Subdiffusive concentration of the graph distance in Bernoulli percolation

Van Hao Can[1] Van Quyet Nguyen[2]

Abstract

Considering supercritical Bernoulli percolation on $\mathbb{Z}^d$, Garet and Marchand [25] proved a diffusive concentration for the graph distance. In this paper, we sharpen this result by establishing the subdiffusive concentration inequality. As a consequence, we revisit a recent result by Dembin [17] on the sublinear variance of the distance.

1 Introduction

1.1. Model and main result. Bernoulli percolation is a simple but well-known probabilistic model for porous material introduced by Broadbent and Hammersley [7]. Let $d \geq 2$ and $\mathbb{E}^d$ be the set of the edges $e = (x, y)$ of endpoints $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{Z}^d$ such that $\|x - y\|_1 := \sum_{i=1}^d |x_i - y_i| = 1$. Given the parameter $p \in (0, 1)$, we let each edge $e \in \mathbb{E}^d$ be *open* with probability p and *closed* otherwise, independently of the state of other edges. The phase transition of model has been well-known since 1960s. Aizenman, Kesten, Newman [1] proved that there exists a critical parameter $p_c(d) \in (0, 1)$, such that there is almost surely a unique infinite open cluster $\mathcal{C}_\infty$ if $p > p_c(d)$, whereas all open clusters are finite if $p < p_c(d)$. In supercritical regime $(p > p_c(d))$ and critical regime $(p > p_c(d))$, the model behavior are more understandable that the critical case. Refer to [27][19] [20] for more detail results and open questions on this fields.

Let $x \in \mathbb{Z}^d$, we denote by x^* the closest point to x in $\mathcal{C}_\infty$ (in $\|.\|_\infty$ distance), call it the regularized point of x. We define the graph distance as: for $x, y \in \mathbb{Z}^d$,

$$\mathrm{D}^*(x, y) = \mathrm{D}(x^*, y^*) = \inf_{\gamma : x^* \to y^*} |\gamma|,$$

where the infimum is taken over all open nearest-neighbor open paths starting at x^* and ending at y^*. Let $\mathbf{e}_1 = (1, 0, \dots, 0)$ and we aim to study the graph distance from 0 to $n\mathbf{e}_1$:

$$\mathrm{D}^*_n = \mathrm{D}^*(0, n\mathbf{e}_1).$$

The linear growth of D^*_n was described by Garet and Marchand [22]: for any $p > p_c(d)$, there exists a constant $\mu(e_1) \in [0, \infty)$ such that,

$$\lim_{n \to \infty} \frac{\mathrm{D}^*_n}{n} = \mu(\mathbf{e}_1) \quad \text{a.e and in } L^1. \tag{1.1}$$

[1]Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam. Email: cvhao@math.ac.vn

[2]Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam. Email: nvquyet@math.ac.vn

The value $\mu(\mathbf{e}_1)$ is called the *time constant.* In fact, they show the stronger result that (1.1) still holds true in a more general context of stationary integrable ergodic fields.

Naturally, the next question we are interested in is the fluctuation and deviation of the graph distance. The moderate deviation of D_n^* (or precisely the concentration with diffusive scale) was established by Garet and Marchand (see [25, Theorem 1.2]): for each $c > 0$, there exist some constants c_1, c_2 such that for all $\lambda \in [c \log n, \sqrt{n}]$,

$$\mathbb{P}[|\mathrm{D}_n^* - \mathbb{E}[\mathrm{D}_n^*]| \geq \sqrt{n}\lambda] \leq c_1 e^{-c_2\lambda}. \tag{1.2}$$

Recently, Dembin give a sublinear bound on the vairance of D_n^* (see [17, Theorem 1.1]): there exists a positive constant C such that

$$\mathrm{Var}(\mathrm{D}_n^*) \leq \frac{Cn}{\log n}. \tag{1.3}$$

The main result of our paper is to sharpen the moderate deviation (1.2) by establishing a sub-diffusive concentration of D_n^*.

Theorem 1.1. *Let $p > p_c(d)$. There exist positive constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ and $\kappa \geq 0$,*

$$\mathbb{P}\left(|\mathrm{D}_n^* - \mathbb{E}[\mathrm{D}_n^*]| \geq \sqrt{\frac{n}{\log n}\kappa}\right) \leq c_1 e^{-c_2\kappa}. \tag{1.4}$$

Consequently, the sublinear bound of the variance (1.3) *holds.*

Remark 1.2. *If $\kappa < 1$, we can take $c_1 = e^{c_2}$ and hence* (1.4) *holds trivially. From now on, we focus on the case $\kappa \geq 1$ throughout this paper.*

1.2. Connection to generalized first passage percolation. We recall here similar results for generalized first passage percolation, a kind of mixed model between first passage percolation and supercritical Bernoulli percolation. For each edge $e \in \mathbb{E}^d$, we assign a random weight t_e taking values in $\mathbb{R}_+ \cup \infty$ such that the family $t = (t_e)_{e\in\mathbb{E}^d}$ is independent and identically distributed with distribution ζ such that $\zeta([0,\infty)) > p_c(d)$, where $p_c(d)$ is the critical point for Bernoulli percolation on $\mathbb{Z}^d$. Thus, the edges with a finite weight are supercritical. If $\zeta([0,\infty)) = 1$, we return to the model of first passage percolation that the edges has finite weight.

The first passage time is defined as follows: for $x, y \in \mathbb{Z}^d$,

$$\mathrm{T}(x,y) = \inf_{\gamma: x\to y} \sum_{e\in\gamma} t_e, \tag{1.5}$$

where infimum is taken over the set of paths from x to y. Then the supercritical Bernoulli percolation can be referred as a particular case of first passage percolation with the distribution

$$\zeta = \zeta_p = p\delta_1 + (1-p)\delta_\infty, \quad p > p_c(d). \tag{1.6}$$

The convergence of the scaled passage time in probability to time constant was obtained by Cerf and Théret [10, Theorem 4], without any moment assumptions: there exists a constant $\mu(\mathbf{e}_1) \in [0,\infty)$ such that

$$\lim_{n\to\infty} \frac{\mathrm{T}(0, n\mathbf{e}_1)}{n} = \mu(\mathbf{e}_1) \qquad \text{in probability.} \tag{1.7}$$

In [23, Remark 1], Garet and Marchand proved if $\mathbb{E}[t^{2+\varepsilon}\mathbb{I}(t<\infty)]<\infty$ with some $\varepsilon>0$, then the convergence in (1.7) holds true almost surely and in L_1. It then has been proved by Damron, Hanson and Sosoe [16] that if $\zeta([0,\infty))=1$ and $\mathbb{E}[t^2\log_+ t]<\infty$, then the sublinear variance holds:

$$\mathrm{Var}(\mathrm{T}(0,n\mathbf{e}_1))\le Cn/\log n. \tag{1.8}$$

This phenomenon was known as superconcentration, coined by Chatterjee [13]. It has been widely conjectured by physicists that the bound in (1.8) should be improved by polynomial. Furthermore, Chatterjee's work establish a more general principle that connects between superconcentration and a chaotic phenomenon of the ground state in many Gaussian disordered systems [11] [12] [13]. Recently, Ahlberg et al. show that that this deep relation holds true in context of first passage percolation under the assumption that $\zeta([0,\infty))=1$ and $\mathbb{E}[t^2]<\infty$. Notice that by (1.8), the first passage time is superconcentrated, so they give first evidence of chaos of the geodesic in this context. It is still open question whether chaotic phenomenon holds in the generalized first passage percolation without any moment conditions.

Additionally, Damron et al.[15] also prove the subdiffusive concentration for $\mathrm{T}(0,n\mathbf{e}_1)$, in similar form as (1.4): for the upper tail inequality (resp. lower tail inequality) the authors require $\mathbb{E}[e^{2\alpha t}]<\infty$ for some $\alpha>0$ (resp. $\mathbb{E}[t^2\log_+ t]<\infty$) and $\zeta([0,\infty))=1$. A lot more was proved concerning generalized first passage percolation, for example, the regularity of time constant [10][26][9], the large deviation [3] [24][14][4][18].

We remark that in supercritical Bernoulli percolation (or the generalized first passage percolation with distribution (1.6) with $\zeta([0,\infty))<1$) the moment conditions $\mathbb{E}(t^2\log_+ t)<\infty$ and $\mathbb{E}[e^{2\alpha t}]<\infty$ do not hold due to the infinite weight. Hence, we need the new discover on this model to apply the techniques in the case first passage percolation with finite edge-weight.

1.3. Method of the proof. In this subsection, we will address main challenge in extending the previous results of Damron, Hanson and Sosoe [15] to Bernoulli percolation and outline our strategy to overcome this issue. In [15], Damron et al. use the ideas of Benaim-Rossignol, to prove the subdiffusive concentration of $\mathrm{T}(0,n\mathbf{e}_1)$, it suffices to estimate the variance of exponential function of $\mathrm{T}(0,n\mathbf{e}_1)$. The remaining step can be derived by combining the geometric averaging trick of Benjamini, Kalai, Schramm with the entropy inequalities [6], following the same sublinear variance strategy for general distribution [16].

In both [15] and [16], we emphasize the importance of imposing moment conditions on the edge-weight distribution. This is crucial for obtaining good control over the impact of resampling an edge. However, in the context of the graph distance in Bernoulli percolation, closing an edge on the geodesic can have a significant impact on the graph distance due to the possibility of infinite edge-weight values. To solve this issue, using a complex multiple-scale renormalization process introduced in [9], Dembin constructs a detour that bypasses a given edge [17] . She then can control the length of these bypasses. In the final step, to prove the sublinear property of variance, Dembin use the concentration inequalities in a similar manner as in [16] with some technical difficulties specific to the graph distance.

In the study of subdiffusive concentration for supercritical Bernoulli percolation, we shall have to deal with the fluctuation of exponential function $e^{\lambda \mathrm{D}_n^*}$, rather than the graph distance D_n^* itself. Hence, the approach of Dembin which is based on multi-scale renormalizations would be much more complex. In this paper, we use a simpler approach, that give a systematic way to tame the effect of resampling a given edge with being of independent interest.

Our mechanism goes as follows: Instead of working with the graph distance D_n^*, we prove sub-diffusive concentration for T_n, a modified graph distance of T_n. Then, we indicate the equivalent of sub-diffusive concentration between these distances. In particular, we sketch the proof of Theorem 1.1 as follows:

Step 1 (Setup). We introduce T_n, a truncated passage time from 0 to $n\mathbf{e}_1$, which is derived from a Bernoulli first passage percolation with truncated edge-weights:

$$t_e = p\delta_1 + (1-p)\delta_{\log^2 n}.$$

Subsequently, in Theorem 5.3, we prove the subdiffusive concentration for T_n, i.e.

$$\forall n \in \mathbb{N}, \kappa \geq 0, \quad \mathbb{P}\left(|\mathrm{T}_n - \mathbb{E}[\mathrm{T}_n]| \geq \sqrt{\frac{n}{\log n}\kappa}\right) \leq c_1 e^{-c_2\kappa}, \tag{1.9}$$

for some positive constant $c_1, c_2 > 0$.

Our argument initially follows the common scheme as in [15]: Considering F_m the averaged version for the passage time T_n (inspired by Benjamini, Kalai, and Schramm in [6]), to prove the subdiffusive concentration, we relies on establishing a connection between bounds on $\mathrm{Var}(e^{\lambda \mathrm{F}_m/2})$ and exponential concentration. Then, we obtain the tails of the true passage time T_n as in (1.9) based on those of F_m (see more details in Section 5.2). That is, we need to prove that (see Theorem 5.3): there exists a constant $c > 0$ such that with $K = \dfrac{cn}{\log n}$,

$$\forall\, |\lambda| < \frac{1}{\sqrt{K}}, \quad \mathrm{Var}\left[e^{\lambda \mathrm{F}_m}\right] \leq \frac{cn}{\log n}\lambda^2 \mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right] < \infty, \tag{1.10}$$

To attain this variance bound, we utilize the Falik-Samorodnitsky inequality (Lemma 5.1). Let us enumerate the edges of $\mathbb{E}^d$ as $\{e_1, e_2, \ldots,\}$ and consider the natural filtration of these as

$$\mathcal{F}_0 = \varnothing, \quad \mathcal{F}_i = \sigma(t_{e_1}, \ldots, t_{e_i}), \forall i \geq 1.$$

We perform a martingale decomposition of the random variable $G = e^{\lambda \mathrm{F}_m}$:

$$G - \mathbb{E}[G] = \sum_{i=1}^{\infty} \Delta_i,$$

where

$$\forall i \geq 1, \quad \Delta_i = \mathbb{E}[G \mid \mathcal{F}_i] - \mathbb{E}[G \mid \mathcal{F}_{i-1}]$$

Then, we have

$$\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] \geq \mathrm{Var}[G] \log\left(\frac{\mathrm{Var}[G]}{\sum_{i=1}^{\infty}(\mathbb{E}[|\Delta_i|])^2}\right). \tag{1.11}$$

Notice that if $\mathrm{Var}[G] < C\lambda^2 n^{15/16}\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right]$ then we obtain (1.10). Otherwise, one has $\mathrm{Var}[G] \geq C\lambda^2 n^{15/16}\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right]$, so

$$\mathrm{Var}[G] \log\left(\frac{C\lambda^2 n^{15/16}\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right]}{\sum_{i=1}^{\infty}(\mathbb{E}[|\Delta_i|])^2}\right) \leq \sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2]. \tag{1.12}$$

As a result, (5.16) reduce to estimate the total influence (the sum in the denominator of left hand side) and total entropy (the sum of right hand side).

Step 2. The bulk of the paper is devoted to bound these quantities, which is done in Proposition 4 and (5), respectively:

$$\sum_{i=1}^{\infty}(\mathbb{E}[|\Delta_i|])^2 \le C\lambda^2\mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big]n^{7/8}, \quad \forall \lambda \in \mathbb{R}; \tag{1.13}$$

$$\sum_{i=1}^{\infty}\mathrm{Ent}[\Delta_i^2] \le C\lambda^2 n\mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big], \quad \forall |\lambda| \le \tfrac{1}{\log^{2(d+11)} n}. \tag{1.14}$$

Using some the martingale computation, we can represent

$$\mathbb{E}[|\Delta_i|] \le C|\lambda|\mathbb{E}\left[e^{\lambda \mathrm{F}_m(t_{e_i},t_{e_i^c})} \times \big(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\big)\right];$$

and by the tensorization property of entropy and the log-Sobolev inequality for the Bernoulli distribution,

$$\sum_{i=1}^{\infty}\mathrm{Ent}[\Delta_i^2] \le 2C|\lambda|^2\sum_{i=1}^{\infty}\mathbb{E}\Big[\Big(e^{2\lambda \mathrm{F}_m(\log^2 n,t_{e_i^c})} + e^{2\lambda \mathrm{F}_m(1,t_{e_i^c})}\Big) \times \big(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\big)^2\Big],$$

where $(r, t_{e_i^c})$ denote the configuration with value $t_{e_i} = r$ and t_{e_j} if $j \neq i$.

That is, we have to control the impact of resampling edge e along the geodesics of the truncated passage times. We estimate this effect by the weight of the bypass that avoids e. The key here is this bypass composes only of 1-weight edges, so its weight can be bounded by using the notion of *effective radius* R_e. Roughly speaking, the effective radius help us find a good path bypassing a given edge e inside the annulus $\mathrm{A}_{R_e}(e) = \Lambda_{3R_e}(e) \setminus \Lambda_{R_e}(e)$ (see Proposition 2). The construction of this radius with appropriate properties is induced from the well-connected properties of infinite cluster, which behaves (in a sense made precise later) like that of $\mathbb{Z}^d$. We refer to Section 3 for more details. Now, we give the radius inequalities (see more details in Subsection 5.3): for all $i \ge 1$,

$$\mathbb{E}[|\Delta_i|] \le \frac{C}{n^{1/4}}|\lambda|\left(\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right]\mathbb{E}\left[\sum_{e\in\gamma_{e_i}} R_e^2\right]\right)^{1/2},$$

where γ_0 is the geodesic of T_n and

$$\gamma_{e_i} = \gamma_0 \cap \{e_i - \Lambda_{n^{1/4}}\}; \quad \{e_i - \Lambda_{n^{1/4}}\} = \{e' = (x_{e_i} - z, y_{e_i} - z) : z \in \Lambda_{n^{1/4}}\},$$

and

$$\sum_{i=1}^{\infty}\mathbb{E}[|\Delta_i|] \le C||\lambda|n\left(\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right]\mathbb{E}\left[\left(\sum_{e\in\gamma_0} R_e\right)^2\right]\right)^{1/2},$$

and for $\lambda \le \frac{1}{\log^{2(d+11)} n}$,

$$\sum_{i=1}^{\infty}\mathrm{Ent}[\Delta_i^2] \le C\lambda^2 n\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right] + C\exp\left(\frac{\rho n}{\log^{2(d+11)} n}\right)\left(\mathbb{P}\left(Y_n \ge Cn\right)\right)^{1/4}, \tag{1.15}$$

where for some constant C_*,

$$Y_n = \sum_{e\in\gamma_0} \hat{R}_e^2; \qquad \hat{R}_e = C_* R_e \wedge \log^2 n.$$

To accomplish the desired influence bound, the remain work is to compute these expectations on the right hand sides, i.e to understand the *total cost* of resampling the edges on random set γ: Cost $= \sum_{e\in\gamma} f(R_e)$ with f a suitable function. We remark that the bound on Cost is highly non-trivial since γ is random and the radii $(R_e)_{e\in\mathbb{E}^d}$ are not independent. Fortunately, we can show that $(R_e)_{e\in\mathbb{E}^d}$ are weakly dependent in the sense that for each e and $t \geq 1$ the event $\{R_e \leq t\}$ is independent of the radii R'_e with $\|e - e'\|_\infty \geq 4t$. Using the technique of greedy animal lattices for dependent weights, we can control the total cost (Corollary 4.3 (i) and (ii)):

$$\sup_{i\geq 1} \mathbb{E}\left[\sum_{e\in\gamma_{e_i}} R_e^2\right] \leq Cn^{1/4}; \quad \mathbb{E}\left[\left(\sum_{e\in\gamma_0} R_e\right)^2\right] \leq Cn.$$

These combine with the above estimates to confirm (1.13).

