The boundedness of non-autonomous nonlinear time-varying delay difference equations subject to external disturbances and its applications

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Abstract

This paper provides some new results on the equi-boundedness and the ultimate boundedness for a general class of non-autonomous nonlinear time-varying delay difference equations subject to external bounded disturbances. The disturbances are assumed to vary within a known interval whose lower bound may be different from zero. First, we employ fixed point theory and compute some difference inequalities to derive some new results on the existence of positive solutions and the equi-boundedness of solutions. Second, we derive a sufficient condition for the ultimate boundedness of solutions. This condition is easy to check and allows us to compute directly both the smallest ultimate upper bound and the largest ultimate lower bound. Third, we apply the obtained results to some discrete population models. Finally, numerical examples are given to illustrate the effectiveness of the proposed results.

Keywords: Fixed point theorem, contraction mapping, nonlinear difference equation, boundedness, time-varying delay.

1. Introduction

The dynamics of species population models has been one of the strong motivations for the impressive development of the theory of continuous dynamical systems as well as discrete dynamical systems. A lot of articles have been written regarding this subject (see, for example, [1]-[23], [26]-[40] and the references therein). In particular, in [1]-[23], [27], [31]-[40] many interesting results on properties of solutions of several discrete models derived from mathematical biology have been reported. Note that, the effect of disturbances was not investigated in these discrete population models. While, as well known, in the real world, the effect of disturbances is a common issue related to the study and analysis of dynamical systems. Disturbances could arise from modelling errors, ageing, uncertainties, and are present in any realistic problem (see, [24], [25]). Therefore, it is very essential to investigate qualitative properties of solutions of non-autonomous discrete population models with time-varying parameters and external disturbances.

Motivated by the above discussion, in this paper, we consider the following general class of non-autonomous nonlinear difference equations with N time-varying delays and bounded distur-

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bances

$$x_{n+1} = \lambda_n x_n + \alpha_n F(n, x_n, x_{n-m_1(n)}, \ldots, x_{n-m_N(n)}) + d_n,\ n \in \mathbb{N}_{n_0}, \quad (1)$$
$$x_\theta = \psi_\theta \in [0,\infty),\ \theta \in D_{n_0}, \quad (2)$$

where $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_{n_0} = \{n_0+1, n_0+2, ...\}$, $n_0 \in \mathbb{N}$, (λ_n) and (α_n) are sequences of positive real numbers, map F maps $\mathbb{N} \times [0,\infty)^{N+1}$ to $[0,\infty)$, time-varying delays $m_i(n),\ i = 1, \ldots, N$ are given integer-valued functions. If $m_i(n)$ are bounded and $0 \le m_1(n) \le \ldots \le m_N(n) \le m_N$ for all $n \in \mathbb{N}$, then for each integer $n_0 \ge 0$, we define D_{n_0} is a set of integers belong to the interval $[-m_N, n_0]$. If $m_i(n)$ are unbounded then D_{n_0} is a set of integers belong to $(-\infty, n_0]$. ψ is the bounded initial value function and $d_n \in [0,\infty)$ is the disturbance varying within a known interval, that is

$$d_* \le d_n \le d^*,\ \forall n \in \mathbb{N}. \quad (3)$$

The equation (1) includes several discrete models derived from mathematical biology such as the Nicholson's blowflies model and the Bobwhite quail population model. Namely, when $d_n \equiv 0$, $m_i(n) = i,\ i = 1, 2, \ldots, N$, $\lambda_n \equiv \lambda$ is a positive constant, $\alpha_n \equiv 1$ and $F(n, x_0, x_1, \ldots, x_N) = \sum_{i=0}^{N} \delta_i x_i e^{-x_i}$, equation (1) is reduced to the model Nicholson's blowflies model (see, for example, [26]); and when $d_n \equiv 0$, $m_i(n) = i,\ i = 1, 2, \ldots, N$, $\lambda_n \equiv \lambda$ is a positive constant, $\alpha_n \equiv 1$ and $F(n, x_0, x_1, \ldots, x_N) = \sum_{i=0}^{N} \frac{\zeta_i x_i}{1+x_i^p}$, equation (1) is reduced to the Bobwhite quail population model (see, for example, [26]).

