Finite-time control analysis for a class of nonlinear fractional-order systems subject to disturbance

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Abstract

This paper deals with finite-time control problem for nonlinear fractional-order systems subject to disturbance. We first derive sufficient conditions for finite-time stabilization based on the Lyapunov function method and linear matrix inequlality technique. Then, we propose a new concept of cost control function for guaranteed cost control problem. In terms of linear matrix inequalities (LMIs), an explicit expression for state feedback controllers is presented to make the closed-loop systems finite-time stable and to guarantee an adequate cost level of performance. With the approaches proposed in this paper, we can analyze and design finite-time control for fractional-order systems with similar way to the integer-order systems. Finally, numerical examples are given to illustrate the validity and effectiveness of the proposed results.

Keywords: Fractional derivative, Finite-time stability, Guaranteed cost control, Disturbances, Lyapunov function, Linear matrix inequalities

1. Introduction

In the last decades, the problem of analysis and synthesis of dynamical systems described by fractional-order differential (or difference) equations has received much attention and found many applications in the fields such as physics, engineering, electrochemistry, dynamics and economics (see [1, 2] and the references therein). The analysis of stability and control for fractional-order systems (FOSs) have been widely investigated, and there have been many interesting results [3–7]. By using the Mittag-Leffler function, Laplace transform and a generalized Gronwall inequality lemma, the authors of [5] derived some sufficient conditions for local asymptotical stability of nonlinear FOSs. Asymptotic stability and stabilization of non-autonomous FOSs were considered in [6]. In [7], the authors established Mittag–Leffler stability criteria of nonlinear FOSs with impulses based on fractional calculus theory and S-procedure. As an efficient and commonly-used approach for stability and control problem for integer-order systems, linear matrix inequality tecniques have been successfully extended to the FOSs [8–10]. It should be noted that these mentioned results considered stability and control problem for FOSs in the sense of Lyapunov stability, which deals with the asymptotic behavior of a system

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over an infinite interval time. Nevertheless, in some practical situations we may be more interested in the finite-time stability, which sustains the trajectories do not exceed a certain threshold during a fixed short time under a given bound on the initial conditions, since most actual systems only act over finite interval time. The original concept of finite-time stability is given in [11–15] for various class of integer-order systems. Like in the integer-order case, the problem of finite-time stability of FOSs was studied in the first time in [16]. The authors in [17] considered finite-time stability of time-delay FOSs by using a generalized Gronwall inequality approach. In [18, 19] some conditions were derived to guarantee the finite-time stability for a class of linear FOSs by using the Mittag–Leffler function, a generalized Gronwall inequality approach with Laplace transform technique. The problem of finite-time stability for FOSs described by neural networks was considered in [20].

On the other hand, from the view of engineering, it is desirable to design control systems which are not only finite-time stable but can also guarantee an adequate level of system performance. This is the problem of guaranteed cost control, which has the advantage of providing an upper bound on a given system performance index and thus the system performance degradation incurred by the uncertainties is guaranteed to be less than this bound. Based on the singular value decomposition approach combining with LMIs technique, the authors in [21] solved the problem of guaranteed cost control for singular linear time-delay systems. The problem of finite-time guaranteed cost control for linear Itô stochastic systems was studied in [22]. A sufficient condition for the problem of finite-time stabilization and guaranteed cost control of delayed neural networks was derived in [23] by employing Wirtinger-based integral inequality and Lyapunov–Krasovskii functional method. It should be noticed that all the metioned above results were considered for integer-order systems. To the best of our knowledge, the problem of finite-time stabilization and guaranteed cost control for FOSs with disturbances has not been fully investigated. The main purpose of the present paper is to fill this gap.

In this paper, we study problem of finite-time control for a class of nonlinear FOSs with disturbances. The main contribution of this paper is as follows. (i) By using Lyapunov function method combined with LMIs technique (see Remark 2), we give sufficient conditions for finite-time stabilization of nonlinear FOSs with disturbance. The derived conditions can be considered as further extensions of the existing results obtained in [18]. (ii) We propose a concept of finite-time guaranteed cost control of nonlinear FOSs, which can be regarded as an extension of the integer-order case. Accordingly, new sufficient conditions are established to guarantee that the closed-loop systems not only finite-time stable but can also guarantee an adequate level of system performance.

The organization of this paper is as follows. In Section 2, we summarize some definitions, notations and give auxiliary lemmas which will be used in the proof of the main results of next section. We present our main results on finite-time stabilization and guaranteed cost control of nonlinear FOSs with disturbances in Section 3 and Section 4, respectively. Numerical examples provided to illustrate the effectiveness of the proposed method are given in Section 5.