One of the complicated point we must address is estimating the total entropy. In particular, to control the the right hand side of (1.15), we need a large deviation estimate for Y_n-the total cost of the truncated radii (taking the value at most $\log^2 n$). Corollary 4.3 (iii) and the large deviation estimate of γ_0 give us,

$$\mathbb{P}(Y_n \geq Cn) \leq \exp\left(-\frac{cn}{\log^{2(d+10)} n}\right), \tag{1.16}$$

for some positive constant C, c. In context of Bernoulli percolation, we can control the total entropy by the total cost of effective radii, $Y'_n = \sum_{e\in\gamma_0} R_e$, using the same strategy. Similarly, we have to deal with proving the large deviation estimate for Y'_n instead of Y_n. However, this issue is not considerably follows by the theory of greedy lattice animals since $(R_e)_{e\in\mathbb{E}^d}$ are unbounded. One explains why we switch to prove the sub-diffusive concentration for the modified graph distance T_n, rather than the graph distance D_n^*. It pays a cost we must show the equivalent between these distance in the next step. We remark that by the large deviation estimate of F_m,

$$\mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big] \geq \exp\left(-\frac{Cn}{\log^{2(d+11)} n}\right), \quad \forall \lambda \geq \frac{-1}{\log^{2(d+11)} n}.$$

Finally, combining two above estimates with (1.15) allows the bound (1.14).
Step 3. We will show that there is not a significant discrepancy between D_n^* and T_n. It is easy to estimate this discrepancy from the following large deviation result (Theorem 2.1): for all $L \geq \log^2 n$,

$$\mathbb{P}\left(|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| \geq L\right) \leq C\exp(-cL/\log L), \tag{1.17}$$

with positive constants c, C.

Using covering argument, the gap between D_n^* and $\mathrm{T}(0^*, (n\mathbf{e}_1)^*)$ are bounded by the total weight of the bypasses avoiding all $\log^2 n$-weight edges on the geodesic γ_n of $\mathrm{T}(0^*, (n\mathbf{e}_1)^*)$. Furthermore, we remark that the weight of bypasses can be simply controlled by using the effective radius. Lemma 6.1 shows that if some certain conditions occur with high probability, there exists a random subset $\Gamma_n \subset \gamma_n$ of $\log^2 n$-weight edges such that

(i) $\forall e, f \in \Gamma_n, \|e-f\|_\infty \geq \max\{R_e, R_f\}$,

(ii) $|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| \leq 2C_* Y_n^*$ with $Y_n^* := \sum_{e \in \Gamma_n} R_e$.

This is essentially done through a selection process of a suitable family of bypasses. We emphasize here that the property (i), in some sense, strengthen the local dependent of $(R_e)_{e\in\Gamma_n}$, so it enable the estimate of the total cost Y_n^*. The remain work we must prove a large deviation estimate for Y_n^* by using the coarse graining argument (see more details in Section 6): for all $L \geq \log^2 n$,

$$\mathbb{P}\left(\exists\, \Gamma_n \subset \gamma_n \text{ satisfying (i) and } Y_n^* \geq \tfrac{L}{2C_*}\right) \leq C^{-1}\exp(-CL/\log L).$$

This deduce (1.17) and complete the proof.

1.4. Organization and notation of this paper. The paper is organized as follows. We introduce the modified graph distance of D_n^* and we deduce Theorem 1.1 from Theorem 2.1 and Theorem 2.2 in Section 2. In Section 3, we present the construction of random effective radius and its application to control on the effect of flipping an edge. We study some moments and large deviations of lattice animal of dependent weight in Section 4. In Section 5, we first revisit concentration inequalities and then prove the subdiffusive concentration of the modified graph distance. Finally, we estimate the discrepancy between the graph distance and its modified version via the covering argument in Section 6.

Finally, we introduce some notations used in the paper.

- *Metric.* We denote by $\|\cdot\|_1, \|\cdot\|_\infty, \|\cdot\|_2$ the l_1, l_∞, l_2 norms, respectively.

- *Box.* Let $x \in \mathbb{Z}^d$ and $N \in \mathbb{N}$, we will denote by $\Lambda_N(x) = x + [-N, N]^d$ the box centered at $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ with side length N. For convenience, we shortly write $\Lambda_N = [-N, N]^d$ for $\Lambda_N(0)$.

- *Set distance.* For $X, Y \subset \mathbb{Z}^d$, we denote $\mathrm{d}_\infty(X, Y)$ the distance between X and Y by
$$\mathrm{d}_\infty(X, Y) = \min\{\|x-y\|_\infty : x \in X, y \in Y\}.$$

- *Edge distance.* For each edge $e \in \mathbb{E}^d$, we pick a deterministic rule to represent $e = (x_e, y_e)$. For any $e, f \in \mathbb{E}^d$, we denote the distance between e and f by
$$\|e-f\|_\infty = \|x_e - x_f\|_\infty.$$

- *$\mathbb{Z}^d$-path.* We say that a sequence $\gamma = (v_0, \ldots, v_n)$ is a $\mathbb{Z}^d$-path if for all $i \in [n], |v_i - v_{i-1}|_1 = 1$. From now on, we shortly write a path replacing of a $\mathbb{Z}^d$-path. In addition, if $v_i \neq v_j$ for $i \neq j$, then we say that γ is self-avoiding. Given $A \subset \mathbb{Z}^d$, let $\mathcal{P}(A)$ be the set of all self-avoiding paths starting in A.

- *Open path, open cluster and crossing cluster.* Given a Bernoulli percolation on $\mathbb{Z}^d$ with parameter p, let $\mathcal{G}_p = (\mathbb{Z}^d, \{e \in \mathbb{E}^d : e \text{ is open}\})$. We say that a path is open if all of its edges are open. A open cluster is a maximal connected component of $\mathcal{G}_p$. A open cluster $\mathcal{C}$ is crossing for a box Λ, if for all d direction, there is an open path in $\mathcal{C} \cap \Lambda$ connecting the two opposite faces of Λ.

- *Diameter*. For $A \subseteq \mathbb{Z}^d$ and $1 \leq i \leq d$, let us define

$$\mathrm{diam}_i(A) = \max_{x,y \in A} |x_i - y_i|,$$

and we thus denote $\mathrm{diam}(A)$ the diameter of A by

$$\mathrm{diam}(A) = \max_{1 \leq i \leq d} \mathrm{diam}_i(A).$$

2 The modified graph distance and proof of Theorem 1.1

Consider a Bernoulli first passage percolation as follows. Let $(t_e)_{e \in \mathbb{E}^d}$ be i.i.d random weights such that

$$t_e = \begin{cases} 1 & \text{with probability } p, \\ \log^2 n & \text{with probability } 1-p. \end{cases}$$

Next, we define a modified graph distance T by

$$\mathrm{T}(x,y) = \inf_{\gamma: x \to y} \sum_{e \in \gamma} t_e, \tag{2.1}$$

and set

$$\mathrm{T}_n = \mathrm{T}(0, n\mathbf{e}_1).$$

We couple this Bernoulli first passage percolation with Bernoulli percolation as follows: each 1-weight edge (resp. $\log^2 n$) is open (resp. closed). Recall that $\mathcal{C}_\infty$ is the infinite cluster of open (resp. 1-weight) edges in Bernoulli percolation and for each $z \in \mathbb{Z}^d$, z^* is the closest point of z in $\mathcal{C}_\infty$.

Our aim is to show the subdiffusive concentration D_n^* via the modified graph distance T_n. The proof is essentially based on two key ingredients. The following theorem is proved in Section 6 help us to control the discrepancy between $\mathrm{T}(0^*, (n\mathbf{e}_1)^*)$ and D_n^*-the chemical distance in Bernoulli percolation.

Theorem 2.1. *There exist positive constants c_1, c_2 such that such that for all $L \geq \log^2 n$,*

$$\mathbb{P}\Big[|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| \geq L\Big] \leq c_1 \exp(-c_2 \tfrac{L}{\log L}). \tag{2.2}$$

As a consequence, we have

$$\mathbb{E}[|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)|] \leq \mathcal{O}(\log^2 n). \tag{2.3}$$

Subsequently, we establish the subdiffusive concentration for T_n as follows.

Theorem 2.2. *There exist positive constants c_1, c_2 such that*

$$\mathbb{P}\left(|\mathrm{T}_n - \mathbb{E}[\mathrm{T}_n]| \geq \sqrt{\tfrac{n}{\log n}}\kappa\right) \leq c_1 e^{-c_2\kappa} \text{ for all } n \in \mathbb{N} \text{ and } \kappa \geq 0. \tag{2.4}$$

We postpone its proof to Section 5 and give the proof of Theorem 1.1.

Proof of Theorem 1.1. By the triangle inequality,

$$|\mathrm{D}_n^* - \mathbb{E}[\mathrm{D}_n^*]| \le |\mathrm{D}_n^* - \mathrm{T}_n| + |\mathrm{T}_n - \mathbb{E}[\mathrm{T}_n]| + |\mathbb{E}[\mathrm{T}_n] - \mathbb{E}[\mathrm{D}_n^*]|. \tag{2.5}$$

Notice also that

$$|\mathrm{T}_n - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| \le \log^2 n(\|0^*\|_1 + \|(n\mathbf{e}_1)^* - n\mathbf{e}_1\|_1). \tag{2.6}$$

Therefore,

$$\begin{aligned} |\mathrm{D}_n^* - \mathrm{T}_n| &\le |\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| + |\mathrm{T}(0^*, (n\mathbf{e}_1)^*) - \mathrm{T}_n| \\ &\le |\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| + \log^2 n(\|0^*\|_1 + \|(n\mathbf{e}_1)^* - n\mathbf{e}_1\|_1). \end{aligned} \tag{2.7}$$

Combining this with triangle inequality, we have

$$\begin{aligned} \mathbb{E}[|\mathrm{D}_n^* - \mathrm{T}_n|] &\le \mathbb{E}[|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)|] + \log^2 n\mathbb{E}[\|0^*\|_1 + \|(n\mathbf{e}_1)^* - n\mathbf{e}_1\|_1] \\ &= \mathbb{E}[|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)|] + 2\log^2 n\mathbb{E}[\|0^*\|_1], \end{aligned} \tag{2.8}$$

where for the last line we used the translation invariance. By Lemma 3.1, there exist positive constants β_1, β_2 such that for $t \ge 0$,

$$\mathbb{P}[\|0^*\|_1 \ge t] \le \beta_1 \exp(-\beta_2 t), \tag{2.9}$$

and thus

$$\mathbb{E}[\|0^*\|_1] = \mathcal{O}(1).$$

Combining this with (2.8), (2.3) and (2.9), we get

$$|\mathbb{E}[\mathrm{D}_n^*] - \mathbb{E}[\mathrm{T}_n]| \le \mathcal{O}(\log^2 n).$$

It follows from the above estimate and (2.5) that for all $\kappa \ge 1$ and n large enough,

$$\mathbb{P}\Big[|\mathrm{D}_n^* - \mathbb{E}[\mathrm{D}_n^*]| \ge \kappa\sqrt{\tfrac{n}{\log n}}\Big] \le \mathbb{P}\Big[|\mathrm{T}_n - \mathbb{E}[\mathrm{T}_n]| \ge \tfrac{\kappa}{4}\sqrt{\tfrac{n}{\log n}}\Big] + \mathbb{P}\Big[|\mathrm{D}_n^* - \mathrm{T}_n| \ge \tfrac{\kappa}{4}\sqrt{\tfrac{n}{\log n}}\Big].$$

By Theorem 2.2,

$$\mathbb{P}\Big[|\mathrm{T}_n - \mathbb{E}[\mathrm{T}_n]| \ge \tfrac{\kappa}{4}\sqrt{\tfrac{n}{\log n}}\Big] \le c_1 \exp(-c_2\kappa/4),$$

for some $c_1, c_2 > 0$. In addition, using (2.7), (2.9) and Theorem 2.1,

$$\begin{aligned} \mathbb{P}\Big[|\mathrm{D}_n^* - \mathrm{T}_n| \ge \tfrac{\kappa}{4}\sqrt{\tfrac{n}{\log n}}\Big] &\le \mathbb{P}\Big[|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| \ge \tfrac{\kappa}{8}\sqrt{\tfrac{n}{\log n}}\Big] \\ &\quad + \mathbb{P}\Big[(\|0^*\|_1 + \|(n\mathbf{e}_1)^* - n\mathbf{e}_1\|_1) \ge \tfrac{\kappa}{8}\tfrac{n^{1/2}}{(\log n)^{5/2}}\Big] \\ &\le \mathbb{P}\Big[|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| \ge \tfrac{\kappa}{8}\sqrt{\tfrac{n}{\log n}}\Big] + 2\mathbb{P}\Big[\|0^*\|_1 \ge \tfrac{\kappa}{16}\tfrac{n^{1/2}}{(\log n)^{5/2}}\Big] \\ &\le c_1 \exp(-c_2\kappa\tfrac{\sqrt{n}}{(\log n)^{3/2}}) + 2\beta_1 \exp\Big(-\beta_2\kappa\tfrac{n^{1/2}}{(\log n)^{5/2}}\Big) \\ &\le c_1' \exp\Big(-c_2'\kappa\tfrac{\sqrt{n}}{(\log n)^{5/2}}\Big), \end{aligned} \tag{2.10}$$

for some $c_1', c_2' > 0$. Finally, combining the last three displays we get Theorem 1.1. □

3 The effect of resampling

As we will see in the next sections, in essence, the problem of sub-diffusive concentration of D_n^* can be reduced to understand the effect of resampling the edges along the geodesic of the modified graph distance. To study this issue, given an edge e, we introduce the *effective radius* R_e that measures how large the change of passage time when flipping the weight of e from 0 to $\log^2 n$ (see Subsection 3.2). Our strategy goes as follows. In the generalized first passage percolation models with a bounded distribution ζ, we define for any $z \in \mathbb{Z}^d$,

$$\mathrm{T}_z := \mathrm{T}(z, z + n\mathbf{e}_1).$$

Our goal is to study how the random variable T_z changes when resampling the value of each single edge e. In particular, the change can be estimated as: for a given edge-weight configuration $(t_e)_{e\in\mathbb{E}^d}$ and edge e, let (r, t_{e^c}) denote the configuration with value $t_e = r$ and $t_{e'}$ if $e' \neq e$. If $b \geq a$, it is easy to check that

$$\forall z \in \mathbb{Z}^d, \quad 0 \leq \mathrm{T}_z(b, t_{e^c}) - \mathrm{T}_z(a, t_{e^c}) \leq (b-a)\mathbb{I}(e \in \gamma_z),$$

where γ_z is the geodesic of $\mathrm{T}(a, t_{e^c})$ from z to $z + n\mathbf{e}_1$. However, this bound becomes less effective when b is much larger than a. To circumvent this difficulty, in our modified model with $\zeta = p\delta_1 + (1-p)\delta_{\log^2 n}$, we will construct a bypass of 1-weight edges avoiding the box centered at e. Thus the cost of resampling t_e can be bounded by the length of the bypass. Furthermore, we can control this length by the effective radius (see Proposition 3). The next question is to estimate the total cost of resampling all the edges in $\mathbb{E}^d$. We shall see that this problem leads to an investigation of total weight in a dependent percolation for which we use greedy lattice animal theory to deal with, see more in Section 4.

3.1. Connectivity properties of the cluster. In this section, we consider a Bernoulli percolation with parameter $p > p_c(d)$. We introduce the notion of good annulus, which plays an important role to construct suitable modified paths. We first review some properties of percolation related to the graph distance and crossing cluster.

Thanks to [28, Theorem 2], we can control the size of the holes in the infinite cluster:

Lemma 3.1. *Let $p > p_c(d)$. There exists a constant $c = c(p) > 0$ such that*

$$\forall t > 0, \quad \mathbb{P}(\Lambda_t \cap \mathcal{C}_\infty = \emptyset) \leq c^{-1}\exp(-ct). \tag{3.1}$$

Consequently, for all $x \in z^d$ and $t > 0$,

$$\mathbb{P}(\|x - x^*\|_\infty \geq t) \leq c^{-1}\exp(-ct). \tag{3.2}$$

The existence of open crossing clusters for boxes with high probability is proved in [27, Theorem 7.68].

Lemma 3.2. *There exist a constant $c = c(p) > 0$ such that for all $t > 0$,*

$$\mathbb{P}(\Lambda_t \textit{ has a open crossing cluster}) \geq 1 - c^{-1}\exp(-ct).$$

The following lemma which is a result of Antal and Pisztora [3, (4.49)] that provides the large deviation estimates for graph distance between two connected points.