Many interesting results on the asymptotic stability of solutions of non-autonomous nonlinear difference equations of the form (1) without the the effect of the disturbance d_n, that is, $d_n \equiv 0$, have been reported in the literature (see, for example, [1], [6], [9], [11], [18], [21] and the references therein). However, as far as we know, it is very difficult to achieve asymptotic stability for dynamical systems perturbed by unknown-but bounded disturbances. Instead, the convergence of the system's trajectories within a bounded set after a large enough time can be guaranteed. To day and to the best of our knowledge, the equi-boundedness and the ultimate boundedness of solutions of equations of the form (1) have still not been studied elsewhere, which motivate the present study.

The remainder of this paper is organized as follows. In Section 2, we present the main results. In Section 3, we apply the obtained results in Section 2 to determine conditions for the equi-boundedness and the ultimate boundedness of solutions of the Nicholson's blowflies model and the Bobwhite quail population model. Some numerical examples are given in Section 4. Finally, a conclusion is drawn in Section 5.

2. Main results

To obtain the main results, we will use the following definitions and lemma. For the sake of convenience, we adopt the notation $\sum_{k=a}^{b} x_k = 0$, $\prod_{k=a}^{b} x_k = 1$ for any $a > b$.

Definition 1. *By a solution of (1), we mean a sequence $(x_{n,\psi,d})$ such that $x_n := x_{n,\psi,d} = \psi_n$ on D_{n_0} and $(x_{n,\psi,d})$ satisfies (1) for $n \in \mathbb{N}_{n_0}$.*

Clearly, equation (1) has a unique non-negative solution $(x_{n,\psi,d})$ with the given initial condition ψ.

Definition 2. *A solution $(x_{n,\psi,d})$ of (1) is said to be bounded if there exists a $B(n_0,\psi,d) > 0$ such that $x_n \leq B(n_0,\psi,d)$ for $n \geq n_0$.*

Definition 3. *The solutions of (1) are said to be equi-bounded if for any n_0 and any $B_1 > 0$ there exists $B_2 = B_2(n_0,B_1,d) > 0$ such that $\psi_n \leq B_1$ on D_{n_0} implies $x_n \leq B_2$ for $n \geq n_0$.*

Definition 4. *A positive solution $(x_{n,\psi,d})$ of (1) is called ultimately bounded if for any initial condition ψ and for any disturbance d_n satisfying (3), there exist positive constants q^* and q_* (which are called ultimate upper bound and ultimate lower bound of system (1), respectively) such that*

$$q_* \leq \liminf_{n\to\infty} x_{n,\psi,d} \leq \limsup_{n\to\infty} x_{n,\psi,d} \leq q^*. \tag{4}$$

Lemma 1 $(x_{n,\psi,d})$ *is a solution of equation (1) if and only if*

$$x_n = x_{n_0}\prod_{s=n_0}^{n-1}\lambda_s + \sum_{t=n_0}^{n-1}\Big[\alpha_t F(t,x_t,x_{t-m_1(t)},\ldots,x_{t-m_N(t)}) + d_t\Big]\prod_{s=t+1}^{n-1}\lambda_s. \tag{5}$$

Proof. Indeed, it is not hard to see that equation (1) is equivalent to the following equation

$$\Delta\Big(x_n\prod_{s=n_0}^{n-1}\lambda_s^{-1}\Big) = [\alpha_n F(n,x_n,x_{n-m_1(n)},\ldots,x_{n-m_N(n)}) + d_n]\prod_{s=n_0}^{n}\lambda_s^{-1}, \tag{6}$$

where $\Delta x_n = x_{n+1} - x_n$. Summing equation (6) from n_0 to $n-1$ gives

$$\sum_{t=n_0}^{n-1}\Delta\Big(x_t\prod_{s=n_0}^{t-1}\lambda_s^{-1}\Big) = \sum_{t=n_0}^{n-1}[\alpha_t F(t,x_t,x_{t-m_1(t)},\ldots,x_{t-m_N(t)}) + d_t]\prod_{s=n_0}^{t}\lambda_s^{-1}$$
$$x_n = x_{n_0}\prod_{s=n_0}^{n-1}\lambda_s + \sum_{t=n_0}^{n-1}[\alpha_t F(t,x_t,x_{t-m_1(t)},\ldots,x_{t-m_N(t)}) + d_t]\prod_{s=t+1}^{n-1}\lambda_s.$$

The proof is complete.

