KÄHLER SOLITONS, CONTACT STRUCTURES, AND ISOPARAMETRIC FUNCTIONS

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Abstract. Let (M, g, f, J, λ) be a complete Kähler gradient Ricci soliton. Our first theorem classifies such a structure in real dimension four when f has a geodesic gradient, a notion weaker than isoparametric. The soliton must be either a product metric or of cohomogeneity one with deformed homogeneous Sasakian orbits. The second result is a partially reverse statement in any dimension. Suppose that each regular level set of f is a deformed contact metric structure then the soliton is totally determined by a regular Sasakian model, which is a Riemannian submersion, with circle fibers, over a Kähler-Einstein manifold. In particular, f must be isoparametric. An important ingredient of the proof is a characterization of a deformed Sasakian structure which generalizes a classical result.

1. Introduction

A gradient Ricci soliton (GRS) (M, g, f, λ) is a Riemannian manifold with metric g, potential function f, and a constant λ such that, for Rc denoting the Ricci curvature,

$$\mathrm{Rc} + \mathrm{Hess} f = \lambda g. \tag{1.1}$$

Such a structure is a self-similar solution to the Ricci flow and plays a crucial role in its analysis. The general theory was introduced by R. Hamilton [40] and has several celebrated applications including [52, 53, 8, 10, 11, 16].

A Kähler GRS (M, g, f, J, λ) is a GRS such that (M, g, J) is Kähler for a complex structure J. That is, $g(JX, JY) = g(X, Y)$ and the Kähler form $\omega := g(\cdot, J\cdot)$ is closed. The subject has an extensive literature; see, for examples, [64, 68, 15, 22, 47, 12, 26, 29, 32]. In particular, tremendous recent efforts lead to the classification of all Kähler GRS surfaces with $\lambda > 0$ [30, 28, 2, 44].

This paper reveals connections between a Kähler GRS, an isoparametric function, and contact structures. The study of isoparametric functions and their level sets in space forms was motivated by questions in geometric optics and initial contributions were given in [61, 17, 43]. The classification in an ambient round sphere, formulated as Question 34 in S.T. Yau's list [69], is remarkably deep. It has attracted enormous interest and important developments are given by, for example, [49, 48, 1, 19, 24]; see [25] for a recent survey. Certain aspects of the theory can be extended to a general Riemannian manifold [67, 39, 46, 56, 33].

In the study of a GRS, the potential function being isoparametric arises as a consequence of constant scalar curvature [37, 23]. Also, it is observed that ∇f is a eigenvector

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of Rc if and only if f is rectifiable or has a geodesic gradient, a slightly weaker notion than isoparametric; see Subsection 2.4. Our first theorem classifies all Kähler GRS in real dimension four with such a potential function.

Theorem 1.1. *Let (M, g, J, f) be a complete connected Kähler GRS of real dimension four. If ∇f is an eigenvector of* Rc *whenever $\nabla f \neq \vec{0}$ then the manifold must be either*

- *a product of a constant curvature surface with a $2D$ Kähler GRS, or*
- *of cohomogeneity one, f is invariant by its action, and each principal orbit is a connected deformed homogeneous Sasakian structure.*

Remark 1.1. *The classification includes all $U(2)$-invairant metrics constructed earlier by* [42, 13, 20, 14, 36, 51]. *The dimension assumption is crucial as* [31] *describes a general construction in higher dimensions such that f satisfies the above property.*

Remark 1.2. *Corresponding to space forms, there are three models of simply connected Sasakian structures with constant holomorphic sectional curvature* [63]*. If simply-connected is replaced by compact then there is also a classification* [3, 9]*. It must be a quotient by a discrete subgroup of the connected component of the isometry group of either $\mathbb{S}^3$, $\widetilde{SL}(2, \mathbb{R})$ the universal cover of $SL(2, \mathbb{R})$, and Nil^3, the Heisenberg group, diffeomorphic to $\mathbb{R}^3$, of 3×3 nilpotent real matrices. There are standard Sasakian structures on these models with constant holomorphic sectional curvature $1, -4, -3$ respectively. Each is a circle bundle over a compact Riemann surface.*

We then deduce an immediate consequence.

Corollary 1.2. *Let (M, g, J, f) be a simply connected irreducible non-compact Kähler GRS of real dimension four. If ∇f is an eigenvector of* Rc *whenever $\nabla f \neq \vec{0}$ then each deformed Sasakian orbit is a Riemannian submersion, with circle fibers, over a simply connected surface of constant curvature. Consequently, the isometry group is of dimension four.*

Remark 1.3. *Such a Kähler GRS is of maximal irreducible symmetry by* [66].

The results above suggest a connection between a Kähler GRS and a Sasakian structure, a fundamental notion of contact and almost contact geometry. This arguably is an odd-dimensional counterpart of symplectic geometry and both lie at the heart of classical mechanics. In general, one can consider either the even-dimensional phase space of a mechanical system or odd-dimensional constant-energy hypersurfaces. Thus, it has rich literature; for example, contact transformation was studied by S. Lie [45]. Here we focus on a direction with more attention to an associated Riemannian metric. The current theory owns much to the foundational work of S. Sasaki, D. Blair, and others [59, 58, 60, 5, 6]; see also a recent book [9] and a survey [7].

An odd-dimensional Riemannian manifold (P, g_P)- with a vector field ζ, an 1-form η, and a tensor field of type $(1,1)$ Φ- is called an almost contact metric structure if

$$\eta(\zeta) = 1, \ \Phi^2 = -\mathrm{Id} + \zeta \otimes \eta, \text{ and } g_P(\Phi(X), \Phi(Y)) = g_P(X, Y) - \eta(X)\eta(Y).$$

An almost contact metric structure $(M, \zeta, \eta, \Phi, g)$ is called a deformed contact metric structure if one further assumes, for some constant $a \neq 0$,

$$d\eta(X, Y) = a g(X, \Phi(Y)).$$

Naturally, it is called contact if $a = 1$. Moreover, a Sasakian structure is a contact metric structure $(M, \zeta, \eta, \Phi, g)$ such that the cone $C(M) = M \times \mathbb{R}^+$ with the cone metric $r^2 g + dr^2$ is Kähler.

It is well-known that a real hypersurface of a Kähler manifold is naturally endowed with an almost contact metric structure [9]. Let (M, g, J, f) be a Kähler GRS. Let M_c be a regular level set of f and

$$(1.2)\quad V := \frac{\nabla f}{|\nabla f|},\ g_c := g_{|M_c},\ \zeta_c := -J(V),\ \eta_c(\cdot) := g(\cdot, \zeta_c),\ \Phi_c(\cdot) := -\eta_c(\cdot)V + J(\cdot).$$

Together, $(M_c, \zeta_c, \eta_c, \Phi_c, g_c)$ is an almost contact metric structure. Our next theorem shows that going from almost contact to deformed contact imposes significant restriction on the soliton structure leading to a full classification.

Theorem 1.3. *Let (M, g, J, f, λ) be a complete connected Kähler GRS. For each regular value c, supposed that $(M_c, \zeta_c, \eta_c, \Phi_c, g_c)$ is a deformed contact structure. Then the soliton is totally determined by a connected Sasakian model $(P, \eta, \zeta, \Phi, g_P)$ which is a Riemannian submersion, with circle fibers, over a Kähler-Einstein manifold (N, g_N, J_N) with $\mathrm{Rc}_N = kg_N$. That is, there is submersion map $\pi : P \mapsto N$ such that*

$$g_P = \eta \otimes \eta + \pi^* g_N,\ d\eta = \pi^* \omega_N.$$

There is an interval I with coordinate s such that there is a diffeomorphism $\phi : I \times P \mapsto M_o$, a dense subset of M, $f \circ \phi = Bs + C$, and

$$\phi^* g = \frac{ds^2}{\alpha(s)} + \alpha(s)\eta \otimes \eta + (2s + A)\pi^* g_N. \tag{1.3}$$

Here A, B, C are constant and α solves a first order equation, for $n = \dim_{\mathbb{C}} N$,

$$\lambda(2s + A) = k - \frac{d\alpha}{ds} - \frac{2n\alpha}{2s + A} + B\alpha.$$

There is a boundary point of I such that, $\alpha \to 0$. Furthermore, if $(2s + A) \to 0$ towards that end point, then $(P, \eta, \zeta, \Phi, g_P)$ is the standard Sasakian sphere and (N, g_N, J_N) is, up to homothety, isomorphic to a standard complex projective space.

Remark 1.4. *Equation (1.3) is a version of the Calabi ansatz. Also, at each finite end point of I, α must satisfy certain conditions to make the metric smooth* [31, 66].

Remark 1.5. *There are also other studies which construct Kähler GRS from Sasaki-Einstein manifolds* [38, 7].

Remark 1.6. *The main content of the theorem is about rectifiable f, constant Ricci, and the regular Sasakian structure.*

Indeed, the proofs of Theorem 1.1 and 1.3 both rely on understanding a deformed contact structure. The following result might be of independent interest.

Theorem 1.4. *An almost contact metric manifold $(M, \zeta, \eta, \Phi, g)$ satisfies*

$$(\nabla_X \Phi)(Y) = bg(X, Y)\zeta - \eta(Y)bX. \tag{1.4}$$

for every vector fields X and Y if and only if

(i) for $b \neq 0$, it is a deformed Sasakian structure;
(ii) for $b = 0$, it is a product of a line or circle with a Kähler manifold whose almost complex structure is induced by Φ.

Remark 1.7. *The case $b = 1$ is a classical result* [9].

Theorem 1.4 has an important consequence. Let (N, g, J) be a Kähler manifold and M a real hypersurface with unit normal V. Let $\zeta = -JV$, η the dual 1-form, and

$$\Phi(\cdot) := J(\cdot) - \eta(\cdot)V.$$

It is immediate that $(M, \zeta, \eta, \Phi, g)$ is an almost contact metric structure. We recall the shape operator

$$LX = \nabla_X V.$$

Theorem 1.5. *$L = \alpha Id + \beta\zeta \otimes \eta$ for a constant α iff $(M, \eta, \zeta, \Phi, g)$ is*
(i) for $\alpha \neq 0$, a deformed-Sasakian structure.
(ii) for $\alpha = 0$, a Riemannian product of a line or a circle with a Kähler manifold.

The organization of the paper is as follows. Section 2 will recall fundamental definitions and preliminary results. The proofs of Theorems 1.4 and 1.5 will be given in Section 3. It is basically about tracking how the covariant derivative of Φ changes under a deformation. Our arguments are based on meticulous analysis of the transverse metric using foliation theory and the toolkit developed by B. O'Neill [50]. Then Section 4 investigates the case each level set of a Kähler GRS is a deformed contact structure and proves Theorem 1.3. The first consequence is that the potential f must be rectifiable and the metric can be written in an explicit way. Then, analysis of the metric's smoothness leads to the conclusion. Finally, the proofs of Theorem 1.1 and Corollary 1.2 will be given in Section 5 combining our earlier developments.