Lemma 3.3. *There exist $\rho = \rho(p), \rho_1 = \rho_1(p), \rho_2 = \rho_2(p) > 0$ such that for every $x \in \mathbb{Z}^d$ and $t \geq \rho\|x\|_\infty$,*

$$\mathbb{P}(\infty > \mathrm{D}(0,x) \geq t) \leq \rho_1^{-1}\exp(-\rho_1 t), \tag{3.3}$$

and consequently,

$$\mathbb{P}(\mathrm{D}^*(0,x) \geq t) \leq \rho_2^{-1}\exp(-\rho_2 t). \tag{3.4}$$

For each $N \geq 1$, we define the family of N-annuli:

$$\forall e \in \mathbb{E}^d, \quad \mathrm{A}_N(e) = \Lambda_{3N}(e) \setminus \Lambda_N(e). \tag{3.5}$$

Next, we introduce properties of good annuli. Roughly speaking, a good annulus possesses the geometry of its percolation cluster so similar to Euclidean space that guarantees the feasibility of constructing a modified path. Before defining what is good annulus, let us give some definitions. Fix ρ as in Lemma 3.1 and $N_\rho = \lfloor \frac{N}{8\rho^2} \rfloor$. We now divide the annulus $\mathrm{A}_N(e)$ into a family of sub-boxes of side-length N_ρ such that two adjacent sub-boxes have only one shared face (see Figure 1a). We enumerate these sub-boxes as $(\Lambda^i_{N_\rho})_{i\in[m_\rho]}$ with some constant $m_\rho \leq (6N/N_\rho)^d = (48\rho^2)^d$ and write

$$\mathrm{A}_N(e) = \bigcup_{i=1}^{m_\rho} \Lambda^i_{N_\rho}.$$

For each $A, B \subset \mathbb{Z}^d$, we define

$$\mathrm{D}(A,B) := \inf\{\mathrm{D}(x,y) : x \in A,\, y \in B\} = \inf\{|\gamma| : x \in A,\, y \in B,\, \gamma \text{ is a open path from } x \text{ to } y\}.$$

In addition, for $A \subseteq \mathbb{Z}^d$, let D^A be the graph distance using only open paths inside A.

Definition 3.4. *For each $e \in \mathbb{E}^d$, we say that the annulus $\mathrm{A}_N(e)$ is **good** if the following hold:*

(i) There exists a open cluster $\mathcal{C}$ in $\mathrm{A}_N(e)$ which is crossing all sub-boxes $(\Lambda^i_{N_\rho})_{i\in[m_\rho]}$;

(ii) For all $x, y \in \mathrm{A}_N(e)$ such that $\mathrm{d}_\infty(\{x,y\}, \partial \mathrm{A}_N(e)) \geq N/2$, $\|x-y\|_\infty \leq 2N_\rho$, if $\mathrm{D}^{\mathrm{A}_N(e)}(x,y) < \infty$, then $\mathrm{D}^{\mathrm{A}_N(e)}(x,y) = \mathrm{D}(x,y)$.

(iii) If $\mathcal{D}$ is a connected component in $\mathrm{A}_N(e)$ such that $diam(\mathcal{D}) \geq N_\rho$, then $\mathcal{D} \cap \mathcal{C} \neq \emptyset$.

Remark 3.5. *The event $\{\mathrm{A}_N(e)$ is **good**$\}$ depends only on the state of the edges in $\Lambda_{4N}(e)$.*

We would like to control the probability of being **good** for $\mathrm{A}_N(e)$.

Lemma 3.6. *Let $p > p_c(d)$. There exists $c = c(p) > 0$ such that for all $N \geq 1$,*

$$\mathbb{P}(\mathrm{A}_N(e) \textit{ is } \textbf{good}) \geq 1 - c^{-1}\exp(-cN).$$

Before proving this lemma, we need the following result, whose proof is postponed until Appendix A. For $N \geq 1$ and $\varepsilon > 0$, we define

$$E_N = \{\text{there exists a crossing } \mathcal{C} \text{ and a connected component } \mathcal{D} \text{ in } \Lambda_N \text{ such that } \mathrm{diam}(\mathcal{D}) \geq \varepsilon N \text{ and } \mathcal{D} \cap \mathcal{C} = \emptyset\}$$

$$L_N = \{\text{there exist two disjoint open clusters of diameter at least } \varepsilon N \text{ in } \Lambda_N\}.$$

Lemma 3.7. *Let $p > p_c(d)$. There exists $c_1 = c_1(\varepsilon, p) > 0$ such that for all $N \geq 1$.*

$$\mathbb{P}(E_N) + \mathbb{P}(L_N) \leq c_1^{-1} \exp(-c_1 N).$$

Proof of Lemma 3.6. We first observe that the property (ii) directly follows from the following:

(ii') For all $x, y \in \mathrm{A}_N(e)$ such that $\mathrm{d}_\infty(\{x,y\}, \partial \mathrm{A}_N(e)) \geq N/2, \|x-y\|_\infty \leq 2N_\rho$,

if $\mathrm{D}^{\mathrm{A}_N(e)}(x,y) < \infty$ then $\mathrm{D}(x,y) \leq 4\rho N_\rho$.

Thanks to the union bound and Lemma 3.3,

$$\mathbb{P}(\mathrm{A}_N(e) \text{ does not satisfy (ii')}) \leq c_3^{-1} |\mathrm{A}_N(e)|^2 \exp(-4c_3 \rho N_\rho) \leq c_3^{-1} \exp(-2c_3 \rho N_\rho),$$

for some positive constant $c_3 = c_3(p)$. Therefore,

$$\mathbb{P}(\mathrm{A}_N(e) \text{ satisfies (ii)}) \geq 1 - c_3^{-1} \exp(-2c_3 \rho N_\rho). \tag{3.6}$$

To deal with (i), let us define

$$Crb := \{\Lambda^i_{N_\rho} \text{ has a open crossing cluster } \mathcal{C}_i, \, \forall i \in [m_\rho]\}.$$

We note that

$$\mathbb{P}(\mathrm{A}_N(e) \text{ satisfies (i)}) \geq \mathbb{P}(Crb \cap \{\mathcal{C}_i \cap \mathcal{C}_{i+1} \neq \emptyset, \, \forall i \in [m_\rho - 1]\}). \tag{3.7}$$

By Lemma 3.2, there exist a constant $c_4 = c_4(p) > 0$ such that

$$\mathbb{P}(Crb) \geq 1 - c_4^{-1} \exp(-c_4 N_\rho). \tag{3.8}$$

Remark that two consecutive sub-boxes always belong to a box of side-length $2N_\rho$ in $\mathrm{A}_N(e)$. Thus by Lemma 3.7,

$$\mathbb{P}(\mathcal{C}_i \cap \mathcal{C}_{i+1} \neq \emptyset \, \forall i \in [m_\rho - 1] \mid Crb) \geq 1 - c_5^{-1} \exp(-c_5 N_\rho),$$

for some constant $c_5 = c_5(p) > 0$. Hence, combining this with (3.8) and (3.7) gives

$$\mathbb{P}(\mathrm{A}_N(e) \text{ satisfies (i)}) \geq 1 - (c_4 c_5)^{-1} \exp(-c_4 c_5 N_\rho). \tag{3.9}$$

Suppose now that $\mathrm{A}_N(e)$ satisfies (i) but not (iii). Let $\mathcal{D}$ be the connected component in $\mathrm{A}_N(e)$ with $\mathrm{diam}(\mathcal{D}) \geq N_\rho$ and $\mathcal{C}$ be the open cluster that crosses all sub-boxes $(\Lambda^i_{N_\rho})_{i \in [m_\rho]}$, such that $\mathcal{D} \cap \mathcal{C} = \emptyset$. Then, there exists a sub-box Λ of side-length N_ρ in $\mathrm{A}_N(e)$ such that $\mathcal{D}'$-the largest connected component of $\mathcal{D} \cap \Lambda$ satisfies $\mathrm{diam}(\mathcal{D}') \geq \frac{N_\rho}{m_\rho}$ and $\mathcal{D}' \cap \mathcal{C} \cap \Lambda = \emptyset$. Hence, thanks to Lemma 3.7, there exists $c_6 = c_6(p) > 0$ such that for all $N \geq 1$

$$\begin{aligned}
&\mathbb{P}(\mathrm{A}_N(e) \text{ satisfies (i) but not (iii)}) \\
&\leq \mathbb{P}(\text{there exists a sub-box } \Lambda \text{ of side-length } N_\rho \text{ and a connected component } \mathcal{D}' \subseteq \Lambda \\
&\qquad \text{satisfying } \mathrm{diam}(\mathcal{D}') \geq \frac{N_\rho}{m_\rho} \text{ and } \mathcal{D}' \cap \mathcal{C} \cap \Lambda = \emptyset) \\
&\leq c_6^{-1} N^d \exp(-c_6 N_\rho).
\end{aligned}$$

Combining this estimate with (3.6) and (3.9) yields that

$$\begin{aligned}
\mathbb{P}[\mathrm{A}_N(e) \text{ is } \mathbf{good}] &\geq 1 - \mathbb{P}(\mathrm{A}_N(e) \text{ does not satisfy one of the three properties (i)–(iii)}) \\
&\geq 1 - c^{-1} \exp(-cN),
\end{aligned}$$

for some constant $c = c(p) > 0$.

3.2. Effective radius and its application. Given γ, the path inside $\mathrm{A}_N(e)$, we say that γ is a crossing path of $\mathrm{A}_N(e)$ if it joins $\partial\Lambda_N(e)$ and $\partial\Lambda_{3N}(e)$. Let $\mathscr{C}(\mathrm{A}_N(e))$ be the collection of all crossing paths of $\mathrm{A}_N(e)$. Let C_* be a fixed positive constant chosen in Proposition 1. To each $e \in \mathbb{E}^d$, $\mathrm{A}_N(e)$ is called C_*-connected if $\mathcal{V}_N^1(e) \cap \mathcal{V}_N^2(e)$ occur where

$$\mathcal{V}_N^1(e) := \{\forall \gamma_1, \gamma_2 \in \mathscr{C}(\mathrm{A}_N(e)),\ \mathrm{D}^{\mathrm{A}_N(e)}(\gamma_1, \gamma_2) \le C_* N\},$$
$$\mathcal{V}_N^2(e) := \{\forall x, y \in \mathrm{A}_N(e) \text{ with } \mathrm{D}^{\Lambda_{3N}(e)}(x, y) < \infty,\ \mathrm{D}^{\Lambda_{4N}(e)}(x, y) \le C_* N\}.$$

For each $e \in \mathbb{E}^d$, we define the C_*-**effective radius** of e as

$$R_e := \inf\{N \ge 1 : \mathrm{A}_N(e) \text{ is } C_*\text{-connected}\}.$$

Remark 3.8. *By the construction of the effective radius, for any $e \in \mathbb{E}^d$ and $t \ge 1$ the event $\{R_e = t\}$ depends only on the state of edges in the box $\Lambda_e(4t)$.*

The following propositions give a large deviation estimate for the effective radius.

Proposition 1. *Let $p > p_c(d)$. There exist $C_* = C_*(d, p) \ge 1, \alpha = \alpha(p) > 0$, such that for all $e \in \mathbb{E}^d$ and $t \ge 1$,*

$$\mathbb{P}[R_e \ge t] \le \alpha^{-1} \exp(-\alpha t).$$

As a consequence, all $(R_e)_{e\in\mathbb{E}^d}$ are finite almost surely.

Proof. Fix $e \in \mathbb{E}^d$. By the definition of R_e, we have for each $t \ge 2$

$$\mathbb{P}(R_e \ge t) \le 1 - \mathbb{P}(\mathcal{V}_{t-1}^1(e) \cap \mathcal{V}_{t-1}^2(e)) \le \mathbb{P}((\mathcal{V}_{t-1}^1(e))^c) + \mathbb{P}((\mathcal{V}_{t-1}^2(e))^c).$$

Thus we only need to show that there exist some positive constants $C_* = C_*(d, p), \beta = \beta(p)$, such that for N large enough

$$\max\{\mathbb{P}((\mathcal{V}_N^1(e))^c), \mathbb{P}((\mathcal{V}_N^2(e))^c)\} \le \beta^{-1} \exp(-\beta N). \tag{3.10}$$

We first consider $\mathbb{P}((\mathcal{V}_N^1(e))^c)$. Recall that ρ is the constant as in Lemma 3.3 and $N_\rho = \lfloor \frac{N}{8\rho^2} \rfloor$. We now show that

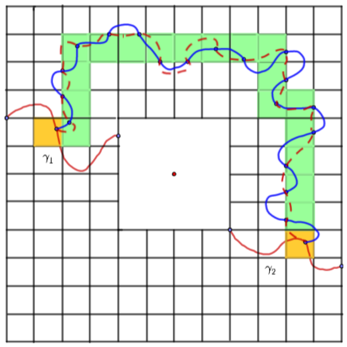
$$(\mathcal{V}_N^1(e))^c \cap \{\mathrm{A}_N(e) \text{ is } \textbf{good}\} \subseteq E_1, \tag{3.11}$$

where

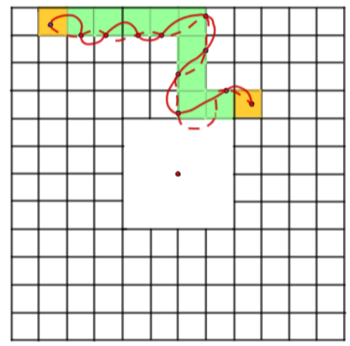
$$E_1 := \{\text{there exists } (x_j)_{j=1}^m \subset \mathrm{A}_N(e) \text{ with } m \le m_\rho, \text{ such that } \|x_j - x_{j+1}\|_\infty \le 2N_\rho \quad \forall j \in [m-1]$$
$$\text{and } \infty > \sum_{j=1}^{m-1} \mathrm{D}(x_j, x_{j+1}) > C_* N\}.$$

Suppose that $(\mathcal{V}_N^1(e))^c$ occurs, i.e. there exist $\gamma_1, \gamma_2 \in \mathscr{C}(\mathrm{A}_N(e))$ such that $\mathrm{D}^{\mathrm{A}_N(e)}(\gamma_1, \gamma_2) > C_* N$. The delicate part of the proof is the construction of a short path inside $\mathrm{A}_N(e)$ that joins γ_1 to γ_2. For each $j \in \{1, 2\}$, let π_j is the largest connected path of $\gamma_j \cap \left\{\Lambda_{2N+\frac{N_\rho}{2}}(e) \setminus \Lambda_{2N-\frac{N_\rho}{2}}(e)\right\}$. It is straightforward to check that

$$\forall j \in \{1, 2\}, \quad \operatorname{diam}(\pi_j) \ge |\pi_j| \ge N_\rho,\ \mathrm{d}_\infty(\pi_j, \partial\mathrm{A}_N(e)) \ge 3N/4.$$



(a) Illustration of the short path (dashed red curve) joining γ_1 to γ_2 inside annulus $\mathrm{A}_N(e)$



(b) Illustration of the short path (dashed red curve) joining x to y inside $\Lambda_{4N}(e)$

Suppose in addition that $\mathrm{A}_N(e)$ is **good**. Then by Definition 3.4 (iii), we have $\pi_1 \cap \mathcal{C} \neq \emptyset$ and $\pi_2 \cap \mathcal{C} \neq \emptyset$. Therefore, there exist $\ell_1 = \ell_1(\gamma_1, \mathcal{C}), \ell_2 = \ell_2(\gamma_2, \mathcal{C}) \in [m_\rho]$ such that

$$\forall j \in \{1,2\}, \quad \pi_j \cap \mathcal{C} \cap \Lambda_{N_\rho}^{\ell_j} \neq \emptyset;\ \mathrm{d}_\infty(\Lambda_{N_\rho}^{\ell_j}, \partial \mathrm{A}_N(e)) \geq \frac{N}{2}. \tag{3.12}$$

As consequence, we can take a tuple $(i_j)_{j=1}^m \subseteq [m_\rho]$ for some $m \leq m_\rho$ with $i_1 = \ell_1, i_m = \ell_2$, such that $(\Lambda_{N_\rho}^{i_j})_{j=1}^m$ is the sequence of consecutive sub-boxes satisfying

$$\forall j \in [m], \quad \mathrm{d}_\infty(\Lambda_{N_\rho}^{i_j}, \partial \mathrm{A}_N(e)) \geq N/2. \tag{3.13}$$

Using Definition 3.4 (i) of being **good** of $\mathrm{A}_N(e)$, we can take a sequence $(x_j)_{j=1}^m \subseteq \mathrm{A}_N(e)$ such that $x_1 \in \gamma_1 \cap \mathcal{C} \cap \Lambda_{N_\rho}^{i_1}, x_m \in \gamma_2 \cap \mathcal{C} \cap \Lambda_{N_\rho}^{i_m}, x_j \in \mathcal{C} \cap \Lambda_{N_\rho}^{i_j}$ for all $j = 2, \ldots, m-1$. We remark that

$$\infty > \sum_{j=1}^{m-1} \mathrm{D}^{\mathrm{A}_N(e)}(x_j, x_{j+1}) \geq \mathrm{D}^{\mathrm{A}_N(e)}(\gamma_1, \gamma_2) > C_* N.$$

Moreover, notice that for all $j \in [m-1]$, $\|x_j - x_{j+1}\|_\infty \leq 2N_\rho$ and for all $j \in [m]$, $\mathrm{d}_\infty(x_j, \partial \mathrm{A}_N(e)) \geq N/2$, since $\Lambda_{N_\rho}^{i_j}$ and $\Lambda_{N_\rho}^{i_{j+1}}$ are two consecutive sub-boxes satisfying (3.13). Hence, it follows from Definition 3.4 (ii) of being **good** of $\mathrm{A}_N(e)$ that for all $j \in [m-1]$

$$\mathrm{D}^{\mathrm{A}_N(e)}(x_j, x_{j+1}) = \mathrm{D}(x_j, x_{j+1}).$$

In conclusion, the sequence $(x_j)_{j=1}^m \subset \mathrm{A}_N(e)$ satisfies $m \leq m_\rho$, $\forall j \in [m-1], \|x_j - x_{j+1}\|_\infty \leq 2N_\rho$ and $\infty > \sum_{j=1}^{m-1} \mathrm{D}(x_j, x_{j+1}) > C_* N$. We complete the proof of (3.11).