2.1. The existence of positive solutions

The following theorem gives a sufficient for the existence of positive solutions of (1).

Theorem 1. *Assume that the following conditions are satisfied:*

i) For each $n \in \mathbb{N}$, $F(n,0,0,\ldots,0) = 0$ and $F(n,x_0,\ldots,x_n)$ is L_n^i-locally Lipschitz in x_i ($i = 0,1,\ldots,N$). That is, there is a $K > 0$ such that if $0 \leq x_i \leq K,\ 0 \leq y_i \leq K,\ i = 0,1,\ldots,N$, then

$$|F(n,x_0,\ldots,x_n) - F(n,y_0,\ldots,y_N)| \leq \sum_{i=0}^{N} L_n^i|x_i - y_i| \tag{7}$$

for positive constants L_n^i $(i = 0, 1, \dots, N)$.

ii) There exist $\sigma \in (0,1)$ and $n_1 \in \mathbb{N}_{n_0}$ such that

$$\sum_{t=n_0}^{n-1} \sum_{i=0}^{N} L_t^i \alpha_t \prod_{s=t+1}^{n-1} \lambda_s \le \sigma, \; n \ge n_1. \tag{8}$$

Then (1) has a unique positive solution for every initial value function ψ satisfying $\psi_n > 0$ on D_{n_0}.

Proof. Let ψ be an initial value function satisfying $\psi_n > 0$ on D_{n_0}. Define

$$S_0 = \{\varphi : D_{n_0} \cup \mathbb{N}_{n_0} \longrightarrow (0,\infty) |\; \varphi_n = \psi_n \text{ on } D_{n_0}\}, \tag{9}$$

where $\|\varphi\| = \max_{n \in D_{n_0} \cup \mathbb{N}_{n_0}} |\varphi_n|$. It is not hard to check that $(S_0, ||.||)$ is a complete metric space. Next, we suppose that (φ^ℓ) is a Cauchy sequence in S_0. We have

$$\forall \varepsilon > 0, \exists \ell_0 \in \mathbb{N} : \forall k, \ell \ge \ell_0 : \left\|\varphi^\ell - \varphi^k\right\| < \varepsilon$$

or

$$\forall \varepsilon > 0, \exists \ell_0 \in \mathbb{N} : \forall k, \ell \ge \ell_0 : \max_{n \in D_{n_0} \cup \mathbb{N}_{n_0}} \left|\left(\varphi^\ell - \varphi^k\right)_n\right| < \varepsilon$$

or

$$\forall \varepsilon > 0, \exists \ell_0 \in \mathbb{N} : \forall k, \ell \ge \ell_0 : \left|\left(\varphi^\ell - \varphi^k\right)_n\right| < \varepsilon, \forall n \in D_{n_0} \cup \mathbb{N}_{n_0}.$$

Fixed n, (φ_n^ℓ) is a Cauchy sequence in $[0,\infty) \subset \mathbb{R}$. In view of $\mathbb{R}$ is a complete metric space,

$$\exists \varphi_n \in [0,\infty) : \varphi_n = \lim_{\ell \to \infty} \varphi_n^\ell.$$

It is not hard to see that $\varphi \in S_0$ and hence $(S_0, ||.||)$ is a complete metric space.

