Constructing D-Efficient Mixed-Level Foldover Designs Using Hadamard Matrices

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May 15, 2018

Abstract

This paper introduces a new class of Hadamard matrix-based mixed-level foldover designs (MLFODs) and an algorithm which facilitates the construction of MLFODs. Our new MLFODs were constructed by converting some 2-level columns of a Hadamard matrix to 3-level columns. Like the 2-level foldover designs (FODs), the new MLFODs requires $2m$ runs where m is the total number of 3- and 2-level factors. Our Hadamard-matrix based MLFODs are compared with the conference matrix-based FODs of Johns & Nachtsheim (2013) in terms of D-efficiency and the maximum of

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the correlation coefficients in terms of the absolute value among the columns of the model matrix. Like the latter, our designs are also *definitive* in the sense that the estimates of all main effects are unbiased with respect to any active second order effects. In addition, they require less runs and can be used to study the presence of the second-order effects more efficiently. Examples illustrating the use of our new MLFODs are given.

Keywords: Augmented designs; Conference matrix; Definitive screening designs; D-efficiency; Interchange algorithm; Plackett-Burman designs.

1 Introduction

Foldover is an experimental design technique used when a fractional factorial design (FFD) such as a resolution III FFD, a Hadamard matrix or a Plackett-Burman design produces ambiguous results. Let us call this fraction $\mathbf{H}$. A foldover design (FOD) constructed from this half fraction is of the form:

$$\begin{pmatrix} \mathbf{H} \\ -\mathbf{H} \end{pmatrix}. \tag{1}$$

FODs form an important class of screening designs. The 2-level FODs have been discussed in most textbooks on design of experiments (DOEs) such as Mee (2009) and Montgomery (2017). This paper also provides some useful references on 2-level FODs.

The 3-level FODs and mixed-level FODs (MLFODs) for screening experiments based

(a)	(b)	(c)	(d)	(e)
0+++++++	0+++++++	0+++++++	0+++++++	0+++++++
+0++-+--	+-++-+--	+0++-+--	+0++-+--	+0++-+--
+-0++-+-	+--++-+-	+-+++-+-	+-0++-+-	+-0++-+-
+--0++-+	+---++-+	+--+++-+	+--+++-+	+--0++-+
++--0++-	++---++-	++--+++-	++--+++-	++--+++-
+-+--0++	+-+---++	+-+--+++	+-+--+++	+-+---++
++-+--0+	++-+---+	++-+--++	++-+--++	++-+---+
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Figure 1: (a) A conference matrix of size 8 and (b)-(e) The half fraction of ADSDs for 1-4 3-level factors in 18 runs constructed from the conference matrix in (a).

on *conference* matrices have appeared in the DOEs literature in the past few years. A $(0, \pm 1)$-matrix with zero diagonal $\mathbf{C}$ is a conference matrix of order m if $\mathbf{C}'\mathbf{C} = (m-1)\,\mathbf{I}_m$, where $\mathbf{I}_m$ is the identity matrix of order m. Conference matrices for even m and $m \leq 50$ are given in Xiao et al. (2012) and Nguyen & Stylianou (2013).

The conference matrix-based FODs include the definitive screening designs (DSDs) introduced by Jones & Nachtsheim (2011) and the mixed-level foldover designs constructed by the DSD-augment method of Jones & Nachtsheim (2013), hereafter called ADSDs. In the case of DSDs, the half fraction in (1) is a conference matrix and a row of 0's is added to (1). In the case ADSDs, the half fraction in (1) is derived from a conference matrix. The detailed method of constructing ADSDs by the DSD-augment method is given in Jones & Nachtsheim (2013).

Figure 1 (a) shows a conference matrix of size 8. Figures 1 (b) to 1 (e) show the half fraction of ADSDs for 1-4 3-level factors constructed from the conference matrix in Figure 1 (a). The ADSDs have the following properties:

(i) All main effects (MEs) are orthogonal to the 2-factor interactions (2FIs);

(ii) All MEs are orthogonal to the quadratic effects.

This paper introduces a class of Hadamard matrix-based MLFODs formed by converting some 2-level columns of a Hadamard matrix or a Plackett-Burman design (Plackett & Burman, 1946) or a D-optimal 2-level design to 3-level columns. Like the ADSDs, this general class of MLFODs retains the two above-mentioned properties.

2 Structure of the $\mathbf{X}'\mathbf{X}$ matrix of an MLFOD

Let $\mathbf{X}$ be the expanded design matrix of an MLFOD with m factors in $2n$ runs for the design in (1) with m_3 3-level factors and m_2 2-level factors, (i.e. $m = m_3 + m_2$) for the pure quadratic model with p $(= 1 + m_3 + m)$ parameters. Now, let the u-th row of $\mathbf{X}$ be written as $(1, h_{u1}^2, \ldots, h_{um_3}^2, h_{u1}, \ldots, h_{um_3}, h_{u(m_3+1)}, \ldots, h_{um})$. Due to the foldover structure of the design, its information matrix $\mathbf{M}$ $(= \mathbf{X}'\mathbf{X})$ will be of the form

$$2 \begin{pmatrix} \mathbf{A} & \mathbf{0}' \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \tag{2}$$

where $\mathbf{A}$ is a matrix of size $(1 + m_3) \times (1 + m_3)$, $\mathbf{B}$ a matrix of size $m \times m$ and $\mathbf{0}$ a matrix of 0's.

Assuming each of the m_3 3-level columns of the half fraction has a fixed number x of

0's and $b = n - x$ number of ± 1's, the matrix $\mathbf{A}$ will be of the form

$$\begin{pmatrix} n & b\mathbf{1}' \\ b\mathbf{1} & \mathbf{A}^* \end{pmatrix}. \tag{3}$$

where $\mathbf{1}$ is a column vector of 1's and $\mathbf{A}^*$ the core of $\mathbf{A}$ in (2), i. e. the matrix $\mathbf{A}$ without its first row and first column.