Next, we estimate $\mathbb{P}[E_1]$. By the union bound,

$$\begin{aligned} \mathbb{P}[E_1] &\leq \mathbb{P}[\text{there exist } x, y \in \mathrm{A}_N(e) \text{ such that } \|x-y\|_\infty \leq 2N_\rho, \infty > \mathrm{D}(x_j, x_{j+1}) > C_* \tfrac{N}{m}] \\ &\leq (6N)^d \max_{\substack{x,y \in \mathrm{A}_N(e) \\ \|x-y\|_\infty \leq 2N_\rho}} \mathbb{P}\left[\infty > \mathrm{D}(x,y) > C_* \tfrac{N}{m_\rho}\right]. \end{aligned} \tag{3.14}$$

Taking $C_* = 48^d \rho^{2d-1}/2$ such that $C_* N / m_\rho \geq \frac{N}{2\rho} \geq 2\rho N_\rho$, thanks to Lemma 3.3, if $\|x-y\|_\infty \leq 2N_\rho$ then

$$\mathbb{P}\left[\infty > \mathrm{D}(x,y) > C_* \tfrac{N}{m}\right] \leq \exp(\tfrac{\rho_2}{2\rho} N). \tag{3.15}$$

Combining this with (3.11) and Lemma 3.6 give us for all N large enough,

$$\begin{aligned}\mathbb{P}((\mathcal{V}_N^1(e))^c) &\le \mathbb{P}((\mathcal{V}_N^1(e))^c \cap \{\mathrm{A}_N(e) \text{ is } \mathbf{good}\}) + c_1\exp(-c_2N)\\ &\le \mathcal{O}(N^d)\exp(-\tfrac{\rho_2}{2\rho}N) + c_1\exp(-c_2N)\\ &\le \exp(-\nu_1 N),\end{aligned} \tag{3.16}$$

for some positive constant $\nu_1 = \nu_1(p)$.

Now by similar but simpler arguments as in (3.16), we also have for N large enough

$$\mathbb{P}[(\mathcal{V}_N^2(e))^c] \le \exp(-\nu_2 N), \tag{3.17}$$

for some positive constant $\nu_2 = \nu_2(p)$. Hence, (3.10) follows by combining (3.16) and (3.17) and taking $\beta = \nu_1 \wedge \nu_2$.

□

For any path γ, we denote by $\mathrm{clo}(\gamma)$ the set of all closed edges in γ. The following propositions help us build a bypass for a given edge in an arbitrary path, with economical cost (comparable to the effective radius).

Proposition 2. *Let C_* be the constant as in Proposition 1. The following holds.*

- *(i) For any path γ between x and y with $x, y \in \mathbb{Z}^d$, if an edge $e \in \gamma$ satisfies $x, y \notin \Lambda_{3R_e}(e)$, then there exists another path η_e between x and y such that:*
  - *(i-a) $\eta_e \cap \Lambda_{R_e}(e) = \emptyset$ and $\eta_e \setminus \gamma$ consists only of open edges;*
  - *(i-b) $|\eta_e \setminus \gamma| \le C_* R_e$;*
- *(ii) Let γ be a path between x and y with $x, y \in \mathcal{C}_\infty$. Suppose that an closed edge $e \in \gamma$ satisfies either $x \notin \Lambda_{3R_e}(e)$ or $y \notin \Lambda_{3R_e}(e)$. Then there exists a path η_e between x and y, such that:*
  - *(ii-a) $\mathrm{clo}(\eta_e) \cap \gamma \cap \Lambda_{R_e}(e) = \emptyset$ and $\eta_e \setminus \gamma$ consists only of open edges;*
  - *(ii-b) $|\eta_e \setminus \gamma| \le 2C_* R_e$.*

Proof. Let us first consider (i). Assume that γ is a arbitrary path between x and y with $x, y \in \mathbb{Z}^d$. If $e \in \gamma$ and $x, y \notin \Lambda_{3R_e}(e)$, then γ crosses the annulus $\mathrm{A}_{R_e}(e)$ at least twice. Let γ_1 and γ_2 be theses first and last crossing, in the order from x to y. Notice that both γ_1 and γ_2 belong to $\mathscr{C}(\mathrm{A}_N(e))$. Furthermore, by the definition of R_e, the event $\mathcal{V}_{R_e}^1(e)$ occurs, and so $\mathrm{D}^{\mathrm{A}_{R_e}(e)}(\gamma_1, \gamma_2) \le C_* R_e$. Let $\tilde{\eta}_e$ be a geodesic of $\mathrm{D}^{\mathrm{A}_{R_e}(e)}(\gamma_1, \gamma_2)$. Then $\tilde{\eta}_e$ consists of only open edges and satisfies

$$|\tilde{\eta}_e| = \mathrm{D}^{\mathrm{A}_{R_e}(e)}(\gamma_1, \gamma_2) \le C_* R_e.$$

Suppose that $\tilde{\eta}_e$ intersects with γ_1 and γ_2 at z_1 and z_2, respectively. We define

$$\eta_e = \gamma_{x,z_1} \cup \tilde{\eta}_e \cup \gamma_{z_2,y},$$

where for any path γ and $u, v \in \gamma$ we write $\gamma_{u,v}$ the sub-path of γ from u to v. Notice that $|\eta_e \setminus \gamma| = |\tilde{\eta}_e| \le C_* R_e$. Moreover, since γ_1 and γ_2 are the first and last crossing path of the annulus $A_{R_e}(e)$, one has $\gamma_{x,z_1} \cap \Lambda_{R_e}(e) = \varnothing$ and $\gamma_{z_2,y} \cap \Lambda_{R_e}(e) = \varnothing$. In addition, $\tilde{\eta} \cap \Lambda_{R_e}(e) = \varnothing$ since $\tilde{\eta}_e \subset \mathrm{A}_{R_e}(e)$. Hence, $\eta_e \cap \Lambda_{R_e}(e) = \varnothing$ and so we get (i).

For (ii), assume that γ is a arbitrary path between x and y with $x, y \in \mathcal{C}_\infty$. Let $e \in \gamma$ such that either $x \notin \Lambda_{3R_e}(e)$ or $y \notin \Lambda_{3R_e}(e)$. We consider two cases:

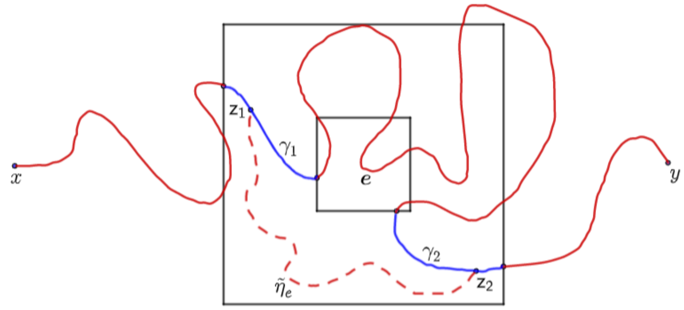


Figure 2 – A bypass consists only of open edges (dashed red curve) avoiding the box centered at e

- **Case 1:** $\{x, y\} \notin \Lambda_{3R_e}(e)$. Then by Proposition 1 (iii), there exists a path η_e from x to y, such that $\eta_e \setminus \gamma$ consists of only open edges, $\eta_e \cap \Lambda_{R_e}(e) = \varnothing$, and $|\eta_e \setminus \gamma| \leq C_* R_e$. Hence, we get (ii).

- **Case 2:** there is only $x \in \Lambda_{3R_e}(e)$ or $y \in \Lambda_{3R_e}(e)$. We suppose that $x \in \Lambda_{3R_e}(e)$, the proof for the remaining case is similar and omitted. The path γ crosses the annulus $\mathrm{A}_{R_e}(e)$ at least once. We call the last crossing path by γ_1. Since $x \in \mathcal{C}_\infty$, there exists a open path $\xi_{x,\infty}$ joining x to ∞.

  <u>Case 2a</u>: $\xi_{x,\infty}$ crosses the annulus $\mathrm{A}_{R_e}(e)$. Let $\gamma_2 \subset \xi_{x,\infty}$ be the first crossing path of $\mathrm{A}_{R_e}(e)$, so $\gamma_2 \in \mathscr{C}(\mathrm{A}_{R_e}(e))$. Since the event $\mathcal{V}^1_{R_e}(e)$ occurs, there exists a geodesic of $\mathrm{D}^{\mathrm{A}_{R_e}(e)}(\gamma_1, \gamma_2)$ inside $\mathrm{A}_{R_e}(e)$, denoted by $\tilde{\eta}_e$, that consists of only open edges and satisfies $|\tilde{\eta}_e| \leq C_* R_e$. Suppose that $\tilde{\eta}_e$ intersects with γ_1 and γ_2 at z_1 and z_2, respectively. By the definition of γ_2 and z_2, ξ_{x,z_2}-the sub-path of $\xi_{x,\infty}$ from x to z_2 is open and satisfies $\xi_{x,z_2} \subset \Lambda_{3R_e}(e)$. Thus $\mathrm{D}^{\Lambda_{3R_e}(e)}(x, z_2) < \infty$. Hence, by the definition of $\mathcal{V}^2_{R_e}(e)$, one has $\mathrm{D}^{\Lambda_{4R_e}(e)}(x, z_2) \leq C_* R_e$. Let us denote by $\tilde{\xi}_{x,z_2}$ the geodesic of $\mathrm{D}^{\Lambda_{4R_e}(e)}(x, z_2)$ and define

  $$\eta_e = \tilde{\xi}_{x,z_2} \cup \tilde{\eta}_e \cup \gamma_{z_1,y}.$$

  We observe that $\eta_e \setminus \gamma \subseteq \tilde{\xi}_{x,z_2} \cup \tilde{\eta}_e$ consists of only open edges, and

  $$|\eta_e \setminus \gamma| \leq |\gamma_{x,z_2}| + |\tilde{\eta}_e| \leq 2C_* R_e, \quad \mathrm{clo}(\eta_e) \cap \gamma \cap \Lambda_{R_e}(e) = \varnothing,$$

  since $\eta_e \cap \gamma \cap \Lambda_{R_e} \subset \tilde{\xi}_{x,z_2}$, which is open and $\mathrm{clo}(\eta_e)$ is closed. Hence, (ii) follows.

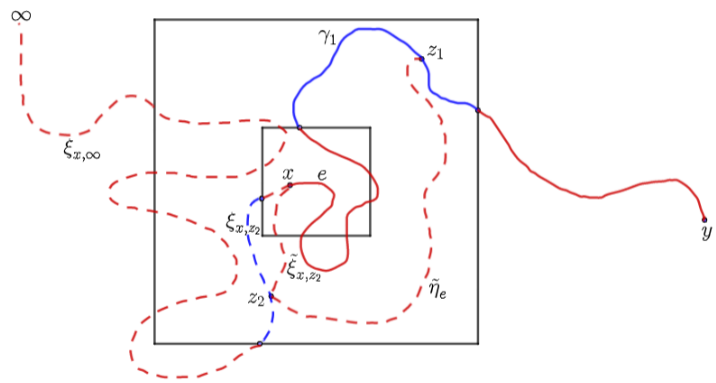
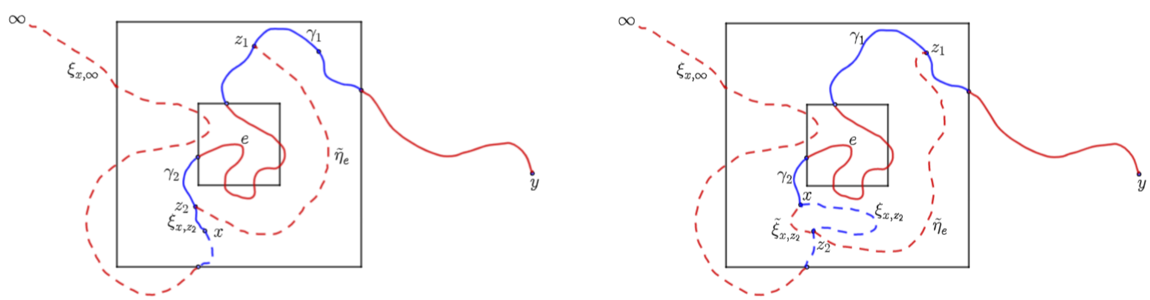


Figure 3 – Illustration of Case 2a: $\xi_{x,\infty}$ crosses the annulus $\mathrm{A}_{R_e}(e)$

Case 2b: $\xi_{x,\infty}$ does not cross the annulus $\mathrm{A}_{R_e}(e)$. Let $\gamma_{e,x}$ be the sub-path of γ joining e to x. Hence, $\eta_{e,\infty} := \gamma_{e,x} \cup \xi_{x,\infty}$ crosses the annulus $\mathrm{A}_{R_e}(e)$. Let $\gamma_2 \subset \eta_{e,\infty}$ be the first crossing path of $\mathrm{A}_{R_e}(e)$, and so $\gamma_2 \in \mathscr{C}(\mathrm{A}_{R_e}(e))$. By the definition of $\mathcal{V}^1_{R_e}(e)$, there exists a geodesic of $\mathrm{D}^{\mathrm{A}_{R_e}(e)}(\gamma_1, \gamma_2)$ inside $\mathrm{A}_{R_e}(e)$, denoted by $\tilde{\eta}_e$, that consists of only q-open edges and satisfies $|\tilde{\eta}_e| \le C_* R_e$. Suppose that $\tilde{\eta}_e$ intersects with γ_1 and γ_2 at z_1 and z_2, respectively.



(a) Sub-case 2b: $z_2 \in \tilde{\eta}_e \cap \gamma_{e,x}$

(b) Sub-case 2b: $z_2 \in \tilde{\eta}_e \cap \xi_{x,\infty}$

Figure 4 – Illustration of Case 2b: $\xi_{x,\infty}$ does not cross the annulus $\mathrm{A}_{R_e}(e)$

If $z_2 \in \tilde{\eta}_e \cap \gamma_{e,x}$, we set

$$\eta_e = \gamma_{x,z_2} \cup \tilde{\eta}_e \cup \gamma_{z_1,y},$$

where γ_{x,z_2} is the sub-path of γ from x to z_2. Therefore, $\eta_e \setminus \gamma = \tilde{\eta}_e$, consists of only q-open edges, and

$$|\eta_e \setminus \gamma| = |\tilde{\eta}_e| \le C_* R_e, \quad \eta_e \cap \Lambda_{R_e}(e) = \varnothing.$$

If $z_2 \in \tilde{\eta}_e \cap \xi_{x,\infty}$, we have $\mathrm{D}^{\Lambda_{3R_e}(e)}(x, z_2) < \infty$. Using similar arguments as in Case 2a, we have $\mathrm{D}^{\Lambda_{4R_e}(e)}(x, z_2) \le C_* R_e$. We then call $\tilde{\xi}_{x,z_2}$ the geodesic of $\mathrm{D}^{\Lambda_{4R_e}(e)}(x, z_2)$ and define

$$\eta_e = \tilde{\xi}_{x,z_2} \cup \tilde{\eta}_e \cup \gamma_{z_1,y}.$$

Hence, $\eta_e \setminus \gamma \subset \tilde{\xi}_{x,z_2} \cup \tilde{\eta}_e$ consists of only open edges, and

$$|\eta_e \setminus \gamma| \le |\tilde{\xi}_{x,z_2}| + |\tilde{\eta}_e| \le 2C_* R_e, \quad \mathrm{clo}(\eta_e) \cap \gamma \cap \Lambda_{R_e}(e) = \varnothing,$$

since $\eta_e \cap \gamma \Lambda_{R_e} \subset \tilde{\xi}_{x,z_2}$ is open and $\mathrm{clo}(\eta_e)$ is closed. Hence, we get (ii).

□

As a consequence, we can control the effect of resampling an edges on the geodesic of the modified graph distance by the following result.