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2. Preliminaries

In this section, we recall preliminary results and certain observations which will be used throughout the article.

2.1. **GRS and Kählerity.** Here we recall the definition of a gradient Ricci soliton, the Kähler setup, and some identities. A Riemannian manifold (M^n, g) with a potential function f is called a GRS if, for Rc denoting the Ricci curvature and $\mathcal{L}$ the Lie derivative,

$$\mathrm{Rc} + \mathcal{L}_{\nabla f} g = \mathrm{Rc} + \mathrm{Hess} f = \lambda g.$$

Taking the trace yields, for $\mathrm{S} = \mathrm{tr}_g \mathrm{Rc}$ the scalar curvature,

$$\mathrm{S} + \triangle f = n\lambda. \tag{2.1}$$

Due to its symmetry, the Ricci curvature is frequently considered as an endormorphism on TM. Thus, via the second Bianchi's identity, we deduce that

$$\mathrm{Rc}(\nabla f) = \frac{1}{2}\nabla \mathrm{S} = \delta \mathrm{Rc}. \tag{2.2}$$

Here δ is the divergence operator or the co-differential, for an orthonormal basis,

$$\delta \mathrm{Rc}(X) = \sum_i g((\nabla_{e_i}\mathrm{Rc})X, e_i).$$

Consequently, the following is considered as a conservation law,

$$\mathrm{S} + |\nabla f|^2 - 2\lambda f = \text{constant}. \tag{2.3}$$

Here is another interesting identity [27],

$$\triangle \mathrm{S} + 2|\mathrm{Rc}|^2 = \langle \nabla f, \nabla \mathrm{S}\rangle + 2\lambda \mathrm{S}. \tag{2.4}$$

Remark 2.1. *If $\lambda \geq 0$, then $\mathrm{S} \geq 0$ by the maximum principle and equation* (2.4). *Moreover, such a complete GRS has positive scalar curvature unless it is isometric to the flat Euclidean space* [70, 21].

In the presence of a complex structure, there are further observations. When a manifold M is of an even dimension, an almost complex structure is defined to be a smooth section J of the bundle of endormorphisms $\mathrm{End}(TM)$ such that

$$J^2 = -\mathrm{Id}.$$

J is said to be integrable if it is genuinely induced from an atlas of complex charts with holomorphic transition functions.

(M, g, J) is called an almost Hermitian manifold and g a Hermitian metric if

$$g(JX, JY) = g(X, Y).$$

The fundamental 2-form or Kähler form is given by

$$\omega_g(X, Y) = g(X, JY).$$

(M, g, J) is called almost Kähler if $d\omega = 0$. When J is integrable, one upgrades an almost Hermitian to Hermitian and almost Kähler to Kähler. For a Riemannian manifold to be Kähler, the following is well-known.

Proposition 2.1. [9, Proposition 3.1.9] *Let (M, g, J) be an almost Hermitian real manifold. The followings are equivalent:*

(i) $\nabla J = 0$,
(ii) $\nabla \omega_g = 0$,
(iii) (M, g, J) *is Kähler.*

Definition 2.1. *(M, g, J, f) is a Kähler GRS if (M, g, f) is a GRS and (M, g, J) is a Kähler manifold.*

It is crucial to observe that, on a Kähler manifold (M, g, J), Rc is J-invariant. Thus, for a Kähler GRS, so is $\mathrm{Hess}f$ which leads to the followings.

Lemma 2.2. *Let (M, g, J) be a Kähler manifold and $f : M \mapsto \mathbb{R}$ such that* $\mathrm{Hess}f$ *is J-invariant. Then, we have the followings:*

(i) $J(\nabla f)$ *is a Killing vector field.*
(ii) ∇f *is an infinitesimal automorphism of J.*

Proof. We compute, since J is parallel, for any vector fields Y and Z,

$$\begin{aligned} g(\nabla_Y(J(\nabla f)), Z) + g(\nabla_Z(J(\nabla f)), Y) &= g(J\nabla_Y(\nabla f), Z) + g(J\nabla_Z(\nabla f), Y), \\ &= -g(\nabla_Y(\nabla f), JZ) - g(\nabla_Z(\nabla f), JY), \\ &= -(\text{Hess f})(Y, JZ) - (\text{Hess f})(Z, JY) = 0. \end{aligned}$$

The last equality follows because Hess$f$ is J-invariant. For the second statement, we consider

$$\begin{aligned} (\mathcal{L}_{\nabla f} J)Y &= [\nabla f, JY] - J([\nabla f, Y]) \\ g((\mathcal{L}_{\nabla f} J)Y, Z) &= g([\nabla f, JY], Z) - g(J([\nabla f, Y]), Z) \\ &= g(J\nabla_{\nabla f} Y, Z) - \text{Hess}f(JY, Z) + g(\nabla_{\nabla f} Y - \nabla_Y \nabla f, JZ) \\ &= -\text{Hess}f(JY, Z) - \text{Hess}f(Y, JZ) = 0. \end{aligned}$$

□

2.2. **Foliation.** In this subsection, we recall the concept of a foliation and related properties. The references are [65, 9]. A p-dimensional foliation $\mathcal{F}$ of an n-dimensional manifold M refers to the partition of M into a union of disjoint p-dimensional immersed submanifolds $\{L_\alpha\}_{\alpha \in A}$, called leaves, with the following property. Every point is on a chart U with coordinates $(x_1, ...x_p;\ y_1, ...y_q)$, $p + q = n$, such that, for each L_α, a connected component of $U \cap L_\alpha$ is described by the equations

$$y_1 = \text{constant}, ..., y_q = \text{constant}.$$

Consequently, it is called a foliated coordinate chart.

Equivalently, one considers a p-dimensional distribution which is a choice of p-dimensional sub-bundle E of the tangent bundle TM. E is associated with a foliation if each point has a foliated coordinate chart U with coordinates $(x_1, ...x_p; y_1, ...y_q)$, $p+q = n$, such that $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_p})$ spans E and the Jacobian matrix of the change of foliated charts belongs to the group that stabilizes E given by

$$GL(p; q; R) = \{ \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \mid A \in GL(p; R); B \in GL(q; R) \}.$$

By the classical Frobenius Theorem, a p-dimensional distribution is integrable if and only if it involutive, that is the Lie bracket of any sections of the subbundle E is also a section of E.

A vector field X is said to be foliate with respect to foliation $\mathcal{F}$ if for every vector field Y tangential to leaves of $\mathcal{F}$, $\mathcal{L}_X Y$ is also tangential. That is, in a local foliated coordinate chart $(x_1, ...x_p; y_1, ...y_q)$ as above, a foliate vector field takes the form

$$X = \sum_{i=1}^{p} A^i(x_1, ...x_p; y_1, ..., y_q) \frac{\partial}{\partial x_i} + \sum_{j=1}^{q} B^j(y_1, ..., y_q) \frac{\partial}{\partial y_j}.$$

A Riemannian metric g induces an orthogonal decomposition $TM = E \oplus E^\perp$. A section of $E^\perp$ is said to be horizontal.

Definition 2.2. *A Riemannian metric is said to be bundle-like with respect to a foliation $\mathcal{F}$ if for any foliate horizontal vector fields X, Y and a tangential V,*

$$Vg(X,Y) = 0.$$

In that case, the foliation is said to be Riemannian.

Remark 2.2. *Locally a foliation looks like a submersion and a Riemannian foliation corresponds to a Riemannian submersion.*

The followings will be important to our investigation.

Proposition 2.3. [9, Prop. 2.6.7 and 2.6.9] *An 1-dimensional foliation induced by a Killing vector field is Riemannian. Also, a Riemannian foliation whose orbits are geodesics is isometric.*

For a Riemannian foliation, the toolkit originally developed by B. O'Neill's to study submersion [50] will play a crucial role. We let π and $\pi^\perp$ be the projections from TM onto E and $E^\perp$ accordingly. Let $\nabla^\perp$ denote the following induced connection, for Y a smooth section of $E^\perp$:

$$\nabla^\perp_X Y = \begin{cases} \pi^\perp(\nabla_X Y) \text{ if } X \text{ is a smooth section of } E^\perp \\ \pi^\perp[X,Y] \text{ if } X \text{ is a smooth section of } E. \end{cases}$$

It is verified that $\nabla^\perp$ is the unique Levi-Civita connection with respect to the transverse metric

$$g^\perp := g_{|_{E^\perp}}.$$

Furthermore, we make the simplifying assumption that each leaf of $\mathcal{F}$ is totally geodesic. Let U be a smooth section of E and X, Y be ones of $E^\perp$. Then tensor A is given as follows:

$$\begin{aligned} A_X U &= \pi^\perp(\nabla_X U), \\ A_X Y &= \pi(\nabla_X Y) = -A_Y X = \frac{1}{2}\pi([X,Y]). \end{aligned}$$

We collect useful identities:

$$\begin{aligned} g(A_X U, Y) &= -g(U, A_X Y), \\ \nabla_U X &= \pi^\perp(\nabla_U X), \\ \nabla_X U &= \pi(\nabla_X U) + A_X U, \\ \nabla_X Y &= A_X Y + \nabla^\perp_X Y. \end{aligned}$$

2.3. **Almost Contact and Contact Structures.** In this subsection, we give a brief introduction to almost contact geometry. The reference is [9].

Definition 2.3. *An odd dimensional manifold M is called almost contact if there exists a triple (ζ, η, Φ) where ζ is a vector field, η is a 1-form, Φ is a tensor field of type $(1,1)$, and they satisfy, everywhere on M,*

$$\eta(\zeta) = 1 \text{ and } \Phi^2 = -\mathrm{Id} + \zeta \otimes \eta.$$

It is immediate from the definition that ζ is no-where vanishing and

$$\Phi(\zeta) = 0, \ \eta \circ \Phi = 0.$$

Additionally, by Frobenius theorem, ζ generates an 1-dimensional foliation $\mathcal{F}$ since an 1-dim sub-bundle of the tangent bundle is always involutive.

In the presence of a Riemannian metric g, (M, ζ, η, Φ) is called an almost contact metric structure if g is compatible with Φ. That is,

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y).$$

Immediately, ζ has unit length as

$$0 = g(\zeta, \zeta) - \eta(\zeta)\eta(\zeta) = |\zeta|^2 - 1.$$

It is of great interest to determine when leaves of $\mathcal{F}$ are geodesics.

Lemma 2.4. *Let (M, ζ, η, Φ) be an almost contact metric structure. Then the followings are equivalent:*

(i) Integral curves of ζ are geodesics.
(ii) η is invariant along the flow of ζ.
(iii) $(\nabla_\zeta \Phi)\zeta = 0$.
(iv) $d\eta(\zeta, \cdot) = 0$.

Proof. As observed earlier, ζ has unit length.

Claim: $(i) \leftrightarrow (ii)$.