From (3), $|\mathbf{M}|$, the determinant of $\mathbf{M}$ for the pure quadratic model, can be computed as

$$|\mathbf{M}| = 2^{1+m_3+m}\ n\ |\mathbf{A}^* - \frac{b^2}{n}\ \mathbf{J}|\ |\mathbf{B}| \tag{4}$$

In this paper, we will maximize $|\mathbf{M}|$ in (4) by maximizing $|\mathbf{A}^* - \frac{b^2}{n}\ \mathbf{J}|$ and $|\mathbf{B}|$ by the approximate approach. Let $\lambda_1, \ldots, \lambda_{m_3}$ be the eigenvalues of a square matrix of full rank. Note that the trace of this matrix is $\sum \lambda_i$ and sum of the squares of the elements of this matrix is $\sum \lambda_i^2$. Thus if the trace of this matrix is constant, minimizing the sum of the squares of the off-diagonal elements of this matrix is the same as making the λ_i as equal as possible with $\sum \lambda_i$ being constant. This is the same as maximizing the determinant of this matrix, i. e. $\Pi\lambda_i$. Since the diagonal elements of $\mathbf{A}^* - \frac{b^2}{n}\ \mathbf{J}$ and $\mathbf{B}$ and their traces $m_3\ (b - \frac{b^2}{n})$ and $m_3 b + m_2 n$ respectively are constant, we can maximize $|\mathbf{A}^* - \frac{b^2}{n}\ \mathbf{J}|$ and $|\mathbf{B}|$ by minimizing the sum of the squares of the off-diagonal elements of these two matrices. This is also the approach which Nguyen (1996) used to obtain efficient supersaturated designs.

For the main effect model, the expanded design matrix $\mathbf{X}$ does not include the pure quadratic effects, and the determinant of $\mathbf{M}$ $(= \mathbf{X}'\mathbf{X})$ can be computed as

$$|\mathbf{M}| = 2^{1+m}\ n\ |\mathbf{B}|. \tag{5}$$

In this paper, we use MLFOD* to denote MLFOD having the matrix $\mathbf{A}^*$ of the form $d\,\mathbf{I} + c\,\mathbf{J}$, where $\mathbf{I}$ is the identity matrix and $\mathbf{J}$ the matrix of 1's. Clearly, ADSD is a special case of MLFOD*. $\mathbf{A}^{-1}$, the inverse of the matrix $\mathbf{A}$ of an MLFOD*, will have the form

$$\begin{pmatrix} p & q\mathbf{1}' \\ q\mathbf{1} & t\mathbf{I} + s\mathbf{J} \end{pmatrix}, \tag{6}$$

where $p = (d + mc)/R,\ q = b/R,\ t = 1/d,\ s = (b^2 - nc)/dR$ and $R = n(d + mc) - b^2 m$. Note that if $c = b^2/n$ then $s = 0$ and the quadratic effects are orthogonal to one another.

3 The FOLDOVER algorithm

Our approach for constructing an MLFOD with m_3 3-level factors and m_2 2-level factors is to select m columns $(m = m_3 + m_2)$ of a Hadamard matrix $\mathbf{H}$ in n runs $(n \geq m)$ and then convert the first m_3 2-level columns of this matrix to 3-level columns for use in (1).

We use the u-th row of $\mathbf{H}$ to construct the vector $\mathbf{J}_u$ of length $2\binom{m_3}{2} + m_3 m_2$. The first $\binom{m_3}{2}$ entries of $\mathbf{J}_u$ are $h_{u1}^2 h_{u2}^2, \ldots, h_{u(m_3-1)}^2 h_{um_3}^2$. The next $\binom{m_3}{2}$ entries are $h_{u1}h_{u2}, \ldots, h_{u(m_3-1)}$ h_{um_3}. The last $m_3 m_2$ entries are $h_{u1}h_{u(m_3+1)},\ \ldots, h_{um_3}h_{um}$.

Now, let $\mathbf{J} = \sum \mathbf{J}_u$. Subtract $\frac{b^2}{n}$ from each of the first $\binom{m_3}{2}$ elements of $\mathbf{J}$. (Recall that each of the m_3 3-level columns of the half fraction will have a fixed number x of 0's and

$b = n - x$ number of ± 1's.) Let A_1 be the sum of the squares of these $\binom{m_3}{2}$ elements and A_2 the sum of the squares of the remaining elements of $\mathbf{J}$.

Following are the steps of the FOLDOVER algorithm for constructing the half fraction in (1) of an MLFOD with m_3 3-level factors and m_2 2-level factors using vector $\mathbf{J}$:

1. Randomly select m columns from a Hadamard matrix $\mathbf{H}$ of order m $(m \leq n)$. Randomly multiply some rows of this matrix with -1. Randomly insert x 0's into each of the first m_3 columns of $\mathbf{H}$. Calculate each $\mathbf{J}_u,\ u = 1, \ldots, n$ and $\mathbf{J} = \sum \mathbf{J}_u$. Subtract $\frac{b^2}{n}$ from each of the first $\binom{m_3}{2}$ elements of $\mathbf{J}$. Calculate the pair $(A_1,\ A_2)$ values.

2. Search for a pair of entries in column j $(j = 1 \ldots, m_3)$ of $\mathbf{H}$ such that the sign swap of these two entries will result in the biggest reduction in A_1 (or A_2 if A_1 cannot be reduced further). If the search is successful, update $A_1,\ A_2,\ \mathbf{J}$ and this column. If A_1 (or A_2 if A_1 cannot be reduced further) cannot be reduced further, repeat this step with the next 3-level column. Repeat this step until the $(A_1,\ A_2)$ values cannot be reduced further by any further swaps.

