$.

(G) For any $\phi \in L^1$, there exists a $G'(\phi) \in \mathcal{L}(L^1, L^1)$ such that
$$G(\phi + h) = G(\phi) + G'(\phi)h + o_G(h), \quad h \in L^1,$$
where $o_G\colon L^1 \to \mathbb{X}$, $\|o_G(h)\|_{L^1} \leq b_G(r)\|h\|_{L^1}$ for $\|h\|_{L^1} \leq r$, and $b_G : [0,\infty) \to [0,\infty)$ is a continuous increasing function satisfying $b_G(0) = 0$; and there exists a continuous increasing function $d_G\colon [0,\infty) \to [0,\infty)$ such that
$$\|G'(\phi) - G'(\psi)\|_{\mathcal{L}(L^1,L^1)} \leq d_G(r)\|\phi - \psi\|_{L^1},$$
for $\|\phi\|_{L^1} \leq r$, $\|\psi\|_{L^1} \leq r$.

Let $\bar{u}$ a be a stationary solution of (5.8), i.e., $\bar{u} \in W^{1,1} = W^{1,1}(0,\infty;\mathbb{R}^n)$, $\bar{u} = F(\bar{u})$, and $\bar{u}' = G(\bar{u})$ where "$'$" stands for $d/da$ when the variable of functions in $W^{1,1}$ is represented by $a$. Fix $r_0 > 0$ such that $\|\bar{u}\|_{L^1} < r_0$. Then define the radial truncations $F_0$ and $G_0$ by

$$F_0(\phi) = \begin{cases} F(\phi) & \text{if } \|\phi\|_{L^1} \leq r_0, \\ F(r_0\phi/\|\phi\|_{L^1}) & \text{if } \|\phi\|_{L^1} > r_0, \end{cases} \tag{5.9}$$

and

$$G_0(\phi) = \begin{cases} G(\phi) & \text{if } \|\phi\|_{L^1} \le r_0, \\ G(r_0\phi/\|\phi\|_{L^1}) & \text{if } \|\phi\|_{L^1} > r_0. \end{cases} \tag{5.10}$$

Then, the functions $F_0$ and $G_0$ are globally Lipschitz continuous and continuously Fréchet differentiable on the ball $B_{r_0}$ in $L^1$.

Now define an operator $A$ on $L^1$ by

$$A\phi = \phi' - G_0(\phi), \qquad \text{for } \phi \in D(A) := \left\{\phi \in W^{1,1} : \phi(0) = F_0(\phi)\right\}. \tag{5.11}$$

Obviously, operator $A$ is not included in Examples 5.1 and 5.2 even though the formula defining $A$ in (5.11) would suggests it is. This is due to the fact that the domain $D(A)$ is not the whole space $W^{1,1}$ (see [13, 28] for more information on the matter).

The properties of operator $A$ are summarized in the following proposition:

**Proposition 5.5.**

(1) *With* $\omega = \|F_0\|_{\mathrm{Lip}} + \|G_0\|_{\mathrm{Lip}}$, $A + \omega I$ *is a densely defined m-accretive operator in* $L^1$.
(2) *With* $\omega_u = \|F'(u)\|_{\mathcal{L}(L^1,\mathbb{X})} + \|G'(u)\|_{\mathcal{L}(L^1,L^1)}$, *operator* $\partial A(u) + \omega_u I$ *is m-accretive in* $L^1$.
(3) *For* $u \in D(A) \cap B_r(\bar{u})$, $\partial A(u)$ *exists and*

$$\operatorname{graph}(\partial A(u)) = \lim_{t\downarrow 0} t^{-1}[\operatorname{graph}(A) - (u, Au)]. \tag{5.12}$$

(4) *Operator* $\partial A(u) + \omega I$ *is m-accretive in* $L^1$ *for* $u \in D(A) \cap B_r(\bar{u})$.
(5) *There exist* $\lambda_{\bar{u}}$ *and a nondecreasing* $L_{\bar{u}} \colon [0,\infty) \to [0,\infty)$ *such that*

$$\left\| J_\lambda^{\partial A(z)} v - J_\lambda^{\partial A(u)} v \right\|_{L^1} \le \lambda \|z - u\|_{L^1} L_{\bar{u}}(\|v\|_{L^1}) \tag{5.13}$$

*for* $0 < \lambda < \lambda_{\bar{u}}$, $z$, $u \in B_{\delta_{\bar{u}}}(\bar{u}) \cap D(A)$ *and* $v \in L^1$.

*Proof.* See Kato [15, Propositions 5.2, 5.4, 5.5 and 5.6]. □

By (4.32) and (5.13) we have

$$d_Y(\partial A(z), \partial A(0)) = \frac{1}{\lambda} \left\| J_\lambda^{\partial A(z)} v - J_\lambda^{\partial A(u)} v \right\|_{L^1} \le \|z - u\|_{L^1} L_{\bar{u}}(\|v\|_{L^1}). \tag{5.14}$$

This implies that

$$\lim_{z\to 0} d_Y(\partial A(z), \partial A(0) = 0. \tag{5.15}$$

This means that the condition (4.56) is fulfilled. Therefore, all conditions listed in Assumption 3 are satisfied. Applying Theorems 4.19 and 4.20, the age-dependent population model has a local stable, unstable manifold near $\bar{u}$ if the linear system $x'(t) = -\partial A(\bar{u})x$ has an exponential dichotomy.