Let us define a mapping $P : S_0 \longrightarrow S_0$ by $(P\varphi)_n = \psi_n$ on D_{n_0} and

$$(P\varphi)_n = \psi_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \Big[\alpha_t F(t, \varphi_t, \varphi_{t-m_1(t)}, \dots, \varphi_{t-m_N(t)}) + d_t\Big] \prod_{s=t+1}^{n-1} \lambda_s, \tag{10}$$

for $n \in \mathbb{N}_{n_0}$. Since $\psi_n > 0$ on D_{n_0}, (λ_n) and (α_n) are sequences of positive real numbers and map F maps $\mathbb{N} \times [0,\infty)^{N+1}$ to $[0,\infty)$, we have $(P\varphi)_n > 0$ for all $n \in D_{n_0} \cup \mathbb{N}_{n_0}$. Hence P maps from S_0 to itself. Moreover, let $\varphi, \eta \in S_0$, we get for $n \ge n_1$,

$$|(P\varphi)_n - (P\eta)_n| \;\; \le \;\; \sum_{t=n_0}^{n-1} \sum_{i=0}^{N} L_t^i \alpha_t \prod_{s=t+1}^{n-1} \lambda_s \|\varphi - \eta\| \le \sigma \|\varphi - \eta\|.$$

Therefore, P is a contraction map. By the contraction mapping principle, P has a unique fixed point $\varphi^* \in S_0$, which satisfies $\varphi_n^* = \psi_n$ for $n \in D_{n_0}$ and

$$\varphi_n^* = \varphi_{n_0}^* \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \Big[\alpha_t F(t, \varphi_t^*, \varphi_{t-m_1(t)}^*, \dots, \varphi_{t-m_N(t)}^*) + d_t\Big] \prod_{s=t+1}^{n-1} \lambda_s, \; \forall n \in \mathbb{N}_{n_0}$$

i.e., (φ_n^*) is a solution of (1). The proof is complete.

2.2. The equi-boundedness

The following theorem provides conditions for solutions of (1) to be equi-bounded.

Theorem 2. *Assume that condition (3) and condition i) of Theorem 1 are satisfied and there exist $b > 0$, $\beta \in (0,1)$, $n_1 \in \mathbb{N}_{n_0}$ satisfying the following condition:*

$$\prod_{s=n_0}^{n-1} \lambda_s \leq b, \ \Big[\sum_{t=n_0}^{n-1}\sum_{i=0}^{N} L_t^i \alpha_t + d^*\Big] \prod_{s=t+1}^{n-1} \lambda_s \leq \beta, \ n \geq n_1. \tag{11}$$

Then the solutions of (1) are equi-bounded.

Proof. Let c_1 be a positive constant. Choose $c_2 > 1$ such that $c_2 \geq \frac{bc_1}{1-\beta}$.

Let ψ be a bounded initial function satisfying $\psi_n \leq c_1$ on D_{n_0}. Define

$$S_1 = \Big\{\varphi : D_{n_0} \cup \mathbb{N}_{n_0} \longrightarrow (0,\infty) |\ \varphi_n = \psi_n \text{ on } D_{n_0} \text{ and } ||\varphi|| \leq c_2\Big\}. \tag{12}$$

Now, based on the similar lines of work in the proof of Theorem 1, we see that $(S_1, ||.||)$ is a complete metric space.

Define mapping $P : S_1 \longrightarrow S_1$ by (10). We will show that P maps from S_1 to S_1. Indeed, we have

$$\begin{aligned}(P\varphi)_n &= \psi_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \Big[\alpha_t F(t, \varphi_t, \varphi_{t-m_1(t)}, \dots, \varphi_{t-m_N(t)}) + d_t\Big] \prod_{s=t+1}^{n-1} \lambda_s \\ &\leq \psi_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \Big[\alpha_t \sum_{i=0}^{N} L_t^i \varphi_{t-m_i(t)} + d^*\Big] \prod_{s=t+1}^{n-1} \lambda_s.\end{aligned}$$

Since $||\varphi|| \leq c_2, \varphi_{t-m_i(t)} \leq c_2$. Hence,

$$(P\varphi)_n \leq c_1 b + \Big[c_2 \sum_{t=n_0}^{n-1}\sum_{i=0}^{N} L_t^i \alpha_t + d^*\Big] \prod_{s=t+1}^{n-1} \lambda_s \leq c_1 b + \beta c_2 \leq c_2.$$