Proposition 3. *Let $p > p_c(d)$. Assume that $(R_e)_{e\in\mathbb{E}^d}$ be the sequence of C_*-effective radii as in Proposition 1. The following holds for all $z \in \mathbb{Z}^d$ and $e \in \mathbb{E}^d$,*

$$0 \le \mathrm{T}_z(\log^2 n, t_{e^c}) - \mathrm{T}_z(1, t_{e^c}) \le (\log^2 n \mathbb{I}(\mathcal{U}_{z,e}) + \hat{R}_e)\mathbb{I}(e \in \gamma_z), \tag{3.18}$$

where γ_z is a geodesic of $\mathrm{T}_z(1, t_{e^c})$,

$$\mathcal{U}_{z,e} = \{R_e \ge r_{z,e}\}, \quad r_{z,e} = \frac{1}{3}\left(\|e - z\|_\infty \wedge \|e - (z + n\mathbf{e}_1)\|_\infty\right),$$

and

$$\hat{R}_e = C_* R_e \wedge \log^2 n.$$

Proof. Since T_z is increasing, the first inequality in (3.18) is trivial. By the definition of the modified graph distance, if $e \notin \gamma_z$ then

$$\mathrm{T}_z(\log^2 n, t_{e^c}) - \mathrm{T}_z(1, t_{e^c}) = 0. \tag{3.19}$$

If $e \in \gamma_z$ and $\mathcal{U}_{z,e}$ occurs (or equivalently either z or $z + n\mathbf{e}_1$ is in $\Lambda_{3R_e}(e)$), we use the trivial bound

$$(\mathrm{T}_z((\log n)^2, t_{e^c}) - \mathrm{T}_z(1, t_{e^c}))\mathbb{I}(e \in \gamma_z; \mathcal{U}_{z,e}) \le (\log^2 n)\mathbb{I}(e \in \gamma_z; \mathcal{U}_{z,e}). \tag{3.20}$$

On the other hand, suppose that $e \in \gamma_z$ and $\mathcal{U}^c_{z,e}$ occurs (or neither z nor $z + n\mathbf{e}_1$ is in $\Lambda_{3R_e}(e)$). Applying Proposition 1 (iii) to $\gamma = \gamma_z$, we have there exists a path η_e between z and $z + n\mathbf{e}_1$ such that

$$\mathrm{T}_z(\log^2 n, t_{e^c}) - \mathrm{T}_z(1, t_{e^c}) \le \mathrm{T}(\eta_e \setminus \gamma_z) \le C_* R_e.$$

Moreover, we always have $\mathrm{T}_z(\log^2 n, t_{e^c}) - \mathrm{T}_z(1, t_{e^c}) \le \log^2 n$. Therefore,

$$(\mathrm{T}_z((\log n)^2, t_{e^c}) - \mathrm{T}_z(1, t_{e^c}))\mathbb{I}(e \in \gamma_z; \mathcal{U}^c_{z,e}) \le (C_* R_e \wedge \log^2 n)\mathbb{I}(e \in \gamma_z). \tag{3.21}$$

Combining this estimate with (3.20), we arrive at

$$\mathrm{T}_z(\log^2 n, t_{e^c}) - \mathrm{T}_z(1, t_{e^c}) \le (\log^2 n \mathbb{I}(\mathcal{U}_{z,e}) + (C_* R_e \wedge \log^2 n))\mathbb{I}(e \in \gamma_z), \tag{3.22}$$

with together with (3.19) implies the desirable result. □

4 Lattice animals of dependent weight

We first recall the result derived from the theory of greedy lattice animals that helps us control the maximal weight of paths in locally dependent percolation.

Given an integer $M \ge 1$ and positive constants a, A, suppose that $(I_{e,M})_{e\in\mathbb{E}^d}$ is a collection of Bernoulli random variables satisfying

(E1) $(I_{e,M})_{e\in\mathbb{E}^d}$ are $aM-$dependent, i.e. for all $e \in \mathbb{E}^d$, the variable $I_{e,M}$ is independent of all variables $\{I_{e',M} : e' \notin \Lambda_{aM}(e)\}$.

(E2)

$$q_M = \sup_{e\in\mathbb{E}^d} \mathbb{E}[I_{e,M}] \leq AM^{-d}.$$

For any path γ, we define

$$N(\gamma) = \sum_{e\in\gamma} I_{e,M}, \quad N_{L,M} = \max_{\gamma\in\Xi_L} N(\gamma),$$

where for $L \geq 1$,

$$\Xi_L = \{\gamma : \gamma \text{ is a set of edges in } \Lambda_L;\ |\gamma| \leq L\}.$$

Lemma 4.1. [8, Lemma 2.6] *Let $M \geq 1, a, A > 0$ and $(I_{e,M})_{e\in\mathbb{E}^d}$ be a collection of random variables satisfying (E1) and (E2). Then, there exists a positive constant $C = C(a, A, d)$ such that*

(i) For all $L \in \mathbb{N}$,

$$\mathbb{E}[N_{L,M}] \leq CLq_M^{1/d} M^{d+1}.$$

(ii) If $t \geq CM^d \max\left(1, MLq_M^{1/d}\right)$, then

$$\mathbb{P}(N_{L,M} \geq t) < 2^d \exp(-t/(16M)^d).$$

We aim at extending this result to general weight distributions. Let a, A be positive constants. Suppose that $(X_e)_{e\in\mathbb{E}^d}$ is a collection of non-negative random variables satisfying the following: for all $M \geq 1$

(P1) for all $e \in \mathbb{E}^d$, the event $\{M - 1 < X_e \leq M\}$ is independent of the state of all edges $\{e' : e' \notin \Lambda_{aM}(e)\}$,

(P2) there exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(M) \leq AM^{-d}$ and

$$q_M = \sup_{e\in\mathbb{E}^d} \mathbb{P}(M - 1 < X_e \leq M) \leq \phi(M).$$

Lemma 4.2. *Let $X = (X_e)_{e\in\mathbb{E}^d}$ be a family of random variables satisfying (P1) and (P2) and let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function satisfying*

$$\text{(H)} \qquad \sum_{M=1}^{\infty} \left(f(M) + f^2(M) + f^4(M)\right) M^{d+1}\phi(M)^{1/d} < \infty.$$

Then there exits a positive constant $C = C(a, A, f)$, such that the following holds.

(i) For all $L \geq 1$,

$$\mathbb{E}\left[\left(\max_{\gamma\in\Xi_L} \sum_{e\in\gamma} f(X_e)\right)^2\right] \leq CL^2.$$

(ii) *Let γ be a random path starting from 0 in the same probability space of X. Then for all $L \geq 1$,*

$$\mathbb{E}\left[\left(\sum_{e\in\gamma} f(X_e)\right)^2\right] \leq CL^2 + C\sum_{\ell\geq L} \ell^2(\mathbb{P}(|\gamma| = \ell))^{1/2}.$$

(iii) *Let $m \geq 1$ and γ be a random set of edges of $B(m)$ in the same probability space of X. Then for all $L \geq 1$,*

$$\mathbb{E}\left[\left(\sum_{e\in\gamma} f(X_e)\right)^2\right] \leq C(L+m)^2 + C\sum_{\ell\geq L} (\ell+m)^2(\mathbb{P}(|\gamma| = \ell))^{1/2}.$$

(iv) *There exist constants $\alpha, \beta > 0$ such that for any random path γ starting from 0 in the same probability space of X and integers L, K satisfying $LM\phi(M)^{1/d} \geq 1$ for all $K \geq M \geq 1$,*

$$\mathbb{P}\left(\sum_{e\in\gamma} f(\hat{X}_e) \geq \alpha L\right) \leq \sum_{M=1}^{K} \exp(-\beta LM\phi(M)^{1/d}) + \mathbb{P}(|\gamma| \geq L),$$

where $\hat{X}_e = X_e \wedge K$.

Proof. We first prove (i). By Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left(\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} f(X_e)\right)^2\right] \leq \mathbb{E}\left[\max_{\gamma\in\Xi_L}|\gamma|\sum_{e\in\gamma} f^2(X_e)\right] \leq L\mathbb{E}\left[\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} f^2(X_e)\right], \tag{4.1}$$

since $|\gamma| \leq L$ for all $\gamma \in \Xi_L$. For set of edges γ, we define

$$A_M^\gamma := |\{e \in \gamma : M-1 < X_e \leq M\}| = \sum_{e\in\gamma} I_{e,M},$$

where

$$I_{e,M} := \mathbb{I}(M-1 < X_e \leq M).$$

Since f is increasing,

$$\sum_{e\in\gamma} f^2(X_e) \leq \sum_{M\geq 1} f^2(M)A_M^\gamma + f^2(0)|\gamma|, \tag{4.2}$$

and hence

$$\begin{aligned}\mathbb{E}\left[\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} f^2(X_e)\right] &\leq \mathbb{E}\left[\sum_{M\geq 1} f^2(M)\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} I_{e,M}\right] + f^2(0)L\\ &= \sum_{M\geq 1} f^2(M)\mathbb{E}\left[N_{L,M}\right] + f^2(0)L,\end{aligned} \tag{4.3}$$

where

$$N_{L,M} = \max_{\gamma\in\Xi_L}\sum_{e\in\gamma} I_{e,M}.$$

By (P1) and (P2), the conditions (E1) and (E2) are satisfied for all $M \geq 1$. Now using Lemma 4.1 (i), we obtain that for all $M \geq 1$,

$$\mathbb{E}[N_{L,M}] = \mathcal{O}(1) L M^{d+1} \phi(M)^{1/d}. \tag{4.4}$$

This together with (4.3) implies that

$$\mathbb{E}\left[\max_{\gamma \in \Xi_L} \sum_{e \in \gamma} f^2(X_e)\right] = \mathcal{O}(1) L \sum_{M \geq 1} f^2(M) M^{d+1} (\phi(M))^{1/d} = \mathcal{O}(L),$$

using (H). Finally, combining the above estimate with (4.1), we obtain (i). Next, to prove (ii), we decompose

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{e \in \gamma} f(X_e)\right)^2\right] &= \mathbb{E}\left[\left(\sum_{e \in \gamma} f(X_e)\right)^2 \mathbb{I}(|\gamma| < L)\right] + \mathbb{E}\left[\left(\sum_{e \in \gamma} f(X_e)\right)^2 \mathbb{I}(|\gamma| \geq L)\right] \\
&\leq \mathbb{E}\left[\left(\max_{\gamma \in \Xi_L} \sum_{e \in \gamma} f(X_e)\right)^2\right] + \sum_{\ell=L}^{\infty} \mathbb{E}\left[|\gamma| \sum_{e \in \gamma} f^2(X_e) \mathbb{I}(|\gamma| = \ell)\right] \\
&\leq \mathcal{O}(L^2) + \sum_{\ell=L}^{\infty} \ell \mathbb{E}\left[\max_{\gamma \in \Xi_\ell} \sum_{e \in \gamma} f^2(X_e) \mathbb{I}(|\gamma| = \ell)\right], \qquad (4.5)
\end{aligned}$$

by using (i). Moreover, by Cauchy-Schwarz inequality and (i),

$$\begin{aligned}
\mathbb{E}\left[\max_{\gamma \in \Xi_\ell} \sum_{e \in \gamma} f^2(X_e) \mathbb{I}(|\gamma| = \ell)\right] &\leq \mathbb{E}\left[\left(\max_{\gamma \in \Xi_\ell} \sum_{e \in \gamma} f^2(X_e)\right)^2\right]^{1/2} \mathbb{E}\left[\mathbb{I}(|\gamma| = \ell)\right]^{1/2} \\
&\leq \mathcal{O}(\ell) (\mathbb{P}(|\gamma| = \ell))^{1/2}.
\end{aligned}$$

Combining the last two displays yields (ii). We can easily prove (iii) by using the same arguments as for (ii) and the fact that if $|\gamma| \leq t$ then $\gamma \in \Xi_{t+m}$ for all $t \geq 1$.

Finally, we show (iv). Using $\hat{X}_e \leq K$ and the similar estimate as in (4.2), we have for any $\alpha \geq 2f(0)$

$$\begin{aligned}
\mathbb{P}\left(\sum_{e \in \gamma} f(\hat{X}_e) \geq \alpha L\right) &\leq \mathbb{P}\left(\sum_{e \in \gamma} f(\hat{X}_e) \geq \alpha L, |\gamma| \leq L\right) + \mathbb{P}(|\gamma| \geq L) \\
&\leq \mathbb{P}\left(\sum_{M=1}^{K} f(M) N_{L,M} \geq \alpha L/2\right) + \mathbb{P}(|\gamma| \geq L), \qquad (4.6)
\end{aligned}$$

Furthermore, the conditions (E1) and (E2) are satisfied for all $M \geq 1$. Let C be the constant as in Lemma 4.1, and set

$$\alpha = 2f(0) + 2C \sum_{M=1}^{\infty} f(M) (\phi(M))^{1/d} M^{d+1}.$$

Note that $\alpha \in (0,\infty)$ by (H). Using Lemma 4.1 (ii),

$$\begin{aligned}\mathbb{P}\left(\sum_{M=1}^{K} f(M)N_{L,M} \geq \alpha L/2\right) &\leq \mathbb{P}\left(\sum_{M=1}^{K} f(M)N_{L,M} \geq \sum_{M=1}^{K} Cf(M)\phi(M)^{1/d}LM^{d+1}\right)\\ &\leq \sum_{M=1}^{K} \mathbb{P}\left(N_{L,M} \geq C\phi(M)^{1/d}LM^{d+1}\right)\\ &\leq \sum_{M=1}^{K} \exp(-cLM\phi(M)^{1/d}),\end{aligned}$$

with $c = c(d)$ a positive constant. Combining this with (4.6), we derive (iv). □

In the next section, we shall apply Lemma 4.2 to the sequence of effective radii. Let C_* be the constant and $(R_e)_{e\in\mathbb{E}^d}$ be the collection of effective radii as in Proposition 1. We define also

$$\hat{R}_e = \min\{C_* R_e, \log^2 n\}.$$

Corollary 4.3. *There exists a positive constant C such that the following holds for all n, m sufficiently large.*

(i) Let γ be a random path in the same probability space of $(R_e)_{e\in\mathbb{E}^d}$, starting from 0 and satisfying $\mathbb{P}(|\gamma| = \ell) \leq C\ell^{-7}$ for all $l \geq Cn$. Then

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} \hat{R}_e\Big)^2\Big] \leq Cn^2.$$

(ii) Let γ be a random set of edges of $B(m)$ for some $m \geq 1$ in the same probability space of $(R_e)_{e\in\mathbb{E}^d}$ satisfying $\mathbb{P}(|\gamma| = \ell) \leq C\ell^{-7}$ for all $\ell \geq Cm$. Then

$$\mathbb{E}\Big[\sum_{e\in\gamma} \hat{R}_e^2\Big] \leq Cm.$$

(iii) Let γ be a random path starting from 0 in the same probability space of $(R_e)_{e\in\mathbb{E}^d}$. Then

$$\mathbb{P}\left(\sum_{e\in\gamma} \hat{R}_e^2 \geq Cn\right) \leq C\log^2 n \exp\left(-\frac{n}{C\log^{2(d+10)} n}\right) + \mathbb{P}(|\gamma| \geq Cn).$$

Proof. Let $\phi(x) = x^{-(d^2+11d)}$ for $x > 0$. We can take $f(x) = x$ for the proof (i) and $f(x) = x^2$ for the proofs of (ii) and (iii). It is easy to check that the condition (H) holds true. By Proposition 1 (iii), the radii $(R_e)_{e\in\mathbb{E}^d}$ satisfy (P1). Moreover, the condition (P2) is verified using Proposition 1 (ii). Hence, the corollary directly follows from Lemma 4.2. □

5 Subdiffusive concentration of T_n

The proof strategy for Theorem 2.2 relies on establishing a connection between bounds on $\mathrm{Var}(e^{\lambda \mathrm{T}_n/2})$ and exponential concentration ([5, Lemma 4.1]). To attain the required variance bound (Theorem 5.3), we apply the Falik-Samorodnitsky inequality (Lemma 5.1) to a martingale decomposition of the random variable $e^{\lambda \mathrm{F}_m}$, where F_m represents an averaged version of the passage time. This approach was initially introduced by Benaïm and Rossignol in [5] and later used by Damron, Hanson, and Sosoe in [15]. Finally, we estimate the tails of the true passage time T_n based on those of F_m (see more details in Section 5.2).

5.1. Variance bound via entropy inequality. Let us enumerate the edges $\mathbb{E}^d$ as $\{e_1, e_2, \ldots\}$ and let $a, b \in \mathbb{R} \cup \{+\infty\}$. Assume that $(t_{e_i})_{i\geq 1}$ are i.i.d. random variables with the same distribution as

$$\zeta = p\delta_a + (1-p)\delta_b.$$

Let $g : \{a, b\}^{\mathbb{E}^d} \to \mathbb{R}$ be a function of $(t_{e_i})_{i\geq 1}$. Fix $\lambda \in \mathbb{R}$ and define

$$G = G_\lambda = e^{\lambda g}.$$

We write

$$G = G(t_{e_i}, t_{e_i^c})$$

to emphasize G is the function of the random variables t_{e_i} and $t_{e_i^c} = (t_{e_j})_{j\neq i}$. We define the natural filtration of these random variables as

$$\mathcal{F}_0 = \varnothing, \quad \mathcal{F}_i = \sigma(t_{e_1}, \ldots, t_{e_i}),$$

for each $i \geq 1$. Now we consider the martingale increments

$$\forall i \geq 1, \quad \Delta_i = \mathbb{E}[G \mid \mathcal{F}_i] - \mathbb{E}[G \mid \mathcal{F}_{i-1}] = \mathbb{E}[G(t'_{e_i}, t_{e_i^c}) - G(t_{e_i}, t_{e_i^c}) \mid \mathcal{F}_{i-1}],$$

where t'_{e_i} is an independent copy of t_{e_i}. We will bound the variance of G based on the following entropy inequality by Falik and Samorodnitsky [21, Lemma 2.3].

Lemma 5.1 (Falik-Samorodnitsky)**.** *If* $\mathbb{E}[G^2] < \infty$ *then*

$$\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] \geq \mathrm{Var}[G] \log \frac{\mathrm{Var}[G]}{\sum_{i=1}^{\infty} (\mathbb{E}[|\Delta_i|])^2}, \tag{5.1}$$

where Ent *denotes the entropy operator: if* X *is a non-negative random variable such that* $\mathbb{E}[X] < \infty$*, then*

$$\mathrm{Ent}[X] = \mathbb{E}\left[X \log \frac{X}{\mathbb{E}[X]}\right].$$

The following estimate on the total entropy is derived from the tensorization property of entropy and the log-Sobolev inequality for the Bernoulli distribution. Notably, the proof of this result follows the same approach as in [16, Lemma 6.3], albeit in a simpler context.