Proof. It is sufficient to observe, for any horizontal vector X,

$$g(\nabla_\zeta \zeta, X) = (\mathcal{L}_\zeta \eta)(X) - \frac{1}{2} X|\zeta|^2.$$

Therefore, integral curves of ζ are geodesics if and only if $(\mathcal{L}_\zeta \eta)(X) = 0$.

Claim: $(i) \leftrightarrow (iii)$.

Proof. We compute

$$(\nabla_\zeta \Phi)(\zeta) = \nabla_\zeta(\Phi(\zeta)) - \Phi(\nabla_\zeta \zeta) = -\Phi(\nabla_\zeta \zeta).$$

Since Φ is non-degenerate on the sub-bundle perpendicular to ζ, the conclusion follows.

Claim: $(i) \leftrightarrow (iv)$.

Proof. We compute

$$\begin{aligned}(d\eta)(\zeta, X) &= (\nabla_\zeta \eta)(X) - (\nabla_X \eta)(\zeta), \\ &= \zeta g(\zeta, X) - g(\zeta, \nabla_\zeta X) - X g(\zeta, \zeta) + g(\zeta, \nabla_X \zeta), \\ &= g(\nabla_\zeta \zeta, X) - \frac{1}{2} X|\zeta|^2.\end{aligned}$$

□

Furthermore, the tensor A can be computed immediately.

Lemma 2.5. *Let (M, ζ, η, Φ) be an almost contact metric structure such that g is bundle-like with respect to the foliation generated by ζ. Then, for horizontal vector fields X and Y,*

$$\begin{aligned} 2g(A_X Y, \zeta) &= -2d\eta(X, Y) = g([X, Y], \zeta), \\ g(A_X \zeta, Y) &= d\eta(X, Y). \end{aligned}$$

Proof. By our convention of the exterior derivative,

$$\begin{aligned} 2d\eta(X, Y) &= (\nabla_X \eta)(Y)) - (\nabla_Y \eta)X \\ &= \nabla_X(\eta(Y)) - \eta(\nabla_X Y) - \nabla_Y(\eta(X)) + \eta(\nabla_Y X) \\ &= \big(\nabla_X g(\zeta, Y) - g(\zeta, [X, Y]) - \nabla_Y g(\zeta, X)\big) \\ &= -g([X, Y], \zeta) = -2g(A_X Y, \zeta). \end{aligned}$$

The bundle-like assumption is used for the last equality. The first statement then follows. Similarly,

$$g(A_X \zeta, Y) = -g(\nabla_X Y, \zeta) = -g(A_X Y, \zeta) = d\eta(X, Y).$$

□

The Φ-sectional curvature of an almost contact manifold (M, ζ, η, Φ) is defined on the horizontal sub-bundle (perpendicular to ζ), for unit length X,

$$K_\Phi(X) = K(X, \Phi(X)).$$

Next, we discuss the notion of a contact structure.

Definition 2.4. *A $(2n + 1)$-dimensional manifold M is a contact manifold if there exists a 1-form η, called a contact 1-form, on M such that*

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. A contact structure is an equivalence class of such 1-forms.

On a contact manifold, there is a unique vector field ζ, called the Reeb vector field [57], such that

$$\eta(\zeta) = 1 \text{ and } d\eta(\zeta, \cdot) = 0.$$

Furthermore, η gives the contact bundle, the kernel of η, the setup of a symplectic vector bundle via $d\eta$. An almost complex structure J in such bundle is said to be compatible with the symplectic form $d\eta$ if, for all vector fields X and Y,

$$d\eta(X, Y) = d\eta(JX, JY).$$

J can then be extended trivially to Φ acting on the whole tangent space. It is straightforward to check that (ζ, η, Φ) defines an almost contact structure.

Definition 2.5. *An almost contact metric structure $(M, \zeta, \eta, \Phi, g)$ is called a contact metric structure if one further assumes*

$$g(X, \Phi(Y)) = d\eta(X, Y).$$

In that case, it is called a contact metric manifold.

The convention here is to be consistent with our definition of the Kähler form. It is immediate to check that a contact metric structure is indeed a contact manifold by the above definition. Additionally, a contact metric manifold $(M, \zeta, \eta, \Phi, g)$ is called K-contact if ζ is Killing; that is,

$$\mathcal{L}_\zeta g = 0.$$

We are interested in certain K-contact structures which give a concrete bridge from almost contact (contact) to almost complex (complex).

Definition 2.6. *A Sasakian structure is a contact metric structure $(M, \zeta, \eta, \Phi, g)$ such that the cone $C(M) = M \times \mathbb{R}^+$ with the cone metric $r^2 g + dr^2$ is Kähler. Moreover, the Kähler form is given by $d(r^2\eta)$.*

Remark 2.3. *Equivalently, a Sasakian structure is a K-contact structure whose almost CR-structure is integrable.*

Next we define a transformation which will play crucial roles later. This is essentially the combination of two transformations defined in [62].

Definition 2.7. *Let $(M, \zeta, \eta, \Phi, g)$ be an almost contact metric structure. For $H, F \in \mathbb{R}^+$, a $\pm(H, F)$-deformation is given by*

$$\zeta^* = \frac{\zeta}{H},\ \eta^* = H\eta,\ \Phi^* = \pm\Phi,\ g^* = F^2 g + (H^2 - F^2)\eta \otimes \eta.$$

Lemma 2.6. *Let $(M, \zeta, \eta, \Phi, g)$ be an almost contact metric structure and $(M, g^*, \zeta^*, \eta^*, \Phi^*)$ be its $\pm$ (H, F) deformation. Let $\mathcal{F}$ be the foliation generated by η. Then we have the followings:*

(i) $(M, g^*, \zeta^*, \eta^*, \Phi^*)$ *is an almost contact metric structure.*
(ii) g *is bundle-like with respect to* $\mathcal{F}$ *if and only if so is* g^*.
(iii) $\mathcal{L}_\zeta g = 0 \leftrightarrow \mathcal{L}_{\zeta^*} g^* = 0$.

Proof. The proof is via straightforward verification. For example, for any vector fields X and Y,

$$\begin{aligned}
g^*(\Phi^* X, \Phi^* X) &= (F^2 g + (H^2 - F^2)\eta \otimes \eta)(\Phi X, \Phi X) \\
&= F^2(g(X, Y) - \eta(X)\eta(Y)), \\
&= (F^2 g + (H^2 - F^2)\eta \otimes \eta)(X, Y) - H^2\eta(X)\eta(Y), \\
&= g^*(X, Y) - \eta^*(X)\eta^*(Y).
\end{aligned}$$

□

Remark 2.4. *A $\pm(H, F)$-deformation of a contact metric is not necessarily contact. If $d\eta(X, Y) = g(X, \Phi Y)$ then,*

$$d\eta^*(X, Y) = H d\eta(X, Y) = H g(X, \Phi Y) = \pm\frac{H}{F^2} g^*(X, \Phi^* Y).$$

Thus, a (H, F) transformation preserves the contact structure if and only if $H = F^2$. That motivates the following definition.

Definition 2.8. *A deformed contact metric structure is obtained via an* $\pm(1,F)$-*deformation of a contact metric structure.*

It is of interest to relate the curvature of a $\pm(H,F)$ deformation with one of the original metric. Towards that end, we recall, for $g = g^{\perp} + \eta \otimes \eta$,

$$g^* = H^2 \eta \otimes \eta + F^2 g^{\perp}.$$

In case of a K-contact structure, the calculation is relatively simple.

Proposition 2.7. *Let* $(M, \zeta, \eta, \Phi, g)$ *be a K-contact structure and* $(M, g^*, \zeta^*, \eta^*, \Phi^*)$ *be its* $\pm\ (H,F)$ *deformation. Then the curvatures are related by, for orthonormal horizontal vectors* X *and* Y

$$\begin{aligned} K^*(X,Y) &= \frac{1}{F^2} K^{\perp}(FX, FY) - 3\frac{H^2}{F^4} g^*(X, \Phi^* Y)^2, \\ K^*(X, \zeta^*) &= \frac{H^2}{F^4}, \\ \mathrm{Rc}^*(X,Y) &= \mathrm{Rc}^{\perp}(X,Y) - 2\frac{H^2}{F^4} g^*(X,Y) \\ \mathrm{Rc}^*(\zeta^*, \zeta^*) &= \frac{H^2}{F^4}(dim(M) - 1). \end{aligned}$$

Here $K^{\perp}$ *and* $\mathrm{Rc}^{\perp}$ *are the sectional and Ricci curvature of* $(g^{\perp}, \nabla^{\perp})$.

Proof. Since ζ is a Killing vector field, its generated foliation $\mathcal{F}$ is Riemannian by Prop. 2.3. By Lemma 2.6, $(M, g^*, \zeta^*, \eta^*, \Phi^*)$ is an almost contact metric structure, g^* is bundle-like with respect to $\mathcal{F}$, and $\mathcal{L}_{\zeta} g = 0 = \mathcal{L}_{\zeta^*} g^*$. Therefore, by Lemma 2.4, orbits of $\mathcal{F}$ are geodesics with respect to either g or g^*.

For a Riemannian totally geodesic foliation, the curvature can be computed via tensor A; see [4, Chapter 9] or [9, Theorem 2.5.16]. By Lemma 2.5,

$$g^*(A^*_X \zeta^*, Y) = d\eta^*(X,Y) = H d\eta(X,Y) = H g^{\perp}(X, \Phi Y) = \frac{-H}{F^2}(g^*)^{\perp}(\Phi X, Y).$$

Thus,

$$A^*_X \zeta^* = \frac{-H}{F^2} \Phi(X).$$

Then,

$$\begin{aligned} \mathrm{Rc}^*(X,Y) &= (\mathrm{Rc}^*)^{\perp}(X,Y) - 2g^*(A^*{}_X, A^*{}_Y), \\ &= (\mathrm{Rc}^*)^{\perp}(X,Y) - 2g^*(A^*{}_X \zeta^*, A^*{}_X \zeta^*), \\ &= \mathrm{Rc}^{\perp}(X,Y) - 2\frac{H^2}{F^4} g^*(X,Y) \end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{Rc}^*(\zeta^*,\zeta^*) &= g^*(A^*\zeta^*, A^*\zeta^*),\\ &= \sum_i g^*(A^*_{X_i}\zeta^*, A^*_{X_i}\zeta^*),\\ &= \frac{H^2}{F^4}(\dim(M)-1).\end{aligned}$$

□

2.4. **Rectifiable, Transnormal, and Isoparametric Functions.** In this subsection, for a complete connected Riemannian manifold (M, g), we consider a smooth function $f : M^n \mapsto \mathbb{R}$. First, we recall the definition of rectifiable which appears in [55, 18].

Definition 2.9. *f is rectifiable if $|\nabla f|$ is constant along every regular connected component of the level sets of f.*

The following observation gives a geometric interpretation.

Lemma 2.8. *The followings are equivalent:*

(i) f is rectifiable.
(ii) Integral curves of ∇f are geodesic after reparametrization.
(iii) The gradient of f is an eigenvector of its Hessian.

