KÄHLER GRADIENT RICCI SOLITONS WITH LARGE SYMMETRY

HUNG TRAN

Abstract. Let (M, g, J, f) be an irreducible non-trivial Kähler gradient Ricci soliton of real dimension $2n$. We show that its group of isometries is of dimension at most n^2 and the case of equality is characterized. As a consequence, our framework shows the uniqueness of $U(n)$-invariant Kähler gradient Ricci solitons constructed earlier. There are corollaries regarding the groups of automorphisms or affine transformations and a general version for almost Hermitian GRS. The approach is based on a connection to the geometry of an almost contact metric structure.

1. Introduction

Let (M, g) be an orientable connected Riemannian manifold. In [30], R. Hamilton introduced the Ricci flow equation, for Rc denoting the Ricci curvature,

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Rc}(t). \tag{1.1}$$

The theory has been utilized to solve fundamental problems; see [50, 52, 51, 5, 9, 10]. As a weakly parabolic system, it generically develops singularities and the study of such models is essential in any potential applications. Gradient Ricci solitons (GRS) are self-similar solutions to (1.1) and arise naturally in that context. Consequently, there have been numerous efforts to study them; see [31, 21, 52, 46, 43, 16, 55, 7, 8, 17, 40, 45] and references therein.

A GRS (M, g, f) is a Riemannian manifold such that, for a constant λ,

$$\mathrm{Rc} + \mathrm{Hess} f = \lambda g. \tag{1.2}$$

It is called shrinking, steady, or expanding depending on the sign of λ being positive, zero, or negative. Clearly, any Einstein manifold is an example with $\mathrm{Hess} f \equiv 0$ and λ being the Einstein constant. Moreover, the Gaussian soliton refers to $(\mathbb{R}^m, g_{Euc}, \lambda\frac{|x|^2}{2})$ for g_{Euc} the Euclidean metric. It is natural to combine these examples and, in that case, a soliton is called rigid, namely isometric to a quotient of $N^{n-k} \times \mathbb{R}^k$ with $f = \frac{|x|^2}{2}$ on the Euclidean factor. A soliton is called non-trivial (or non-rigid) if at least a factor in its de Rham decompsotion is non-Einstein.

Many non-trivial examples are Kähler and the topic receives tremendous interest; see, for examples, [61, 63, 15, 19, 44, 12, 20, 39, 24, 27]. In particular, significant efforts lead to the classification of all Kähler Ricci shrinker surfaces [25, 23, 1, 41]. For $m = 2n$, (M^m, g, J) is called an almost Hermitian manifold if g is compatible with an

Date: June 13, 2023.

almost complex structure $J : TM \mapsto TM$, $J^2 = -\mathrm{Id}$. The associated Kähler two-form is defined as, for tangential vector fields X and Y,

$$\omega(X, Y) = g(X, JY).$$

Consequently, (M^{2n}, g, J) is called almost Kähler if ω is closed and Kähler if, additionally, J is a complex structure. A Kähler GRS (M, g, J, f) is simultaneously a Kähler manifold and a gradient Ricci soliton.

In this paper, we propose an investigation based on the group of symmetry. An isometry on (M^m, g) is a diffeomorphism preserving the metric g. The dimension of the group of isometries is at most $\frac{m(m+1)}{2}$ [36] and it is attained iff the manifold is simply-connected of constant curvature: the round spheres or the real projective space, the Euclidean space, or the hyperbolic space.

For an almost Hermitian manifold, using the terminology of [35], an automorphism is an isometry which preserves J. S. Tanno [59] showed that the maximal dimension of the automorphism group is $n(n+2)$. Additionally, the maximal case is characterized as the manifold must be homothetic to one of the followings: the unitary space $\mathbb{C}^n = \mathbb{R}^{2n}$ with g_{Euc}, a complex projective space $\mathbb{CP}^n$ with a Fubini-Study metric, or an open ball $B^n_{\mathbb{C}}$ with a Bergman metric. These models play an important role in our work.

Definition 1.1. *Let $\mathbb{N}^n(k)$ be a simply connected Kähler manifold of real dimension $2n$ with constant holomorphic sectional curvature and normalized Ricci curvature k.*

From the above discussion, it is immediate that a Gaussian soliton has $n(2n+1)$ isometries and $n(n+2)$ automorphisms. Also, P. Petersen and W. Wylie showed that a homogeneous GRS must be rigid and if the Riemannian metric is reducible then the soliton structure is reduced accordingly [54]. It is, thus, interesting to ponder the next best scenario. It is noted that, many non-trivial Kähler GRS's, see [13, 14, 38, 18, 29], are $U(n)$-invariant and $\dim(U(n)) = n^2$. According to [26], their metrics all belong to the following cohomogeneity one structure:

An Ansatz: *Let $N^{n-1}(k)$ be a Kähler-Einstein manifold with $\mathrm{Rc}_N = k\mathrm{Id}$, I be an interval, and functions $H, F : I \mapsto \mathbb{R}^+$. (P, g_t) is a Riemann submersion of a line or circle bundle with coordinate z over $(N, F^2 g_N)$ and a bundle projection $\pi : P \mapsto \mathbb{N}$. η is the one-form dual of ∂_z such that $d\eta = q\pi^*\omega_{\mathbb{N}}$ for $q \in \mathbb{Z}$. If $N = \mathbb{CP}^{n-1}$ and $q = 1$ then $P = \mathbb{S}^{2n-1}$ and one recovers the Hopf fibration. If $N = \mathbb{N} \neq \mathbb{CP}^{n-1}$, the bundle is trivial. The metric on $I \times P$ is given by*

$$g = dt^2 + g_t = dt^2 + H^2(t)\eta \otimes \eta + F^2(t)\pi^* g_{\mathbb{N}}. \tag{1.3}$$

Our first result asserts the uniqueness.

Theorem 1.2. *Let $(M^{2n}, g, J, f, \lambda)$ be an irreducible non-trivial complete Kähler GRS. Its group of isometries is of dimension at most n^2 and equality happens iff it is smoothly constructed by the ansatz 1.3 for $N = \mathbb{N}(k)$ and $q = 1$. If $\lambda \geq 0$ then $\mathbb{N} = \mathbb{CP}^{n-1}$.*

Remark 1.1. *When $N = \mathbb{CP}^{n-1}$, the construction of complete Kähler GRS with ansatz 1.3 is considered in* [26]. *By their analysis, there must be exactly one or two singular orbits (two only if $\lambda > 0$); to smoothly compactify each, one must collapse either the* *whole sphere (both H and F going to zero) or just the fiber (H going to zero). Here are all possible configurations:*

- $\lambda = 0$, $I = [0, \infty)$, *the singular orbit is either a point (M is topologically $\mathbb{C}^n$) or $\mathbb{CP}^{n-1}$ ((M is $\mathbb{C}^n$ blowing up at one point)* [13, 18].
- $\lambda < 0$, $I = [0, \infty)$, *the original construction is due to* [14, 18, 29].
- $\lambda > 0$, $I = [0, 1]$, *each singular orbit is $\mathbb{CP}^{n-1}$* [38, 13, 18].
- $\lambda > 0$, $I = [0, \infty)$, *the singular orbit is $\mathbb{CP}^{n-1}$* [29] *(if the singular orbit is a point, it recovers a Gaussian soliton).*

Remark 1.2. *The metric in Theorem 1.2 has each (P, g_t) being a deformed homogenous Sasakian structure with constant holomorphic sectional curvature.*

For the reducible case, the group of isometries is potentially skewed by a Gaussian soliton factor of a large dimension. Thus, it is natural to consider the following.

Corollary 1.3. *Let $(M^{2n}, g, J, f, \lambda)$ be a complete simply connected non-trivial Kähler GRS. Its group of automorphisms is of dimension at most n^2 and equality happens iff it is either irreducible as in Theorem 1.2 or isometric to*

(i) a product of a Gaussian soliton and a Hamilton's cigar ($\lambda = 0$);
(ii) a product of $\mathbb{N}(k)$ ($k \leq 0$) with a complete Kähler expanding GRS in real dimension two ($\lambda < 0$).

Remark 1.3. *There is a list of all models of GRS in real dimension two* [3].

Under certain conditions, an infinitesimal isometry is closely related to conformal [57] and affine vector fields [36]. For example, following [37, Chapter 9], one recalls that a Kähler manifold is non-degenerate if the restricted linear holonomy group at $x \in M$ contains the endormorphism J_x for an arbitrary $x \in M$.

Corollary 1.4. *Let (M^{2n}, g, J, f) be a non-degenerate complete simply connected Kähler GRS. If f is non-constant, then the group of affine transformations is of dimension at most n^2 and equality happens iff it is either irreducible as in Theorem 1.2 or a product of $\mathbb{N}(k)$ ($k < 0$) with a complete Kähler expanding GRS in real dimension two.*

Indeed, Theorem 1.2 follows from a more general version for (possibly incomplete) almost Hermitian GRS $(M^{2n}, g, J, f, \lambda)$. These structures are compatible:

$$g(X, Y) = g(JX, JY),$$
$$\mathrm{Hess} f(X, Y) = \mathrm{Hess} f(JX, JY).$$

The group of symmetry is to preserve all g, J, and f.

Theorem 1.5. *Let (M^{2n}, g, J, f) be an almost Hermitian GRS with symmetry group G. If f is non-constant then $dim(G) \leq n^2$ and equality happens iff locally it is either*

(i) constructed by the ansatz 1.3 for $N = \mathbb{N}(k)$ and $q = 0, 1$.
(ii) a product of a line/circle with a hyperbolic space.