Lemma 5.2. *Assume that* $\mathbb{E}[G^4] < \infty$*. Then, there exists a positive constant* C *depending on* p *such that*

$$\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] \leq C \sum_{i=1}^{\infty} \mathbb{E}[(G(b, t_{e_i^c}) - G(a, t_{e_i^c}))^2]. \tag{5.2}$$

5.2. Proof of Theorem 2.2. Instead of directly showing the subdiffusive concentration of T_n, we will employ a strategy inspired by Benjamini, Kalai, and Schramm [6], known as the BKS trick. This approach involves proving the subdiffusive concentration for a geometric average of passage times, a notion previously used in both [2] and [29]. It is expected that the majority of edges in the lattice have a low probability of lying in the geodesic of T_n, meaning they have a small influence. However, this does not hold for edges very close to the origin, and the BKS trick provides a way to circumvent this challenge.

First of all, let us define a spatial average of the first passage time,

$$\mathrm{F}_m = \frac{1}{|\Lambda_m|} \sum_{z \in \Lambda_m} \mathrm{T}_z, \tag{5.3}$$

where

$$\mathrm{T}_z := \mathrm{T}(z, z + n\mathbf{e}_1), \quad m = \lfloor n^{1/4} \rfloor.$$

To prove Theorem 2.2, it now suffices to show the following variance bound.

Theorem 5.3. *There exists a constant $c > 0$ such that*

$$\forall\, |\lambda| < \frac{1}{\sqrt{K}}, \quad \mathrm{Var}\left[e^{\lambda \mathrm{F}_m}\right] \le K\lambda^2 \mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right] < \infty, \tag{5.4}$$

where $K = \dfrac{cn}{\log n}$.

The following result is a direct consequence of Theorem 5.3. We refer the reader to [5, Lemma 4.1] for a proof.

Corollary 5.4. *There exist positive constants c_1', c_2' such that*

$$\mathbb{P}\left(|\mathrm{F}_m - \mathbb{E}[\mathrm{F}_m]| \ge \sqrt{\frac{n}{\log n}\kappa}\right) \le c_1' e^{-c_2'\kappa}, \quad \forall \kappa \ge 0. \tag{5.5}$$

Next, we prepare a simple large deviation estimate for the first passage time which will be used to compare T_n and F_m.

Lemma 5.5. *There exist positive constants ρ, ρ_1, ρ_2 such that for all $x, y \in \mathbb{Z}^d$ and $t \ge \rho\|x-y\|_\infty$,*

$$\mathbb{P}(\mathrm{T}(x,y) \ge t) \le \rho_1 \exp(-\rho_2 t / \log^2 n). \tag{5.6}$$

Proof. Observe that $\mathrm{T}(u,v) \le \log^2 n \|u - v\|_1$ for all $u, v \in \mathbb{Z}^d$. Therefore, by the triangle inequality

$$\begin{aligned}\mathbb{P}\left(\mathrm{T}(x,y) \ge t\right) &\le \mathbb{P}\left(\mathrm{T}(x,x^*) + \mathrm{T}(y,y^*) + \mathrm{T}(x^*,y^*) \ge t\right) \\ &\le \mathbb{P}\left(\mathrm{D}^*(x,y) \ge t/2\right) + \mathbb{P}\left(\|x - x^*\|_1 \ge \tfrac{t}{4\log^2 n}\right) + \mathbb{P}\left(\|y - y^*\|_1 \ge \tfrac{t}{4\log^2 n}\right),\end{aligned}$$

where we recall that z^* is the closest point of z in the infinite cluster $\mathcal{C}_\infty$. The last two terms are bounded by $\beta_1 \exp\left(\frac{-\beta_2 t}{\log^2 n}\right)$, for some positive constants β_1, β_2 using Lemma 3.1, whereas by Lemma 3.3, the first term is bounded by $\rho_2^{-1}\exp(-\rho_2 t)$ when $t \ge \rho\|x - y\|_\infty$ with some $\rho, \rho_1, \rho_2 > 0$. Hence, the result follows. □

Proof of Theorem 2.2. Since $\mathbb{E}[\mathrm{F}_m] = \mathbb{E}[\mathrm{T}_n]$,

$$\begin{aligned}|\mathrm{T}_n - \mathbb{E}[\mathrm{T}_n]| = |\mathrm{F}_m - \mathbb{E}[\mathrm{T}_n] + \mathrm{T}_n - \mathrm{F}_m| &= |\mathrm{F}_m - \mathbb{E}[\mathrm{F}_m] + \mathrm{T}_n - \mathrm{F}_m| \\ &\le |\mathrm{F}_m - \mathbb{E}[\mathrm{F}_m]| + |\mathrm{T}_n - \mathrm{F}_m|.\end{aligned} \tag{5.7}$$

Thus, for all $M \geq 1$, using the union bound, we have

$$\mathbb{P}\left(|\mathrm{T}_n - \mathbb{E}[\mathrm{T}_n]| \geq 4M\right) \leq \mathbb{P}\left(|\mathrm{F}_m - \mathbb{E}[\mathrm{F}_m]| \geq 2M\right) + \mathbb{P}\left(|\mathrm{T}_n - \mathrm{F}_m| \geq 2M\right). \tag{5.8}$$

By subadditivity property,

$$\begin{aligned} |\mathrm{T}_n - \mathrm{F}_m| = \Big|\mathrm{T}_n - \frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m} \mathrm{T}_z\Big| &\leq \frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m} |\mathrm{T}(0, n\mathbf{e}_1) - \mathrm{T}(z, z+n\mathbf{e}_1)| \\ &\leq \frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m} (\mathrm{T}(0,z) + \mathrm{T}(n\mathbf{e}_1, n\mathbf{e}_1 + z)). \end{aligned} \tag{5.9}$$

Observe that if the event $\left\{\frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m}(\mathrm{T}(0,z) + \mathrm{T}(n\mathbf{e}_1, n\mathbf{e}_1 + z)) \geq 2M\right\}$ occurs,

$$\max_{z\in\Lambda_m} \mathrm{T}(0,z) \geq M \text{ or } \max_{z\in\Lambda_m} \mathrm{T}(n\mathbf{e}_1, n\mathbf{e}_1 + z) \geq M. \tag{5.10}$$

Combining this with union bound, it yields that

$$\begin{aligned} &\mathbb{P}\left(\frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m} (\mathrm{T}(0,z) + \mathrm{T}(n\mathbf{e}_1, n\mathbf{e}_1 + z)) \geq 2M\right) \\ &\qquad \leq \mathbb{P}\left(\max_{z\in\Lambda_m} \mathrm{T}(0,z) \geq M\right) + \mathbb{P}\left(\max_{z\in\Lambda_m} \mathrm{T}(n\mathbf{e}_1, n\mathbf{e}_1 + z) \geq M\right) \\ &\qquad = 2\mathbb{P}\left(\max_{z\in\Lambda_m} \mathrm{T}(0,z) \geq M\right) \\ &\qquad \leq 2|\Lambda_m| \max_{z\in\Lambda_m} \mathbb{P}\left(\mathrm{T}(0,z) \geq M\right), \end{aligned} \tag{5.11}$$

where for the equation we have used the translation invariance.

Let $M = \frac{1}{4}\sqrt{\frac{n}{\log n}}\kappa$. Since $m = o(M)$, Lemma 5.5 shows that

$$\max_{z\in\Lambda_m} \mathbb{P}\left(\mathrm{T}(0,z) \geq M\right) \leq \rho_1 e^{-\rho_2 M/\log^2 n}, \tag{5.12}$$

for some positive constants ρ_1, ρ_2. Using this estimate, (5.9), and (5.11) yields

$$\mathbb{P}\left(|\mathrm{T}_n - \mathrm{F}_m| \geq \frac{\kappa}{2}\sqrt{\frac{n}{\log n}}\right) \leq \mathcal{O}(m^d)\exp\Big(-\rho_2 \frac{\sqrt{n}}{4\sqrt{\log^5 n}}\kappa\Big).$$

Combining this with Corollary 5.4 and (5.8), it follows that

$$\mathbb{P}\left(|\mathrm{T}_n - \mathbb{E}[\mathrm{T}_n]| \geq \sqrt{\frac{n}{\log n}}\kappa\right) \leq c_1 e^{-c_2\kappa}, \qquad \forall \kappa \geq 0, \tag{5.13}$$

for some positive constants c_1, c_2. □

Proof of Theorem 5.3. According to the Lemma 5.1, the variance bound relies on two crucial factors: the estimate of total influence and total entropy. These keys are presented in the following results.

Proposition 4. *Let $d \geq 2$. There exists a positive constant C such that*

$$\sum_{i=1}^{\infty}(\mathbb{E}[|\Delta_i|])^2 \leq C\lambda^2\mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big]n^{(9-d)/8}, \quad \forall \lambda \in \mathbb{R}.$$

Proposition 5. *Let $d \geq 2$. There exists a positive constant C such that*

$$\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] \leq C\lambda^2 n \mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big], \quad \forall |\lambda| \leq \tfrac{1}{\log^{2(d+11)} n}. \tag{5.14}$$

By Lemma 5.1, Propositions 4 and 5, we have

$$\mathrm{Var}\left[e^{\lambda \mathrm{F}_m}\right] \leq C\left(\log \frac{\mathrm{Var}[e^{\lambda \mathrm{F}_m}]}{C\lambda^2 n^{(9-d)/8}\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right]}\right)^{-1} \lambda^2 n \mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big]. \tag{5.15}$$

We can assume that

$$\mathrm{Var}[e^{\lambda \mathrm{F}_m}] \geq C\lambda^2 n^{15/16}\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right], \tag{5.16}$$

since otherwise there is nothing left to prove. By (5.15) and (5.16), there exist constants $c, C > 0$ such that for any $\lambda \leq \frac{C}{\log^{2(d+11)} n}$,

$$\mathrm{Var}\left[e^{\lambda \mathrm{F}_m}\right] \leq c\lambda^2 \frac{n}{\log n}\mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big].$$

This concludes the proof of Theorem 5.3 by substituting $\lambda/2$ for λ. □

In the rest of Section 5, we prove Propositions 4 and 5 in subsections 5.3 and 5.4 respectively.

5.3. Bound on the total influence: Proof of Proposition 4. Proposition 4 is a direct consequence of the following lemma with the notice that $m = \lfloor n^{1/4} \rfloor$.

Lemma 5.6. *Let $d \geq 2$. There exists a constant $C > 0$ such that the following holds.*

(i)

$$\sup_{i \geq 1} \mathbb{E}[|\Delta_i|] \leq C|\lambda| m^{(1-d)/2}\big(\mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big]\big)^{1/2}, \quad \forall \lambda \in \mathbb{R}. \tag{5.17}$$

(ii)

$$\sum_{i=1}^{\infty} \mathbb{E}[|\Delta_i|] \leq C|\lambda| n \big(\mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big]\big)^{1/2}, \quad \forall \lambda \in \mathbb{R}. \tag{5.18}$$

5.3.1. Proof of Lemma 5.6 (i). Fix $i \geq 1$ and consider

$$\Delta_i = \mathbb{E}[G|\mathcal{F}_i] - \mathbb{E}[G|\mathcal{F}_{i-1}] = \mathbb{E}\left[G(t'_{e_i}, t_{e_i^c}) - G(t_{e_i}, t_{e_i^c})|\mathcal{F}_{i-1}\right]. \tag{5.19}$$

We have

$$\mathbb{E}\left[|\Delta_i|\right] \leq \mathbb{E}\left[|G(t'_{e_i}, t_{e_i^c}) - G(t_{e_i}, t_{e_i^c})|\right] = 2\mathbb{E}\left[\left(e^{\lambda \mathrm{F}_m(t'_{e_i}, t_{e_i^c})} - e^{\lambda \mathrm{F}_m(t_{e_i}, t_{e_i^c})}\right)_+\right],$$

where t'_{e_i} is the independent copy of t_{e_i}. Furthermore, using the inequality that $(e^{\lambda a} - e^{\lambda b})_+ \leq |\lambda|(e^{\lambda a} + e^{\lambda b})|a - b|$, we get

$$\begin{aligned}
\mathbb{E}[|\Delta_i|] &\leq 2|\lambda|\mathbb{E}\left[\left(e^{\lambda \mathrm{F}_m(t'_{e_i}, t_{e_i^c})} + e^{\lambda \mathrm{F}_m(t_{e_i}, t_{e_i^c})}\right)\left|\mathrm{F}_m(t'_{e_i}, t_{e_i^c}) - \mathrm{F}_m(t_{e_i}, t_{e_i^c})\right|\right] \\
&= 4|\lambda|\mathbb{E}\left[e^{\lambda \mathrm{F}_m(t_{e_i}, t_{e_i^c})}\left|\mathrm{F}_m(t'_{e_i}, t_{e_i^c}) - \mathrm{F}_m(t_{e_i}, t_{e_i^c})\right|\right] \\
&\leq 8|\lambda|\mathbb{E}\left[e^{\lambda \mathrm{F}_m(t_{e_i}, t_{e_i^c})}\left(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\right)\right], \qquad (5.20)
\end{aligned}$$

where for the last line we have used $\left|\mathrm{F}_m(t'_{e_i}, t_{e_i^c}) - \mathrm{F}_m(t_{e_i}, t_{e_i^c})\right| \leq 2\left(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\right)$. For each $z \in \mathbb{Z}^d$, let γ_z be a geodesic of $\mathrm{T}_z(1, t_{e_i^c})$. Let C_* be the constant and $(R_{e_i})_{i\geq 1}$ be the collection of random radii obtained in Proposition 1. By Proposition 3,

$$\mathrm{T}_z(\log^2 n, t_{e_i^c}) - \mathrm{T}_z(1, t_{e_i^c}) \leq (\log^2 n \mathbb{I}(\mathcal{U}_{z,e_i}) + \hat{R}_{e_i})\mathbb{I}(e_i \in \gamma_z),$$

where

$$\mathcal{U}_{z,e_i} = \{R_{e_i} \geq r_{z,e_i}\}, \quad r_{z,e_i} = \tfrac{1}{3}\|e_i - z\|_\infty \wedge \|e_i - (z + n\mathbf{e}_1)\|_\infty,$$

and

$$\hat{R}_{e_i} = C_* R_{e_i} \wedge \log^2 n.$$

Therefore,

$$\begin{aligned}\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c}) &= \frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m}(T_z(\log^2 n, t_{e_i^c}) - T_z(1, t_{e_i^c})) \\ &\leq \frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m}\left(\log^2 n\mathbb{I}(\mathcal{U}_{z,e_i}) + \hat{R}_{e_i}\right)\mathbb{I}(e_i \in \gamma_z).\end{aligned}$$

Observe that if the event $\mathcal{U}_{z,e_i} \cap \{r_{z,e_i} \geq \log^3 n\}$ occurs, then $C_* R_{e_i} \geq \log^2 n$ and so $\hat{R}_{e_i} = \log^2 n$. Therefore, the above estimate implies that

$$0 \leq \mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c}) \leq A_i, \tag{5.21}$$

where

$$A_i = \frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m}\left(2\hat{R}_{e_i} + \log^2 n\mathbb{I}(r_{z,e_i} \leq \log^3 n)\right)\mathbb{I}(e_i \in \gamma_z). \tag{5.22}$$

Combining this with (5.20) and Cauchy-Schwarz inequality yields

$$\mathbb{E}[|\Delta_i|] \leq 8|\lambda|\mathbb{E}\left[e^{\lambda \mathrm{F}_m} A_i\right] \tag{5.23}$$

$$\leq 8|\lambda|\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right]^{1/2}\mathbb{E}\left[A_i^2\right]^{1/2}. \tag{5.24}$$

Here for the first line, we remark that $\mathrm{F}_m(t_{e_i}, t_{e_i^c}) = \mathrm{F}_m$.

Next we will estimate $\mathbb{E}[A_i^2]$. Notice that for all edges $e \in \mathbb{E}^d$ and $\Lambda \subset \mathbb{Z}^d$,

$$|\{z \in \Lambda : r_{z,e} \leq t\}| \leq \left|\{z \in \mathbb{Z}^d : r_{z,e} \leq t\}\right| = \mathcal{O}(t^d). \tag{5.25}$$

Therefore,

$$\frac{1}{\Lambda_m}\sum_{z\in\Lambda_m}\mathbb{I}(r_{z,e_i} \leq \log^3 n) \leq \frac{\mathcal{O}(\log^{3d} n)}{|\Lambda_m|} = \mathcal{O}(m^{1-d}), \tag{5.26}$$

since $m = \lfloor n^{1/4}\rfloor$. Thus, by Cauchy-Schwarz inequality,

$$A_i^2 \leq \frac{8}{\Lambda_m}\sum_{z\in\Lambda_m}\hat{R}_{e_i}^2\mathbb{I}(e_i \in \gamma_z) + \mathcal{O}(m^{2-2d}). \tag{5.27}$$

Combining this estimate and the translation invariance, we have

$$\begin{aligned}\mathbb{E}[A_i^2] &\leq \frac{8}{|\Lambda_m|}\mathbb{E}\left[\sum_{z\in\Lambda_m}\hat{R}^2_{e_i-z}\mathbb{I}(e_i - z\in\gamma_0)\right] + \mathcal{O}(m^{2-2d})\\ &= \frac{8}{|\Lambda_m|}\mathbb{E}\left[\sum_{e\in\gamma}\hat{R}^2_{e}\right] + \mathcal{O}(m^{2-2d}),\end{aligned} \tag{5.28}$$

where

$$\gamma = \gamma_0\cap\{e_i - \Lambda_m\},\quad \{e_i-\Lambda_m\} = \{e' = (x_{e_i}-z, y_{e_i}-z) : z\in\Lambda_m\}.$$

Observe that if $|\gamma|\geq \ell$ then there exist $x,y\in V(e_i,m)$-the vertex set of $e_i-\Lambda_m$ such that $\mathrm{T}(x,y)\geq\ell$. Therefore, using the union bound and Lemma 5.5 we have for all $l\geq\rho m$ with ρ a sufficiently large constant,

$$\begin{aligned}\mathbb{P}(|\gamma|\geq\ell) &\leq \mathbb{P}\left(\exists\, x,y\in V(e_i,m) : \mathrm{T}(x,y)\geq\ell\right)\\ &\leq (2m+1)^{2d}\max_{x,y\in V(e_i,m)}\mathbb{P}\left(\mathrm{T}(x,y)\geq\ell\right)\\ &\leq \mathcal{O}(1)\exp\left(-\frac{cl}{\log^2 n}\right),\end{aligned} \tag{5.29}$$

with some positive constant $c>0$. Here, notice that to apply Lemma 5.5, we have used $\|x-y\|_\infty\leq 2m$ for all $x,y\in V(e_i,m)$.