Proof. Integral curves of ∇f are geodesic if and only if, when $\nabla f \neq 0$,

$$\begin{aligned}0 &= \frac{1}{|\nabla f|}\nabla_{\nabla f}\frac{\nabla f}{|\nabla f|}\\ &= \frac{1}{|\nabla f|^3}(|\nabla f|\nabla_{\nabla f}\nabla f - (\nabla_{\nabla f}|\nabla f|)\nabla f).\end{aligned}$$

Thus, it is equivalent to that, for any unit vector field $E_i \perp \nabla f$,

$$0 = g(\nabla_{\nabla f}\nabla f, E_i) = \mathrm{Hess} f(\nabla f, E_i) = \frac{1}{2}\nabla_{E_i}|\nabla f|^2.$$

Equivalently, $|\nabla f|^2$ is constant on each connected component of a regular level set. □

Remark 2.5. *Consequently, it justifies the terminology that a rectifiable function is also called one with a geodesic gradient* [33, 34].

A priori, $|\nabla f|$ might vary between different connected components. Thus, it is useful to have a global condition.

Definition 2.10. *f is called transnormal if there is a continuous function $b : \mathbb{R} \mapsto \mathbb{R}$ such that*

$$|\nabla f|^2 = b(f).$$

Furthermore, a transnormal function is called isoparametric if there is a continuous function $a : \mathbb{R} \mapsto \mathbb{R}$ such that

$$\Delta f = a(f).$$

Remark 2.6. *Generally speaking, the former condition corresponds to equidistant level sets while the latter implies that each regular one has constant mean curvature.*

Remark 2.7. *The level sets of a transnormal f could be formulated as a singular Riemannian foliation. Thus, its study is closely related to the theory of a transnormal system and polar foliations* [46].

The followings is immediate.

Lemma 2.9. *f is rectifiable and each level set is connected then f is transnormal.*

Proof. It follows immediately from the definitions and Sard's theorem about regular values being dense in the range of f. ☐

Definition 2.11. *A geodesic segment $\gamma : [\alpha, \beta] \mapsto M$ is called an f-segment if $\gamma'(t) = \frac{\nabla f}{|\nabla f|}$ whenever $|\nabla f| \neq 0$.*

Remark 2.8. *At a point p such that $\nabla f(p) \neq 0$, it is possible to reparametrize an f-segment γ via translation such that $\gamma \circ f = \mathrm{Id}$ in a neighborhood of p. If $f(p)$ is a local minimum or maximum, one can only reparametrize to obtain such a property in an one-sided neighborhood.*

For $c \in f(M)$, denote $M_c = f^{-1}(c)$.

Lemma 2.10. *If f is transnormal and $[\alpha, \beta] \subset f(M)$ contains no critical value of f then for any $x \in M_\alpha$, $y \in M_\beta$, we have*

(i) $d(x, M_\beta) = d(M_\alpha, y) = \int_\alpha^\beta \frac{df}{\sqrt{|\nabla f|}}$;

(ii) the integral curves of ∇f after reparametrization are f-segments;

(iii) the f-segments are the shortest curves among all curves connecting M_α and M_β.

Proof. See [67, Lemma 1]. ☐

Obviously, there is a local version when f is rectifiable. Thus, a Riemannian manifold with a rectifiable function f is locally foliated by equidistant hypersurfaces. Following [35, Section 2], let P be a differentiable manifold corresponding to a regular connected component. There is a local diffeomorphism $\phi : I \times P \mapsto M$ such that the metric can be written as

$$\phi^* g = dt^2 + g_t.$$

Here t is a parametrization of $f(M)$ with unit tangent vector (see Remark 2.8) and g_t is an one-parameter family of metrics P such that, for $\phi_c : \{c\} \times P \mapsto M_c$ the restriction of ϕ to a slice,

$$\phi_c^* g_{|M_c} = g_c.$$

That is, g_t is equal to the pullback of the induced metric on the corresponding nearby connected component of level sets.

For a level set M_c, we denote the normal exponential map

$$\Pi_c : T^\perp M_c \mapsto M.$$

When c is regular, Π_c induces a diffeomorphism between nearby regular connected components of level sets.

$$\Pi_c^\epsilon = (\Pi_c)_{|V = \phi_*(\epsilon \partial_t)} : M_c \mapsto M_{c+\epsilon}.$$

Lemma 2.11. *Let I be a continuous open interval with regular values of f. Π_c^ϵ is just the identification by the diffeomorphism ϕ.*

Proof. For p a point in P and $c \in I$ such that $\phi(p,c) = q \in M_c$. Let $\gamma(t)$ be the curve given by, for $t \in (c-\epsilon, c+\epsilon)$

$$\gamma(t) = \phi(p,t).$$

It is readily verified that

$$\frac{d}{dt}\gamma = \phi_* \partial_t.$$

Thus γ is a geodesic segment and by the uniqueness of a geodesic given initial conditions,

$$\Pi_c^\epsilon(q) = \gamma(\epsilon) = \phi(p,\epsilon).$$

The result follows.

□

A soliton structure comes with a potential function and it is intriguing to determine exactly if or when it is transnormal and isoparametric. We have the following observations. It is our convention that $\vec{0}$ is parallel with any vector.

Proposition 2.12. *Let (M,g,f) be a GRS with a non-constant f. The followings are equivalent:*

(i) f is rectifiable,
(ii) whenever $\nabla f \neq 0$, ∇f is an eigenvector of Rc,
(iii) $\nabla f \parallel \nabla \mathrm{S}$ everywhere.

Proof. We break down the proof into several claims.

Claim: $(ii) \leftrightarrow (iii)$.
Proof: It is due to the equation (2.2).

Claim: $(i) \implies (iii)$.
Proof: If f is rectifiable, $|\nabla f|$ is constant on each connected component of a regular level set and, by equation (2.3), so is S. Thus, $\nabla f \parallel \nabla \mathrm{S}$ on each such component. By Sard's theorem, the set of regular values is dense and, by continuity, the result follows.

Claim: $(iii) \implies (i)$.
Proof: On each connected component M_c of regular level set of f, ∇S is perpendicular to TM_c and, consequently, S is constant on M_c. Because of equation (2.3), so is $|\nabla f|$. The result then follows.

□

Proposition 2.13. *Let (M,g,f) be a GRS with a non-constant f. The followings are equivalent:*

(i) f is transnormal,
(ii) f is isoparametric,
(iii) f is transnormal and b is C^∞.

Proof. The proof is based on the following claims.

Claim: $(i) \leftrightarrow (ii)$.

Proof: If f is transnormal, then $|\nabla f|$ is constant on each level set and, by equation (2.3), so is S. Consequently, Δf is also constant on each level set due to equation (2.1).

Claim: $(i) \leftrightarrow (iii)$. Let $\gamma : \mathbb{R} \mapsto M$ be an f-segment. Since γ is a geodesic, it is smooth and so is $\mathrm{S} \circ (f \circ \gamma)$. By the remark following Definition 2.11, one may assume $f \circ \gamma = \mathrm{Id}$ up to a critical value of f. Thus, S, as a function of f on $f(M)$, is smooth and so is $|\nabla f|^2$ by equation (2.3).

$\square$

Remark 2.9. *Here we only show that $b : f(M) \mapsto \mathrm{R}$ is smooth at a regular value. At a critical one, it is only smooth in the sense of one-sided limits.*

Remark 2.10. *When f is rectifiable, b can be defined locally and the smoothness also follows.*

Furthermore, the local foliation of equidistant hypersurfaces allows one to rewrite the soliton equation as follows [31]. For $N = \partial_t$, the shape operator is given as,

$$LX := \nabla_X N.$$

Denoting the ordinary derivative $\frac{d}{dt}$ by $'$, it follows that

$$\begin{aligned} g' &= 2g \circ L, \\ \nabla_N L &= L'. \end{aligned} \tag{2.5}$$

Due to Gauss, Codazzi, and Riccati equations, the Ricci curvature of (M, g) of is totally determined by that of (M_t, g_t) and the shape operator. Precisely, for tangential vectors X and Y,

$$\begin{aligned} \mathrm{Rc}(X,Y) &= \mathrm{Rc}_t(X,Y) - \mathrm{tr}(L_t) g_t(LX,Y) - g_t(L'(X),Y), \\ \mathrm{Rc}(X,N) &= -\nabla_X \mathrm{tr}(L_t) - g_t(\delta L, X), \\ \mathrm{Rc}(N,N) &= -\mathrm{tr}(L') - \mathrm{tr}(L^2). \end{aligned} \tag{2.6}$$

Here Rc_t denotes the Ricci curvature of (M_t, g_t) and $\mathrm{tr}T = \mathrm{tr}_{g_t} T_t$. Next, we recall $\mathrm{Hess}f(X,Y) = g(\nabla_X \nabla f, Y)$ and $\nabla f = \frac{df}{dt} N = f'$. Consequently,

$$\begin{aligned} \mathrm{Hess}f(X,N) &= 0, \\ \mathrm{Hess}f(N,N) &= f'', \\ \mathrm{Hess}f(X,Y) &= f' g_t(L_t(X),Y). \end{aligned} \tag{2.7}$$

The gradient Ricci soliton equation $\mathrm{Rc} + \mathrm{Hess}f = \lambda g$ is reduced to a system for each t,

$$\begin{aligned} 0 &= -(\delta L) - \nabla \mathrm{tr} L, \\ \lambda &= -\mathrm{tr}(\dot{L}) - \mathrm{tr}(L^2) + f'', \\ \lambda g(X,Y) &= \mathrm{Rc}(X,Y) - (\mathrm{tr}L) g(LX,Y) - g(L'(X),Y) + f' g(LX,Y). \end{aligned} \tag{2.8}$$

3. Deformed Sasakian

In this section, we give a characterization for an almost contact metric manifold to be a deformed Sasakian structure, proving Theorem 1.4. Consequently, it is possible to detect such a structure on real hypersurface of a Kähler manifold via examining its shape operator as in Theorem 1.5. Thus, they generalizes a classical result and might be of independent interest.

Let $(M, \zeta, \eta, \Phi, g)$ be an almost contact metric manifold and $\mathcal{F}$ be the foliation generated by ζ.