Hence P maps from S_1 to itself. Then, based on the similar lines of work in the proof of Theorem 1, we can verify that P is a contraction under the supremum norm. Thus, by the contraction mapping principle, P has a unique fixed point $\varphi^* \in S_1$. We have

$$(P\varphi^*)_n = \varphi_n^* = \psi_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \Big[\alpha_t F(t, \varphi^*_{t-m_1(t)}, \dots, \varphi^*_{t-m_N(t)}) + d_t\Big] \prod_{s=t+1}^{n-1} \lambda_s.$$

Since $n_0 \in D_{n_0}$ and $\varphi^* \in S_1$, $\psi_{n_0} = \varphi^*_{n_0}$. Hence

$$\varphi_n^* = \varphi^*_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \Big[\alpha_t F(t, \varphi_t^*, \varphi^*_{t-m_1(t)}, \dots, \varphi^*_{t-m_N(t)}) + d_t\Big] \prod_{s=t+1}^{n-1} \lambda_s,$$

i.e, φ_n^* is a solution of (1). This prove that solutions of (1) are equi-bounded. The proof is complete.

2.3. The ultimate-boundedness

In this subsection, we assume that $0 \le m_1(n) \le \ldots \le m_N(n) \le m_N$ for all $n \in \mathbb{N}$, where m_N is a known integer and D_0 is a set of integers belong to the interval $[-m_N, 0]$. We will derive the smallest ultimate upper bound q^* and the largest ultimate lower bound q_* for equation (1).

Theorem 3. *Assume that condition (3) and the following conditions are satisfied,*

$$0 < \lambda_* \le \lambda_n \le \lambda^* < \infty,\ 0 < \alpha_* \le \alpha_n \le \alpha^* < \infty,\ \forall n \in \mathbb{N}, \tag{13}$$

$$\sum_{i=0}^{N} \beta_i x_i \le F(n, x_0, x_1, \ldots, x_N) \le \sum_{i=0}^{N} \gamma_i x_i \tag{14}$$

and

$$0 < \lambda_+ = \lambda^* + \sum_{i=0}^{N} \alpha^* \gamma_i < 1,\ 0 < \lambda_- = \lambda_* + \sum_{i=0}^{N} \alpha_* \beta_i < 1, \tag{15}$$

where β_i, γ_i $(i = 0, 1, \ldots, N)$ are nonnegative numbers. Then every solution of (1) is ultimately bounded with the smallest ultimate upper bound $q^* = \frac{d^*}{1-\lambda_+}$ and the largest ultimate lower bound $q_* = \frac{d_*}{1-\lambda_-}$.

Proof. From (1), (13)-(15) we have the following inequalities

$$x_{n+1} \le (\lambda^* + \alpha^* \gamma_0) x_n + \sum_{i=1}^{N} \alpha^* \gamma_i x_{n-m_i(n)} + d_n \tag{16}$$

and

$$x_{n+1} \ge (\lambda_* + \alpha_* \beta_0) x_n + \sum_{i=1}^{N} \alpha_* \beta_i x_{n-m_i(n)} + d_n. \tag{17}$$

Now, we consider the following linear difference equations

$$\bar{x}_{n+1} = (\lambda^* + \alpha^* \gamma_0) \bar{x}_n + \sum_{i=1}^{N} \alpha^* \gamma_i \bar{x}_{n-m_i(n)} + d_n, \tag{18}$$

$$\bar{x}_\theta = \bar{\phi}_\theta \ge 0,\ \theta \in D_0 = \{-m_N, -m_N + 1, \ldots, 0\} \tag{19}$$

and

$$\underline{x}_{n+1} = (\lambda_* + \alpha_* \beta_0) \underline{x}_n + \sum_{i=1}^{N} \alpha_* \beta_i \underline{x}_{n-m_i(n)} + d_n, \tag{20}$$

$$\underline{x}_\theta = \underline{\phi}_\theta \ge 0,\ \theta \in D_0 = \{-m_N, -m_N + 1, \ldots, 0\}. \tag{21}$$