Remarks:

1. Steps 1 and 2 make up a *try*. Among all tries we select the one with the highest $|\mathbf{X}'\mathbf{X}|$ and lowest $r_{\max}$, the maximum of the correlation coefficients in terms of the absolute value among the columns of the model matrix.

2. If n is not a multiple of four and a Hadamard matrix or a Plackett-Burman design is not available, FOLDOVER could use the matrices of maximal determinant in `http://www.indiana.edu/~maxdet/`.

Figure 2 shows the steps for constructing a half fraction of an MLFOD* for three 3-level factors and four 2-level factors in 16 runs $(m_3 = 3,\ m_2 = 4,\ n = 8)$ with $x = 2$. Figures 2

(a)	(b)	(c)	(d)	(e)
+++++++	-------	--0----	--0----	--0----
--+-+-+	++-+-+-	++-+-+-	0+-+-+-	0+-+-+-
+++----	+++----	++0----	++0----	++0----
--++-+-	--++-+-	0-++-+-	0-++-+-	0-++-+-
+-+--++	+-+--++	+-+--++	+-+--++	+0+--++
-+++--+	-+++--+	-+++--+	-+++--+	-+++--+
+-+++--	+-+++--	+0+++--	+0+++--	+0+++--
-++-++-	+--+--+	00-+--+	+0-+--+	+--+--+

Figure 2: Steps of the FOLDOVER algorithm in constructing an MLFOD* for three 3-level factors and four 2-level factors in 16 runs.

(a), 2 (b) and 2 (c) correspond to Step 1. Figures 2 (a) shows seven columns of a Hadamard matrix of order eight, which are randomly selected to form the matrix $\mathbf{H}$. Figure 2 (b) shows some rows of $\mathbf{H}$ multiplied with -1. Figure 2 (c) shows two 0's randomly inserted into each of the first three columns of $\mathbf{H}$. At the end of Step 1 $(A_1,\ A_2)$ values are (0.75, 21). Figures 2 (d) and 2 (e) correspond to Step 2. Figure 2 (d) shows the swap of the second and the eighth elements of the first column, after which $(A_1,\ A_2)$ values have been changed to (0.75, 12). Figures 2 (e) shows the swap of the fifth and the eighth elements of the second column, after which $(A_1,\ A_2)$ values have been changed to (0.75, 0). After these two swaps, the matrix $\mathbf{A}^*$ of the half fraction in Figures 2 (e) will have the form $2\mathbf{I} + 4\mathbf{J}$ and the matrix $\mathbf{B}$ will have the form $\mathrm{diag}(6, 6, 6, 8, 8, 8, 8)$.

4 Results and Discussion

Table 1 compares the 204 MLFODs and ADSD alternatives. The MLFODs were constructed from the Hadamard matrices of sizes $n = 8$ to 28 and the matrices of maximal determinant of sizes 10, 14, 18, 22, 26 and 30 (see `http://www.indiana.edu/~maxdet/`). For each matrix, two MLFODs with $m_3 = 1, \ldots, \frac{n}{2} - 1$ and $m_2 = n - m_3$ were given. The first MLFOD uses $x = \text{round}(\frac{n}{5})$ and the second MLFOD $x = \text{round}(\frac{n}{3})$. To ease our discussion, we use MLFOD($\frac{n}{5}$) to denote the former and MLFOD($\frac{n}{3}$) the latter. N, the run size of the designs in Table 1, is $2n^*$, where n^* is the size of the half fraction in (1). For MLFODs and ADSDs, $n^* = n$ and $n^* = n + 1$ repectively, where n is the size of the Hadamard matrix or matrix of maximal determinant used to construct the MLFOD or the conference matrix used to construct the ADSD.

The comparison is in terms of d_1, the first-order D-efficiency; d_2, second-order D-efficiency and $r_{\max}$. The D-efficiencies are calculated as

$$|\mathbf{X}'\mathbf{X}|^{1/p}/N, \tag{7}$$

where $\mathbf{X}$ is the model matrix, p the number of parameters in the model and N the run size of the MLFOD.

It can be seen in Table 1 that while the d_1 values of ADSDs are larger than those of MLFODs in general, their d_2 values are substantially smaller than those of MLFODs. With few exceptions, the $r_{\max}$ values of ADSDs are much larger than those of MLFODs. Out of 204 MLFODs in Table 1, 200 are MLFOD*'s. Recall that for MLFOD*'s, the $\mathbf{A}^*$ matrices in (3) are of the form $d\mathbf{I} + c\mathbf{J}$.

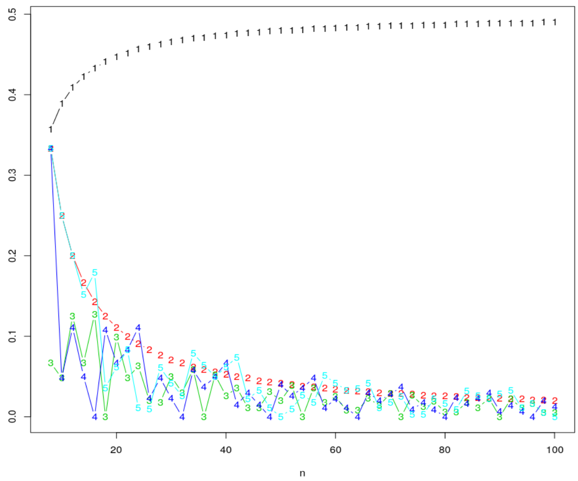

Figure 3: Correlation between two quadratic effects in terms of absolute value of (1) ADSDs, (2) MLFOD(2), (3) MLFOD($\frac{n}{3}$), (4) MLFOD($\frac{n}{4}$) and (5) MLFOD($\frac{n}{5}$) for various values of $n = 8$ to 100

The correlation between two quadratic effects of an MLFOD* is computed as:

$$r = \frac{c - \frac{b^2}{n^*}}{b - \frac{b^2}{n^*}} \tag{8}$$

Recall that n^* is the size of the half fraction in (1) which is n for our MDSD*'s and $n+1$ for ADSDs. Figure 3 shows the correlations between the two quadratic effects in terms of the absolute value of the (1) ADSDs, (2) MLFOD*(2), (3) MLFOD*($\frac{n}{3}$), (4) MLFOD*($\frac{n}{4}$) and (5) MLFOD*($\frac{n}{5}$) for various values of $n = 8, 10, \ldots, 100$. It can be seen that the correlations of ADSDs are increasing until they reach the limit, i. e. 0.5 and those of MLFODs are decreasing until they reach 0. Like ADSDs, MLFOD*(2)'s have two 0's in each of the 3-level columns of the half fraction in (1).