Remark 1.4. *The metric in case (ii) above can be written as, for non-zero constants H and A,*

$$g = dt^2 + g_t = dt^2 + H^2 dz^2 + e^{2Az} g_{\mathbb{C}^{n-1}},$$
$$\lambda = \frac{\partial^2 f}{\partial t^2} = -2(\frac{A}{H})^2(n-1).$$

Remark 1.5. *For Ansatz 1.3, equation (1.2) is equivalent to an ODE system:*

$$\lambda = -\frac{H''}{H} - \frac{(2n-2)F''}{F} + f'' = \frac{H^2q^2(2n-2)}{F^4} - \frac{H''}{H} - \frac{2(n-1)H'F'}{HF} + f'\frac{H'}{H}$$
$$= \frac{k}{F^2} - \frac{2H^2q^2}{F^4} - \frac{F''}{F} - (2n-3)(\frac{F'}{F})^2 - \frac{H'F'}{FH} + f'\frac{F'}{F}. \tag{1.4}$$

The almost Kähler condition is equivalent to

$$FF' = qH.$$

Remark 1.6. *It is possible to construct local solutions for (1.4) giving (possibly incomplete) manifolds with maximal symmetry. For $\mathbb{N} = \mathbb{CP}^{n-1}$, generalized versions of (1.4) were investigated by* [26] *and* [11].

Remark 1.7. *The Gaussian soliton $(\mathbb{R}^{2n}, g_{Euc}, f = \lambda\frac{|x|^2}{2}, \lambda)$ for $\lambda \neq 0$ belongs to family $q = 1$ with P being the round sphere, $H = F = t$, and $k = 2n$. For $\lambda = 0$, the soliton $(\mathbb{R}^{2n}, g_{Euc}, f = ax_i + b)$ belongs to family $q = 0$ with P being the Euclidean space.*

To illustrate the dimension n^2, let's consider the case of a Gaussian soliton for $\lambda \neq 0$. The isometry group consists of $2n$ translations and $\frac{2n(2n-1)}{2}$ rotations. With a standard coordinate $\{x_i, y_i\}_{i=1}^n$, one specifies an almost complex structure such that

$$J(\partial_{x_i}) = \partial_{y_i},\ J(\partial_{y_i}) = -\partial_{x_i}.$$

Then it is clear that not all rotations preserve this tensor field. That's why the automorphism group is only of dimension $n(n+2)$. Among those, the translations do not preserve the potential function $f = \lambda\frac{|x|^2}{2}$. Consequently, the group of symmetry preserving g, J and f is of dimension n^2.

The paper is organized as follows. Section 2 recalls general and useful preliminaries while Section 3 is devoted to calculation about ansatz 1.3. Afterward, we'll discuss the relation between the symmetry of an almost Hermitian GRS and one of its level sets determined by f. The key idea is that a symmetry group on (M, g, J, f) induces a symmetry group of regular level sets considered as almost contact metric structures. In Section 5, the rigidity of a maximal dimension is examined and proofs of all theorems are collected. Finally the appendix explains our convention and recalls submersion.

1.1. **Acknowledgment.** H. T was partially supported by grants from the Simons Foundation [709791], the National Science Foundation [DMS-2104988], and the Vietnam Institute for Advanced Study in Mathematics. He also benefits greatly from discussion with Profs. McKenzie Wang, Catherine Serle, and Ronan Conlon.

2. Preliminaries

We'll recall fundamental concepts and useful results about an almost complex structure, a gradient Ricci soliton, group actions on a manifold, an almost contact structure, and certain model spaces.

2.1. **Almost Complex Structure.**

Let M be a smooth manifold of dimension $2n$.

Definition 2.1. *An almost complex structure is a smooth section J of the bundle of endormorphisms $End(TM)$ such that*

$$J^2 = -\mathrm{Id}.$$

One can immediately extend J to be an endormorphism on the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$ via $\mathbb{C}$-linearity. An almost complex structure is said to be integrable if M admits an atlas of complex charts with holomorphic transition functions such that J corresponds to the induced complex multiplication on $TM \otimes_{\mathbb{R}} \mathbb{C}$. A real differentiable manifold with an integrable almost complex structure is, by definition, a complex manifold. Thanks to the work of Newlander and Nirenberg [47], the integrability of J is equivalent to the vanishing of the Nijenhuis tensor

$$N_J(X,Y) = [JX, JY] - [X,Y] - J[X,JY] - J[JX,Y].$$

Definition 2.2. *Let (M^{2n}, g) be a Riemannian manifold with an almost complex structure J. (M, g, J) is called an almost Hermitian manifold and g a Hermitian metric if*

$$g(JX, JY) = g(X, Y).$$

The fundamental 2-form or Kähler form is given by

$$\omega(X,Y) = g(X, JY).$$

(M, g, J) is called almost Kähler if $d\omega = 0$. When J is integrable, we upgrade an almost Hermitian to Hermitian and almost Kähler to Kähler.

For a Riemannian manifold to be Kähler, the following is well-known.

Proposition 2.3. [6, Proposition 3.1.9] *Let (M, g, J) be an almost Hermitian (real) manifold. The followings are equivalent:*

(1) $\nabla J = 0$,
(2) $\nabla \omega_g = 0$,
(3) *(M, g, J) is Kähler.*

On a Kähler manifold, one observes that

$$J(\mathrm{R}(X,Y)Z) = \mathrm{R}(X,Y)JZ,$$
$$\mathrm{R}(X,Y,JZ,JW) = g(\mathrm{R}(X,Y)JZ, JW) = g(\mathrm{R}(X,Y)Z, W) = \mathrm{R}(X,Y,Z,W).$$

Naturally, it leads to the notion of the Ricci form.

Definition 2.4. *The Ricci form ρ is the image of ω_g via the curvature operator:*

$$\rho(X,Y) = g(\mathrm{R}(\omega_g)(X), Y).$$

A priori, it is not clear how the Ricci form ρ is related to the Ricci curvature tensor.

Proposition 2.5. [4, Proposition 2.45] *On a Kähler manifold* (M, g, J, ω_g)*, we have*

$$\mathrm{Rc}(X, Y) = \rho(X, JY).$$

Corollary 2.6. *On a Kähler manifold* (M, g, J)*,* Rc *is* J*-invariant.*

2.2. **Gradient Ricci Solitons.** In this subsection, we recall how a GRS is compatible with a complex setup.

Definition 2.7. (M, g, J, f) *is an almost Hermitian GRS if* (M, g, f) *is a GRS,* (M, g, J) *is an almost Hermitian manifold, and* $\mathcal{L}_{\nabla f} g$ *is* J*-invariant.*

Because of (1.2), $\mathcal{L}_{\nabla f} g$ is J-invariant if and only if Rc is J-invariant. Thus, the assumption is automatic for Kähler manifolds.

Definition 2.8. (M, g, J, f) *is a Kähler GRS if* (M, g, f) *is a GRS,* (M, g, J) *is Kähler manifold.*

In a complex coordinate system, the J-invariant property is equivalent to ∇f being a holomorphic vector field. That is,

$$\mathcal{L}_{\nabla f} g(\partial_{z_i}, \partial_{z_j}) = \mathcal{L}_{\nabla f} g(\partial_{\overline{z_i}}, \partial_{\overline{z_j}}) = 0.$$

2.3. **Group Actions on a manifold.** In this subsection, we review the basic setup and properties of group actions on a manifold. The standard texts are [36, 37, 34]. Let G be a topological group. An action of G on a manifold M is a homomorphism from G to the group of homomorphisms on M

$$g \mapsto A_g \text{ such that } A_g : M \mapsto M, x \mapsto g.x$$

The action is *continuous/smooth* if the map $G \times M \mapsto M$, given by $(g, x) \mapsto g \cdot x$ is continuous/smooth (for smoothness, it requires G to be a Lie group). The action is said to be *proper* if the associated map $G \times M \mapsto M \times M$, given by $(g, x) \mapsto (x, g \cdot x)$ is proper (that is, the inverse of any compact set is compact).

For each $x \in G$, the subgroup $G_x = \{g \in G, g \cdot x = x\}$ is called the *isotropy* subgroup or the *stabilizer*. The orbit through x is an immersed sub-manifold and there is a natural identification

$$G \cdot x = \{y \in M, y = g \cdot x, g \in G\} \equiv G/G_x.$$

Orbits are also classified based on the relative size of associated isotropy groups. In particular, principal orbits correspond to the smallest possible groups and singular ones have isotropy groups of higher dimensions.

At the infinitesimal level, a smooth vector field X on M generates a (local) one-parameter family of maps between domains in M. If the vector field is complete, then it generates global differmorphisms. If the corresponding maps preserve certain geometric quantities and structures then the vector field is called a (local) infinitesimal transformation of the same property. A vector field preserves a tensor T if and only if

$$\mathcal{L}_X T = 0.$$

It is also noted that the set of all vector fields can be seen as a Lie algebra $\mathfrak{X}(M)$ by the natural bracket

$$[X, Y] = XY - YX.$$

Since

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X,$$

the set of all infinitesimal transformations preserving a tensor T is always a Lie subalgebra of $\mathfrak{X}(M)$. It is noted that the group of transformations preserving a tensor T is not necessarily a Lie group.

Nevertheless, on a Riemnnian manifold, the group of isometries (preserving the Riemannian metric) is a Lie group [36, Chapter 6. Theorem 3.4]. The infinitesimal transformation corresponding to a subgroup of isometries is called a Killing vector field. That is,

$$\mathcal{L}_X g = 0.$$

The Lie algebra of such vector fields corresponds to the Lie algebra of the Lie group of all isometries on M. It is well-know that a Killing vector field is totally determined by its zero and first order values at a point $(X_p, (\nabla X)_p)$ [36, Chapter VI].

The Kähler and GRS structures impose rigidity on the Riemannian manifold as the followings are well-known [29].

Lemma 2.9. *Let (M, g, J, f) be a Kähler gradient Ricci soliton. Then, we have the followings:*

(1) $J(\nabla f)$ is a Killing vector field.
(2) $\mathcal{L}_{\nabla f} J \equiv 0$.

On an almost Hermitian GRS (M, g, J, f), one may consider transformations and vector fields preserving each individual structure: the metric g, the almost complex structure J. and the potential function f. The group of such symmetry is clearly a closed subgroup of the group of isometry and, thus, is a Lie group.