2. Preliminaries

The following notations will be used in this paper. $\mathbb{Z}^+$ denotes the set of on-negative integers, $\mathbb{R}^n$ denotes the $n-$dimensional linear vector space over the reals with the Euclidean norm

$\|.\|$ given by $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}, x = (x_1, \ldots, x_n) \in \mathbb{R}^n$; $\mathbb{R}^{n\times m}$ denotes the space of $n \times m$ matrices. For a real matrix $A, \lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximal and the minimal eigenvalue of A, respectively. The norm of a real matrix A is defined by $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$. A matrix P is positive definite $(P > 0)$ if $x^T P x > 0, \forall x \neq 0; P > Q$ means $P - Q > 0$. The symmetric term in a matrix is denoted by $*$.

We first introduce some definitions of fractional calculus.

Definition 1. ([1]) The Riemann-Liouville integral of order $\alpha > 0$ is defined by

$$D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 2. ([1]) The Caputo fractional derivative is defined by

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t \geq 0,\ n-1 < \alpha \leq n,$$

where $n \in \mathbb{N}, \Gamma(.)$ is the gamma function, $\Gamma(s) = \int\limits_0^{\infty} e^{-t} t^{s-1} dt, s > 0$. In particular, for $0 < \alpha < 1$, we have

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{f}(s)}{(t-s)^{\alpha}} ds, \quad t \geq 0.$$

Lemma 1. *([24]) If $x(t) \in C^n([0,+\infty), \mathbb{R})$ and $n-1 < \alpha < n, (n \geq 1, n \in \mathbb{Z}^+)$, then*

$${}_0I_t^{\alpha}(D_t^{\alpha} x(t)) = x(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} x^{(k)}(0).$$

In particular, when $0 < \alpha < 1$, we have

$${}_0I_t^{\alpha} D_t^{\alpha} x(t) = x(t) - x(0).$$

Lemma 2. *([25]) Let $x(t) \in \mathbb{R}^n$ be a diffrentiable function, $P \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Then, for any time instant $t \geq t_0$, the following condition holds*

$$\frac{1}{2} {}_{t_0}^{C} D_t^{\alpha} \left(x^T(t) P x(t) \right) \leq x^T(t) P\, {}_{t_0}^{C} D_t^{\alpha} x(t), \quad \forall \alpha \in (0,1), \forall t \geq t_0 \geq 0.$$

We now consider the following nonlinear fractional-order system with disturbance:

$$\begin{cases} D_t^{\alpha} x(t) = Ax(t) + Bu(t) + W\omega(t) + f(t, x(t), u(t), \omega(t)), t \geq 0, \\ x(0) = x_0, \end{cases} \tag{1}$$

where $\alpha \in (0,1), x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control, $\omega(t) \in \mathbb{R}^p$ is the disturbance satisfying

$$\exists d > 0: \qquad \omega^T(t)\omega(t) \leq d, \forall t \in [0,T]; \tag{2}$$

A, B, W are given real constant matrices of appropriate dimensions. The nonlinear function $f(.)$ satisfying the following condition

$$f^T(t,x,u,\omega) f(t,x,u,\omega) \leq x^T E_1^T E_1 x + u^T E_2^T E_2 u + \omega^T E_3^T E_3 \omega, \tag{3}$$

for all $(t,x,u,\omega) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, $E_i, i = 1,2,...,3$ are given constant matrices.

Lemma 3. *([26]) Under the assumption* (3) *the system* (1) *has a unique solution on* $[0,+\infty)$.

Definition 3. For given positive numbers c_1, c_2, T_f and a symmetric positive definite matrix R, the systems (1) is robustly finite-time stabilizable w.r.t (c_1, c_2, T_f, R, d) if there exists a feedback control $u(t) = Kx(t)$ such that the solution of the closed-loop system

$$\begin{cases} D_t^{\alpha}x(t) = [A+BK]x(t) + W\omega(t) + f(t,x(t),Kx(t),\omega(t)), \quad t \geq 0, \\ x(0) = x_0 \end{cases} \tag{4}$$

satisfies the following relation

$$x_0^T R x_0 \leq c_1 \Longrightarrow x^T(t)Rx(t) < c_2, \quad t \in [0, T_f],$$

for all disturbances $\omega(t) \in \mathbb{R}^p$ satisfying (2).

Given a positive number $T_f > 0$, we consider the following quadratic cost function for system (1):

$$J(u) = \frac{1}{\Gamma(\alpha)} \int_0^{T_f} (T_f - s)^{\alpha-1}(x^T(s)Q_1x(s) + u^T(s)Q_2u(s))ds, \tag{5}$$

where $Q_1 \in \mathbb{R}^{n\times n}, Q_2 \in \mathbb{R}^{m\times m}$ are given symmetric positive definite matrices.