Lemma 3.1. *If $\nabla_X \Phi(Y) = bg(X,Y)\zeta - b\eta(Y)X$ for a real number b and any vector fields X and Y then g is bundle-like with respect to $\mathcal{F}$ and ζ is Killing.*

Proof. The assumption implies that

$$\begin{aligned} b(\eta(X)Y - \eta(Y)X) &= \nabla_X \Phi(Y) - \nabla_X \Phi(Y), \\ &= \nabla_X(\Phi Y) - \nabla_Y(\Phi X) - \Phi([X,Y]) \end{aligned}$$

Let $X = \zeta$ and Y be a foliate horizontal vector field then $[X, Y]$ is tangential and, thus,

$$\nabla_Y(\Phi X) = \Phi([X,Y]) = 0.$$

Consequently, $\nabla_\zeta(\Phi Y) = bY$. Applying the assumption again yields

$$\begin{aligned} 0 = bg(\zeta, \Phi Y)\zeta - \eta(\Phi(Y))b\zeta &= \nabla_\zeta \Phi(\Phi Y), \\ = \nabla_\zeta \Phi^2(Y) - \Phi(\nabla_\zeta(\Phi Y)) &= \nabla_\zeta(-Y) - b\Phi(Y). \end{aligned}$$

Thus, for foliate horizontal vector fields Y and Z,

$$\begin{aligned} \zeta g(Y,Z) &= g(\nabla_\zeta Y, Z) + g(Y, \nabla_\zeta Z), \\ &= -bg(\Phi Y, Z) - bg(Y, \Phi Z) = 0. \end{aligned}$$

The last equality is due to the compatibility of g and Φ. Therefore, g is bundle-like by Definition 2.2 and $\mathcal{F}$ is a Riemannian foliation. Furthermore, Lemma 2.4 is also applicable since

$$\nabla_\zeta \Phi(\zeta) = b\zeta - b\zeta = 0.$$

Thus $\mathcal{F}$ is a Riemannian foliation whose orbits are geodesics. By Lemma 2.3, ζ is a Killing vector field.

□

Lemma 3.2. *For a horizontal vector field X, the followings are equivalent*

(i) $(\nabla_\zeta \Phi)X = 0$
(ii) $[\zeta, \Phi(X)] - \Phi([\zeta, X]) = \Phi(A_X\zeta) - A_{\Phi X}\zeta$.

Proof. We compute, using the notation from Section 2,

$$\begin{aligned} (\nabla_\zeta \Phi)(X) &= \nabla_\zeta(\Phi X) - \Phi(\nabla_\zeta X) \\ &= [\zeta, \Phi X] - \Phi([\zeta, X]) - \Phi(\nabla_X \zeta) + \nabla_{\Phi X}\zeta, \\ &= [\zeta, \Phi X] - \Phi([\zeta, X]) - \Phi(A_X\zeta) + A_{\Phi X}\zeta. \end{aligned}$$

□

Moreover, since $\mathcal{F}$ is Riemannian, by Lemma 2.5, the tensor A can be computed as, for horizontal vector fields X and Y,

$$\begin{aligned} 2A_XY &= -2d\eta(X,Y)\zeta = g([X,Y],\zeta)\zeta \\ g(A_X\zeta, Y) &= d\eta(X,Y). \end{aligned}$$

Lemma 3.3. *For horizontal vector fields X, Y, the followings are equivalent*

(i) $(\nabla_X\Phi)Y = bg(X,Y)\zeta$

(ii) $A_X(\Phi Y) = bg^{\perp}(X,Y)\zeta$ *and* $\nabla^{\perp}_X(\Phi Y) = \Phi(\nabla^{\perp}_X Y)$

Proof. We compute,

$$\begin{aligned} (\nabla_X\Phi)(Y) &= (\nabla_X(\Phi Y) - \Phi(\nabla_X Y) \\ &= A_X(\Phi Y) + \pi^{\perp}(\nabla_X(\Phi Y)) - \Phi(A_XY + \pi^{\perp}(\nabla_X Y)) \\ &= A_X(\Phi Y) + \nabla^{\perp}_X(\Phi Y)) - \Phi(\nabla^{\perp}_X Y). \end{aligned}$$

□

Lemma 3.4. *For horizontal vector fields X, Y, the followings are equivalent*

(i) $(\nabla_X\Phi)\zeta = -bX$

(ii) $A_X\zeta = -b\Phi X$.

Proof. We compute

$$\begin{aligned} (\nabla_X\Phi)(\zeta) &= (\nabla_X(\Phi(\zeta)) - \Phi(\nabla_X\zeta) \\ &= -\Phi(A_X\zeta). \end{aligned}$$

□

Thus, it is possible to track how the covariant derivative of Φ changes under a deformation.

Theorem 3.5. *Let $(M, \zeta, \eta, \Phi, g)$ be an almost contact metric structure such that, for every vector fields X and Y,*

$$\nabla_X\Phi(Y) = bg(X,Y)\zeta - \eta(Y)bX.$$

Let $(M, \zeta', \eta', \Phi', g')$ be an $\pm(1,c)$-deformation then

$$\nabla'_X\Phi'(Y) = \pm\frac{b}{c^2}(g(X,Y)\zeta' - \eta(Y)bX).$$

Proof. Let $\mathcal{F}$ be the foliation generated by ζ. By Lemma 3.1, $\mathcal{F}$ is Riemannian and ζ is a Killing vector field. We write the metric as

$$g = g^{\perp} + \eta\otimes\eta.$$

By Lemmas 3.3 and 3.4, for horizontal vector fields X, Y,

$$\begin{aligned} A_X\Phi(Y) &= bg(X,Y)\zeta, \\ \nabla^{\perp}_X\Phi(Y) &= \Phi(\nabla^{\perp}_X Y), \\ A_X\zeta &= -b\Phi(X). \end{aligned} \tag{3.1}$$

We will assume $(M, \zeta', \eta', \Phi', g')$ be an $(1, c)$-deformation as the minus case can be done similarly. Thus, $\eta' = \eta$, $\zeta' = \zeta$, $\Phi' = \Phi$ and

$$g' = c^2 g + (1 - c^2)\eta \otimes \eta = c^2 g^{\perp} + \eta \otimes \eta.$$

By Lemma 2.6, $(M, g', \eta', \zeta', \Phi')$ is an almost contact metric structure; $\zeta = \zeta'$ is a Killing unit vector field with respect to g'; and g' is bundle-like with respect to $\mathcal{F}$. By Lemma 2.5 and equation (3.1), we have

$$\begin{aligned} A'_X(\Phi' Y) &= -d\eta'(X, \Phi'(Y))\zeta' = d\eta(X, \Phi(Y))\zeta = A_X \Phi(Y) \\ &= b g^{\perp}(X, Y)\zeta = \frac{b}{c^2} g'(X, Y)\zeta'. \end{aligned}$$

Similarly,

$$\begin{aligned} g'(A'_X \zeta', Y) &= d\eta'(X, Y) = d\eta(X, Y) = g(A_X \zeta, Y) \\ &= -g(b\Phi(X), Y) = -\frac{b}{c^2} g'(\Phi(X), Y). \end{aligned}$$

Thus, $A'_X \zeta = -\frac{b}{c^2}\Phi(X)$. As the Levi-Civita connection is scaling invariant, $\nabla^{c^2 g^{\perp}} = \nabla^{g^{\perp}} = \nabla^{\perp}$. By Lemma 3.2, $\nabla_{\zeta}\Phi(X) = 0$ iff

$$[\zeta, \Phi(X)] - \Phi([\zeta, X]) = \Phi(A_X \zeta) - A_{\Phi X}\zeta = \Phi(-b\Phi(X)) + b\Phi(\Phi(X)) = 0.$$

As the left-hand side is independent of the metric, $\nabla'_{\zeta'}\Phi'(X) = 0$. Together with Lemmas 3.3 and 3.4, we obtain, for any vector fields V and W,

$$\nabla'_V \Phi'(W) = \frac{b}{c^2}\Big(g'(V, W)\zeta' - \eta'(W)V\Big).$$

□

The following is classical.

Theorem 3.6. [9] *An almost contact metric manifold $(M, \zeta, \eta, \Phi, g)$ is Sasakian if and only if, for every vector fields X and Y,*

$$\nabla_X \Phi(Y) = g(X, Y)\zeta - \eta(Y)X. \tag{3.2}$$

Corollary 3.7. *Let $(M, \zeta, \eta, \Phi, g)$ be a Sasakian manifold and $(M, \zeta', \eta', \Phi', g')$ be an $\pm(H, F)$-deformation. Then*

$$\nabla'_X \Phi'(Y) = \frac{\pm H}{F^2}(g(X, Y)\zeta' - \eta(Y)bX).$$

Proof. By Remark 2.4, a $(H, \sqrt{H})$-deformation is Sasakian. Thus, applying Theorem 3.5 for $b = 1, c = \frac{F}{\sqrt{|H|}}$ in combination with Theorem 3.6 yields the result. □

We are ready to give the proof of Theorem 1.4.

Proof of Theorem 1.4. For part (i), without loss of generality, we assume $b > 0$ (if $b < 0$ then we consider $(M, \zeta, \eta, -\Phi)$. One direction follows from Corollary 3.7. The other is deduced by applying Theorem 3.5 for $c = \sqrt{b}$.

For part (ii), $b = 0$ means Φ is parallel since, for every vector fields X and Y

$$\nabla_X \Phi(Y) = 0.$$

By Lemmas 2.4 and 3.1, ζ is a Killing and geodesic vector field. By Lemma 3.4, for any horizontal vector field X,

$$\nabla_X \zeta = A_X \zeta = 0.$$

Thus, ζ is a global unit-length parallel vector field and, accordingly, the manifold splits into a Riemannian product. The factor perpendicular to ζ has Φ as an almost complex structure. Its compatibility with g makes the induced metric almost Herminian. Since Φ is parallel, the metric is Kähler by Prop. 2.1. □

Proof of Theorem 1.5. We have

$$\nabla_X Y = (\nabla^M_X Y) - g(LX, Y)V.$$

Since J is parallel with respect to ∇, $\nabla_X(JY) = J(\nabla_X Y)$. Thus,

$$\nabla^M_X(\Phi Y) - g(LX, \Phi Y)V + X\eta(Y)V + \eta(Y)LX = \Phi(\nabla^M_X Y) + \eta(\nabla_X Y)V + g(LX, Y)\zeta.$$

Consequently, equating the normal and tangent components give

$$\begin{aligned}(\nabla^M_X \Phi)Y &= -\eta(Y)LX + g(LX, Y)\zeta,\\ -g(LX, \Phi Y) + X\eta(Y) &= \eta(\nabla_X Y)\end{aligned}$$

For the first equation, substituting $L = \alpha\mathrm{Id} + \beta\zeta \otimes \eta$ yields

$$\begin{aligned}(\nabla^M_X \Phi)Y &= -\eta(Y)\Big(\alpha X + \beta\zeta\eta(X)\Big) + g\Big(\alpha X + \beta\zeta\eta(X), Y\Big)\zeta\\ &= -\alpha\eta(Y)X + \alpha g(X, Y)\zeta.\end{aligned}$$

Similarly, for the second equation, we have

$$\begin{aligned}\eta(\nabla_X Y) &= -g(\alpha X + \beta\zeta\eta(X), \Phi Y) + X\eta(Y),\\ \leftrightarrow g(\nabla_X \zeta, Y) &= \alpha g(X, \Phi(Y)) = \alpha g(\Phi(X), -Y),\\ \leftrightarrow \nabla_X \zeta &= -\alpha\Phi(X).\end{aligned}$$

By Lemmas 3.1 and 3.4, the second equation is implied by the first. The result then follows from Theorem 1.4. □

4. Kähler Solitons and Contact Structures

In this section, we'll study a Kähler GRS (M, g, J, f) under the assumption that each regular connected component level set of f is a deformed contact structure. Thus, a proof of Theorem 1.3 will be provided. For $c \in f(M)$, we recall the construction (1.2), for a regular value c,

$$V := \frac{\nabla f}{|\nabla f|},\ g_c := g_{|M_c},\ \zeta_c := -J(V),\ \eta_c(\cdot) := g(\cdot, \zeta_c),\ \Phi_c(\cdot) := -\eta_c(\cdot)V + J(\cdot).$$

Together, $(M_c, \zeta_c, \eta_c, \Phi_c, g_c)$ is an almost contact metric structure. Additionally, the shape operator is given by

$$g(LX, Y) = g(\nabla_X V, Y) = g(\nabla_X \frac{\nabla f}{|\nabla f|}, Y) = \frac{|\nabla f|\mathrm{Hess} f(X, Y) - X|\nabla f|g(\nabla f, Y)}{|\nabla f|^2}.$$

Thus,

$$\mathrm{Hess}f(X,Y) = |\nabla f| g(LX,Y) + \frac{X|\nabla f|}{|\nabla f|} g(\nabla f, Y).$$

We first observe the following.