2.4. **Almost Contact Structure.** In this subsection, we recall important notions about an almost contact structure following the book by C. Boyer and K. Galicki [6].

Definition 2.10. *A $(2n+1)$-dimensional manifold M is an almost contact manifold if there exists a triple (ζ, η, Φ) where ζ is a vector field, η is a 1-form, Φ is a tensor field of type $(1,1)$, and they satisfy, everywhere on M,*

$$\eta(\zeta) = 1 \text{ and } \Phi^2 = -\mathrm{Id} + \zeta \otimes \eta.$$

Definition 2.11. *An almost contact manifold (M, ζ, η, Φ) with a Riemannian metric g is called an almost contact metric structure if*

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y).$$

Definition 2.12. *The holomorphic or Φ-sectional curvature of an almost contact manifold (M, ζ, η, Φ) is given by, for $\eta(X) = 0$ and $g(X, X) = 1$,*

$$K_\Phi(X) = K(X, \Phi(X)).$$

Closely related is the notion of a contact structure.

Definition 2.13. *A $(2n+1)$-dimensional manifold M is a contact manifold if there exists a 1-form η, called a contact 1-form, on M such that*

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. A contact structure is an equivalence class of such 1-forms.

Definition 2.14. *An almost contact metric structure $(M, \zeta, \eta, \Phi, g)$ is called a contact metric structure if one further assumes*

$$g(X, \Phi(Y)) = d\eta(X, Y).$$

It is immediate to check that a contact metric structure is indeed a contact manifold by the above definition. As ζ and Φ are uniquely determined by η and g, we also denote a contact metric manifold by (M, η, g).

Definition 2.15. *A contact metric structure (M, g, η) is called Sasakian if the cone $C(M) = M \times \mathbb{R}^+$ with the cone metric $r^2 g + dr^2$ is Kähler.*

Next we recall certain transformations which will play crucial roles.

Definition 2.16. *Let $(M, \zeta, \eta, \Phi, g)$ be an almost contact metric structure. For $a > 0$, a transverse a-homothety deformation is given by*

$$\hat{\zeta} = \frac{1}{a}\zeta,\ \ \hat{\eta} = a\eta,\ \ \hat{\Phi} = \Phi,\ \ \hat{g} = ag + (a^2 - a)\eta \otimes \eta.$$

If $(M, \zeta, \eta, \Phi, g)$ is Sasakian, then so is its homothety transformation.

Definition 2.17. *Let $(M, \zeta, \eta, \Phi, g)$ be an almost contact metric structure. For $a > 0$, a $\pm a$-deformation is given by*

$$\zeta^* = \zeta,\ \ \eta^* = \eta,\ \ \Phi^* = \pm\Phi,\ \ g^* = ag + (1 - a)\eta \otimes \eta.$$

A $\pm a$-deformation of a Sasakian manifold is not necessarily Sasakian.

2.5. **Model Spaces.** Using the submersion toolkit, we can describe several model spaces that will appear in our classification. First, the unitary space is the complex formulation of the Euclidean space $\mathbb{C}^n = \mathbb{R}^{2n}$ with standard coordinates $\{x_1, y_1, ..., x_n, y_n\}$. The metric, the almost complex structure, and the fundamental 2-form are as follows:

$$g = \sum_i (dx^i)^2 + (dy^i)^2,$$

$$J = \sum_i (\partial_{y_i} \otimes dx^i - \partial_{x_i} \otimes dy^i),$$

$$\omega_{\mathbb{C}^n} = -2 \sum_i dx^i \wedge dy^i.$$

The flat Sasakian space $(P, g_P) = \mathbb{R}^{2n+1}(-3)$: the total space of a real line bundle over $\mathbb{C}^n$ with coordinates $\{x_1, y_1, ..., x_n, y_n, z\}$. For $\eta = dz + 2\sum_i y_i dx_i$, one considers:

$$g_P = \sum_i \big(((dx_i)^2 + (dy_i)^2\big) + \eta \otimes \eta,$$

$$\Phi = \sum_i \big(\partial_{y_i} \otimes dx^i - (\partial_{x_i} - 2y_i\partial_z) \otimes dy^i.\big).$$

It is readily verified, by Lemma 6.3, that $(P, g_P, \eta, \partial_z, \Phi)$ is Sasakian with constant Φ-sectional curvature -3.

The spherical Sasakian $(P, g_P) = \mathbb{S}^{2n+1}(a)$: For simplicity, we utilize the ambient coordinates of $\mathbb{R}^{2n+2}$, $\{x_1, y_1, ..., x_{n+1}, y_{n+1}\}$. All tensors described below are understood as their restriction to the unit sphere. With the induced metric, the canonical Sasakian structure on $\mathbb{S}^{2n+1}$ is given by

$$\zeta = \sum_i (y_i \partial_{x_i} - x_i \partial_{y_i}),$$
$$\eta = \sum_i (y_i dx^i - x_i dy^i),$$
$$\Phi = \sum_{i,j} (x_i x_j - \delta_{ij}) \partial_{x_i} \otimes dy_j - (y_i y_j - \delta_{ij}) \partial_{y_i} \otimes dx_j + x_j y_i \partial_{y_i} \otimes dy_j - x_i y_j \partial_{x_i} \otimes dx_j$$

Let $\pi : \mathbb{S}^{2n+1} \mapsto N = \mathbb{CP}^n$ be the Hopf fibration and g_N the Fubini-Study metric. The Sasakian metric can be realized as

$$g = \pi^* g_N + \eta \otimes \eta, \; d\eta = \pi^* \omega_N.$$

Via a homothetic deformation (Definition 2.16), if g_N is scaled to have Ricci curvature $k\mathrm{Id}$, $k > 0$, then the constant Φ-sectional curvature a of g_P is, by Lemma 6.3,

$$a = \frac{4k}{n+1} - 3 > -3.$$

The hyperbolic Sasakian $(P, g_P) = \mathbb{SB}^{2n+1}(a)$: Let g_0 be the Bergman metric of constant sectional curvature -1 in the unit ball in $\mathbb{C}^n$. One then scales it to have Ricci curvatutre $k\mathrm{Id}$, for $k < 0$ and denote such construction by $N = B^n_{\mathbb{C}}(k)$ with metric g_N. Let ω_N be the corresponding Kähler form and, since $B^n_{\mathbb{C}}(k)$ is simply connected, there exists 1-form α such that $d\alpha = \omega_N$. On the total space of the line bundle $P = B^n_{\mathbb{C}}(k) \times \mathbb{R}$ with natural projection π, one considers:

$$g_P = \pi^* g_N + \eta \otimes \eta,$$
$$\eta = dz + \pi^* \alpha.$$

By Lemma 6.3, the Φ-sectional curvature of (P, g_P) is

$$a = \frac{k}{2n-1} - 3 < -3.$$

Theorem 2.18. [60] *Let $(M^{2n+1}, g, \eta, \Phi, \zeta)$ be a simply connected complete Sasakian manifold with constant Φ-sectional curvature H then it must be isometric to:*

(i) ($H > -3$) the Sasakian sphere $\mathbb{S}^{2n+1}(H)$,
(ii) ($H = -3$) the flat Sasakian space $\mathbb{R}^{2n+1}(-3)$,
(iii) ($H < -3$) the Sasakian disk model $\mathbb{SB}^{2n+1}(H)$.

As described earlier, Sasakian manifolds belong to the family of almost contact metric structures which also include certain Riemannian products and the following. For $P = \mathbb{R} \times \mathbb{C}^n$ and a constant A,

$$g_P = dz^2 + e^{2Az} g_{\mathbb{C}^{n-1}}.$$

One realizes it as a hyperbolic metric $\mathbb{H}^{2n+1}(-A^2)$.

Lemma 2.19. *The sectional and Ricci curvature of g_P, for orthonormal vectors X, Y on $\mathbb{C}^n$ and ∂_z along $\mathbb{R}$,*

$$K(\partial_z, X) = -A^2 = K(X, Y)$$
$$\mathrm{R}(\partial_z, \partial_z) = -2nA^2 = \mathrm{Rc}(X_i, X_i).$$

All aforementioned models appear in the following result. Let $(P^{2n+1}, g, \eta, \zeta, \Phi)$ be an almost contact metric structure. The symmetry group is supposed to preserve both g, η, ζ and Φ.

Theorem 2.20. [58] *The maximum dimension of the symmetry group is $(n+1)^2$. It is attained iff the sectional curvature for 2-planes which contain ζ is a constant C and the manifold is one of the following spaces:*

(i) $C > 0$: an $\pm b$ deformation of a homogeneous Sasakian manifold with constant Φ-sectional curvature H or, precisely,
- *$H > -3$: the Sasakian sphere $\mathbb{S}^{2n+1}(H)$ or its quotient by a finite group generated by exp$(t\zeta)$ for $2\pi/t$ being an integer,*
- *$H = -3$: the flat Sasakian space $\mathbb{R}^{2n+1}(-3)$ or its quotient by a cyclic group generated by exp$(t\zeta)$,*
- *$H < -3$: the Sasakian disk model $\mathbb{SB}(H)$ or its quotient by a cyclic group generated by exp$(t\zeta)$.*

(ii) $C = 0$: six global Riemannian product $X \times \mathbb{CP}^{n-1}(k)$, $X \times \mathbb{C}^{n-1}$, $X \times B^{n-1}_{\mathbb{C}}(k)$ where X is a line or a circle;

(iii) $C < 0$ the hyperbolic space $\mathbb{H}^{2n+1}(C)$.