Remark 1. It should be noted that when $\alpha = 1$ the quadratic cost function (5) is turned into the cost function in integer-order case, which was considered in the literature [21–23].

Definition 4. If there exists a feedback control law $u^*(t) = Kx(t)$ and a positive number J^* such that the closed-loop system

$$\begin{cases} D_t^{\alpha}x(t) = (A+BK)x(t) + W\omega(t) + f(t,x(t),Kx(t),\omega(t)), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{6}$$

is robustly finite-time stable and the cost function satisfies $J(u^*) \leq J^*$, then the value J^* is a guaranteed cost value and the control $u^*(t)$ is a guaranteed cost controller.

Now, we present the following auxiliary lemma, which will be used in the proof of the main results.

Lemma 4. *(Schur complement lemma [27]). Given constant matrices X, Y, Z with appropriate dimensions satisfying $Y = Y^T > 0, X = X^T$, then $X + Z^TY^{-1}Z < 0$ if and only if*

$$\begin{bmatrix} X & Z^T \\ Z & -Y \end{bmatrix} < 0.$$

3. Finite-time stabilization

In this section we give sufficient conditions for finite-time stabilization of system (1). Let us denote

$$M_{11} = AP + PA^T + BY + Y^TB^T + I + WW^T,$$
$$\overline{P} = R^{-\frac{1}{2}}P^{-1}R^{-\frac{1}{2}}, \lambda_1 = \lambda_{\min}(\overline{P}), \lambda_2 = \lambda_{\max}(\overline{P}), \beta = \lambda_{\max}(E_3^TE_3).$$

Theorem 5. *Given positive numbers c_1, c_2, T_f and a symmetric positive definite matrix R, the system* (1) *is robustly finite-time stabilizable w.r.t* (c_1, c_2, T_f, R, d) *by the state feedback control* $u(t) = YP^{-1}x(t), t \in [0, T_f]$, *if there exist a symmetric positive definite matrix P, a matrix Y with appropriate dimensions satisfying the following conditions:*

$$\begin{bmatrix} M_{11} & PE_1^T & Y^T E_2^T \\ * & -I & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{7a}$$

$$\lambda_2 c_1 + \frac{d(1+\beta)}{\Gamma(\alpha+1)} T_f^{\alpha} < \lambda_1 c_2. \tag{7b}$$

Proof. With the feedback control matrix $K = YP^{-1}$, we consider the following non-negative quadratic function for the closed-loop system (6):

$$V(x(t)) = x^T(t) P^{-1} x(t).$$

From Lemma 2 the Caputo derivative of $V(x(t))$ along the solution of the system (6) is defined as

$$\begin{aligned} D_t^{\alpha} V(x(t)) &\leq 2x^T(t) P^{-1} D_t^{\alpha} x(t) \\ &= x^T(t) \left[P^{-1}A + A^T P^{-1} + P^{-1}BK + K^T B^T P^{-1} \right] x(t) \\ &\quad + 2x^T(t) P^{-1} f(.) + 2x^T(t) P^{-1} W \omega(t). \end{aligned} \tag{8}$$

By using the Cauchy matrix inequality, we have the following estimates

$$\begin{aligned} 2x^T(t)P^{-1}f(.) \leq & x^T(t)P^{-1}P^{-1}x(t) + f^T(.)f(.) \\ \leq & x^T(t)P^{-1}P^{-1}x(t) + [x^T(t)E_1^T E_1 x(t) + x^T(t) K^T E_2^T E_2 K x(t) \\ & + \omega^T(t) E_3^T E_3 \omega(t)], \\ 2x^T(t)P^{-1}W\omega(t) \leq & x^T(t)P^{-1}WW^T P^{-1}x(t) + \omega^T(t)\omega(t). \end{aligned} \tag{9}$$

From (8)–(9), we obtain

$$D_t^{\alpha} V(x(t)) \leq x^T(t) \Omega x(t) + (1+\beta)\omega^T(t)\omega(t), \tag{10}$$

where

$$\begin{aligned} \Omega = & P^{-1}A + A^T P^{-1} + P^{-1}BK + K^T B^T P^{-1} + P^{-1}P^{-1} \\ & + K^T E_2^T E_2 K + P^{-1}WW^T P^{-1} + E_1^T E_1. \end{aligned}$$