## References


1. J.P. Aubin, I. Ekeland. *Applied Nonlinear Analysis.* John Wiley & Sons, Inc., New York, 1984.
2. V. Barbu. *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Noordhoff International Publishing, Leiden, 1976.
3. P.W. Bates, K. Lu, and C. Zeng. Existence and persistence of invariant manifolds for semiflows in Banach space. *Memoirs of the American Mathematical Society* **135** (1998), no. 645, viii+129 pp.
4. P.W. Bates, K. Lu, and C. Zeng. Approximately invariant manifolds and global dynamics of spike states. *Inventiones Mathematicae* **174** (2008), no. 2, 355-433.
5. S.-N. Chow, H. Leiva. Unbounded perturbation of the exponential dichotomy for evolution equations. *Journal of Differential Equations* **129** (1996) no. 2, 509-531.
6. S.-N. Chow, K. Lu. Invariant manifolds for flows in Banach spaces. *Journal of Differential Equations* **74** (1988), 355-385.
7. W.A. Coppel. *Dichotomies in Stability Theory.* Lecture Notes in Mathematics **629**. Springer-Verlag, Berlin-New York, 1978.

8. M.G. Crandall. Nonlinear semigroups and evolution equations governed by accretive operators. *Proceedings of Symposia in Pure Mathematics*, #45, Part 1. American Mathematical Society, 1986, 305-337.
9. M.G. Crandall, T.M. Liggett. Generation of semi-groups of nonlinear transformations on general Banach spaces. *American Journal of Mathematics* **93** (1971), 265-298.
10. G. Da Prato, A. Lunardi. Stability, instability and center manifold theorem for fully nonlinear autonomous parabolic equations in Banach space. *Archive for Rational Mechanics and Analysis* **101** (1998), 115-141.
11. Ju.L. Daleckii, M.G. Krein. *Stability of Solutions of Differential Equations in Banach Spaces*. American Mathematical Society Translations, 1974.
12. K.J. Engel, R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Graduate Texts in Mathematics **194**, Springer, 2000.
13. J. Hale. *Theory of Functional Differential Equations* (second edition). Applied Mathematical Sciences **3**, Springer-Verlag, New York-Heidelberg, 1977.
14. M.-L. Hein, J. Prüss. The Hartman-Grobman theorem for semilinear hyperbolic evolution equations. *Journal of Differential Equations* **261** (2016), no. 8, 4709-4727.
15. N. Kato. A principle of linearized stability for nonlinear evolution equations. *Transactions of the American Mathematical Society* **347** (1995), 2851-2868.
16. A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser, Basel, 1995.
17. A. Lunardi. Stability in fully nonlinear parabolic equations. *Archive for Rational Mechanics and Analysis* **130** (1995), 1-24.
18. K. Lu, W. Zhang, and W. Zhang. $C^1$ Hartman theorem for random dynamical systems. *Advances in Mathematics* **375** (2020), 107375, 46 pp.
19. N.V. Minh, J. Wu. Invariant manifolds of partial functional differential equations. *Journal of Differential Equations* **198** (2004), 381-421.
20. S. Oharu, T. Takahashi. Characterization of nonlinear semigroups associated with semilinear evolution equations, *Transactions of the American Mathematical Society* **311** (1989), 593-619.
21. A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences **44**, Springer-Verlag, New York, 1983.
22. J. Prüss. Perturbations of exponential dichotomies for hyperbolic evolution equations. *Operator Semigroups Meet Complex Analysis, Harmonic Analysis and Mathematical Physics*, 453-461, Operator Theory: Advances and Applications **250**, Birkhäuser/Springer, Cham, 2015.
23. R.T. Rockafellar. Proto-differentiability of set-valued mappings and its applications in optimization. Analyse non lineaire (Perpignan, 1987). *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* **6** (1989), suppl., 449-482.
24. G.R. Sell, Y. You. Inertial manifolds: The non-self-adjoint case. *Journal of Differential Equations* **96** (1992), no. 2, 203-255.
25. C.C. Travis, G.F. Webb. Existence and stability for partial functional differential equations. *Transactions of the American Mathematical Society* **200** (1974), 394-418.
26. X. Wang, K. Lu, and B. Wang. The Wong-Zakai approximations and attractors for stochastic reaction-diffusion equations on unbounded domains. *Journal of Differential Equations* **264** (2018), no. 1, 378-424.
27. G.F. Webb. Asymptotic stability for abstract nonlinear functional differential equations. *Proceedings of the American Mathematical Society* **54** (1976), 225-230.
28. G.F. Webb. *Theory of Nonlinear Age-Dependent Population Dynamics*. Monographs and Textbooks in Pure and Applied Mathematics **89**. Marcel Dekker, Inc., New York, 1985.
29. T. Yosida. *Functional Analysis*. Springer-Verlag, Berlin, 1995.

Faculty of Fundamental Sciences, PHENIKAA University, Hanoi 12116, Vietnam

*Email address*: `quang.buixuan@phenikaa-uni.edu.vn`

Department of Mathematics and Statistics, University of Arkansas at Little Rock, 2801 S University Ave, Little Rock, AR 72204, USA

*Email address*: `mvnguyen1@ualr.edu`