For a Sasakian model with submersion $\pi : P \mapsto N$ with $N = \mathbb{N}(k)$, the metric can always be written as

$$g = g_N + \eta \otimes \eta, \;\; d\eta = \pi^* \omega_N.$$

Lemma 2.21. *If $(M, \zeta', \eta', \Phi', g')$ is obtained via a transverse a-homothety and a $\pm b$-deformation then*

$$g' = bag_N + a^2 \eta \otimes \eta, \;\; \zeta' = \frac{1}{a}\zeta, \;\; , \eta' = a\eta, \;\; \Phi' = \pm\Phi.$$

Proof. Via a transerver a-homothety transformation:

$$g^* = ag + (a^2 - a)\eta \otimes \eta = ag_N + a^2 \eta \times \eta = ag_N + \eta^* \otimes \eta^*;$$
$$\eta^* = a\eta; \;\; \zeta^* = \frac{1}{a}\zeta; \;\; \Phi^* = \Phi.$$

Via a $\pm b$-deformation:

$$\begin{aligned} g' &= bg^* + (1-b)\eta^* \otimes \eta^* = bag_N + b\eta^* \otimes \eta^* + (1-b)\eta^* \otimes \eta^* \\ &= bag_T + \eta^* \otimes \eta^* = bag_N + a^2 \eta \otimes \eta \end{aligned}$$
$$\eta' = \eta^* = a\eta, \;\; \zeta' = \zeta^* = \frac{1}{a}\zeta, \;\; \Phi' = (\pm)\Phi.$$

□

3. Cohomogeneity One Ansatz

Here we assume the cohomogeneity one condition and collect calculation related to ansatz 1.3. The setup follows [26] closely. Let G be a Lie group acting isometrically on a Riemannian manifold (M, g). Supposed that there is a dense subset $M_0 \subset M$ such that, locally, there is a G-equivariant diffeomorphism:

$$\Psi : I \times P \mapsto M_0 \text{ given by } \Psi(t, hK) = h \cdot \gamma(t).$$

Here, I is an interval; $\gamma(t)$ is a unit speed geodesic intersecting all orbits orthogonally; $P = G/K$ where K is the istropy group along $\gamma(t)$. It follows that

$$\Phi^*(g) = g = dt^2 + g_t$$

where g_t is a one-parameter family of G-invariant metrics on G/K. For unit vector fields $N = \Phi_*(\partial_t)$, let L denote the shape operator

$$L(X) = \nabla_X N.$$

We will consider $L_t = L_{|_{\Psi(t\times P)}}$ to be a one-parameter family of endormorphisms on TP via identification $T(\Psi(t \times P)) = TP$. Following [26], one observes

$$\partial_t g = 2g_t \circ L_t.$$

Thanks to Gauss, Codazzi, and Riccati equations, the Ricci curvature of (M_0, g) is totally determined by the geometry of the shape operator and how it evolves. Thus, the gradient Ricci soliton equation $\mathrm{Rc} + \mathrm{Hess} f = \lambda g$ is reduced to

$$\begin{aligned} 0 &= -(\delta L) - \nabla \mathrm{tr} L, \\ \lambda &= -\mathrm{tr}(L') - \mathrm{tr}(L^2) + f'', \\ \lambda g_t(X, Y) &= \mathrm{Rc}_t(X, Y) - (\mathrm{tr} L) g_t(LX, Y) - g_t(L'(X), Y) + f' g_t(LX, Y). \end{aligned} \tag{3.1}$$

Here Rc_t denotes the Ricci curvature of (P, g_t), $\delta L = \sum_i \nabla_{e_i} L(e_i)$ for an orthonormal basis and $\mathrm{tr} T = \mathrm{tr}_{g_t} T_t$.

We are particularly interested in the metric given by Ansatz 1.3. For convenience, let $m = n - 1$, the dimension of N. We recall

$$\begin{aligned} g = dt^2 + g_t = dt^2 + F(t)^2 \pi^* g_N + H(t)^2 \eta \otimes \eta, \\ \eta = (dz + q\pi^*\alpha),\ d\alpha = \omega_{\mathbb{N}}. \end{aligned}$$

We have

$$2g_t L_t = g_t' = 2\frac{H'}{H} H^2 \eta \otimes \eta + 2\frac{F'}{F} F^2 \pi^* g_N.$$

Thus, for Id denoting the identity operator on the horizontal subspace of TP, which is g_t- perpendicular to ∂_z,

$$\begin{aligned} L_t &= \frac{H'}{H} \partial_z \otimes \eta + \frac{F'}{F} \mathrm{Id}. \\ L'_t &= \Big(\frac{H''}{H} - (\frac{H'}{H})^2\Big) \partial_z \otimes \eta + \Big(\frac{F''}{F} - (\frac{F'}{F})^2\Big) \mathrm{Id}. \end{aligned}$$

Consequently,

$$\begin{aligned}
\mathrm{tr}L_t &= \frac{H'}{H} + (2m)\frac{F'}{F},\\
\mathrm{tr}L_t^2 &= (\frac{H'}{H})^2 + (2m)\frac{(F')^2}{F^2},\\
\mathrm{tr}L_t' &= \frac{H''}{H} + (2m)\frac{F''}{F} - \frac{(H')^2}{H^2} - (2m)\frac{(F')^2}{F^2}.
\end{aligned}$$

Lemma 3.1. *The gradient Ricci soliton equation becomes*

$$\begin{aligned}
\lambda &= -\frac{H''}{H} - (2m)\frac{F''}{F} + f'' = \frac{H^2q^2}{F^4}(2m) - \frac{H''}{H} - 2m\frac{H'F'}{HF} + f'\frac{H'}{H}\\
&= \frac{k}{F^2} - \frac{H^2q^2}{F^4}2 - \frac{F''}{F} - (2m-1)(\frac{F'}{F})^2 - \frac{H'F'}{FH} + f'\frac{F'}{F}.
\end{aligned}$$

Proof. By Lemma 6.3 and system 3.1, we obtain

$$\begin{aligned}
0 &= -\delta L - \nabla(\mathrm{tr}L) = -0 - 0 = 0;\\
\lambda &= -\mathrm{tr}L_t' - \mathrm{tr}L_t^2 + f''\\
&= -\frac{H''}{H} - (2m)\frac{F''}{F} + \frac{(H')^2}{H^2} + (2m)\frac{(F')^2}{F^2} - \frac{(H')^2}{H^2} - (2m)\frac{(F')^2}{F^2} + f''\\
&= -\frac{H''}{H} - (2m)\frac{F''}{F} + f'';\\
H^2\lambda &= \mathrm{Rc}(\partial_z,\partial_z) - \mathrm{tr}Lg(L\partial_z,\partial_z) - g(L'\partial_z,\partial_z) + f'g(L\partial_z,\partial_z)\\
&= \frac{H^4q^2}{F^4}(2m) - (\frac{H'}{H} + (2m)\frac{F'}{F})H^2\frac{H'}{H} - H^2\Big(\frac{H''}{H} - (\frac{H'}{H})^2\Big) + f'H^2\frac{H'}{H}\\
&= H^2\Big(\frac{H^2q^2}{F^4}(2m) - \frac{H''}{H} - (2m)\frac{F'}{F}\frac{H'}{H} + f'\frac{H'}{H}\Big);\\
F^2\lambda &= k - 2\frac{H^2q^2}{F^2} - (\frac{H'}{H} + (2m)\frac{F'}{F})F^2\frac{F'}{F} - F^2\Big(\frac{F''}{F} - (\frac{F'}{F})^2\Big) + f'F^2\frac{F'}{F}\\
&= F^2\Big(\frac{k}{F^2} - \frac{H^2q^2}{F^4}2 - \frac{F''}{F} - \frac{F'}{F}\frac{H'}{H} - (2m-1)\Big(\frac{F'}{F}\Big)^2 + f'\frac{F'}{F}\Big).
\end{aligned}$$

□

The almost complex structure on $I \times P$ is constructed from one on (N, g_N):

$$J = \partial_t \otimes H\eta - \frac{1}{H}\partial_z \otimes dt + \pi^* J_N.$$

Thus, the Kähler form becomes:

$$\begin{aligned}
\omega &= 2dt \wedge H\eta + F^2\pi^*\omega_N,\\
d\omega &= -2qH dt \wedge \pi^*\omega_N + 2FF'dt \wedge \pi^*\omega_N.
\end{aligned}$$

Thus the metric is almost Kähler if and only if

$$FF' = qH \tag{3.2}$$

Following [26], we consider the change of variables:

$$ds = Hdt,\ \alpha(s) := H^2(t),\ \beta(s) = F^2(t), \varphi(s) := f(t). \tag{3.3}$$

Consequently, for $\dot{X} = \partial_s X$,

$$\begin{aligned}
\dot\alpha &= 2H', & \ddot\alpha &= \frac{2H''}{H} \\
\dot\beta &= \frac{2FF'}{H}, & \ddot\beta &= \frac{2F'^2 + 2FF''}{H^2} - \frac{2FF'H'}{H^3}, \\
\dot\varphi &= \frac{f'}{H}, & \ddot\varphi &= \frac{f''}{H^2} - \frac{f'H'}{H^3}.
\end{aligned}$$

The almost Kählerity of g implies $\beta(s) = 2s + A$, $\varphi(s) = Bs + C$. The soliton equation becomes

$$\lambda = -\frac{\alpha''}{2} + \frac{2m\alpha}{(2s+A)^2} - \frac{m\alpha'}{2s+A} + B\frac{\alpha'}{2} = -\frac{\alpha''}{2} + B\frac{\alpha'}{2} - \big(\frac{m\alpha}{2s+A}\big)'$$
$$\lambda(2s+A) = k - \alpha' - \frac{2(m-1+q^2)\alpha}{2s+A} + B\alpha.$$

We summarize the above calculation in the following lemma.

Lemma 3.2. *Let $(I \times P, g)$ be given as in Ansatz 1.3. g is almost Kähler GRS if and only if, under transformation (3.3), we have:*

$$\lambda = -\frac{\ddot\alpha}{2} + B\frac{\dot\alpha}{2} - \frac{d}{ds}\big(\frac{m\alpha}{2s+A}\big)$$
$$\lambda(2s+A) = k - \dot\alpha - \frac{2(m-1+q^2)\alpha}{2s+A} + B\alpha$$

It order to obtain a global complete metric, one needs to smoothly extend the construction to singular orbits (if any). The following follows from the proof of [28, Lemma 1.1]. We provide a direct proof as our ansatz (1.3) is fairly explicit.