Now, pre- and post-multiply both sides Ω by P and letting $K = YP^{-1}$, we have

$$\Phi = P\Omega P = AP + PA^T + BY + Y^T B^T + I + WW^T + PE_1^T E_1 P + Y^T E_2^T E_2 Y.$$

Note that $\Omega < 0$ is equivalent to $\Phi < 0$. Using the Schur complement lemma (Lemma 4), we have $\Phi < 0$ is equivalent to (7a). Therefore, from the conditions (7a), (10), we have

$$D_t^{\alpha} V(x(t)) \leq (1+\beta)\omega^T(t)\omega(t), \quad \forall t \in [0, T_f]. \tag{11}$$

Integrating with order α both sides of (11) from 0 to $t(0 < t < T_f)$ and using Lemma 1, we have

$$\begin{aligned} x^T(t)P^{-1}x(t) &\leq x^T(0)P^{-1}x(0) + {}_0I_t^{\alpha}((1+\beta)\omega^T(t)\omega(t)) \\ &= x^T(0)P^{-1}x(0) + \frac{1+\beta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega^T(s)\omega(s)ds \\ &\leq x^T(0)P^{-1}x(0) + \frac{d(1+\beta)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\ &\leq x^T(0)P^{-1}x(0) + \frac{d(1+\beta)}{\Gamma(\alpha+1)}T_f^{\alpha}. \end{aligned} \tag{12}$$

On the other hand, we have

$$\begin{aligned} x^T(t)P^{-1}x(t) =& x^T(t)R^{\frac{1}{2}}\overline{P}R^{\frac{1}{2}}x(t) \\ \geq& \lambda_{\min}(\overline{P})x^T(t)Rx(t) \\ =& \lambda_1 x^T(t)Rx(t), \end{aligned} \tag{13}$$

and

$$\begin{aligned} x^T(0)P^{-1}x(0) =& x^T(0)R^{\frac{1}{2}}\overline{P}R^{\frac{1}{2}}x(0) \\ \leq& \lambda_{\max}(\overline{P})x^T(0)Rx(0) \\ =& \lambda_2 x^T(0)Rx(0) \leq \lambda_2 c_1. \end{aligned} \tag{14}$$

From (12) -(14), we have

$$\lambda_1 x^T(t)Rx(t) \leq V(x(t)) = x^T(t)P^{-1}x(t) \leq \lambda_2 c_1 + \frac{d(1+\beta)}{\Gamma(\alpha+1)}T_f^{\alpha}.$$

Condition (7b) implies that $x^T(t)Rx(t) < c_2$. Thus, the system (1) is robustly finite-time stabilizable w.r.t (c_1, c_2, T_f, R), which completes the proof of the theorem. □

Remark 2. Since the condition (7a) is a linear matrix inequality condition, we can solve the condition by using Matlab's LMI Control Toolbox in [28]. Therefore, from Theorem 5, we have the following procedure for solving finite-time stabilization problem of system (1):
Step 1. Solve the linear matrix inequalities (7a) and obtain symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times n}$.
Step 2. Compute $\overline{P} = R^{-\frac{1}{2}}PR^{-\frac{1}{2}}$, $\lambda_1 = \lambda_{\min}(\overline{P})$, $\lambda_2 = \lambda_{\max}(\overline{P})$.
Step 3. Check condition (7b) in Theorem 5. If they hold, enter Step 4; else return to Step 1.
Step 4. The closed-loop system (6) is finite-time stable with respect to (c_1, c_2, T_f, R, d).

In the sequel, we apply the obtained result (Theorem 5) to the following uncertain linear FOSs:

$$\begin{cases} D_t^{\alpha}x(t) = [A+\Delta A(t)]x(t) + [B+\Delta B(t)]u(t) + [W+\Delta W(t)]\omega(t), \quad t \geq 0, \\ x(0) = x_0, \end{cases} \tag{15}$$

where

$$\Delta A(t) = G_1F_1(t)H_1, \Delta B(t) = G_2F_2(t)H_2, \Delta W(t) = G_3F_3(t)H_3,$$

where $G_i, H_i (i = 1,2,3)$ are given constant matrices, the unknown perturbations $F_i(t)(i = 1,2,3)$ satisfy the following condition

$$F_i^T(t)F_i(t) \leq I, \quad t \geq 0, \quad (i = 1,2,3).$$

In this case the nonlinear perturbation is

$$f(.) = G_1F_1(t)H_1x(t) + G_2F_2(t)H_2u(t) + G_3F_3(t)H_3\omega(t).$$

By using some simple computations, we have the nonlinear perturbation $f(.)$ satisfying condition (3) with

$$E_1 = \sqrt{a(a+b+c)}I, E_2 = \sqrt{b(a+b+c)}I, E_3 = \sqrt{a(a+b+c)}I,$$

where $a = \|G_1\|\|H_1\|, b = \|G_2\|\|H_2\|, c = \|G_3\|\|H_3\|$. From Theorem 5, we have the following result.