Bullington et al. (1993) described a thermostat experiment with 11 2-level factors in 12 runs conducted to identify the cause of early failures in thermostats manufactured by the Eaton Corporation. The 11 factors are: (**A**) Diaphragm plating rinse (clean/dirty); (**B**) Current density (min @ amps (5 @ 60/10 @ 15); (**C**) Sulfuric acid cleaning in seconds (3/30); (**D**) Diaphragm electro clean in minutes (2/12); (**E**) Beryllium copper grain size in inches (0.008/0.018); (**F**) Stress orientation to steam weld (perpendicular/parallel); (**G**) Diaphragm condition after brazing (wet/air-dried); (**H**) Heat treatment in hours @ 600°F (0.75/4); (**I**) Brazing machine water and flux (none/extra); (**J**) Power element electro clean time (short/long) and (**K**) Power element plating rinse (clean/dirty). This experiment was also reported in Mee (2009) p. 202. There is no good reason why we have to treat factors **B**, **C**, **D** and **H** as 2-level factors instead of 3-level (quantitative) factors. Let us assume that the experimenters are looking for the designs that treat these factors as 3-level.

Figure 4 shows the half fractions of the two candidate designs: (a) our 24-run MLFOD* and (b) 26-run ADSD for four 3-level factors and eight 2-level factors. The half fraction of our MLFOD* was constructed by converting the first four 2-level columns of a Hadamard matrix of order 12 to 3-level columns. We use an extra 2-level factor as the blocking factor. The $(d_1, d_2, r_{\max})$ values of our 24-run MLFOD* are (0.852, 0.617, 0.204) and those of the 26-run ADSD are (0.925, 0.557, 0.409). Having additional 0's (i. e. mid-levels) in each 3-level column of the MLFOD* improves the second-order efficiency at the cost of reducing the first-order efficiency. Note that if we set x to be two instead of four as in Figure 6 (a), the $(d_1, d_2, r_{\max})$ values will be (0.926, 0.58, 0.2).

Figure 5 shows the correlation cell plots (CCPs) of our MLFOD* and ADSD whose half fractions are in Figure 4. These CCPs, advocated by Jones & Nachtsheim (2011) show the pairwise (absolute) correlation between the two terms under study as a coloured square. Each plot in Figure 5 has 82 rows, 82 columns and 82^2 coloured cells ($82=4+12+\binom{12}{2}$)).

(a)

```
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000++++---++
+0--++---+-+
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-0++++-+-+--
+-0++--++--+
---0+-+-----
0-+-++--+-+-
0++0+-+-++-+
++00+++++---
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(b)

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+-+-++--+--+
+++++--+----
++-0+-+--+-+
-++-+-+-+-+-
+-0++++---+-
0+--++-+--++
-0+-++++-+--
+---+--++++-
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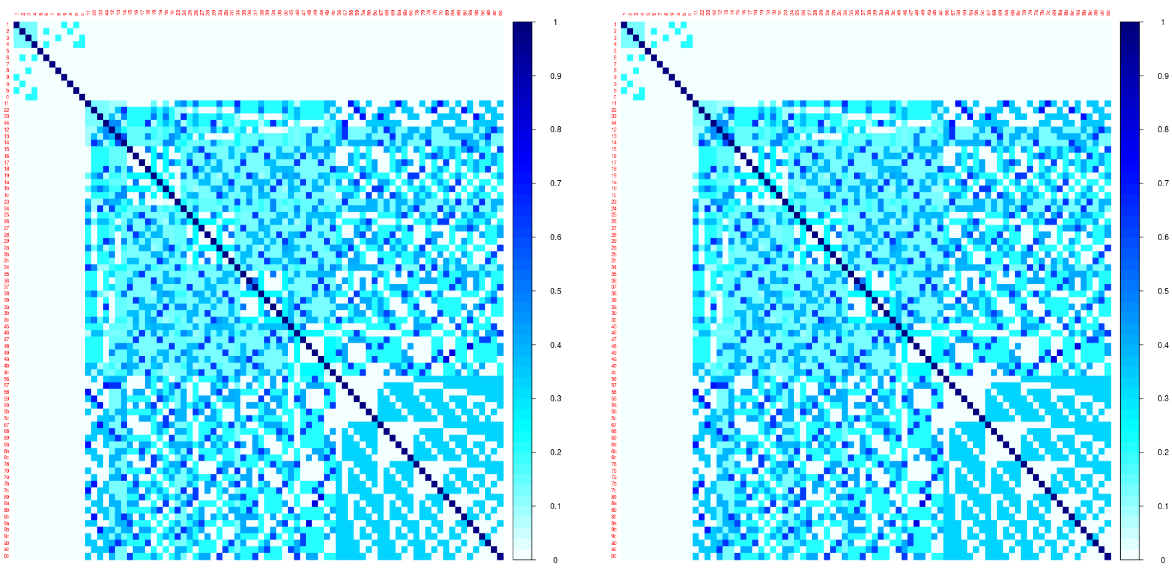
Figure 4: Half fractions of (a) our 24-run MLFOD* and (b) 26-run ADSD for four 3-level factors and eight 2-level factors.



(a) (b)

Figure 5: Correlation Cell Plots of our 24-run MLFOD* and 26-run ADSD for four 3-level factors and eight 2-level factors.