Lemma 3.3. *Let $I = (0, r)$ for $r > 0$ and $(I \times P, g)$ is given by ansatz 1.3 for $H(0) = 0$, $F(0) > 0$. The metric can be extended smoothly to $t = 0$ if and only if, for any non-zero integer k,*

$$H'(0) = 1,\ H^{(2k)}(0) = 0 = F^{(2k+1)}(0).$$

Proof. We rewrite the metric in polar coordinates, for $x = t\cos(z)$ and $y = t\sin(z)$,

$$\begin{aligned}
dt &= \frac{x}{t}dx + \frac{y}{t}dy, \\
dz &= \frac{-y}{t^2}dx + \frac{x}{t^2}dy
\end{aligned}$$

Then,

$$\begin{aligned}
g &= dt^2 + H^2(dz+q\alpha)\otimes(dz+q\alpha) + F^2 g_N \\
&= t^{-2}(x^2dx^2 + y^2dy^2 + xydx\otimes dy + xydy\otimes dx) \\
&+ \frac{H^2}{t^4}(y^2dx^2 + x^2dy^2 - xydx\otimes dy - xydy\otimes dx) \\
&+ \frac{-qyH^2}{t^2}(\alpha\otimes dx + dx\otimes\alpha) + \frac{qxH^2}{t^2}(\alpha\otimes dy + dy\otimes\alpha) \\
&+ H^2q^2\alpha\otimes\alpha + F^2g_N, \\
&= dx^2(\frac{H^2y^2}{t^4} + \frac{x^2}{t^2}) + dy^2(\frac{H^2x^2}{t^4} + \frac{y^2}{t^2}) + (dx\otimes dy + dy\otimes dx)(-\frac{H^2xy}{t^4} + \frac{xy}{t^2}) \\
&+ \frac{-qyH^2}{t^2}(\alpha\otimes dx + dx\otimes\alpha) + \frac{qxH^2}{t^2}(\alpha\otimes dy + dy\otimes\alpha) \\
&+ H^2q^2\alpha\otimes\alpha + F^2g_N.
\end{aligned}$$

Thus, the metric can be extended smoothly to $t=0$ if and only if the metric components

$$\frac{y^2}{t^2}(\frac{H^2}{t^2}-1), \frac{x^2}{t^2}(\frac{H^2}{t^2}-1), \frac{xy}{t^2}(\frac{H^2}{t^2}-1), \frac{qyH^2}{t^2}, \frac{qxH^2}{t^2}, F^2$$

can be smoothly extended to $x=y=0$. According to [33, Prop. 2.7], a function $\tilde{f}(x,y) = f(t,z)$ is smooth if and only if

- $f(t,z) = f(-t, z+\pi)$ for all t, z.
- $t^k(\frac{\partial^k f}{\partial t^k}(0,\theta))$ is a homogeneous polynomial of degree k in x and y,

Applying such criteria to our case yields

- $H'(0) = 1$ and $H^{(2n)}(0) = 0$;
- $F^{(2n+1)}(0) = 0$.

□

Additionally, the hyperbolic case gives rise to the following.

Lemma 3.4. *Let $I\times P$ be equipped with the metric*

$$dt^2 + g_t = dt^2 + e^{2A(t)z}\pi^*g_N + H^2(t)\eta\otimes\eta$$

for $\eta = dz$ and $\mathrm{Rc}_N = 0$. The gradient Ricci soliton equation becomes

$$A' = 0 = H',$$

$$\lambda = -(\frac{A}{H})^2(2m) = f''.$$

Proof. We have

$$\begin{aligned}
2g_tL_t = g_t' &= 2\frac{H'}{H}H^2dz^2 + 2zA'e^{2Az}\pi^*g_N, \\
L_t &= \frac{H'}{H}\partial_z\otimes dz + zA'\mathrm{Id}, \\
L_t' &= (\frac{H''}{H} - (\frac{H'}{H})^2)\partial_z\otimes dz + zA''\mathrm{Id}.
\end{aligned}$$

Consequently,

$$\begin{aligned}\mathrm{tr}L_t &= \frac{H'}{H} + zA'(2m),\\ \mathrm{tr}L_t^2 &= (\frac{H'}{H})^2 + 2mz^2(A')^2,\\ \mathrm{tr}L_t' &= (\frac{H''}{H} - (\frac{H'}{H})^2) + (2m)zA''.\end{aligned}$$

By the first equation of (3.1), one deduces that $A' = 0$ or A is constant. By the third equation of (3.1) and Lemma 2.19, H is constant. The result then follows. □

4. Induced Symmetry

Let (M, g, J, f) be an almost Hermitian GRS. In this section, we examine how the symmetry of (M, g, J, f) induces certain symmetry on level sets of function f. For each $c \in f(M)$, $M_c := f^{-1}(c)$ is called a level set of f. By the regular level set theorem [62], if c is a regular value, then the level set is a smooth submanifold of codimension one. From now on, we assume c is a regular value unless stated otherwise.

As $V = \frac{\nabla f}{|\nabla f|}$ is well-defined on M_c, let $\zeta = -J(V)$ and η be the dual 1-form to ζ. We define Φ on TM_c by

$$\Phi X + \eta(X)V = JX.$$

Proposition 4.1. *Let (M, g, J, f) be an almost Hermitian GRS. If c is a regular value of f, then $(M_c, g, \zeta, \eta, \Phi)$ is an almost contact metric structure.*

Proof. If $X = a\zeta + X_1$ for X_1 a section of TM_c and $X_1 \perp \zeta$, then it is immediate that $\Phi(X) = JX_1$ is also a section of TM_c. We check

$$\begin{aligned}\Phi^2(X) &= \Phi(J(X_1)) = J(J(X_1)) - \eta(J(X_1))V,\\ &= -X_1 - g(\zeta, J(X_1))V = -X_1 + g(J(\zeta), X_1)V = -X_1 + g(V, X_1)V,\\ &= -X_1 = -X + a\zeta = -X + \eta(X)\zeta.\end{aligned}$$

Therefore, $\Phi^2 = -\mathrm{Id} + \zeta \otimes \eta$ and (M_c, Φ, ζ, η) is an almost contact structure. Next, for $X = a\zeta + X_1$ and $Y = b\zeta + Y_1$, we compute

$$\begin{aligned}g(\Phi X, \Phi Y) &= g(J(X_1), J(Y_1)) = g(X_1, Y_1) = g(X - a\zeta, Y - b\zeta)\\ &= g(X, Y) - ag(\zeta, Y) - bg(X, \zeta) + abg(\zeta, \zeta)\\ &= g(X, Y) - 2ab + ab = g(X, Y) - \zeta(X)\zeta(Y).\end{aligned}$$

Thus, (M_c, Φ, ζ, η) is an almost contact metric structure. □

We will collect useful observations.

Lemma 4.2. *Suppose that $\mathcal{L}_X g = 0$.*

(i) $\mathcal{L}_X f = 0 \iff \mathcal{L}_X \nabla f = 0 \iff \mathcal{L}_X |\nabla f|^2 = 0$.
(ii) Let γ be the 1-form dual to a vector field Z. $\mathcal{L}_X Z = 0 \iff \mathcal{L}_X \gamma = 0$.

Proof. We only show the first statement as others follow from similar calculation. One computes

$$\begin{aligned}(\mathcal{L}_X\nabla f)Y &= g(\nabla_X\nabla f - \nabla_{\nabla f}X, Y)\\ &= \text{Hess}f(X,Y) + g(\nabla_Y X, \nabla f) - \mathcal{L}_X g(Y,\nabla f)\\ &= \text{Hess}f(X,Y) - g(X,\nabla_Y\nabla f) + Y(\mathcal{L}_X f) - \mathcal{L}_X g(Y,\nabla f)\\ &= Y(\mathcal{L}_X f) - \mathcal{L}_X g(Y,\nabla f).\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathcal{L}_X|\nabla f|^2 &= 2g(\nabla_X\nabla f,\nabla f)\\ &= 2g([X,\nabla f],\nabla f) - 2g(\nabla_{\nabla f}X,\nabla f)\\ &= 2g([X,\nabla f],\nabla f) - (\mathcal{L}_X g)(\nabla f,\nabla f).\end{aligned}$$

The conclusion then follows. □

Lemma 4.3. *Suppose that $\mathcal{L}_X V = 0$ and $\mathcal{L}_X\eta = 0$. $\mathcal{L}_X\Phi = 0$ if and only if $\mathcal{L}_X J = 0$.*

Proof. We compute

$$\begin{aligned}(\mathcal{L}_X\Phi)Y &= [X,\Phi(Y)] - \Phi([X,Y])\\ &= [X, J(Y) - \eta(Y)V] - J([X,Y]) + \eta([X,Y])V\\ &= (\mathcal{L}_X J)Y - [X,\eta(Y)V] + \eta([X,Y])V\\ &= (\mathcal{L}_X J)Y - \eta(Y)[X,V] - \nabla_X(\eta(Y))V + \eta([X,Y])V\\ &= (\mathcal{L}_X J)Y - \eta(Y)\mathcal{L}_X V - (\mathcal{L}_X\eta)(Y)V\end{aligned}$$

□

Proposition 4.4. *On each M_c, if $\mathcal{L}_X g = \mathcal{L}_X J = \mathcal{L}_X f = 0$ then we have*

$$\mathcal{L}_X\zeta = 0, \qquad \mathcal{L}_X\eta = 0, \qquad \mathcal{L}_X\Phi = 0.$$

Proof. It follows from Lemmas 4.2 and 4.3. □

Proposition 4.5. *Suppose that $\mathcal{L}_X g = 0$ and $\mathcal{L}_X\nabla f = 0$. If $X_{|M_c} \equiv 0$ then $X \equiv 0$.*

Proof. We compute

$$\begin{aligned}\nabla_{\nabla f}X &= -[X,\nabla f] + \nabla_X\nabla f\\ &= \text{Hess}f(X,\cdot) = 0.\end{aligned}$$

Since a Killing vector field is completely determined by its zero and first order values at a point, X must be trivial. □

Theorem 4.6. *Let (M^{2n}, g, J, f) be an almost Hermitian GRS with a non-trivial f and G be a group of symmetry preserving g, J, and f. Then G is also a group of symmetry for $(M_c, g, \zeta, \eta, \Phi)$ as an almost contact metric structure.*

Proof. Let $u : M \mapsto M$ be an isometry preserving J and f. As $f(u(a)) = f(a)$, $u(M_c) = M_c$ and u induces a map $u_c : M_c \mapsto M_c$. The proof will follow from the following claims.