Corollary 6. *Given positive numbers c_1, c_2, T_f and a symmetric positive definite matrix R, the system* (15) *is robustly finite-time stabilizable w.r.t (c_1, c_2, T_f, R, d) by the state feedback control $u(t) = YP^{-1}x(t), t \in [0, T_f]$, if there exist a symmetric positive definite matrix P, a matrix Y with appropriate dimensions satisfying the following conditions:*

$$\begin{bmatrix} N_{11} & N_{12} & N_{13} \\ * & -I & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{16a}$$

$$\lambda_2 c_1 + \frac{d(1+\beta)}{\Gamma(\alpha+1)} T_f^{\alpha} < \lambda_1 c_2, \tag{16b}$$

where

$$\begin{aligned}
&a = \|G_1\|\|H_1\|, b = \|G_2\|\|H_2\|, c = \|G_3\|\|H_3\|, \\
&N_{11} = AP + PA^T + BY + Y^TB^T + I + WW^T, \\
&N_{12} = \sqrt{a(a+b+c)}I, N_{13} = \sqrt{b(a+b+c)}, \\
&\overline{P} = R^{-\frac{1}{2}}P^{-1}R^{-\frac{1}{2}}, \lambda_1 = \lambda_{\min}(\overline{P}), \lambda_2 = \lambda_{\max}(\overline{P}), \beta = a(a+b+c).
\end{aligned}$$

4. Finite-time guaranteed cost control

In this section we give sufficient conditions for the finite-time guaranteed cost control of system (1). Let us denote

$$\begin{aligned}
&\mathscr{M}_{11} = AP + PA^T + BY + Y^TB^T + I + WW^T, \\
&\overline{P} = R^{-\frac{1}{2}}P^{-1}R^{-\frac{1}{2}}, \lambda_1 = \lambda_{\min}(\overline{P}), \lambda_2 = \lambda_{\max}(\overline{P}), \beta = \lambda_{\max}(E_3^TE_3),
\end{aligned}$$

Theorem 7. *Given positive numbers c_1, c_2, T_f and a symmetric positive definite matrix R, if there exist a symmetric positive definite matrix P, a matrix Y with appropriate dimensions satisfying the following conditions:*

$$\begin{bmatrix} \mathscr{M}_{11} & PE_1^T & Y^T E_2^T & PQ_1 & Y^T Q_2 \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -Q_1 & 0 \\ * & * & * & * & -Q_2 \end{bmatrix} < 0, \tag{17a}$$

$$\lambda_2 c_1 + \frac{d(1+\beta)}{\Gamma(\alpha+1)} T_f^{\alpha} < \lambda_1 c_2, \tag{17b}$$

then

$$u(t) = YP^{-1}x(t), \quad t \in [0, T_f]$$

is a guaranteed cost controller for the system (1) *and the guaranteed cost value is*

$$J^* = \frac{d(1+\beta)}{\Gamma(\alpha+1)} T_f^{\alpha} + \lambda_2 c_1.$$

Proof. We choose the non-negative quadratic function defined as in Theorem 5. We have obtained that

$$D_t^{\alpha} V(x(t)) \leq x^T(t)\overline{\Omega}x(t) + (1+\beta)\omega^T(t)\omega(t) - x^T(t)\left[Q_1 + K^T Q_2 K\right]x(t), \tag{18}$$

where

$$\begin{aligned} \overline{\Omega} =& P^{-1}A + A^T P^{-1} + P^{-1}BK + K^T B^T P^{-1} + P^{-1}P^{-1} \\ &+ K^T E_2^T E_2 K + P^{-1}WW^T P^{-1} + E_1^T E_1 + Q_1 + K^T Q_2 K. \end{aligned}$$

Now, pre- and post-multiply both sides $\overline{\Omega}$ by P and letting $K = YP^{-1}$, we have

$$\begin{aligned} \overline{\Phi} = P\overline{\Omega}P = AP + PA^T + BY + Y^T B^T + I + WW^T + PE_1^T E_1 P \\ + Y^T E_2^T E_2 Y + PQ_1 P + Y^T Q_2 Y. \end{aligned}$$