The above estimate verifies the condition in Corollary 4.3 (ii) and thus

$$\mathbb{E}\Big[\sum_{e\in\gamma}\hat{R}^2_e\Big] = \mathcal{O}(m). \tag{5.30}$$

Combining (5.28) and (5.30), it yields that for all $i\geq 1$,

$$\mathbb{E}[A_i^2] = \mathcal{O}(m^{1-d}). \tag{5.31}$$

Finally, we conclude from (5.24) and (5.31) that

$$\sup_{i\geq 1}\mathbb{E}[|\Delta_i|] \leq \mathcal{O}(1)|\lambda|\big(\mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big]\big)^{1/2} m^{(1-d)/2}, \tag{5.32}$$

and the result follows. □

5.3.2. Proof of Lemma 5.6 (ii). Using (5.23) and Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}\sum_{i=1}^{\infty}\mathbb{E}[|\Delta_i|] &\leq 8|\lambda|\mathbb{E}\Big[e^{\lambda\mathrm{F}_m}\sum_{i=1}^{\infty}A_i\Big]\\ &\leq 8|\lambda|\big(\mathbb{E}\big[e^{2\lambda\mathrm{F}_m}\big]\big)^{1/2}\Big(\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}A_i\Big)^2\Big]\Big)^{1/2},\end{aligned} \tag{5.33}$$

where A_i is defined as in (5.22). Notice that

$$\begin{aligned}\sum_{i=1}^{\infty} A_i &= \frac{1}{|\Lambda_m|} \sum_{z\in\Lambda_m} \sum_{i=1}^{\infty} \big(2\hat{R}_{e_i} + \log^2 n \mathbb{I}(r_{z,e_i} \le \log^3 n)\big)\mathbb{I}(e_i \in \gamma_z) \\ &= \frac{2}{|\Lambda_m|} \sum_{z\in\Lambda_m} \sum_{e\in\gamma_z} \hat{R}_e + \frac{\log^2 n}{|\Lambda_m|} \sum_{z\in\Lambda_m} \sum_{e\in\gamma_z} \mathbb{I}(r_{z,e} \le \log^3 n) \\ &= \frac{2}{|\Lambda_m|} \sum_{z\in\Lambda_m} \sum_{e\in\gamma_z} \hat{R}_{e_i} + \mathcal{O}(\log^{3d+2} n),\end{aligned}$$

by using (5.25). Therefore, by Cauchy-Schwarz inequality,

$$\Big(\sum_{i=1}^{\infty} A_i\Big)^2 \le \frac{8}{|\Lambda_m|} \sum_{z\in\Lambda_m} \Big(\sum_{e\in\gamma_z} \hat{R}_e\Big)^2 + \mathcal{O}(\log^{6d+4} n). \tag{5.34}$$

It follows from Lemma 5.5 that

$$\mathbb{P}(|\gamma_z| \ge \rho n) \le \rho_1 \exp(-\rho_2 n/\log^2 n).$$

Then applying Corollary 4.3 (i) gives

$$\mathbb{E}\left[\left(\sum_{e\in\gamma_z} \hat{R}_e\right)^2\right] = \mathcal{O}(n^2). \tag{5.35}$$

Combining (5.34) with (5.35) yields

$$\mathbb{E}\left[\left(\sum_{i=1}^{\infty} A_i\right)^2\right] = \mathcal{O}(n^2), \tag{5.36}$$

which together with (5.33) implies that

$$\sum_{i=1}^{\infty} \mathbb{E}[|\Delta_i|] \le \mathcal{O}(1)|\lambda| n \big(\mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big]\big)^{1/2}.$$

□

5.4. Entropy bound: Proof proposition 5 . Using Lemma 5.2, the total entropy is bounded by

$$\begin{aligned}\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] &\le C \sum_{i=1}^{\infty} \mathbb{E}\Big[(G(\log^2 n, t_{e_i^c}) - G(1, t_{e_i^c}))^2\Big] \\ &= C \sum_{i=1}^{\infty} \mathbb{E}\Big[\Big(e^{\lambda \mathrm{F}_m(\log^2 n, t_{e_i^c})} - e^{\lambda \mathrm{F}_m(1, t_{e_i^c})}\Big)^2\Big] \\ &\le 2C|\lambda|^2 \sum_{i=1}^{\infty} \mathbb{E}\Big[\Big(e^{2\lambda \mathrm{F}_m(\log^2 n, t_{e_i^c})} + e^{2\lambda \mathrm{F}_m(1, t_{e_i^c})}\Big) \\ &\qquad \times \big(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\big)^2\Big].\end{aligned} \tag{5.37}$$

Here for the last line we used the inequality that $|e^{\lambda a} - e^{\lambda b}| \leq |\lambda|(e^{\lambda a} + e^{\lambda b})(a-b)$ for all $a \geq b$. We remark that $\mathrm{F}_m(\log^2 n, t_{e_i^c})$ and $\mathrm{F}_m(1, t_{e_i^c})$ are independent of t_{e_i}, and hence

$$\begin{aligned}
&\mathbb{E}\Big[e^{2\lambda \mathrm{F}_m(\log^2 n, t_{e_i^c})}\big(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\big)^2\Big] \\
&= \frac{1}{1-p}\mathbb{E}\Big[e^{2\lambda \mathrm{F}_m(\log^2 n, t_{e_i^c})}\big(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\big)^2 \mathbb{I}(t_{e_i} = \log^2 n)\Big] \\
&\leq \frac{1}{1-p}\mathbb{E}\Big[e^{2\lambda \mathrm{F}_m}\big(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\big)^2\Big].
\end{aligned}$$

Similarly,

$$\mathbb{E}\Big[e^{2\lambda \mathrm{F}_m(1, t_{e_i^c})}\big(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\big)^2\Big] \leq \frac{1}{p}\mathbb{E}\Big[e^{2\lambda \mathrm{F}_m}\big(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\big)^2\Big].$$

Combining these inequalities with (5.37) and (5.21), we get

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] &\leq \mathcal{O}(1)\lambda^2 \sum_{i=1}^{\infty} \mathbb{E}\Big[e^{2\lambda \mathrm{F}_m}\big(\mathrm{F}_m(\log^2 n, t_{e_i^c}) - \mathrm{F}_m(1, t_{e_i^c})\big)^2\Big] \\
&\leq \mathcal{O}(1)\lambda^2 \sum_{i=1}^{\infty} \mathbb{E}\big[e^{2\lambda \mathrm{F}_m} A_i^2\big]. \qquad (5.38)
\end{aligned}$$

By Cauchy-Schwarz inequality and (5.25),

$$\begin{aligned}
\sum_{i=1}^{\infty} A_i^2 &\leq \frac{1}{|\Lambda_m|} \sum_{z \in \Lambda_m} \sum_{i=1}^{\infty} \big(8\hat{R}_{e_i}^2 + 2\log^4 n \mathbb{I}(r_{z,e_i} \leq \log^3 n)\big)\mathbb{I}(e_i \in \gamma_z) \\
&= \frac{8}{|\Lambda_m|} \sum_{z \in \Lambda_m} \sum_{e \in \gamma_z} \hat{R}_e^2 + \frac{2\log^4 n}{|\Lambda_m|} \sum_{z \in \Lambda_m} \sum_{e \in \gamma_z} \mathbb{I}(r_{z,e} \leq \log^3 n) \\
&= \frac{8}{|\Lambda_m|} \sum_{z \in \Lambda_m} Y_z + \mathcal{O}(\log^{3d+4} n), \qquad (5.39)
\end{aligned}$$

where

$$Y_z = \sum_{e \in \gamma_z} \hat{R}_e^2.$$

Combining the last two estimates, we obtain

$$\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] \leq \frac{\mathcal{O}(1)\lambda^2}{|\Lambda_m|} \sum_{z \in \Lambda_m} \mathbb{E}\big[e^{2\lambda \mathrm{F}_m} Y_z\big] + \mathcal{O}(\log^{3d+4} n)\lambda^2 \mathbb{E}\big[e^{2\lambda \mathrm{F}_m}\big]. \qquad (5.40)$$

By Lemma 5.5, there exist positive constants c_1, c_2 and C such that

$$\begin{aligned}
\mathbb{P}(|\gamma_z| \geq Cn) &\leq \mathbb{P}(\mathrm{T}(z, z + n\mathbf{e}_1) \geq Cn) \\
&= \mathbb{P}(\mathrm{T}_n \geq Cn) \leq c_1 \exp(-c_2 n / \log^2 n). \qquad (5.41)
\end{aligned}$$

Using this estimate and Corollary 4.3 (iii),

$$\mathbb{P}(Y_z \geq Cn) \leq \mathcal{O}(\log^2 n) \exp\left(-c\frac{n}{\log^{2(d+10)} n}\right), \qquad (5.42)$$

where c is the positive constant. Moreover,

$$
\begin{aligned}
\mathbb{E}\left[e^{2\lambda \mathrm{F}_m} Y_z\right] &\le Cn\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right] + \mathbb{E}\left[e^{2\lambda \mathrm{F}_m} Y_z \mathbb{I}(Y_z \ge Cn)\right] \\
&\le Cn\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right] + (\mathbb{E}[e^{8\lambda \mathrm{F}_m}])^{1/4}(\mathbb{P}[Y_z \ge Cn])^{1/4}(\mathbb{E}[Y_z^2])^{1/2},
\end{aligned} \tag{5.43}
$$

using Cauchy-Schwarz inequality. Thanks to (5.42),

$$\mathbb{E}[Y_z^2] = \mathcal{O}(n^2).$$

It follows from Jensen inequality, the transition invariance, and Lemma 5.5 that for all $\lambda \le \frac{1}{\log^{2(d+11)} n}$,

$$\mathbb{E}[e^{8\lambda \mathrm{F}_m}] \le \frac{1}{|\Lambda_m|} \sum_{z \in \Lambda_m} \mathbb{E}[e^{8\lambda \mathrm{T}_z}] = \mathbb{E}[e^{8\lambda \mathrm{T}_n}] \le \mathcal{O}(1) \exp\left(\frac{8\rho n}{\log^{2(d+11)} n}\right).$$

Notice that by Lemma 5.5, there exist ρ, ρ_1, ρ_2 such that for any $t \ge \rho n$,

$$\mathbb{P}(\mathrm{F}_m \ge t) = \mathbb{P}(\exists z \in \Lambda_m : \mathrm{T}_z \ge t) \le \rho_1 \exp(-\rho_2 t / \log^2 n).$$

This implies that for all $|\lambda| \le \frac{1}{\log^{2(d+11)} n}$,

$$\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right] \ge \exp\left(-\frac{\rho n}{\log^{2(d+11)} n}\right).$$

Combining the last five display equations yields

$$
\begin{aligned}
\mathbb{E}\left[e^{2\lambda \mathrm{F}_m} Y_z\right] &\le Cn\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right] + \mathcal{O}(1)n^2 \exp\left(\frac{\rho n}{\log^{2(d+11)} n}\right) \exp\left(-c\frac{n}{4\log^{2(d+10)} n}\right) \\
&\le \mathcal{O}(n)\mathbb{E}\left[e^{2\lambda \mathrm{F}_m}\right],
\end{aligned}
$$

which together with (5.40) implies the desired result. □

6 Comparison of the graph distance and the first passage time

To control the difference between D_n^* and $\mathrm{T}(0^*, (n\mathbf{e}_1)^*)$, we remove all $\log^2 n$-weight edges on the geodesic of $\mathrm{T}(0^*, (n\mathbf{e}_1)^*)$ by constructing the family of bypasses using only 1-weight edges. As a result, this discrepancy can be bounded from above by the total weight (or total length) of these bypasses. Furthermore, we remark that the lengths of bypasses can be simply controlled by using the effective radius, as defined in Proposition 1.

The following lemma is key result to prove Theorem 2.1.

Lemma 6.1. *Let C_* be the constant and $(R_e)_{e \in \mathbb{E}^d}$ be the effective radii as in Proposition 1. Let γ_n be a geodesic of $\mathrm{T}(0^*, (n\mathbf{e}_1)^*)$ satisfying $\{0^*, (n\mathbf{e}_1)^*\} \not\subset \Lambda_{3R_e}(e)$ for all $e \in \gamma_n$. Then there exists a subset $\Gamma_n \subset \gamma_n$ such that*

(i) if $e \in \Gamma_n$, then $t_e = \log^2 n$ and $R_e \ge \frac{\log^2 n}{2C_*}$,

(ii) for all $e, e' \in \Gamma_n$,

$$\|e - e'\|_\infty \ge \max\{R_e, R_{e'}\},$$

(iii)

$$|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| \le 2C_* \sum_{e\in\Gamma_n} R_e.$$

Proof. We recall the coupling between Bernoulli first passage percolation and Bernoulli percolation with parameter p: each 1-weight edge (resp. $\log^2 n$) is open (resp. closed). We construct Γ_n by the following process: Notice that $\mathrm{clo}(\gamma_n) = \{e \in \gamma_n : t_e = \log^2 n\}$. Define

$$e = \arg\max\{R_e : e \in \mathrm{clo}(\gamma_n)\}. \tag{6.1}$$

In the case that there several maximize edges, we choose one of them in a deterministic rule. By Proposition 2 (ii), we obtain a modified path of γ_n, namely η_e, such that the following holds.

(a) $\mathrm{clo}(\eta_e) \subset \mathrm{clo}(\gamma_n)$, $\mathrm{clo}(\eta_e) \cap \gamma_n \cap \Lambda_{R_e}(e) = \emptyset$;

(b) $\eta_e \setminus \gamma_n$ consists only of 1-weight (open) edges;

(c) $\mathrm{T}(\eta_e \setminus \gamma_n) = |\eta_e \setminus \gamma_n| \le 2C_* R_e$.

Therefore,

$$\mathrm{T}(\gamma_n) \le \mathrm{T}(\eta_e) \le \mathrm{T}(\gamma_n) + \mathrm{T}(\eta_e \setminus \gamma_n) - t_e \le \mathrm{T}(\gamma_n) + 2C_* R_e - \log^2 n,$$

which implies that $R_e \ge \frac{\log^2 n}{2C_*}$. We now update

$$\gamma_n := \eta_e; \quad \mathrm{clo}(\gamma_n) := \mathrm{clo}(\gamma_n) \setminus \Lambda_{R_e}(e) = \mathrm{clo}(\eta_e).$$

We iteratively repeat this process until $\mathrm{clo}(\gamma_n)$ is empty. We call the final set of $\log^2 n$-weight (closed) edges revealed along this process by Γ_n, and the final path composing only of 1-weight (open) edges by $\tilde{\gamma}_n$.