It is not hard to see that

$$\underline{x}_{n,\underline{\phi}_\theta,d} \le x_{n,\psi,d} \le \bar{x}_{n,\bar{\phi}_\theta,d},\ n \in \mathbb{N}. \tag{22}$$

Since $\lambda_- < 1$ and $\lambda_+ < 1$, there exist $\eta_* > 0$, $\eta^* > 0$ such that

$$\lambda_- \eta_* < \eta_*, \lambda_+ \eta^* < \eta^*, \underline{\phi}_\theta \ge q_* - \eta_* = \underline{\psi}_\theta, \bar{\phi}_\theta \le q^* + \eta^* = \bar{\psi}_\theta, \theta \in D_0. \tag{23}$$

Denote $e_n = \bar{y}_n - \bar{x}_n$ and $\epsilon_n = d^* - d_n$, where $\bar{y}_n$ is the solution of the following equation

$$\bar{y}_{n+1} = (\lambda^* + \alpha^*\gamma_0)\bar{y}_n + \sum_{i=1}^{N} \alpha^*\gamma_i \bar{y}_{n-m_i(n)} + d^*, \tag{24}$$

$$\bar{y}_\theta = \bar{\psi}_\theta,\ \theta \in D_0. \tag{25}$$

Then, we obtain

$$e_{n+1} = (\lambda^* + \alpha^*\gamma_0)\bar{e}_n + \sum_{i=1}^{N} \alpha^*\gamma_i \bar{e}_{n-m_i(n)} + \epsilon_n, \tag{26}$$

$$e_\theta = \bar{\psi}_\theta - \bar{\phi}_\theta \geq 0,\ \theta \in D_0. \tag{27}$$

Since system (26)-(27) is positive, we have

$$\bar{x}_{n,\underline{x}_{n,\underline{\phi}_\theta,d}} \leq \bar{y}_{n,\bar{\psi}_\theta,d},\ \forall n \in \mathbb{N}. \tag{28}$$

Similarly, we can prove that

$$\underline{y}_{n,\underline{\psi}_\theta,d} \leq \underline{x}_{n,\underline{\phi}_\theta,d},\ \forall n \in \mathbb{N}, \tag{29}$$

where $\underline{y}_{n,\underline{\psi}_\theta,d}$ is the solution of the following equation

$$\underline{y}_{n+1} = (\lambda_* + \alpha_*\beta_0)\underline{y}_n + \sum_{i=1}^{N} \alpha_*\beta_i \underline{y}_{n-m_i(n)} + d_*, \tag{30}$$

$$\underline{y}_\theta = \underline{\psi}_\theta,\ \theta \in D_0. \tag{31}$$

From (22), (28) and (29) we obtain

$$\underline{y}_{n,\underline{\psi}_\theta,d} \leq \underline{x}_{n,\underline{\phi}_\theta,d} \leq x_{n,\psi,d} \leq \bar{x}_{n,\bar{\phi}_\theta,d} \leq \bar{y}_{n,\bar{\psi}_\theta,d},\ n \in \mathbb{N}. \tag{32}$$

In the following, we will prove that

$$q_* \leq \liminf_{n\to\infty} x_{n,\psi,d} \leq \limsup_{n\to\infty} x_{n,\psi,d} \leq q^*. \tag{33}$$

In order to prove inequality $\liminf_{n\to\infty} x_{n,\psi,d} \geq q_*$, we define the set

$$J_k = \{k(m_N + 1) + j, j = 1, 2, \ldots, m_N + 1\},\ \forall k \in \mathbb{N}. \tag{34}$$

We shall show that

$$\underline{y}_{n,\underline{\psi}_\theta,d} \geq q_* - \lambda_-^k \eta_*,\ \forall k \in \mathbb{N},\ \forall n \in J_k. \tag{35}$$