Therefore, by similar arguments used in the proof of Theorem 5 we have the system (1) is robustly finite-time stabilizable w.r.t (c_1, c_2, T_f, R) if conditions (17a) and (17b) are satisfied. Next, we will find the guaranteed cost value of the cost function (5). Note that $\overline{\Omega} < 0$ is equivalent to $\overline{\Phi} < 0$. Using the Schur Complement Lemma (Lemma 4), we have $\overline{\Phi} < 0$ is equivalent to (17a). From conditions (17a) and (18), we have

$${}_0^C D_t^{\alpha} V(x(t)) \leq (1+\beta)\omega^T(t)\omega(t) - x^T(t)\left[Q_1 + K^T Q_2 K\right]x(t), \quad \forall t \in [0, T_f]. \tag{19}$$

Integrating with order α both sides of (19) from 0 to T_f and using Lemma 1, we obtain

$$V(x(T_f)) - V(x(0)) \leq {}_0I_{T_f}^{\alpha}\left((1+\beta)\omega^T(t)\omega(t)\right) - J(u). \tag{20}$$

Hence

$$J(u) \leq {}_0I_{T_f}^{\alpha}\left((1+\beta)\omega^T(t)\omega(t)\right) + V(x(0)) \leq \frac{d}{\Gamma(\alpha+1)} T_f^{\alpha} + \lambda_2 c_1 := J^* \tag{21}$$

due to $V(x(T_f)) = x^T(T_f)P^{-1}x(T_f) \geq 0$, which completes the proof of the theorem. □

We now consider a special case of system (1), where $f(.) = 0$, then system (1) is reduced to the linear FOSs which was considered in [18]:

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + Bu(t) + W\omega(t), \quad t \geq 0, \\ x(0) = x_0. \end{cases} \tag{22}$$

Based on the proof of Theorem 7, we obtain the following result. Let us denote

$$\mathcal{N}_{11} = AP + PA^T + BY + Y^T B^T + WW^T,$$
$$\overline{P} = R^{-\frac{1}{2}} P^{-1} R^{-\frac{1}{2}}, \lambda_1 = \lambda_{\min}(\overline{P}), \lambda_2 = \lambda_{\max}(\overline{P}).$$

Corollary 8. *Given positive numbers c_1, c_2, T_f and a symmetric positive definite matrix R, if there exist a symmetric positive definite matrix P, a matrix Y with appropriate dimensions satisfying the following conditions:*

$$\begin{bmatrix} \mathcal{N}_{11} & PQ_1 & Y^T Q_2 \\ * & -Q_1 & 0 \\ * & * & -Q_2 \end{bmatrix} < 0, \tag{23a}$$

$$\lambda_2 c_1 + \frac{d}{\Gamma(\alpha+1)} T_f^\alpha < \lambda_1 c_2, \tag{23b}$$

then $u(t) = YP^{-1}x(t), t \in [0, T_f]$ is a guaranteed cost controller for the system (22) *and the guaranteed cost value is $J^* = \frac{d}{\Gamma(\alpha+1)} T_f^\alpha + \lambda_2 c_1$.*

5. Numerical examples

In this section, two illustrative examples are implemented to illustrate the validity and effectiveness of the proposed results.

Example 1. (Finite-time stabilization) Consider the Lorenz system [29]

$$\begin{cases} D_t^{0.98} x(t) = Ax(t) + Bu(t) + W\omega(t) + f(t, x(t)), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^3, \end{cases} \tag{24}$$

where

$$A = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}, f(t, x(t)) = \begin{bmatrix} 0 \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{bmatrix},$$
$$B = \begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix}, W = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix},$$

$x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3, u(t) \in \mathbb{R}, \omega(t) = \cos t \in \mathbb{R}$. We have the disturbance $\omega(t)$ satisfying the condition (2) with $d = 1$. The closed-loop system with a state feedback controller $u(t) = Kx(t)$ of system (24) is described by

$$\begin{cases} D_t^{0.98} x(t) = (A + BK)x(t) + W\omega(t) + f(t, x(t)), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^3. \end{cases} \tag{25}$$