```
(a)

-+-00+++--+++
-+0+0---+++++
+00+--++-+++-
-------+-+--+
+0+--0-++-+++
++00+0+++++-+
--++-+++++++-+
0-+-+0+--++++
0--0++-+++++-
+0-0-++-++-++
00--0-+-+-+--
-++-00++++-+-
0-0++-+++--++
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(b)

0++++++++++++
++-0++-+-+---
+-++0--++--+-
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+--+-++++---+
+--++-++-+++-
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+0-+-+----+++
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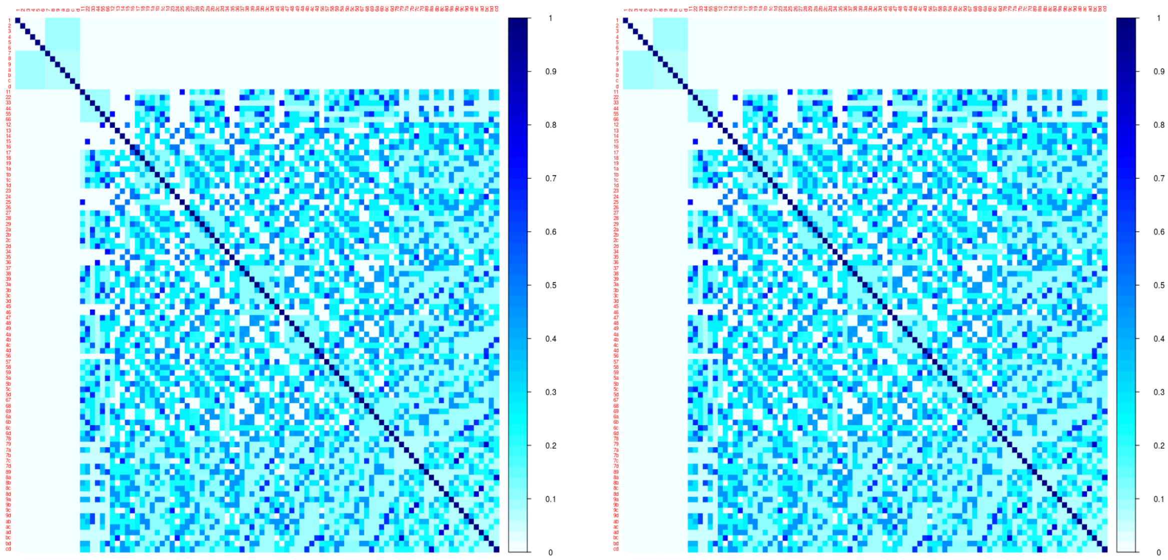
Figure 6: Half fractions of (a) our 26-run MLFOD* and (b) 30-run ADSD for six 3-level factors and seven 2-level factors.

The color of each cell goes from white to dark. The white cells imply no correlation while the dark ones imply a correlation of 1 or close to 1. It can be seen in Figure 5 that while the correlation between any 2-level MEs of our MLFOD* is zero, the correlation between any 3-level MEs of the ADSD is zero. This figure also shows that the magnitude of the correlation among the quadratic effects is more visible in the CCP of the ADSD than that of the MLFOD*.

Irvine et al. (1996) described a 2^{13-9} pulping experiment investigating the best method to remove lignin during the pulping stage without negatively impacting the strength and yield. The 13 factors are: (**A**) Wood chip presoaked (yes/no); (**B**) Chips pre-steamed for 10 min at 110^0C; (**C**) Initial effective alkali level in % (6/12); (**D**) Sulfide level in impregnation in % (3/10); (**E**) Liquor (black/white); (**F**) Liquor/wood ratio (3.5:1/6:1); (**G**) Impregnation temperature in ^{0}C (110/150); (**H**) Impregnation pressure in kPa (190/1140); (**J**) Impregnation time in min (10/40); (**K**) Anthraquinone in % (0.00/0.05); (**L**) Cook



(a) (b)

Figure 7: Correlation Cell Plots of (a) our 26-run MLFOD* and (b) 30-run ADSD for six 3-level factors and seven 2-level factors.

(a)

26	18	18	18	18	18	18	0	0	0	0	0	0	0	0	0	0	0	0	0
18	18	12	12	12	12	12	0	0	0	0	0	0	0	0	0	0	0	0	0
18	12	18	12	12	12	12	0	0	0	0	0	0	0	0	0	0	0	0	0
18	12	12	18	12	12	12	0	0	0	0	0	0	0	0	0	0	0	0	0
18	12	12	12	18	12	12	0	0	0	0	0	0	0	0	0	0	0	0	0
18	12	12	12	12	18	12	0	0	0	0	0	0	0	0	0	0	0	0	0
18	12	12	12	12	12	18	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	18	0	0	0	0	0	2	−2	2	−2	2	2	−2
0	0	0	0	0	0	0	0	18	0	0	0	0	2	−2	2	−2	2	2	−2
0	0	0	0	0	0	0	0	0	18	0	0	0	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	18	0	0	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0	18	0	2	−2	2	−2	2	2	−2
0	0	0	0	0	0	0	0	0	0	0	0	18	2	2	2	2	2	2	2
0	0	0	0	0	0	0	2	2	2	2	2	2	26	2	2	2	2	2	2
0	0	0	0	0	0	0	−2	−2	2	2	−2	2	2	26	2	2	2	2	2
0	0	0	0	0	0	0	2	2	2	2	2	2	2	2	26	2	2	2	2
0	0	0	0	0	0	0	−2	−2	2	2	−2	2	2	2	2	26	2	2	2
0	0	0	0	0	0	0	2	2	2	2	2	2	2	2	2	2	26	2	2
0	0	0	0	0	0	0	2	2	2	2	2	2	2	2	2	2	2	26	2
0	0	0	0	0	0	0	−2	−2	2	2	−2	2	2	2	2	2	2	2	26