Lemma 4.1. *If $(M_c, \zeta_c, \eta_c, \Phi_c, g_c)$ is a deformed connected contact metric structure, then it is a deformed K-contact structure and $|\nabla f|$ is constant along M_c.*

Proof. For convenience, we drop the c dependence. By Lemma 2.2, $W := J(\nabla f) = -|\nabla f|\zeta$ is a Killing vector field. It suffices to show $|\nabla f|$ is constant along M_c. First, we obverse that, since $\mathrm{Hess}f$ is J-invariant:

$$\nabla_W |\nabla f|^2 = 2g(\nabla_W \nabla f, \nabla f) = 2\mathrm{Hess}f(W, \nabla f) = 0.$$

Next, the deformed contact metric structure implies that

$$d\eta(\zeta, \cdot) = 0.$$

By Lemma 2.4, the integral curves of ζ are geodesics. Therefore, and any X tangential to M_c,

$$0 = g(\nabla_{\frac{W}{|W|}} \frac{W}{|W|}, X) = \frac{|W| g(\nabla_W W, X) - W|\nabla f| g(W,X)}{|\nabla f|^3}.$$

Since $W|\nabla f|^2 = 0$ we have, $g(\nabla_W W, X) = 0$. Since W is a Killing vector field,

$$0 = g(\nabla_X W, W) = \frac{1}{2} X|W|^2.$$

As $|W| = |\nabla f|$ the result follows. ∎

Lemma 4.2. *Let $(P, \zeta, \eta, \Phi, \eta\otimes\eta + g^{\perp})$ be a contact metric structure. Suppose the metric and the almost complex structure, on $I \times P$,*

$$g = dt^2 + H^2(t)\eta\otimes\eta + F^2(t) g^{\perp},$$
$$J = \partial_t \otimes H\eta - \frac{1}{H}\zeta\otimes dt + \Phi,$$

are almost Kähler. Then,

$$FF' = H.$$

Proof. The contact structure implies $d\eta = g^{\perp}(\cdot, \Phi(\cdot)) := \omega^{\perp}(\cdot,\cdot)$. Recall $\omega_g(X,Y) = g(X, JY)$, the Kähler form becomes,

$$\omega_g = 2dt \wedge H\eta + F^2\omega^{\perp}.$$

Thus, $d\omega_g = -2H dt\wedge\omega^{\perp} + 2FF' dt\wedge\omega^{\perp}$ and one deduces

$$FF' = H.$$

∎

We are ready to give the main proof of this section.

Proof of Theorem 1.3. By Lemma 4.1, f is rectifiable and each level set is endowed with a deformed K-contact structure. Let P be a differentiable manifold corresponding to a regular connected component of a level set of f. As in Subsection 2.4, there is a local diffeomorphism $\phi : I \times P \mapsto M$ such that

$$\phi^*(g) = dt^2 + g_t.$$

for (P, g_t) an one-parameter family of Riemannian metrics which are equal to the pull-back of induced metric on nearby connected components.

Since each is a deformed K-contact structure, we can write

$$g_t = c_t g_t^{\perp} + \eta_t \otimes \eta_t,$$

for η_t be a family of one-form dual, via g_t, with ζ_t and $g_t^{\perp}$ be a family of transverse metrics on the sub-bundle g_t-perpendicular to ζ_t. We observe, since f is invariant by the Killing vector field $J(\nabla f)$

$$0 = \mathcal{L}_{J\nabla f} \nabla f = -\mathcal{L}_{\nabla f} J(\nabla f).$$

Therefore, $J(\nabla f) = -|\nabla f|_t \zeta_t$ is invariant by the flow generated by ∇f. Thus, by Lemma 2.11, $f'\zeta_t$ is identified with a fixed vector on P. Consequently, by continuity, we can fix a background one-form η and its corresponding transverse metric $g^{\perp}$ such that $(P, \eta, \zeta, \Phi, \eta \otimes \eta + g^{\perp})$ a K-contact structure and, for some constant B,

$$\begin{aligned} g_t &= H^2(t)\eta \otimes \eta + F^2(t) g^{\perp}, \\ H &= f'B \end{aligned}$$

Due to $g_t' = 2g_t \circ L$, we have, for Id denoting the identity operator on the horizontal subspace of TP, which is $(g^{\perp} + \eta \otimes \eta)$- perpendicular to ζ,

$$\begin{aligned} L_t &= \frac{H'}{H}\zeta \otimes \eta + \frac{F'}{F}\mathrm{Id}. \\ L'_t &= \Big(\frac{H''}{H} - (\frac{H'}{H})^2\Big)\zeta \otimes \eta + \Big(\frac{F''}{F} - (\frac{F'}{F})^2\Big)\mathrm{Id}. \end{aligned}$$

By Theorem 1.4, at regular values, $F' \neq 0$ and $(P, \eta_t, \zeta_t, \Phi_t, g_t)$ is a deformed Sasakian structure. Furthermore, for $m = \dim_{\mathbb{C}} M - 1$,

$$\begin{aligned} \mathrm{tr} L_t &= \frac{H'}{H} + (2m)\frac{F'}{F}, \\ \mathrm{tr} L_t^2 &= (\frac{H'}{H})^2 + (2m)\frac{(F')^2}{F^2}, \\ \mathrm{tr} L_t' &= \frac{H''}{H} + (2m)\frac{F''}{F} - \frac{(H')^2}{H^2} - (2m)\frac{(F')^2}{F^2}. \end{aligned}$$

Via equation (2.8) and Prop. 2.7,

$$\mathrm{Rc}^{\perp} - 2\frac{H^2}{F^4} g_t = \lambda g_t + \mathrm{tr}(L) g_t \circ L - g_t \circ L' + f' g_t \circ L_t.$$

Since $g_t = F^2 g^\perp + H^2 \eta\otimes\eta$, one deduces that $\mathrm{Rc}^\perp = kg^\perp$. Consequently, the soliton equation 1.1 becomes

$$\begin{aligned}
\lambda &= -\frac{H''}{H} - (2m)\frac{F''}{F} + f'' \\
&= \frac{H^2}{F^4}(2m) - \frac{H''}{H} - 2m\frac{H'F'}{HF} + f'\frac{H'}{H} \\
&= \frac{k}{F^2} - \frac{H^2}{F^4}2 - \frac{F''}{F} - (2m-1)(\frac{F'}{F})^2 - \frac{H'F'}{FH} + f'\frac{F'}{F}. \qquad (4.1)
\end{aligned}$$

By Lemma 4.2, the ODE system is augmented with additional constraints $FF' = H = f'B$. It was investigated and solved explicitly via a transformation to a modified Calabi's ansatz in [31] (see also [66]). In particular, f is monotonic and one concludes that f is transnormal and each level set is connected.

Furthermore, there must be a finite t where either $H\to 0$ or both $H, F\to 0$. By Lemmas 4.3 and 4.4 below, there is a Riemannian submersion from $(P, \eta, \zeta, \Phi, g^\perp + \eta\otimes\eta)$ to a Kähler-Einstein manifold. Thus, the result follows.

□

For the following results, one assumes the setup as in the proof of Theorem 1.3.

Lemma 4.3. *Let $I = (0, \epsilon)$ for $\epsilon > 0$ and suppose that*

$$\lim_{t\to 0^+} H(t) = \lim_{t\to 0^+} F(t) = 0,$$

and the metric can be extended smoothly to $t = 0$. Then, $(P, \zeta, \eta, \Phi, g^\perp + \eta\otimes\eta)$ is a Sasakian sphere with Hopf fibration over a complex projective space.

The proof models after one for [54, Theorem 4.3.3]. The idea is that, as $H, F\to 0$, the metric must become rounder and rounder. The rigidity of the deformed structure implies that it is actually round.

Proof. Let p denotes the point compactification at $t = 0$. Since it is locally Euclidean around p, each level set of f corresponds to a distance sphere. This also follows from the Morse lemma as equation (2.7) implies

$$\mathrm{Hess} f(p) = \lim_{t\to 0} f''(t) g.$$

Thus, P is diffeomorphic to a sphere. Consequently, there is a diffeomorphism $\hat{\phi}$ from an open ball in $\mathbb{R}^{2n}$ to M such that each Euclidean round sphere is mapped to a level set of f.

Let's fixed a sphere with the standard round metric $(\mathbb{S}^{2n-1}, g_{round})$. In $\mathbb{R}^{2n}$, the intersection of a 2-dimensional plane through the origin with $(\mathbb{S}^{2n-1}, g_{round})$ is a great circle $\mathbb{S}^1$ with coordinate θ. That is,

$$g_{round}(\partial_\theta, \partial_\theta) = 1.$$

Next, we consider the image of that plane via $\hat{\phi}$. Since $\hat{\phi}$ is a diffeomorphism, the image is a submanifold of dimension two with coordinates t and θ. Using polar coordinates

$$\begin{aligned} x &= t\cos\theta, \\ y &= t\sin\theta, \\ \partial_x &= \cos\theta\partial_t - \frac{1}{t}\sin\theta\partial_\theta, \\ \partial_y &= \sin\theta\partial_t + \frac{1}{t}\cos\theta\partial_\theta \end{aligned}$$

Thus, for $g = dt^2 + H^2(t)\eta\otimes\eta + F^2(t)g^\perp$

$$\begin{aligned} g_{xx} &:= g(\partial_x,\partial_x) = \cos^2\theta + \frac{1}{t^2}\sin^2\theta(H^2\eta^2(\partial_\theta) + F^2g^\perp(\partial_\theta,\partial_\theta)), \\ g_{yy} &:= g(\partial_y,\partial_y) = \sin^2\theta + \frac{1}{t^2}\cos^2\theta(H^2\eta^2(\partial_\theta) + F^2g^\perp(\partial_\theta,\partial_\theta)). \end{aligned}$$

Letting $t\to 0$ yields

$$\begin{aligned} g_{xx}(p) &= \cos^2\theta + \sin^2\theta((H'(0))^2\eta^2(\partial_\theta) + (F'(0))^2g^\perp(\partial_\theta,\partial_\theta)), \\ g_{yy}(p) &= \sin^2\theta + \cos^2\theta((H'(0))^2\eta^2(\partial_\theta) + (F'(0))^2g^\perp(\partial_\theta,\partial_\theta)). \end{aligned}$$

Adding them together we deduce that, since $g(p)$ is independent of θ,

$$1 = H'(0))^2\eta^2(\partial_\theta) + (F'(0))^2g^\perp(\partial_\theta,\partial_\theta) = g_{round}(\partial_\theta,\partial_\theta).$$

Since ∂_θ is arbitrary,

$$H'(0)^2\eta\otimes\eta + F'(0)^2g^\perp = g_{round}.$$

By Prop. 2.7, $g^\perp + \eta\otimes\eta$ has constant Φ-holomorphic sectional curvature. By the classification of Tanno [63], it must be the standard round sphere with the contact structure given by the Hopf fibration over a complex projective space.