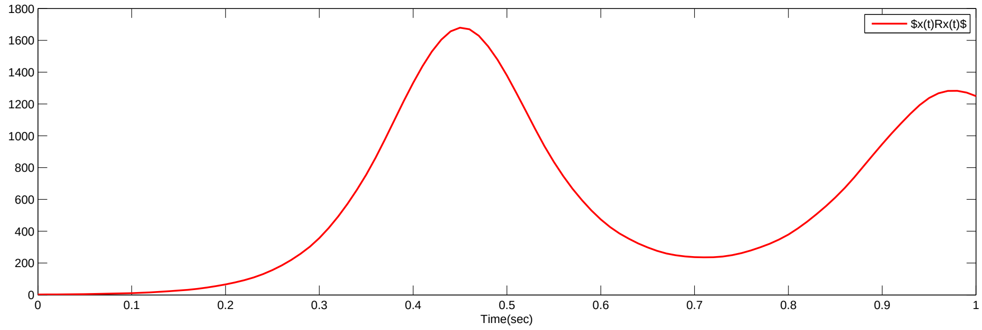

Figure 1: $x^T(t)Rx(t)$ of the open-loop system for $\alpha = 0.95$

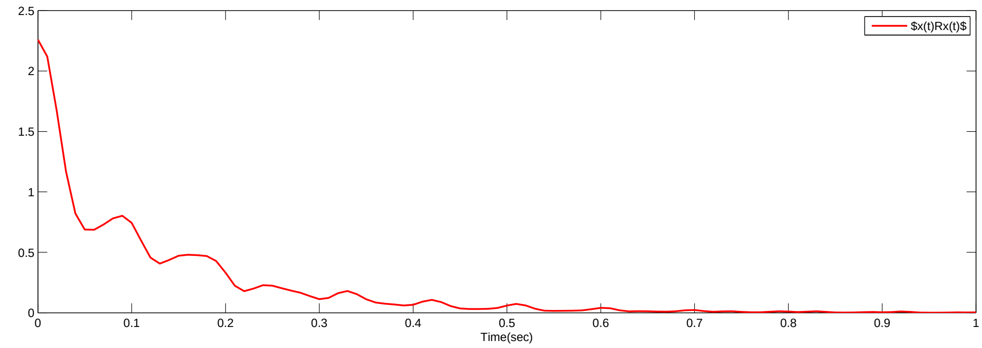

Figure 2: $x^T(t)Rx(t)$ of the closed-loop system for $\alpha = 0.95$

We have

$$\begin{aligned}\|f(t,x(t))\| &= \sqrt{(-x_1(t)x_3(t))^2+(x_1(t)x_2(t))^2}\\ &\le |x_1(t)|\sqrt{x_1^2(t)+x_2^2(t)+x_3^2(t)} \le \kappa\|x(t)\|.\end{aligned}$$

Given $\kappa = 2, c_1 = 1, c_2 = 3.6, T_f = 1$, the function $f(t,x(t))$ satisfies condition (3) with $E_1 = I, E_2 = 0, E_3 = 0$. Moreover, the conditions (7a) and (7b) in Theorem 5 hold with

$$P = \begin{bmatrix} 0.5355 & 0.1275 & -0.0457 \\ 0.1275 & 0.5983 & 0.1273 \\ -0.0457 & 0.1273 & 1.1499 \end{bmatrix}, Y = \begin{bmatrix} 0.2846 & -2.9402 & -0.5131 \end{bmatrix}.$$

By Theorem 5, the closed-loop system (27) is finite-time stable with respect to $(1, 4.1, 1, I, 1)$ by state feedback controller is

$$u(t) = \begin{bmatrix} 1.8241 & -5.3496 & 0.2185 \end{bmatrix} x(t), \quad t \in [0,1].$$