For $k = 0$, $n = 1$, from (30) we obtain

$$\underline{y}_{1,\underline{\psi}_\theta,d} = (\lambda_* + \alpha_*\beta_0)\underline{y}_{0,\underline{\psi}_\theta,d} + \sum_{i=1}^{N} \alpha_*\beta_i \underline{y}_{-m_i(n),\underline{\psi}_\theta,d} + d_* \tag{36}$$

$$\geq (\lambda_* + \alpha_*\beta_0)(q_* - \eta_*) + \sum_{i=1}^{N} \alpha_*\beta_i (q_* - \eta_*) + d_* \tag{37}$$

$$= (\lambda_- q_* + d_*) + \lambda_-\eta_* = q_* - \lambda_-\eta_* \geq q_* - \eta_*. \tag{38}$$

Similarly, $k = 0$, $n = 2, \ldots, m_N + 1$, we get

$$\underline{y}_{n,\underline{\psi}_\theta,d} \geq q_* - \eta_*. \tag{39}$$

For $k = 1$, $n = m_N + 2$, from (30) we obtain

$$\begin{aligned}
\underline{y}_{m_N+2,\underline{\psi}_\theta,d} &= (\lambda_* + \alpha_*\beta_0)\underline{y}_{m_N+1,\underline{\psi}_\theta,d} + \sum_{i=1}^{N} \alpha_*\beta_i \underline{y}_{m_N+1-m_i(n),\underline{\psi}_\theta,d} + d_* &(40)\\
&\geq (\lambda_* + \alpha_*\beta_0)(q_* - \lambda_-\eta_*) + \sum_{i=1}^{N} \alpha_*\beta_i(q_* - \lambda_-\eta_*) + d_* &(41)\\
&= \lambda_- q_* + d_* + \lambda_-^2\eta_* = q_* - \lambda_-^2\eta_* \geq q_* - \lambda_-\eta_*. &(42)
\end{aligned}$$

Similarly, $k = 1$, $n = m_N + 3, \ldots, 2m_N + 1$, we get

$$\underline{y}_{n,\underline{\psi}_\theta,d} \geq q_* - \lambda_-\eta_*. \tag{43}$$

Now, we assume that (35) is true for all $k \leq K$ and for all n that is less than the end of J_k. Then, for $n + 1 \in J_{K+1}$, we have

$$\begin{aligned}
\underline{y}_{n+1,\underline{\psi}_\theta,d} &= (\lambda_* + \alpha_*\beta_0)\underline{y}_{n,\underline{\psi}_\theta,d} + \sum_{i=1}^{N} \alpha_*\beta_i \underline{y}_{n-m_i(n),\underline{\psi}_\theta,d} + d_*\\
&\geq q_* - \lambda^{K+1}\eta_*. &(44)
\end{aligned}$$

Similarly, for $n \in J_{K+1}$, we get

$$\underline{y}_{n,\underline{\psi}_\theta,d} \geq q_* - \lambda_-^K\eta_*. \tag{45}$$

Thus, by induction, (35) is true for all $k \in \mathbb{N}$ and for all $n \in J_k$. Hence, from (32) and (35) we have

$$\liminf_{n\to\infty} x_{n,\psi,d} \geq \liminf_{n\to\infty} \underline{y}_{n,\underline{\psi}_\theta,d} \geq q_*. \tag{46}$$

In order to conclude that q_* is the largest ultimate lower bound of (1), we need to prove that

$$\lim_{n\to\infty} \underline{y}_{n,q_*,d} = q_*. \tag{47}$$

For this, let us denote $\underline{\nu}_n = \underline{y}_n - q_*$. Then, it follows from (30) that

$$\begin{aligned}
\underline{\nu}_{n+1} &= (\lambda_* + \alpha_*\beta_0)\underline{\nu}_n + \sum_{i=1}^{N} \alpha_*\beta_i \underline{\nu}_{n-m_i(n)}, &(48)\\
\underline{\nu}_\theta &= 0,\ \theta \in D_0 &(49)
\end{aligned}$$

and $\underline{\nu}_{n,0} = \underline{y}_{n,q_*} - q_*$, $n \in \mathbb{N}$ is a solution of (48) with initial condition $\underline{\nu}_\theta = 0$. Since $\lambda_- < 1$, equation (48) is exponentially stable, that is, there exist a scalar $0 < \underline{\sigma} < 1$ and a nonnegative function $\underline{\mu}$ such that

$$0 \leq \underline{\nu}_{n,0} \leq \underline{\mu}_{q_*}\underline{\sigma}^n,\ n \in \mathbb{N}. \tag{50}$$