Claim: If u_c is an identity map then so is u.

Proof: An isometry is necessarily an affine transformation which preserves parallelism [36, Chapter VI]. The result then follows.

Claim: $\frac{\nabla f}{|\nabla f|}$ is u-invariant. Consequently, so is $\zeta = -J(\frac{\nabla f}{|\nabla f|})$.

Proof: Since f is u-invariant, so is df. As ∇f is the dual of df via g and each is u-invariant, the first statement follows via Lemma 4.2. The second is because J is u-invariant.

Claim: η and Φ are u-invariant.

Proof: Because η is the dual of an u-invariant vector field and u is an isometry, η is u-invariant. Next, one recalls

$$\Phi(\cdot) = J(\cdot) - \eta(\cdot)\frac{\nabla f}{|\nabla f|}$$

and each component is u-invariant. The result then follows. □

Corollary 4.7. *Let (M^{2n}, g, J, f) be an almost Hermitian GRS with a non-trivial f. The dimension of the group of symmetry is at most n^2.*

Proof. It follows from Theorem 4.6 and the corresponding result for an almost contact metric structure, Theorem 2.20.

□

Under the setup of cohomogeneity one, there is a converse statement. Let (M, g, J, f) be an almost Hermitian GRS such that over a dense subset, the metric is given by the ansatz 1.3. Let X be an infinitesimal automorphism vector field on N. That is,

$$\mathcal{L}_X g_N = \mathcal{L}_X J_N = 0.$$

Since $\omega_N = g(\cdot, J\cdot)$,

$$\mathcal{L}_X \omega_N = 0.$$

Let X^* be its horizontal lift to (P, g_t). By Cartan's formula,

$$3d\omega(W, Y, Z) = (\mathcal{L}_W \omega)(Y, Z) - d(\mathfrak{i}_W \omega)(Y, Z).$$

For $\omega = \pi^*(\omega_N)$, $W = X^*$, $d\omega = 0 = \mathcal{L}_{X^*}\omega$. Thus, $\mathfrak{i}_{X^*}\omega$ is closed. If P is simply connected then there exists a function ℓ such that

$$\mathfrak{i}_{X^*}\pi^*(\omega) = d\ell.$$

Lemma 4.8. *If P is simply connected then the vector field $X^* - \ell\partial_z$ is independent of t and is an infinitesimal symmetry of $(P, g_t, \eta, \zeta, \Phi)$.*

Proof. X^* is independent of t as it only depends on π and the fixed subspace which is g_t perpendicular to ∂_z for all t. ℓ is independent of t as the proof of the Poincare's lemma is topological. The rest is straightforward; see [58, Lemma 5.1] for details. □

Remark 4.1. *If N is simply-connected, one can choose u to be constant on each fiber.*

Since (P, g_t) is complete for each t, it is possible to construct G, the group consisting of automorphisms generated by vector fields of the form $X^* - u\partial_z$ in Lemma 4.8 and the Killing vector field ∂_z.

Proposition 4.9. *If P is simply connected then G is a group of automorphism for (M, g, J, f).*

Proof. Let X_i be either a vector field from Lemma 4.8 or ∂_z. Then, immediately,

$$\mathcal{L}_{X_i} g = \mathcal{L}_{X_i}(dt^2 + g_t) = \mathcal{L}_{X_i} g_t = 0.$$

For $V = \partial_t$, by Lemmas 4.3 and 4.8, $\mathcal{L}_X J = 0$. The result then follows. □

5. Rigidity of the Maximal Dimension

We are now ready to prove main results.

Proof of Theorem 1.5. First, by Corollary 4.7, $\dim(G) \leq n^2$. Suppose that $\dim(G) = n^2$. Since f is non-constant, let $\gamma(t)$ be a unit speed integral curve of ∇f. For each regular value $f(t)$, $(M_t, \zeta_t, \eta_t, \Phi_t)$ is an almost contact metric structure by Prop 4.1. By Theorem 4.6, G is also a group of symmetry for each such almost contact metric structure.

By S. Tanno's Theorem 2.20, each connected component of M_t must be one of the model spaces therein. Furthermore, as G acts transitively on each connected component of M_t, M_t is a principal orbit and the orbit space of G-actions on M is of dimension one. Thus, (M, g, J, f) is of cohomogeneity one and each M_c is connected. Let $P = M_{t_0}$, for some fixed value t_0, which is a total space of a line or circle bundle over $\mathbb{N}(k)$ with the fiber projection π. By Sard's theorem, the set of singular values for $f : M \mapsto \mathbb{R}$ is of measure zero in $\mathbb{R}$. Thus, by continuity, nearby M_t must be obtained from the same model. Locally, the metric can be written as,

$$g = dt^2 + g_t, \ \ f = f(t).$$

Next, we consider cases as described in Theorem 2.20.

Case 1: $(P, g_t, \zeta_t, \eta_t, \Phi_t)$ is a deformation of a homogeneous Sasakian metric with constant Φ-sectional curvature. Thus, g_t is obtained from a standard Sasakian metric via an a-homothety and a $\pm b$ deformation. Thus, by fixing a background η and $\zeta = \partial_z$ on each fiber, By Theorem 2.18 and Lemma 2.21, for $F^2 = ab$ and $H = a$,

$$g_t = F^2(t)\pi^* g_{\mathbb{N}} + H^2(t)\eta \otimes \eta,$$
$$\eta(\partial_z) = 1, \ \ d\eta = \pi^* \omega_N.$$

Case 2: P is a trivial bundle over $\mathbb{N}(k)$. Thus,

$$g_t = H^2(t)dz^2 + F(t)^2 \pi^* g_{\mathbb{N}}.$$

Case 3: (P, g_t) is a hyperbolic metric. That is,

$$g_t = H^2 dz^2 + e^{2A(t)z} g_N.$$

One direction then follows from Lemmas 3.1 (for $q = 1, 0$) and Lemma 3.4.

For the reverse direction, if the soliton is locally constructed by the ansatz 1.3 and P is simply connected, then its automorphism group is the same as the group of symmetry of $(P, g_t, \eta_t, \zeta_t, \Phi_t)$ for each regular value t by Lemma 4.9. For $N = \mathbb{N}(k)$ such group is of dimension n^2 by [58]. The hyperbolic case case is trival as the metric is a product.

□

The next results will pave the way to the proof of Theorems 1.2.

Proposition 5.1. *Let $(M^{2n}, g, J, f, \lambda)$ be a non-trivial almost Kähler GRS with G the group of symmetry. If dim$(G) = n^2$ then either*

(i) the soliton belongs to case (i) of Theorem 1.5 with $q = 1$. Furthermore, H, F and f can be solved as follows:

$$\begin{aligned} ds &= Hdt, \\ F^2(t) = \beta(s) &= 2s + A, \\ f(t) = \phi(s) &= Bs + C, \\ D &= k - \lambda A, \end{aligned}$$
$$\alpha(x)(2x+A)^m e^{-Bx} \,|_{s_0}^{s} = \frac{e^{Bs}}{(2s+A)^m}\int_{s_0}^{s}(-2\lambda x + D)e^{-Bx}(2z+A)^m dx.$$

(ii) the soliton splits as $(M_1, g, J_1, f_1, \lambda) \times (\mathbb{N}, g_{\mathbb{N}}, J_{\mathbb{N}}, f_{\mathbb{N}}, \lambda)$ for $(M_1, g, J_1, f_1, \lambda)$ a Kähler GRS in real dimension two.

Proof. We continue from the proof of Theorem 1.5. For the first case $q = 1$, the result follows from Lemma 3.2. For the case $q = 0$, by equation (3.2), F is constant. Thus, the soliton must splits as a Riemannian product

$$M_1 \times M_2.$$

By [54, Lemma 2.1] and the discussion after Lemma 3.1, each $(M_i, g_i, J_i.f_i, \lambda)$ is a Kähler GRS. As $(M_2, g_2, J_2) = (\mathbb{N}, g_{\mathbb{N}}, J_{\mathbb{N}})$ the result follows. Finally, for the hyperbolic case, the product metric is not almost Kähler.

□

Proof of Theorem 1.2. We first need the following.

Claim 1: The largest connected group of isometries preserves J and f.

Proof: By A. Lichnerowicz [42], for an irreducible Kähler manifold, the largest connected group of isometries preserves the almost complex structure if n is odd or if n is even and Rc does not vanish. As the soliton is non-trivial, $\mathrm{Rc} \neq 0$.

Furthermore, by [54], for a Killing vector field X, either $\mathcal{L}_X f = \nabla_X f \equiv 0$ or $\nabla X \equiv 0$ and the manifold splits off a line or a circle. Since the metric is Kähler,

$$\nabla X \equiv 0 \rightleftarrows \nabla(JX) \equiv 0.$$

Thus, it splits off a line/circle if and only if there is a decomposition with a flat factor with respect to the Kähler structure, which contradicts the irreducibility. The claim follows.