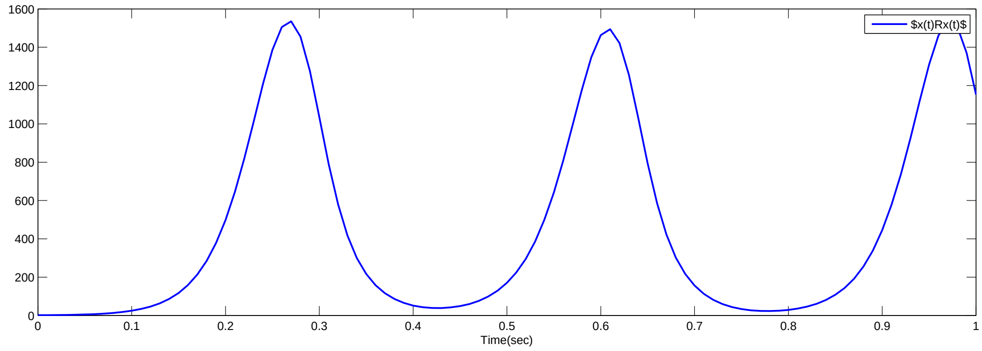

Figure 3: $x^T(t)Rx(t)$ of the open-loop system for $\alpha = 0.95$

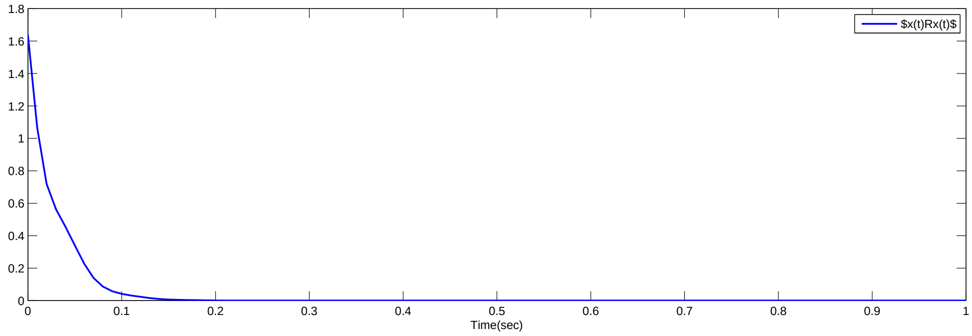

Figure 4: $x^T(t)Rx(t)$ of the closed-loop system for $\alpha = 0.95$

With initial conditions are $x_1(0) = 0.9, x_2(0) = 0.8, x_3(0) = 0.9$, Figure 1 shows the responses of $x^T(t)Rx(t)$ of the open-loop system of system (26), while Figure 2 shows the responses of $x^T(t)Rx(t)$ of the closed-loop system (27). It is clear from the Figure 2 that the closed-loop system is finite-time stable with respect to $(1, 4.1, 1, I, 1)$.

Example 2. (Guaranteed cost control) Let us consider the Chen's fractional-order system [29]

$$\begin{cases} D_t^{0.95}x(t) = Ax(t) + Bu(t) + W\omega(t) + f(t, x(t)), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^3, \end{cases} \tag{26}$$

where

$$A = \begin{bmatrix} -36 & 36 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -3 \end{bmatrix}, f(t, x(t)) = \begin{bmatrix} 0 \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}, W = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix},$$

$x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3, u(t) \in \mathbb{R}, \omega(t) = 0.1\sin t \in \mathbb{R}$. We have the disturbance $\omega(t)$ satisfying the condition (2) with $d = 0.01$. The closed-loop system with a state feedback controller $u(t) = Kx(t)$ of system (26) is described by

$$\begin{cases} D_t^{0.95}x(t) = (A+BK)x(t) + W\omega(t) + f(t,x(t)), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^3. \end{cases} \tag{27}$$

It is very easy to verify that

$$\begin{aligned} \|f(t,x(t))\| &= \sqrt{(-x_1(t)x_3(t))^2 + (x_1(t)x_2(t))^2} \\ &\leq |x_1(t)|\sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t)} \leq \kappa\|x(t)\|. \end{aligned}$$

Given $\kappa = 1, c_1 = 1, c_2 = 3.6, T_f = 1$, the function $f(t,x(t))$ satisfies condition (3) with $E_1 = I, E_2 = 0, E_3 = 0$. The cost function associated with system (26) is given by (5) with

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.1 \end{bmatrix}.$$

We can find that the conditions (17a) and (17b) in Theorem 7 are satisfied with

$$P = \begin{bmatrix} 1.5974 & 0.8426 & 0.4876 \\ 0.8426 & 1.8627 & 0.6119 \\ 0.4876 & 0.6119 & 1.8670 \end{bmatrix}, Y = \begin{bmatrix} -9.5078 & -11.8795 & -3.1079 \end{bmatrix}.$$

By Theorem 7, the closed-loop system (27) is finite-time stable with respect to $(1, 3.6, 1, I, 0.01)$ and the guaranteed cost value is $J^* = 0.5021\|x_0\|^2$. Moreover, the guaranteed cost controller is

$$u(t) = \begin{bmatrix} -3.5599 & -5.0719 & 0.9272 \end{bmatrix} x(t), \quad t \in [0,1].$$

With initial conditions are $x_1(0) = 0.6, x_2(0) = 0.8, x_3(0) = -0.8$, Figure 3 shows the responses of $x^T(t)Rx(t)$ of the open-loop system of system (26), while Figure 4 shows the responses of $x^T(t)Rx(t)$ of the closed-loop system (27). It is easily seen from the Figure 4 that the closed-loop system is finite-time stable with respect to $(1, 3.6, 1, I, 0.01)$.

6. Conclusion

In this paper, the problems of finite-time stabilization and finite-time guaranteed cost control for nonlinear fractional-order systems with order $0 < \alpha \leq 1$ have been investigated. Some sufficient conditions have been established for the control design by using finite-time stability theory and LMIs approach. The effectiveness and advantages of the proposed method in this paper have been demonstrated by two numerical examples with simulation results.

Acknowledgments

This work was completed while the authors were visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM). We would like to thank the Institute for its support and hospitality. This work was supported by the National Foundation for Science and Technology Development, Vietnam, grant number 101.01-2017.300.

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