Therefore, by the property (c)

$$0 \le \mathrm{D}_n^* - \mathrm{T}(0^*, (ne_1)^*) \le |\tilde{\gamma}_n| - \mathrm{T}(0^*, (ne_1)^*) \le 2C_* \sum_{e\in\Gamma_n} R_e. \tag{6.2}$$

We write the set Γ_n as $\{e_1, \dots, e_\ell\}$ in the order of revealing. By its construction, the sequence $(R_{e_i})_{1\le i\le \ell}$ is non-increasing. Moreover, by the property (a) and (6.1),

$$\forall 1 \le j \le k \le \ell, \quad \|e_j - e_k\|_\infty \ge R_{e_j} = \max(R_{e_j}, R_{e_k}),$$

and thus (ii) follows. □

Proof of Theorem 2.1. For the convenience, we recall the desired statement: there exist positive constants C, c, such that for all $L \ge \log^2 n$,

$$\mathbb{P}\left(|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| \ge L\right) \le C\exp(-cL/\log L). \tag{6.3}$$

By Lemma 3.1 and Lemma 3.3, there exist positive constants C and c, such that

$$\min(\mathbb{P}(\mathrm{D}_n^* \le Cn/2), \mathbb{P}(\|0 - 0^*\|_\infty \le n/4)) \ge 1 - C\exp(-cn). \tag{6.4}$$

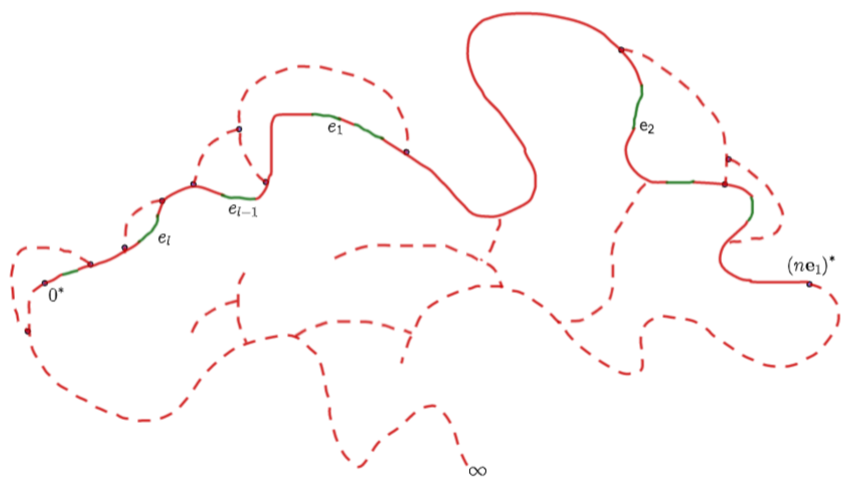


Figure 5 – Illustration of the family of bypasses (the dashed red line) avoiding all $\log^2 n$-weight edges (green line) on the geodesic γ_n

Remark further that $\mathrm{T}(0^*, (n\mathbf{e}_1)^*) \le \mathrm{D}_n^*$, and if $\mathrm{T}(0^*, (n\mathbf{e}_1)^*) \le k$ then $\gamma_n \subset \Lambda_k(0^*)$, since $t_e \ge 1$ for all e. Therefore,

$$\mathbb{P}(\mathcal{E}) \ge 1 - C\exp(-cn), \qquad \mathcal{E} := \{\gamma_n \subset \Lambda_{Cn}\}. \tag{6.5}$$

We define also

$$\mathcal{E}_* = \{\{0^*, (n\mathbf{e}_1)^*\} \not\subset \Lambda_{3R_e}(e)\, \forall\, e \in \gamma_n\}.$$

Using (6.4), (6.5), and Proposition 1 (i), there are positive constants C and c, such that

$$\begin{aligned} \mathbb{P}(\mathcal{E}_*^c) &\le \mathbb{P}(\mathcal{E}_*^c \cap \mathcal{E} \cap \{\|0 - 0^*\|_\infty + \|n\mathbf{e}_1 - (n\mathbf{e}_1)^*\|_\infty \le n/2\}) + \mathbb{P}(\mathcal{E}^c) \\ &\quad + \mathbb{P}(\{\|0 - 0^*\|_\infty + \|n\mathbf{e}_1 - (n\mathbf{e}_1)^*\|_\infty \ge n/2) \\ &\le \mathbb{P}(\exists\, e \in \Lambda_{Cn} : R_e \ge n/12) + 3C\exp(-cn) \\ &\le 4C\exp(-cn). \end{aligned} \tag{6.6}$$

Case 1: $L \ge Cn$. Using Lemma 3.3,

$$\mathbb{P}\left(|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| \ge L\right) \le \mathbb{P}\left(\mathrm{D}_n^* \ge L\right) \le \exp(-cL),$$

and the result follows.

Case 2: $L < Cn$. Using (6.5), (6.6), Proposition 1 and Lemma 6.1,

$$\begin{aligned} &\mathbb{P}\left(|\mathrm{D}_n^* - \mathrm{T}(0^*, (n\mathbf{e}_1)^*)| \ge L\right) \\ &\le \mathbb{P}(\mathcal{E}^c) + \mathbb{P}(\mathcal{E}_*^c) + \mathbb{P}\left(\exists\, e \in [-Cn, Cn]^d : R_e \ge L\right) \\ &\quad + \mathbb{P}\left(\exists\, \Gamma_n \subset \Lambda_{Cn} \text{ satisfying (a)–(c)}\right) \\ &\le c^{-1}\exp(-cL) + \mathbb{P}\left(\exists\, \Gamma_n \subset \Lambda_{Cn} \text{ satisfying (a)–(c)}\right), \end{aligned} \tag{6.7}$$

where c is a positive constant and

(a) $\frac{\log^2 n}{2C_*} \leq R_e \leq L$ for all $e \in \Gamma_n$,

(b) $\|e - e'\|_\infty \geq \max\{R_e, R_{e'}\}$, for all $e, e' \in \Gamma_n$,

(c) $\sum_{e \in \Gamma_n} R_e \geq L/2C_*$.

To estimate the last term of (6.7), we set $M_0 = \lceil \log^{3/2} n \rceil$, and $M_q = M_0 2^q$ for $1 \leq q \leq a_n$ with $a_n = \lceil \log_2 L - \log_2 \log n \rceil$. Remark that $\log^{3/2} n \leq R_e \leq L \leq M_{a_n}$ and thus,

$$\sum_{e \in \Gamma_n} R_e \leq \sum_{q=0}^{a_n} 2M_q N_q,$$

where for each $1 \leq q \leq a_n$,

$$N_q = |\{e \in \Gamma_n : R_e \in [M_q, 2M_q]\}|.$$

Therefore, it follows from the union bound that

$$\mathbb{P}\left(\exists\, \Gamma_n \subset \Lambda_{Cn} \text{ satisfying (a)–(c)}\right) \leq \sum_{q=0}^{a_n} \mathbb{P}\left(\exists\, \Gamma_n \subset \Lambda_{Cn} \text{ satisfying (a)–(c)}; N_q \geq b_q\right), \tag{6.8}$$

where for each $1 \leq q \leq a_n$,

$$b_q := \left\lfloor \frac{L}{4C_* M_q \log_2 L} \right\rfloor.$$

Moreover, by (b), if R_e and $R_{e'}$ are in $[M_q, 2M_q]$ for some $e, e' \in \Gamma_n$, then

$$\|e - e'\|_\infty \geq \max(R_e, R_{e'}) \geq M_q. \tag{6.9}$$

Therefore,

$$\mathbb{P}\left(\exists\, \Gamma_n \subset \Lambda_{Cn} \text{ satisfying (a)–(c)}; N_q \geq b_q\right) \leq S_q, \tag{6.10}$$

where

$$\begin{aligned} S_q := &\mathbb{P}\Big(\exists\, \{e_1, \ldots, e_{b_q}\} \subset \Lambda_{Cn} : R_{e_j} \in [M_q, 2M_q]\, \forall\, 1 \leq j \leq b_q; \\ &\qquad \|e_j - e_k\|_\infty \geq M_q\, \forall\, 1 \leq j \neq k \leq b_q\Big). \end{aligned} \tag{6.11}$$

The following claim is straightforward.

Claim *There exists a constant $c = c(d) > 0$ such that the following holds: for any $M \in \mathbb{N}$ and $\Lambda \subset \mathbb{E}^d$ satisfying*

$$\|u - v\|_\infty \geq M, \quad \forall u, v \in \Lambda,$$

we can find $\Lambda' \subset \Lambda$ such that $|\Lambda'| \geq c|\Lambda|$ and $\|u - v\|_\infty \geq 17M$ for all $u, v \in \Lambda'$.

By this claim, there exists a positive constant c depending on d such that for any $1 \leq q \leq a_n$ if the event in (6.11) occurs then we can find $\Lambda' \subset \{e_1, \ldots, e_{b_q}\}$ such that $|\Lambda'| \geq \lfloor c b_q \rfloor$ and $\|e - e'\|_\infty \geq 17M_q$ for all $e, e' \in \Lambda'$. As a result, we have

$$\begin{aligned} S_q &\leq \mathbb{P}\left(\exists\, \{e'_1, \ldots, e'_{c_q}\} \in \mathcal{T}_q : R_{e'_j} \in [M_q, 2M_q]\, \forall\, 1 \leq j \leq c_q\right) \\ &\leq \sum_{\{e'_1, \ldots, e'_{c_q}\} \in \mathcal{T}_q} \mathbb{P}\left(R_{e'_j} \in [M_q, 2M_q]\, \forall 1 \leq j \leq c_q\right), \end{aligned} \tag{6.12}$$

where $c_q = \lfloor cb_q \rfloor$ and

$$\mathcal{T}_q = \{\{e'_1, \dots, e'_{c_q}\} \subset \Lambda_{Cn} : \|e'_j - e'_k\|_\infty \geq 17M_q \, \forall 1 \leq j,k \leq c_q\}.$$

We remark that the event $R_e \in [M_q, 2M_q]$ only depends on the state of edges in $\Lambda_{8M_q}(e)$. Therefore, given $\{e'_1, \dots, e'_{c_q}\} \in \mathcal{T}_q$, the family of the events $(\{R_{e'_j} \in [M_q, 2M_q]\})_{1 \leq j \leq c_q}$ are independent. Hence,

$$\begin{aligned}\mathbb{P}\left(R_{e'_j} \in [M_q, 2M_q] \, \forall 1 \leq j \leq c_q\right) &= \prod_{j=1}^{c_q} \mathbb{P}\left(R_{e'_j} \in [M_q, 2M_q]\right) \\ &\leq \alpha^{-1} \exp(-\alpha M_q c_q),\end{aligned}$$

where α is a positive constant, by using Proposition 1. This estimate together with (6.12) yields for all $1 \leq q \leq a_n$,

$$\begin{aligned}S_q &\leq \alpha^{-1} |\mathcal{T}_q| \exp(-\alpha c_q M_q) \leq \alpha^{-1} (4Cn)^{dc_q} \exp(-\alpha c_q M_q) \\ &\leq \alpha^{-1} \exp(-\alpha c_q M_q/2) \leq \alpha^{-1} \exp\left(-\frac{c\alpha L}{16 C_* \log_2 L}\right).\end{aligned}$$

Combining the above estimate with (6.10), (6.8) and (6.7), we obtain (6.3). □

A Proof of Lemma 3.7

By Lemma 3.2, there exists a positive constant $c_2 = c_2(p)$ such that

$$\mathbb{P}(L_N) \leq \mathbb{P}(E_N) + \mathbb{P}(\text{there does not exists a crossing cluster in } \Lambda_N) \leq \mathbb{P}(E_N) + c_2^{-1} \exp(-c_2 N).$$

Therefore, it remains to bound $\mathbb{P}(E_N)$.

Case 1: $d = 2$. Let $\mathcal{C}$ be a crossing cluster and let $\mathcal{D}$ be a connected component of Λ_N such that $\mathrm{diam}(\mathcal{D}) \geq \varepsilon N$. Then, there exists a sub-rectangle Λ of size $\varepsilon N \times 2N$ in $\mathrm{A}_N(e)$ such that $\mathcal{D}$ is crossing for two opposite faces of size $2N$ in Λ. It follows from the proof of [27, Theorem 7.61] that

$$\mathbb{P}(\text{there exists a crossing cluster in } \Lambda, \text{ denoted by } Cr(\Lambda)) \geq 1 - c_1^{-1} \exp(-c_1 N),$$

for some $c_1 = c_1(\varepsilon, p)$. Furthermore, by the planar property of $\mathbb{Z}^2$, the crossing cluster $Cr(\Lambda)$ of Λ always intersect with $\mathcal{D}$ and $\mathcal{C}$. Hence, we complete the proof of (i) for $d = 2$.

Case 2: $d \geq 3$. We closely follows the proof of [27, Lemma 7.104]. For $-N \leq t \leq N$ and $1 \leq i \leq d$, let

$$H_t^i = \{x = (x_1, \dots, x_d) \in \Lambda_N : x_i = t\}.$$

and for $-N \leq j_1 < j_2 \leq N$, define

$$K^i_{j_1, j_2} = \{x = (x_1, \dots, x_d) \in \Lambda_N : j_1 \leq x_i \leq j_2\}.$$

Let $U, V, X \subseteq \mathbb{Z}^d$, we denote by $U \overset{X}{\longleftrightarrow} V$ the event that there exists a q-open path between U and V in X. Now for $1 \leq i \leq d$ and $x, y \in \Lambda_N \cap H^i_a$, we define

$$Q^i_{a,k}(x,y) = \{y \overset{K^i_{a,a+k}}{\longleftrightarrow} H^i_{a+k}\} \cap \{\exists \text{ a path } \eta_x \subseteq K^i_{a,a+k} \text{ joins } x \text{ to } H^i_{a+k}; y \overset{K^i_{a,a+k}}{\not\longleftrightarrow} \eta_x\}$$

Suppose that the event $E_{q,N}$ occurs. Then there exist $\mathcal{C}$ a q-crossing cluster and $\mathcal{D}$ a connected component of Λ_N such that $\mathrm{diam}(\mathcal{D}) \geq \varepsilon N$ and $\mathcal{D} \cap \mathcal{C} = \emptyset$. Notice that the diameter of $\mathcal{D}$ is achieved in the i^{th} coordinate for some $1 \leq i \leq d$, i.e.

$$\text{there exist } u, v \in \mathcal{D} \text{ with } v_i - u_i = \mathrm{diam}(\mathcal{D}).$$

Therefore, if the event $E_{q,N}$ occurs, then there exist $1 \leq i \leq d$, $a \in [-N, (1-\varepsilon)N]$, and $x, y \in H_a^i$ such that the event $Q^i_{a,\varepsilon N}(x,y)$ occurs. By the union bound and the symmetry of $\mathbb{Z}^d$,

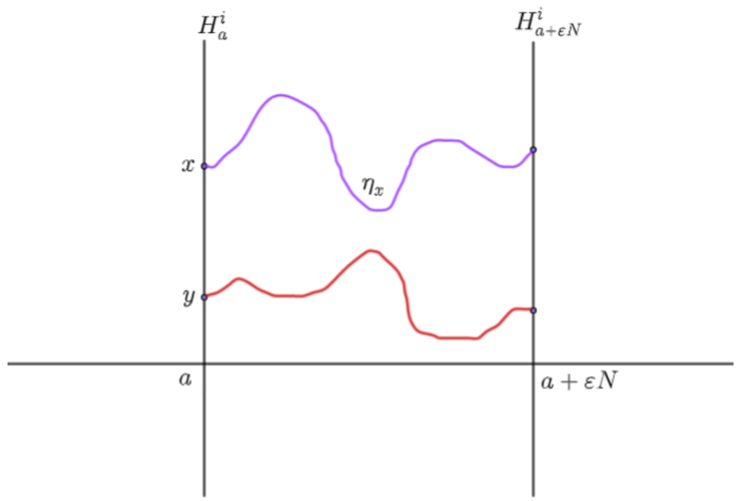


Figure 6 – Illustration the event $Q^i_{a,\varepsilon N}(x,y)$

(A.1)
$$\mathbb{P}(E_{q,N}) \leq \sum_{1\leq i\leq d} \sum_{-N\leq a\leq(1-\varepsilon)N} \sum_{x,y\in H_a^i} \mathbb{P}(Q^i_{a,\varepsilon N}(x,y)) \quad \leq d(2N)^{2d+2} \sup_{x,y\in H_0^1} \mathbb{P}((Q^1_{0,\varepsilon N}(x,y)).$$

Let L be a positive integer chosen later and write

$$\varepsilon N = KL + r, \quad 0 \leq r < L, \tag{A.2}$$

where K is a non-negative integer. We have

$$Q^1_{0,\varepsilon N}(x,y) \subseteq Q^1_{0,KL}(x,y) \subseteq Q^1_{0,(K-1)L}(x,y) \subseteq \cdots \subseteq Q^1_{0,L}(x,y),$$

which implies that

$$\mathbb{P}(Q^1_{0,\varepsilon N}(x,y)(x,y)) \leq \mathbb{P}\left(Q^1_{0,KL}(x,y\right) = \prod_{i=0}^{K-1} \mathbb{P}(Q^1_{0,(i+1)L}(x,y) \mid Q^1_{0,iL}(x,y)). \tag{A.3}$$

For $x, y \in H_0^1$ and $i \geq 0$, let

$$O_i(x,y) := \{u \in H^1_{iL} : \exists \text{ a path } \eta_x \subseteq K^1_{0,iL} \text{ joins } x \text{ to } u; y \stackrel{K^1_{0,iL}}{\not\longleftrightarrow} \eta_x\},$$
$$O_i(y) := \{v \in H^1_{iL} : y \stackrel{K^1_{0,iL}}{\longleftrightarrow} v\}$$

We remark that the sets $O_i(x,y)$ and $O_i(y)$ only depend the state of edges inside $K^1_{0,iL}$. Furthermore, on the event $Q^1_{0,iL}(x,y)$, $O_i(x,y)$ and $O_i(y)$ are non-empty and disjoint. On $Q^1_{0,iL}(x,y)$, if the event $Q^1_{0,(i+1)L}(x,y)$ occurs, then $u \overset{K^1_{iL,(i+1)L}}{\not\longleftrightarrow} v$ for all $u \in O_i(x,y)$ and $v \in O_i(y)$. Therefore, we have

$$\mathbb{P}(Q^1_{0,(i+1)L}(x,y) \mid Q^1_{0,iL}(x,y)) \leq \sup_{u,v \in H^1_{iL}} \mathbb{P}(u \overset{K^1_{iL,(i+1)L}}{\not\longleftrightarrow} v) = \sup_{u,v \in H^1_0} \mathbb{P}(u \overset{K^1_{0,L}}{\not\longleftrightarrow} v)$$

$$= 1 - \min_{u,v \in H^1_0} \mathbb{P}(u \overset{K^1_{0,L}}{\longleftrightarrow} v).$$

It is clear that $\mathbb{P}(u \overset{K^1_{0,L}}{\longleftrightarrow} v)$ is non-decreasing in q. Thus, for all $q \geq q_0$, using [27, Lemma 7.78], there exist $L = L(q_0)$ and $\delta_0 = \delta_0(q_0, L) > 0$ such that for all $u, v \in H^1_0$

$$\mathbb{P}(u \overset{K^1_{0,L}}{\longleftrightarrow} v) \geq \mathbb{P}_{q_0}(u \overset{K^1_{0,L}}{\longleftrightarrow} v) \geq \delta_0.$$

Combining the last two estimates and (A.3), it yields that

$$\mathbb{P}(Q^1_{0,\varepsilon N}(x,y)) \leq (1-\delta_0)^K \leq (1-\delta_0)^{\varepsilon N/L}, \quad \forall x, y \in H^1_0. \tag{A.4}$$

The desired bound for $\mathbb{P}(E_{q,N})$ follows from (A.1) and (A.4). □

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