□

Lemma 4.4. *Suppose that*

$$\lim_{t\to 0^+} H(t) = 0 \text{ and } \lim_{t\to 0^+} F(t) \neq 0,$$

and the metric can be extended smoothly to $t = 0$. Then, $(P,\eta,\zeta,\Phi,g^\perp + \eta\otimes\eta)$ is a Riemannian submersion over a Kähler-Einstein manifold.

Proof. By the ODE system (4.1), the set of points corresponding to $t = 0$ and $H\to 0$ corresponds to a level set, called N, of a critical value of f. Since f is monotonic, N is focal variety and also the zero set of the Killing vector field $J(\nabla f)$. Due to Kobayashi's [41], one deduces that N is a totally geodesic submanifold of co-dimension two. Let g_N be the induced metric of g on N.

Let's recall the normal exponential map and its cousin

$$\Pi_c : T^\perp M_c \mapsto M,\ \Pi_c^\epsilon = (\Pi_c)_{|V=\phi_*(\epsilon\partial_t)} : M_c \mapsto M_{c+\epsilon}.$$

Then, $\Pi_\epsilon^{-\epsilon} : M_\epsilon \mapsto N$ is a smooth focal map. Let X be a vector field on P and we use the same name for its identification on M_ϵ via ϕ. Let $\tilde{X}$ be the push-forward of X via

$\Pi_{\epsilon}^{-\epsilon}$. By continuity and Lemma 2.11, it is just the limit, as $t \to 0$, of t-identifications of the same vector field X via ϕ. Thus,

$$\begin{aligned} g_N(\tilde{X}, \tilde{X}) &= \lim_{t\to 0} g_t(X, X), \\ &= F(0)^2 g^{\perp}(X, X). \end{aligned}$$

Thus, $\tilde{X} = \vec{0}$ if and only if X is a multiple of ζ. Thus, at each point $q \in M_\epsilon$, the differential of $\Pi_{\epsilon}^{-\epsilon}$ is onto with an one-dimensional kernel. Thus, $\Pi_{\epsilon}^{-\epsilon}$ is a submersion and the equation above shows that it is a Riemannian submersion $(N, F(0)^2 g_N)$. It remains to show $(N, F(0)^2 g_N)$ is Kähler-Einstein with a compatible almost complex structure.

Indeed, Φ induces an almost complex structure on N by

$$J_N \tilde{X} := \widetilde{\Phi X}.$$

Claim: $J_N(\tilde{X}) = J(\tilde{X})$.

Proof. Without loss of generality, one may assume that $X \perp \zeta$ and, by the construction of Φ,

$$\widetilde{\Phi X} = \widetilde{JX}.$$

On the other hand, $\tilde{X} = \lim_{t\to 0}(\phi_t)_* X$ and, by continuity,

$$\begin{aligned} J(\tilde{X}) &= J(\lim_{t\to 0}(\phi_t)_* X) = \lim_{t\to 0} J((\phi_t)_* X) \\ &= \lim_{t\to 0}(\phi_t)_*(\phi_t^* J X) = \lim_{t\to 0}(\phi_c)_*(JX) \\ &= \widetilde{JX}. \end{aligned}$$

Claim: (N, g_N, J_N) is a Kähler-Einstein manifold.

Proof. By continuity,

$$\mathrm{Rc}_N(\tilde{X}, \tilde{X}) = F^2(0)\mathrm{Rc}^{\perp}(X, X) = kF^2 g^{\perp}(X, X) = k g_N(\tilde{X}, \tilde{X}).$$

Thus, it is Einstein. Then J_N is parallel since $J_N = J_{|TN}$, J is parallel, and (N, g_N) is totally geodesic.

□

5. Dimension four

We restrict our investigation to real dimension four and great simplification occurs.

Lemma 5.1. *Let (M, g, J, f) be a Kähler GRS in real dimension four. Suppose that f is rectifiable then along each regular connected component,* Hess$f$ *has two constant eigenvalues, each of multiplicity two.*

Proof. Let M_c be a connected component of a regular level set of f. Since $|\nabla f|$ is constant on M_c, for any vector field X tangent to M_c,

$$\text{Hess f}(\nabla f, X) = 0.$$

Following Remark 2.10, $b(f) = |\nabla f|^2$ is locally defined for nearby connected components and it is smooth. Thus,

$$\begin{aligned}\text{Hess f}(\nabla f, \nabla f) &= g(\nabla_{\nabla f}(\nabla f), \nabla f)\\ &= \frac{1}{2}\nabla_{\nabla f}|\nabla f|^2 = \frac{1}{2}\nabla_{\nabla f}|\nabla f|^2\\ &= \frac{1}{2}b'(c)|\nabla f|^2\end{aligned}$$

Therefore, ∇f is an eigenvector with eigenvalue $\frac{1}{2}b'(c)$. As each tangent space is of dimension 4 and Hess f is J-invariant, Hess f has exactly two eigenvalues each of multiplicity two. By equations (2.1) and (2.3),

$$\Delta f = \text{tr}(\text{Hess} f) = n\lambda - |\nabla f|^2 + 2\lambda f - \text{a constant}.$$

Thus, it is also fixed along M_c and the result then follows.

□

We are now ready to classify all Kähler GRS with a rectifiable potential function f.

Proof of Theorem 1.1. Let M_c be a connected component of a regular level set of f. Then, $(M_c, \zeta_c, \eta_c, \Phi_c, g_c)$ via (1.2) is an almost contact metric structure. Along a regular connected component, $|\nabla f|$ is a nonzero constant and we have, for L denoting the shape operator,

$$\begin{aligned}\text{Hess} f(X, Y) &= |\nabla f| g(LX, Y) + \frac{X|\nabla f|}{|\nabla f|} g(\nabla f, Y),\\ &= |\nabla f| g(LX, Y).\end{aligned}$$

By Lemma 5.1, Hess$f$ has constant eigenvalues μ_1, μ_2 with ζ as one of the eigenvector. Therefore,

$$L = \frac{1}{|\nabla f|}((\mu_1 - \mu_2)\zeta \otimes \eta + \mu_2 \text{Id}).$$

Consequently, L satisfies the condition of Theorem 1.5 and $(M_c, \zeta_c, \eta_c, \Phi_c, g_c)$ must be either a deformed Sasakian structure or a product of line or a circle with a Kähler manifold. By continuity, nearby regular connected components must all be of the same type. Thus, we consider two cases.

Case 1: Locally, each regular connected component is a deformed Sasakian structure. By Theorem 1.3, each is a Riemannian submersion over a Kähler-Einstein manifold (N, g_N, J_N). Since N is of real dimension two, it must have constant curvature.

Case 2: Locally, each regular connected component is a product of a line or circle with a Kähler manifold (N, g_N). Thus, locally the metric can be written as

$$g = dt^2 + H^2(t)\eta \otimes \eta + g_N(t),$$

for $(N, g_N(t), J_N(t))$ an one-parameter family of Kähler metrics and $d\eta = 0$. Furthermore, $J_N = \Phi_{|TN} = J_{|TN}$. Since $\omega_g(X, Y) = g(X, JY)$, the Kähler form becomes

$$\omega_g = 2dt \wedge H\eta + \omega_N(t).$$

Thus, $d\omega_g = dt \wedge \frac{d\omega_N(t)}{dt}$ and one deduces that ω_N is independent of t. Comparing covariant and ordinary derivatives yield, for t-independent vector fields X, Y on TN,

$$\begin{aligned} 0 &= \frac{d}{dt}(\omega_N(X,Y)) = \frac{d}{dt}(g_N(X, J_N Y)) = \nabla_{\partial t}(g_N(X, J_N Y)) \\ &= (\nabla_{\partial t} g_N(t))(X, J_N Y) + g_N(X, \nabla_{\partial_t}(JY)) \\ &= (\frac{d}{dt} g_N(t))(X, J_N Y). \end{aligned}$$

As a consequence, g_N is independent of t and the soliton splits into a Riemannian product

$$(M_1, dt^2 + H^2(t)\eta \otimes \eta) \times (N, g_N, J_N).$$

By the soliton system (2.8), $(N, F^2 g_N)$ has constant Ricci curvature and, thus, constant curvature due to its low dimension.

$\square$

Proof of Corollary 1.2. By Theorem 1.1, as the metric is irreducible, (M, g, J, f) must be of cohomogeneity one, f is invariant by the action, and each principal orbit is a connected deformed Sasakian structure on a manifold P. By Theorem 1.3, it is a Riemannian submersion, with circle fibers, over a Kähler-Einstein manifold (N, g_N, J_N).

Claim: If M is non-compact, then N is simply connected.

Proof of the claim: Since M is non-compact M is constructed by collapsing at one end of $P \times [0, \infty]$. If the singular orbit is a point, the result follows immediately from Theorem 1.3. If not, M could be written as a union of an one-sided tubular neighborhood around the singular orbit, diffeomorphic to N, and $P \times (0, \infty)$ and the intersection is diffeomorphic to $P \times (0, \epsilon)$. It is immediate to construct a deformation retract from the tubular neighborhood to N and from each of $P \times (0, \epsilon)$ and $P \times (0, infty)$ to P. Consequently, Seifert-Van Kampen theorem deduces that, for π_1 denoting the fundamental group,

$$\pi_1(M) = \pi_1(N) *_{\pi_1(P)} \pi_1(P).$$

Since M is simply connected, so is N.

Thus, N is simply connected and (N, g_N, J_N) is of constant curvature. Thus, its isometry group is of dimension $2^2 - 1 = 3$. The isometry group of (M, g, J, f) is then constructed by that of N and the circle action generated by $J(\nabla f)$; see [66, Theorem 1.5].

$\square$

6. Statements and Declarations

- On behalf of all authors, the corresponding author states that there is no conflict of interest.
- The manuscript has no associated data

References

[1] U. Abresch. Isoparametric hypersurfaces with four or six distinct principal curvatures. Necessary conditions on the multiplicities. *Math. Ann.*, 264(3):283–302, 1983.