The claim implies that the largest connected group of isometries is contained in the group of symmetry preserving g, J, and f. By Theorem 1.5, the dimension of the group of isometries is at most n^2. The maximal dimension is attained only if, locally, the metric must be constructed from ansatz 1.3 as in Prop. 5.1(i). Since $\alpha(s)$ and $\beta(s)$ do not both approach ∞ as $s \to -A/2$, the metric is only complete if there is a singular orbit and one needs to smoothly compactify such an end. By taking a scaling if necessary, one assumes that the singular orbit is at $s = 0 = t$ and the metric is defined in a neighborhood where $s > 0$. Thus, immediately, $\beta \geq 0$ if and only if $A \geq 0$.

Claim 2: If $\lambda \geq 0$ then $\mathbb{N} = \mathbb{CP}^{n-1}$.

Proof: Assume that $\mathbb{N} \neq \mathbb{CP}^{n-1}$ then it is non-compact. Then, at $t = s = 0$, one can only collapse the fiber; that is, $\alpha(0) = H(0) = 0, \beta(0) = F^2(0) > 0$. By Prop. 3.3, the smoothness of the metric requires

$$\frac{\partial \alpha}{\partial s}(0) = 2\frac{\partial H}{\partial t}(0) = 2.$$

Evaluating the equation $\lambda(2s + A) = k - \dot{\alpha} - \frac{2(m-1+q^2)\alpha}{2s+A} + B\alpha$ at $s = 0$ yields

$$k - \lambda A = D = 2.$$

For $\lambda \geq 0$, it implies that $k > 0$, a contradiction to $\mathbb{N}(k) \neq \mathbb{CP}^{n-1}$.

Finally, the reverse direction follows from Claim 1 and Theorem 1.5.

□

Remark 5.1. *For $\lambda < 0$, Lemma 3.3 shows that it is possible to construct Kähler GRS with $\mathbb{N} \neq \mathbb{CP}^{n-1}$. The details will appear elsewhere.*

Proof Corollary 1.3. If it is irreducible, Theorem 1.2 applies. Otherwise, we argue as in the proof of Theorem 1.2 to obtain a decomposition:

$$(M, g, f, J, \lambda) = \Pi_{i=0}^{k}(M_i^{n_i}, g_i, f_i, J_i, \lambda)$$

with $\sum_i n_i = n$ (complex dimensions), the $i = 0$-factor is Gaussian, and each $i > 0$-one is an irreducible Kähler GRS. The group of automorphisms for the $i = 0$-factor is of dimension at most

$$n_0(n_0 + 2).$$

For the rest, by Theorem 1.2, the group of isometry of the i-factor is of dimension at most n_i^2. As each is simply-connected, the group of the product is equal to the product of groups [36]. Thus, together, the group of isometry is of dimension at most

$$n_1^2 + ...n_k^2 \leq (n_1 + ...n_k)^2 = (n - n_0)^2.$$

It follows that the automorphism group of (M, g, J, f) is of dimension at most, for $k \leq n - 1$,

$$(n - n_0)^2 + n_0(n_0 + 2) = n^2 - 2n_0(n - n_0 - 1) \leq n^2.$$

Equality happens if and only if $k = 1$ and $n_0 = n - 1$ or $n_0 = 0$. The case $n_0 = n - 1$, one factor is a complete Kähler GRS in real dimension two. The result then follows.

□

Proof of Corollary 1.4. By [35], the largest connected group of affine transformations G contains of automorphisms. Non-degeneracy implies the Riemannian metric does not split off any flat factors. Thus, f is G-invariant. The result then follows from Theorem 1.5 and Corollary 1.3. For the reverse direction, the non-degeneracy follows as the Ricci curvature is non-singular at some point [35]. □

6. Appendix

6.1. **Convention.** Here are our conventions:

- $\mathcal{L}$ denotes the Lie derivative.
- The convention of exterior derivative, for an m-form α,

$$(m+1)d\alpha(X_0, ...X_m) = \sum_i (-1)^i X_i(\alpha(X_0, ..., \hat{X}_i, ...X_m)) + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ...X_m).$$

Consequently,

$$(dx^{i_1} \wedge ... \wedge dx^{i_k})(\partial_{x_i}, ...\partial_{x_k}) = \frac{1}{k!}.$$

Remark 6.1. *Our convention agrees with* [22] *but differs from* [4] *by a scaling.*

- The interior product for a m-form is defined as

$$(\mathfrak{i}_X \alpha(Y_1, ..., Y_m)) = m\alpha(X, Y_1, ..., Y_m).$$

The following identity is the so-called Cartan's formula for a differential form

$$\mathcal{L}_X \alpha = (d \circ \mathfrak{i}_X + \mathfrak{i}_X \circ d)\alpha.$$

- On a Riemannian manifold (M, g), there is a unique Levi-Civita connection $\nabla : TM \times C^\infty(TM) \mapsto C^\infty(TM)$. The connection induces a Riemannian curvature via the covariant second derivative:

$$\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}.$$

The Riemann curvature $(3,1)$ tensor and $(4,0)$ tensor are defined as follows,

$$\mathrm{R}(X, Y, Z) = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z$$
$$\mathrm{R}(X, Y, Z, W) = g(\nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z, W).$$

Remark 6.2. *Our sign convention agrees with* [22, 6]. *Our* $(3,1)$ *curvature tensor differs from one of* [4] *by a sign.*

Furthermore, the curvature can be seen as an operator on the space of two forms. For an orthonormal basis $\{e_i\}_i$ and any 2-form α,

$$\mathrm{R}(\alpha)(e_i, e_j) = \sum_{k<l} \mathrm{R}(e_i, e_j, e_k, e_l)\alpha(e_k, e_l).$$

Consequently, due to our exterior derivative convention,

$$\mathrm{R}(X \wedge Y)(Z, W) = \frac{1}{2}\mathrm{R}(X, Y, Z, W).$$

The sectional curvature and Ricci curvature are defined as follows,

$$K(X, Y) = \mathrm{R}(X, Y, Y, X),$$
$$\mathrm{Rc}_{ik} = \sum_j \mathrm{R}_{ijjk}.$$

6.2. **Submersion.** A differentiable map between smooth manifolds $\pi : P \mapsto N$ is a submersion if the pushforward of the tangent space at each point is surjective. That is, for $p \in P$, $\pi_*(T_xP) = T_{\pi(p)}N$. A Riemannian submersion is a submersion between Riemannian manifolds such that the differential above is a linear isometry.

We consider a Riemannian submersion $\pi : (P^{2n+1}, g_p) \mapsto (N^{2n}, g_N)$ such that each fiber is a geodesic line or circle with tangential vector field ζ such that $g(\zeta, \zeta) = 1$. The submersion naturally decomposes TP into vertical and horizontal distributions. Let $(\cdot)^{\mathcal{H}}$ and $(\cdot)^{\mathcal{V}}$ denote the horizontal and vertical parts, respectively, of a vector field on P. The vertical subspace consists of multiples of ζ. Furthermore, for each vector field on N there is a unique horizontal vector field on P such that they are π-related. We'll collect useful lemmas whose proofs can be found in [53, 4, 6] or a straightforward calculation.

Lemma 6.1. *For horizontal vector fields X and Y:*

i) $[\zeta, X]$ *is vertical,*
ii) $g([X,Y],\zeta) = 2g(\nabla_X Y, \zeta) = 2g(\nabla_Y \zeta, X) = -2g(\nabla_\zeta X, Y)$,
iii) ζ *is a Killing vector field.*

The curvature of a submersion can be computed via B. O'Neill's A and T tensors [48]. Since each fiber is totally geodesic, $T \equiv 0$ and only A is non-trivial. One recalls

$$A_X\zeta = (\nabla_X \zeta)^{\mathcal{H}},\ A_X Y = (\nabla_X Y)^{\mathcal{V}}.$$

For horizontal vector fields X, T, Z, W,

$$\mathrm{R}_P(X,Y,Z,W) = \mathrm{R}_N(X,Y,Z,W) + 2g(A_XY, A_ZW) + g(A_XZ, A_YW) - g(A_XW, A_YZ),$$
$$\mathrm{R}_P(X,\zeta,Z,\zeta) = -g(A_X\zeta, A_Z\zeta).$$

Remark 6.3. *These formulas differ from ones of* [4, Chapter 9] *by a sign convention.*

Consequently, there are corresponding identities for the sectional curvature and Ricci curvature. For orthonormal horizontal vectors X, Y

$$K_P(X,Y) = K_N(X,Y) - 3|A_XY|^2;\ K_P(X,\zeta) = |A_X\zeta|^2,$$
$$\mathrm{Rc}_P(X,Y) = \mathrm{Rc}_N(X,Y) - 2g(A_X, A_Y);\ \mathrm{Rc}(X,\zeta) = 0;\ \mathrm{Rc}(\zeta,\zeta) = g(A\zeta, A\zeta).$$

Here, as $\{E_i\}_{i=1}^{2n}$ denotes a local orthonormal frame for the horizontal distribution,

$$g(A_X, A_Y) = \sum_i g(A_XE_i, A_YE_i) = g(A_X\zeta, A_Y\zeta),$$
$$g(A\zeta, A\zeta) = \sum_i g(A_{E_i}\zeta, A_{E_i}\zeta).$$

Next, we restrict to the submersion given by ansatz 1.3. It is immediate that $\zeta = \frac{1}{H}\partial_z$ is a Killing vector field. In this situation, tensor A can be computed immediately.