(b)

30	26	26	26	26	26	26	0	0	0	0	0	0	0	0	0	0	0	0	0
26	26	24	24	24	24	24	0	0	0	0	0	0	0	0	0	0	0	0	0
26	24	26	24	24	24	24	0	0	0	0	0	0	0	0	0	0	0	0	0
26	24	24	26	24	24	24	0	0	0	0	0	0	0	0	0	0	0	0	0
26	24	24	24	26	24	24	0	0	0	0	0	0	0	0	0	0	0	0	0
26	24	24	24	24	26	24	0	0	0	0	0	0	0	0	0	0	0	0	0
26	24	24	24	24	24	26	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	26	0	0	0	0	0	2	2	−2	−2	−2	2	2
0	0	0	0	0	0	0	0	26	0	0	0	0	−2	−2	2	2	−2	2	2
0	0	0	0	0	0	0	0	0	26	0	0	0	2	−2	−2	2	−2	−2	−2
0	0	0	0	0	0	0	0	0	0	26	0	0	−2	2	2	−2	2	−2	−2
0	0	0	0	0	0	0	0	0	0	0	26	0	−2	2	−2	2	2	2	−2
0	0	0	0	0	0	0	0	0	0	0	0	26	−2	−2	−2	−2	−2	−2	−2
0	0	0	0	0	0	0	2	−2	2	−2	−2	−2	30	2	2	2	−2	−2	2
0	0	0	0	0	0	0	2	−2	−2	2	2	−2	2	30	−2	−2	2	2	−2
0	0	0	0	0	0	0	−2	2	−2	2	−2	−2	2	−2	30	−2	2	2	−2
0	0	0	0	0	0	0	−2	2	2	−2	2	−2	2	−2	−2	30	2	2	−2
0	0	0	0	0	0	0	−2	−2	−2	2	2	−2	−2	2	2	2	30	−2	2
0	0	0	0	0	0	0	2	2	−2	−2	2	−2	−2	2	2	2	−2	30	2
0	0	0	0	0	0	0	2	2	−2	−2	−2	−2	2	−2	−2	−2	2	2	30

Figure 8: The $\mathbf{X}'\mathbf{X}$ matrices of (a) our 26-run MLFOD* and (b) 30-run ADSD for six 3-level factors and seven 2-level factors.

temperature in ^{0}C (165/170); (**M**) Water quench (no/yes) and (**N**) Extended alkali wash for 1 hour (no/yes). This experiment was also reported in Mee (2009) p. 183. Again, like the previous experiment, let us assume that the experimenters are looking for the designs that treat the six factors **C**, **D**, **G**, **H**, **J** and **K** as 3-level quantitative factors instead of 2-level factors.

Figure 6 displays the half fractions of (a) our 26-run MLFOD* and (b) 30-run ADSD for six 3-level factors and seven 2-level factors. The half fraction of our MLFOD* was constructed by converting the first six 2-level columns of a 13×13 ± 1 matrix with maximal determinant (`http://www.indiana.edu/~maxdet/d13.html`) to 3-level columns. The $(d_1, d_2, r_{\max})$ values of our 26-run MLFOD* are (0.827, 0.547, 0.092) while the ones of the ADSD are (0.918, 0.455, 0.423). The CCPs of these two designs are in Figure 7. It can be seen in Figure 5 and Figure 7 that all MEs are orthogonal to the 2FIs. Also in these CCPs, the magnitude of the correlations among the quadratic effects of the candidate ADSDs are more severe than those of the MLFOD*s. At the same time, the magnitude of the correlations between the quadratic effects and the 2FIs of the candidate ADSDs is less visible than those of MLFOD*s. The $\mathbf{X}'\mathbf{X}$ matrices of the two designs are in Figure 8.

5 Conclusion

While most screening experiments include the 3-level quantitative factors, 2-level designs such as Plackett-Burman designs and FFDs of resolution III or IV have been used. This paper introduces a new class of MLFODs. This class of design retains the core benefits of ADSDs, i. e. the orthogonality between the MEs and the 2FIs and the orthogonality between the MEs and the quadratic effects. In addition, they have less severe correlation among the quadratic effects, require less runs, are more flexible in the sense that the 3-level columns can include more mid-levels and guarantee the orthogonality among 2-level columns when the input design is a Hadamard matrix or a Plackett-Burman design.

Users using our MLFODs, however, should be aware that, unlike ADSDs, MLFOD*s do not always guarantee the orthogonality among the 3-level factors.

The supplemental materials contain 510 text files of 510 half fractions: 306 half fractions MLFODs and ADSDs in Table 1 and 204 half fractions of MLFOD(2) and MLFOD($\frac{n}{4}$) with $m_3 = 1, \ldots, \frac{n}{2} - 1$ and $m_2 = n - m_3$ using the same input matrices. Also included in the supplemental materials are the java program `hmld` and the associated class files which implement the FOLDOVER algorithm in Section 3.