[2] Richard H Bamler, Charles Cifarelli, Ronan J Conlon, and Alix Deruelle. A new complete two-dimensional shrinking gradient Kähler-Ricci soliton. *arXiv preprint arXiv:2206.10785*, 2022.

[3] Florin Alexandru Belgun. Normal CR structures on compact 3-manifolds. *Math. Z.*, 238(3):441–460, 2001.

[4] Arthur L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

[5] D. E. Blair. Geometry of manifolds with structural group $\mathcal{U}(n) \times \mathcal{O}(s)$. *J. Differential Geometry*, 4:155–167, 1970.

[6] David E. Blair. *Contact manifolds in Riemannian geometry*, volume Vol. 509. Springer-Verlag, Berlin-New York, 1976.

[7] David E. Blair. A survey of Riemannian contact geometry. *Complex Manifolds*, 6(1):31–64, 2019.

[8] Christoph Böhm and Burkhard Wilking. Manifolds with positive curvature operators are space forms. *Ann. of Math. (2)*, 167(3):1079–1097, 2008.

[9] Charles P. Boyer and Krzysztof Galicki. *Sasakian geometry*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.

[10] Simon Brendle and Richard Schoen. Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.*, 22(1):287–307, 2009.

[11] Simon Brendle and Richard M. Schoen. Classification of manifolds with weakly 1/4-pinched curvatures. *Acta Math.*, 200(1):1–13, 2008.

[12] Jacob Cable and Hendrik Süß. On the classification of kähler–ricci solitons on gorenstein del pezzo surfaces. *European Journal of Mathematics*, 4(1):137–161, 2018.

[13] Huai-Dong Cao. Existence of gradient Kähler-Ricci solitons. In *Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994)*, pages 1–16. A K Peters, Wellesley, MA, 1996.

[14] Huai-Dong Cao. Limits of solutions to the Kähler-Ricci flow. *J. Differential Geom.*, 45(2):257–272, 1997.

[15] Huai-Dong Cao. Recent progress on ricci solitons. *Adv. Lect. Math.*, 11:1–38, 2009.

[16] Xiaodong Cao, Matthew J. Gursky, and Hung Tran. Curvature of the second kind and a conjecture of Nishikawa. *Comment. Math. Helv.*, 98(1):195–216, 2023.

[17] Élie Cartan. Familles de surfaces isoparamétriques dans les espaces à courbure constante. *Ann. Mat. Pura Appl.*, 17(1):177–191, 1938.

[18] Giovanni Catino and Lorenzo Mazzieri. Gradient Einstein solitons. *Nonlinear Anal.*, 132:66–94, 2016.

[19] Thomas E. Cecil, Quo-Shin Chi, and Gary R. Jensen. Isoparametric hypersurfaces with four principal curvatures. *Ann. of Math. (2)*, 166(1):1–76, 2007.

[20] Thierry Chave and Galliano Valent. On a class of compact and non-compact quasi-Einstein metrics and their renormalizability properties. *Nuclear Phys. B*, 478(3):758–778, 1996.

[21] Bing-Long Chen. Strong uniqueness of the Ricci flow. *J. Differential Geom.*, 82(2):363–382, 2009.

[22] Qiang Chen and Meng Zhu. On rigidity of gradient Kähler-Ricci solitons with harmonic Bochner tensor. *Proc. Amer. Math. Soc.*, 140(11):4017–4025, 2012.

[23] Xu Cheng and Detang Zhou. Rigidity of four-dimensional gradient shrinking ricci solitons. *arXiv preprint arXiv:2105.10744*, 2021.

[24] Quo-Shin Chi. Isoparametric hypersurfaces with four principal curvatures, IV. *J. Differential Geom.*, 115(2):225–301, 2020.

[25] Quo-Shin Chi. The isoparametric story, a heritage of Élie Cartan. In *Proceedings of the International Consortium of Chinese Mathematicians 2018*, pages 197–260. Int. Press, Boston, MA, [2020] ©2020.

[26] Otis Chodosh and Frederick Tsz-Ho Fong. Rotational symmetry of conical Kähler-Ricci solitons. *Math. Ann.*, 364(3-4):777–792, 2016.

[27] Bennett Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci flow*, volume 77 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2006.

[28] Charles Cifarelli, Ronan J Conlon, and Alix Deruelle. On finite time type i singularities of the Kähler-Ricci flow on compact Kähler surfaces. *arXiv preprint arXiv:2203.04380*, 2022.

[29] Ronan J Conlon and Alix Deruelle. Expanding kähler–ricci solitons coming out of kähler cones. *Journal of Differential Geometry*, 115(2):303–365, 2020.

[30] Ronan J Conlon, Alix Deruelle, and Song Sun. Classification results for expanding and shrinking gradient Kähler-Ricci solitons. *arXiv preprint arXiv:1904.00147*, 2019.

[31] Andrew S. Dancer and McKenzie Y. Wang. On Ricci solitons of cohomogeneity one. *Ann. Global Anal. Geom.*, 39(3):259–292, 2011.

[32] Yuxing Deng and Xiaohua Zhu. Rigidity of κ-noncollapsed steady Kähler-Ricci solitons. *Math. Ann.*, 377(1-2):847–861, 2020.

[33] Andrzej Derdzinski. Killing potentials with geodesic gradients on Kähler surfaces. *Indiana Univ. Math. J.*, 61(4):1643–1666, 2012.

[34] Andrzej Derdzinski and Paolo Piccione. Kähler manifolds with geodesic holomorphic gradients. *Rev. Mat. Iberoam.*, 36(5):1489–1526, 2020.

[35] J.-H. Eschenburg and McKenzie Y. Wang. The initial value problem for cohomogeneity one Einstein metrics. *J. Geom. Anal.*, 10(1):109–137, 2000.

[36] Mikhail Feldman, Tom Ilmanen, and Dan Knopf. Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons. *J. Differential Geom.*, 65(2):169–209, 2003.

[37] Manuel Fernández-López and Eduardo García-Río. On gradient Ricci solitons with constant scalar curvature. *Proceedings of the American Mathematical Society*, 144(1):369–378, 2016.

[38] Akito Futaki and Mu-Tao Wang. Constructing Kähler-Ricci solitons from Sasaki-Einstein manifolds. *Asian Journal of Mathematics*, 15(1):33–52, 2011.

[39] Jianquan Ge and Zizhou Tang. Isoparametric functions and exotic spheres. *J. Reine Angew. Math.*, 683:161–180, 2013.

[40] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.

[41] Shoshichi Kobayashi. Fixed points of isometries. *Nagoya Math. J.*, 13:63–68, 1958.

[42] Norihito Koiso. On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics. In *Recent topics in differential and analytic geometry*, volume 18 of *Adv. Stud. Pure Math.*, pages 327–337. Academic Press, Boston, MA, 1990.

[43] Tullio Levi-Civita. Famiglie di superficie isoparametriche nell'ordinario spazio euclideo. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur.*, 26:355–362, 1937.

[44] Yu Li and Bing Wang. On Kähler Ricci shrinker surfaces. *arXiv preprint arXiv:2301.09784*, 2023.

[45] Sophus Lie. *Geometrie der Berührungs transformationen*. Chelsea Publishing Co., New York, corrected edition, 1977. With editorial assistance by Georg Scheffers.

[46] Reiko Miyaoka. Transnormal functions on a Riemannian manifold. *Differential Geom. Appl.*, 31(1):130–139, 2013.

[47] Ovidiu Munteanu and Jiaping Wang. Topology of Kähler Ricci solitons. *J. Differential Geom.*, 100(1):109–128, 2015.

[48] Hans Friedrich Münzner. Isoparametrische Hyperflächen in Sphären. *Math. Ann.*, 251(1):57–71, 1980.

[49] Katsumi Nomizu. Some results in E. Cartan's theory of isoparametric families of hypersurfaces. *Bull. Amer. Math. Soc.*, 79:1184–1188, 1973.

[50] Barrett O'Neill. The fundamental equations of a submersion. *Michigan Math. J.*, 13:459–469, 1966.

[51] Henrik Pedersen, Christina Tønnesen-Friedman, and Galliano Valent. Quasi-Einstein Kähler metrics. *Lett. Math. Phys.*, 50(3):229–241, 1999.

[52] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv:math.DG/0211159*, 2002.

[53] Grisha Perelman. Ricci flow with surgery on three-manifolds. *arXiv:math.DG/0303109*, 2003.

[54] Peter Petersen. *Riemannian geometry*, volume 171. Springer, 2006.

[55] Peter Petersen and William Wylie. On gradient Ricci solitons with symmetry. *Proc. Amer. Math. Soc.*, 137(6):2085–2092, 2009.

[56] Chao Qian and Zizhou Tang. Isoparametric functions on exotic spheres. *Adv. Math.*, 272:611–629, 2015.

[57] Georges Reeb. Sur certaines propriétés topologiques des trajectoires des systèmes dynamiques. *Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8°*, 27(9):64, 1952.

[58] S. Sasaki and Y. Hatakeyama. On differentiable manifolds with contact metric structures. *J. Math. Soc. Japan*, 14:249–271, 1962.

[59] Shigeo Sasaki. On differentiable manifolds with certain structures which are closely related to almost contact structure. I. *Tohoku Math. J. (2)*, 12:459–476, 1960.

[60] Shigeo Sasaki. Spherical space forms with normal contact metric 3-structure. *J. Differential Geometry*, 6:307–315, 1971/72.

[61] Carlo Somigliana. Sulle relazione fra il principio di huygens e l'ottica geometrica, also. *Atti Acc. Sc. Torino*, pages 974–979, 1918.

[62] Shûkichi Tanno. The automorphism groups of almost contact Riemannian manifolds. *Tohoku Math. J. (2)*, 21:21–38, 1969.

[63] Shûkichi Tanno. Sasakian manifolds with constant ϕ-holomorphic sectional curvature. *Tohoku Math. J. (2)*, 21:501–507, 1969.

[64] Gang Tian and Xiaohua Zhu. Uniqueness of Kähler-Ricci solitons. *Acta Math.*, 184(2):271–305, 2000.

[65] Philippe Tondeur. *Geometry of foliations*, volume 90 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[66] Hung Tran. Kähler gradient Ricci solitons with large symmetry. *submitted*, 2023.

[67] Qi Ming Wang. Isoparametric functions on Riemannian manifolds. I. *Math. Ann.*, 277(4):639–646, 1987.

[68] Xu-Jia Wang and Xiaohua Zhu. Kähler-Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.*, 188(1):87–103, 2004.

[69] Shing-Tung Yau. Open problems in geometry. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 1–28. Amer. Math. Soc., Providence, RI, 1993.

[70] Zhu-Hong Zhang. On the completeness of gradient Ricci solitons. *Proc. Amer. Math. Soc.*, 137(8):2755–2759, 2009.

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