Lemma 6.2. *For horizontal vector fields X and Y*

$$2Hd\eta(X,Y) = -g([X,Y],\zeta) = \frac{Hq}{F^2} g(X,JY),$$
$$A_X Y = -\frac{Hq}{F^2} g(X,JY)\zeta$$
$$A_X \zeta = -\frac{Hq}{F^2} JX.$$

Lemma 6.3. *The sectional and Ricci curvature of (P,g) are given by, for orthonormal horizontal vectors X, Y*

$$K(X,Y) = \frac{1}{F^2} K_N(FX,FY) - 3\frac{H^2q^2}{F^4} g(X,JY)^2,$$
$$K(X,\zeta) = \frac{H^2q^2}{F^4},$$
$$\mathrm{Rc}(X,Y) = \mathrm{Rc}_N(X,Y) - 2\frac{H^2q^2}{F^4} g(X,Y)$$
$$\mathrm{Rc}(\zeta,\zeta) = \frac{H^2q^2}{F^4} 2m.$$

References

[1] Richard H Bamler, Charles Cifarelli, Ronan J Conlon, and Alix Deruelle. A new complete two-dimensional shrinking gradient k\" ahler-ricci soliton. *arXiv preprint arXiv:2206.10785*, 2022.

[2] Shigetoshi Bando. Real analyticity of solutions of Hamilton's equation. *Math. Z.*, 195(1):93–97, 1987.

[3] Jacob Bernstein and Thomas Mettler. Two-dimensional gradient Ricci solitons revisited. *Int. Math. Res. Not. IMRN*, (1):78–98, 2015.

[4] Arthur L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

[5] Christoph Böhm and Burkhard Wilking. Manifolds with positive curvature operators are space forms. *Ann. of Math. (2)*, 167(3):1079–1097, 2008.

[6] Charles P. Boyer and Krzysztof Galicki. *Sasakian geometry*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.

[7] Simon Brendle. Rotational symmetry of self-similar solutions to the ricci flow. *Inventiones mathematicae*, pages 1–34, 2012.

[8] Simon Brendle. Rotational symmetry of Ricci solitons in higher dimensions. *J. Differential Geom.*, 97(2):191–214, 2014.

[9] Simon Brendle and Richard Schoen. Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.*, 22(1):287–307, 2009.

[10] Simon Brendle and Richard M. Schoen. Classification of manifolds with weakly 1/4-pinched curvatures. *Acta Math.*, 200(1):1–13, 2008.

[11] M. Buzano, A. S. Dancer, and M. Wang. A family of steady Ricci solitons and Ricci flat metrics. *Comm. Anal. Geom.*, 23(3):611–638, 2015.

[12] Jacob Cable and Hendrik Süß. On the classification of kähler–ricci solitons on gorenstein del pezzo surfaces. *European Journal of Mathematics*, 4(1):137–161, 2018.

[13] Huai-Dong Cao. Existence of gradient Kähler-Ricci solitons. In *Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994)*, pages 1–16. A K Peters, Wellesley, MA, 1996.

[14] Huai-Dong Cao. Limits of solutions to the Kähler-Ricci flow. *J. Differential Geom.*, 45(2):257–272, 1997.

[15] Huai-Dong Cao. Recent progress on ricci solitons. *Adv. Lect. Math.*, 11:1–38, 2009.
[16] Huai-Dong Cao and Detang Zhou. On complete gradient shrinking Ricci solitons. *J. Differential Geom.*, 85(2):175–185, 2010.
[17] Xiaodong Cao and Hung Tran. The Weyl tensor of gradient Ricci solitons. *Geom. Topol.*, 20(1):389–436, 2016.
[18] Thierry Chave and Galliano Valent. On a class of compact and non-compact quasi-Einstein metrics and their renormalizability properties. *Nuclear Phys. B*, 478(3):758–778, 1996.
[19] Qiang Chen and Meng Zhu. On rigidity of gradient Kähler-Ricci solitons with harmonic Bochner tensor. *Proc. Amer. Math. Soc.*, 140(11):4017–4025, 2012.
[20] Otis Chodosh and Frederick Tsz-Ho Fong. Rotational symmetry of conical Kähler-Ricci solitons. *Math. Ann.*, 364(3-4):777–792, 2016.
[21] Bennett Chow. The Ricci flow on the 2-sphere. *J. Differential Geom.*, 33(2):325–334, 1991.
[22] Bennett Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci flow*, volume 77 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2006.
[23] Charles Cifarelli, Ronan J Conlon, and Alix Deruelle. On finite time type i singularities of the k\" ahler-ricci flow on compact k\" ahler surfaces. *arXiv preprint arXiv:2203.04380*, 2022.
[24] Ronan J Conlon and Alix Deruelle. Expanding kähler–ricci solitons coming out of kähler cones. *Journal of Differential Geometry*, 115(2):303–365, 2020.
[25] Ronan J Conlon, Alix Deruelle, and Song Sun. Classification results for expanding and shrinking gradient k\" ahler-ricci solitons. *arXiv preprint arXiv:1904.00147*, 2019.
[26] Andrew S. Dancer and McKenzie Y. Wang. On Ricci solitons of cohomogeneity one. *Ann. Global Anal. Geom.*, 39(3):259–292, 2011.
[27] Yuxing Deng and Xiaohua Zhu. Rigidity of κ-noncollapsed steady Kähler-Ricci solitons. *Math. Ann.*, 377(1-2):847–861, 2020.
[28] J.-H. Eschenburg and McKenzie Y. Wang. The initial value problem for cohomogeneity one Einstein metrics. *J. Geom. Anal.*, 10(1):109–137, 2000.
[29] Mikhail Feldman, Tom Ilmanen, and Dan Knopf. Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons. *J. Differential Geom.*, 65(2):169–209, 2003.
[30] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.
[31] Richard S. Hamilton. The Ricci flow on surfaces. In *Mathematics and general relativity (Santa Cruz, CA, 1986)*, pages 237–262. Amer. Math. Soc., Providence, RI, 1988.
[32] Thomas A. Ivey. Local existence of Ricci solitons. *Manuscripta Math.*, 91(2):151–162, 1996.
[33] Jerry L. Kazdan and F. W. Warner. Curvature functions for open 2-manifolds. *Ann. of Math. (2)*, 99:203–219, 1974.
[34] Shoshichi Kobayashi. *Transformation groups in differential geometry*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1972 edition.
[35] Shoshichi Kobayashi and Katsumi Nomizu. On automorphisms of a Kählerian structure. *Nagoya Math. J.*, 11:115–124, 1957.
[36] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.
[37] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. II*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1969 original, A Wiley-Interscience Publication.
[38] Norihito Koiso. On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics. In *Recent topics in differential and analytic geometry*, volume 18 of *Adv. Stud. Pure Math.*, pages 327–337. Academic Press, Boston, MA, 1990.
[39] Brett Kotschwar. Kählerity of shrinking gradient Ricci solitons asymptotic to Kähler cones. *J. Geom. Anal.*, 28(3):2609–2623, 2018.
[40] Xiaolong Li, Lei Ni, and Kui Wang. Four-dimensional gradient shrinking solitons with positive isotropic curvature. *Int. Math. Res. Not. IMRN*, (3):949–959, 2018.

[41] Yu Li and Bing Wang. On k\" ahler ricci shrinker surfaces. *arXiv preprint arXiv:2301.09784*, 2023.
[42] André Lichnerowicz. Sur les groupes d'automorphismes de certaines variétés kähleriennes. *C. R. Acad. Sci. Paris*, 239:1344–1346, 1954.
[43] Ovidiu Munteanu and Natasa Sesum. On gradient Ricci solitons. *J. Geom. Anal.*, 23(2):539–561, 2013.
[44] Ovidiu Munteanu and Jiaping Wang. Topology of Kähler Ricci solitons. *J. Differential Geom.*, 100(1):109–128, 2015.
[45] Ovidiu Munteanu and Jiaping Wang. Positively curved shrinking Ricci solitons are compact. *J. Differential Geom.*, 106(3):499–505, 2017.
[46] Aaron Naber. Noncompact shrinking four solitons with nonnegative curvature. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2010(645):125–153, 2010.
[47] A. Newlander and L. Nirenberg. Complex analytic coordinates in almost complex manifolds. *Ann. of Math. (2)*, 65:391–404, 1957.
[48] Barrett O'Neill. The fundamental equations of a submersion. *Michigan Math. J.*, 13:459–469, 1966.
[49] Henrik Pedersen, Christina Tø nnesen Friedman, and Galliano Valent. Quasi-Einstein Kähler metrics. *Lett. Math. Phys.*, 50(3):229–241, 1999.
[50] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv:math.DG/0211159*, 2002.
[51] Grisha Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. *arXiv:math.DG/0307245*, 2003.
[52] Grisha Perelman. Ricci flow with surgery on three-manifolds. *arXiv:math.DG/0303109*, 2003.
[53] Peter Petersen. *Riemannian geometry*, volume 171. Springer, 2006.
[54] Peter Petersen and William Wylie. On gradient Ricci solitons with symmetry. *Proc. Amer. Math. Soc.*, 137(6):2085–2092, 2009.
[55] Peter Petersen and William Wylie. On the classification of gradient Ricci solitons. *Geom. Topol.*, 14(4):2277–2300, 2010.
[56] Georges Reeb. Sur certaines propriétés topologiques des trajectoires des systèmes dynamiques. *Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8°*, 27(9):64, 1952.
[57] R. Schoen. On the conformal and CR automorphism groups. *Geom. Funct. Anal.*, 5(2):464–481, 1995.
[58] Shûkichi Tanno. The automorphism groups of almost contact Riemannian manifolds. *Tohoku Math. J. (2)*, 21:21–38, 1969.
[59] Shûkichi Tanno. The automorphism groups of almost Hermitian manifolds. *Trans. Amer. Math. Soc.*, 137:269–275, 1969.
[60] Shûkichi Tanno. Sasakian manifolds with constant ϕ-holomorphic sectional curvature. *Tohoku Math. J. (2)*, 21:501–507, 1969.
[61] Gang Tian and Xiaohua Zhu. Uniqueness of Kähler-Ricci solitons. *Acta Math.*, 184(2):271–305, 2000.
[62] Loring W. Tu. *An introduction to manifolds*. Universitext. Springer, New York, second edition, 2011.
[63] Xu-Jia Wang and Xiaohua Zhu. Kähler-Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.*, 188(1):87–103, 2004.