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Table 1: Comparison of D-efficiencies and $r_{\max}$ of MLFOD*'s and ADSDs

			MLFOD*($\frac{n}{5}$)			MLFOD*($\frac{n}{3}$)			ADSD		
$N\dagger$	m_3	m_2	d_1	d_2	$r_{\max}$	d_1	d_2	$r_{\max}$	d_1	d_2	$r_{\max}$
16	1	7	0.938	0.799	0.289	0.901	0.787	0.474	0.944	0.797	0.126
16	2	6	0.901	0.670	0.333	0.832	0.661	0.474	0.912	0.666	0.357
16	3	5	0.880	0.572	0.333	0.808	0.592	0.200	0.885	0.572	0.357
20	1	9	0.915	0.791	0.224	0.898	0.795	0.359	0.925	0.795	0.273
20	2	8	0.899	0.686	0.250	0.869	0.698	0.359	0.917	0.684	0.389
20	3	7	0.898	0.610	0.250	0.852	0.631	0.200	0.905	0.597	0.389
20	4	6	0.885	0.535	0.250	0.813	0.566	0.200	0.895	0.529	0.389
24	1	11	0.972	0.846	0.183	0.940	0.848	0.204	0.965	0.836	0.084
24	2	10	0.951	0.734	0.200	0.899	0.745	0.204	0.949	0.719	0.409
24	3	9	0.935	0.648	0.200	0.881	0.678	0.204	0.937	0.629	0.409
24	4	8	0.926	0.580	0.200	0.852	0.617	0.204	0.925	0.557	0.409
24	5	7	0.910	0.514	0.200	0.816	0.562	0.204	0.913	0.499	0.409
28	1	13	0.939	0.843	0.242	0.923	0.846	0.267	0.943	0.827	0.200
28	2	12	0.919	0.751	0.242	0.893	0.761	0.267	0.938	0.725	0.423
28	3	11	0.902	0.680	0.242	0.859	0.689	0.267	0.932	0.640	0.423
28	4	10	0.884	0.620	0.242	0.826	0.630	0.267	0.928	0.573	0.423
28	5	9	0.856	0.565	0.242	0.791	0.579	0.267	0.923	0.517	0.423
28	6	8	0.828	0.519	0.242	0.745	0.531	0.267	0.918	0.470	0.423
32	1	15	0.976	0.880	0.208	0.957	0.881	0.226	0.975	0.861	0.063
32	2	14	0.954	0.785	0.208	0.923	0.791	0.226	0.966	0.755	0.433
32	3	13	0.936	0.710	0.208	0.896	0.722	0.226	0.959	0.669	0.433
32	4	12	0.919	0.648	0.208	0.870	0.664	0.226	0.951	0.598	0.433
32	5	11	0.901	0.595	0.208	0.842	0.613	0.226	0.943	0.539	0.433
32	6	10	0.876	0.547	0.231	0.799	0.563	0.226	0.937	0.490	0.433
32	7	9	0.851	0.506	0.231	0.764	0.522	0.226	0.93	0.449	0.433
36	1	17	0.948	0.870	0.126	0.935	0.871	0.136	0.952	0.848	0.158
36	2	16	0.933	0.795	0.126	0.915	0.800	0.136	0.948	0.753	0.441
36	3	15	0.921	0.733	0.126	0.894	0.739	0.136	0.946	0.674	0.441
36	4	14	0.904	0.678	0.126	0.866	0.684	0.167	0.944	0.608	0.441
36	5	13	0.884	0.629	0.214	0.832	0.632	0.167	0.941	0.551	0.441
36	6	12	0.857	0.583	0.214	0.801	0.589	0.272	0.938	0.504	0.441
36	7	11	0.847	0.552	0.252	0.765	0.548	0.272	0.935	0.463	0.441
36	8	10	0.819	0.516	0.252	0.751	0.523	0.272	0.932	0.428	0.441

†Run size of MLFODs. For the same set of (m_3, m_2) ADSD requires two extra runs.

Table 1: Comparison of D-efficiencies and $r_{\max}$ of MLFOD*'s and ADSDs (Cont.)

			MLFOD*($\frac{n}{5}$)			MLFOD*($\frac{n}{3}$)			ADSD		
$N\dagger$	m_3	m_2	d_1	d_2	$r_{\max}$	d_1	d_2	$r_{\max}$	d_1	d_2	$r_{\max}$
40	1	19	0.979	0.902	0.112	0.960	0.899	0.186	0.971	0.870	0.143
40	2	18	0.961	0.822	0.112	0.933	0.825	0.186	0.966	0.776	0.447
40	3	17	0.947	0.758	0.112	0.920	0.772	0.186	0.969	0.700	0.447
40	4	16	0.934	0.704	0.112	0.895	0.717	0.186	0.964	0.631	0.447
40	5	15	0.915	0.653	0.112	0.864	0.665	0.186	0.959	0.573	0.447
40	6	14	0.894	0.609	0.188	0.816	0.610	0.186	0.955	0.523	0.447
40	7	13	0.883	0.575	0.224	0.775	0.563	0.186	0.951	0.481	0.447
40	8	12	0.866	0.542	0.224	0.736	0.523	0.308	0.946	0.444	0.447
40	9	11	0.840	0.509	0.224	0.730	0.501	0.310	0.942	0.413	0.447
44	1	21	0.961	0.889	0.101	0.947	0.891	0.165	0.955	0.861	0.130
44	2	20	0.950	0.819	0.101	0.929	0.827	0.165	0.951	0.773	0.452
44	3	19	0.939	0.759	0.101	0.912	0.773	0.165	0.951	0.699	0.452
44	4	18	0.929	0.707	0.101	0.890	0.722	0.165	0.949	0.634	0.452
44	5	17	0.916	0.660	0.201	0.866	0.676	0.165	0.947	0.579	0.452
44	6	16	0.903	0.620	0.201	0.835	0.631	0.165	0.944	0.531	0.452
44	7	15	0.883	0.581	0.201	0.777	0.576	0.267	0.941	0.490	0.452
44	8	14	0.866	0.547	0.201	0.741	0.538	0.267	0.94	0.454	0.452
44	9	13	0.852	0.519	0.201	0.755	0.530	0.275	0.938	0.422	0.452
44	10	12	0.829	0.489	0.201	0.727	0.502	0.275	0.936	0.395	0.452
48	1	23	0.981	0.916	0.140	0.968	0.915	0.204	0.984	0.891	0.042
48	2	22	0.966	0.848	0.140	0.944	0.848	0.204	0.974	0.797	0.457
48	3	21	0.955	0.791	0.140	0.928	0.796	0.204	0.976	0.725	0.457
48	4	20	0.946	0.743	0.140	0.906	0.746	0.204	0.967	0.655	0.457
48	5	19	0.931	0.698	0.140	0.881	0.700	0.204	0.964	0.599	0.457
48	6	18	0.913	0.655	0.140	0.850	0.655	0.204	0.961	0.550	0.457
48	7	17	0.896	0.619	0.140	0.819	0.615	0.204	0.962	0.510	0.457
48	8	16	0.884	0.588	0.158	0.775	0.571	0.204	0.959	0.472	0.457
48	9	15	0.869	0.560	0.234	0.764	0.549	0.306	0.956	0.440	0.457
48	10	14	0.844	0.529	0.234	0.722	0.513	0.306	0.953	0.411	0.457
48	11	13	0.822	0.503	0.234	0.549	0.414	0.408	0.951	0.385	0.457
52	1	25	0.966	0.905	0.128	0.954	0.906	0.143	0.962	0.875	0.111
52	2	24	0.956	0.844	0.128	0.936	0.849	0.143	0.961	0.794	0.460
52	3	23	0.947	0.791	0.128	0.918	0.798	0.143	0.96	0.723	0.460
52	4	22	0.937	0.743	0.128	0.898	0.752	0.143	0.959	0.661	0.460
52	5	21	0.924	0.700	0.128	0.873	0.707	0.176	0.958	0.607	0.460
52	6	20	0.915	0.663	0.143	0.844	0.664	0.238	0.957	0.560	0.460

†Run size of MLFODs. For the same set of $(m_3,\ m_2)$ ADSD requires two extra runs.

Table 1: Comparison of D-efficiencies and $r_{\max}$ of MLFOD*'s and ADSDs (Cont.)

			MLFOD*($\frac{n}{5}$)			MLFOD*($\frac{n}{3}$)			ADSD		
N†	m_3	m_2	d_1	d_2	$r_{\max}$	d_1	d_2	$r_{\max}$	d_1	d_2	$r_{\max}$
52	7	19	0.899	0.626	0.143	0.812	0.624	0.238	0.956	0.518	0.460
52	8	18	0.886	0.595	0.214	0.795	0.596	0.238	0.954	0.481	0.460
52	9	17	0.872	0.567	0.214	0.765	0.564	0.238	0.953	0.449	0.460
52	10	16	0.853	0.538	0.214	0.733	0.533	0.238	0.952	0.421	0.460
52	11	15	0.835	0.513	0.214	0.699	0.504	0.294	0.951	0.395	0.460
52	12	14	0.805	0.485	0.214	0.684	0.487	0.294	0.949	0.372	0.460
56	1	27	0.984	0.927	0.161	0.974	0.926	0.130	0.972	0.888	0.103
56	2	26	0.971	0.867	0.161	0.954	0.867	0.130	0.973	0.810	0.463
56	3	25	0.961	0.816	0.161	0.941	0.820	0.130	0.974	0.741	0.463
56	4	24	0.952	0.771	0.161	0.921	0.774	0.158	0.971	0.678	0.463
56	5	23	0.939	0.729	0.161	0.906	0.735	0.217	0.969	0.623	0.463
56	6	22	0.928	0.691	0.161	0.874	0.689	0.217	0.967	0.575	0.463
56	7	21	0.912	0.655	0.161	0.849	0.652	0.217	0.967	0.534	0.463
56	8	20	0.897	0.623	0.161	0.822	0.617	0.217	0.965	0.496	0.463
56	9	19	0.868	0.585	0.161	0.792	0.583	0.217	0.965	0.464	0.463
56	10	18	0.854	0.560	0.227	0.769	0.557	0.263	0.963	0.435	0.463
56	11	17	0.844	0.537	0.227	0.746	0.532	0.263	0.958	0.407	0.463
56	12	16	0.830	0.515	0.242	0.726	0.510	0.303	0.959	0.384	0.463
56	13	15	0.799	0.486	0.242	0.698	0.486	0.303	0.957	0.363	0.463
60	1	29	0.969	0.916	0.149	0.959	0.916	0.163	0.966	0.886	0.097
60	2	28	0.960	0.862	0.149	0.945	0.866	0.163	0.965	0.810	0.466
60	3	27	0.952	0.814	0.149	0.933	0.822	0.163	0.965	0.742	0.466
60	4	26	0.943	0.770	0.149	0.914	0.777	0.163	0.964	0.682	0.466
60	5	25	0.933	0.730	0.149	0.893	0.736	0.163	0.963	0.629	0.466
60	6	24	0.922	0.694	0.149	0.865	0.693	0.163	0.962	0.583	0.466
60	7	23	0.913	0.662	0.149	0.843	0.659	0.245	0.962	0.542	0.466
60	8	22	0.900	0.631	0.149	0.824	0.629	0.245	0.961	0.505	0.466
60	9	21	0.881	0.599	0.149	0.787	0.590	0.245	0.96	0.472	0.466
60	10	20	0.871	0.575	0.224	0.772	0.568	0.250	0.959	0.443	0.466
60	11	19	0.855	0.550	0.224	0.741	0.538	0.250	0.958	0.417	0.466
60	12	18	0.840	0.527	0.224	0.717	0.514	0.327	0.957	0.393	0.466
60	13	17	0.812	0.500	0.224	0.713	0.502	0.327	0.956	0.372	0.466
60	14	16	0.799	0.482	0.224	0.675	0.447	0.327	0.955	0.352	0.466

†Run size of MLFODs. For the same set of (m_3, m_2) ADSD requires two extra runs.