This implies that

$$q_* \leq \underline{y}_{n,q_*} \leq q_* + \underline{\mu}_{q_*} \underline{\sigma}^n, \ n \in \mathbb{N}. \tag{51}$$

Let n tend to infinity in (51) we obtain (47).

The proof of inequality $\limsup_{n\to\infty} x_{n,\psi,d} \leq q^*$ can be obtained similarly as the proof of inequality $\liminf_{n\to\infty} x_{n,\psi,d} \geq q_*$, so we omit it here.

3. Applications

In this section, we apply the results obtained in Section 2 to the Nicholson's blowflies model

$$x_{n+1} = \lambda_n x_n + \alpha_n \sum_{i=0}^{N} \delta_i x_{n-m_i(n)} e^{-q_i(n) x_{n-m_i(n)}} + d_n \tag{52}$$

and the bobwhite quail population model

$$x_{n+1} = \lambda_n x_n + \alpha_n \sum_{i=0}^{N} \zeta_i \frac{x_{n-m_i(n)}}{1 + x^p_{n-m_i(n)}} + d_n, \tag{53}$$

where $(\lambda_n), (\alpha_n), (q_i(n)), i = 0, 1, \ldots, N$ are sequences of positive real numbers, $p, \delta_i, \zeta_i \in (0, \infty)$, $m_0(n) \equiv 0$, $m_i(n)$, $i = 1, 2, \ldots, N$ are sequences of positive integer numbers and $d_n \in [0, \infty)$ is the disturbance satisfying condition (3).

We see that, the model (52) is in the form of equation (1) with

$$F(n, x_0, x_1, \ldots, x_N) = \sum_{i=0}^{N} \delta_i x_i e^{-q_i(n) x_i}.$$

We have

$$\frac{\partial F(n, x_0, x_1, \ldots, x_N)}{\partial x_i} = \delta_i (1 - x_i q_i(n)) e^{-q_i(n) x_i}.$$

Thus, for each $n \in \mathbb{N}$,

$$\Big| \frac{\partial F(n, x_0, x_1, \ldots, x_N)}{\partial x_i} \Big| \leq \delta_i (1 + q_i(n)), \ \forall x_i \geq 0, \ i = 1, 2, \ldots, N,$$

which implies that $F(n, x_0, x_1, \ldots, x_N)$ is L^i_n-locally Lipschitz in x_i with $L^i_n = \delta_i(1 + q_i(n))$, $(i = 1, 2, \ldots, N)$. Hence, from Theorem 2, we obtain the following corollary.

Corollary 1. *Assume that condition (3) is satisfied and there exist $b > 0$, $\beta \in (0, 1)$, $n_1 \in \mathbb{N}_{n_0}$ satisfying the following condition:*

$$\prod_{s=n_0}^{n-1} \lambda_s \leq b, \ \Big[\sum_{t=n_0}^{n-1} \sum_{i=0}^{N} (1 + q_i(t)) \delta_i \alpha_t + d^* \Big] \prod_{s=t+1}^{n-1} \lambda_s \leq \beta, \ n \geq n_1. \tag{54}$$

Then the solutions of (52) are equi-bounded.

On the other hand, for $q_i(n) > 0,\ \forall n \in \mathbb{N}$, we have

$$0 \le F(n, x_0, x_1, \dots, x_N) = \sum_{i=0}^{N} \delta_i x_i e^{-q_i(n)x_i} \le \sum_{i=0}^{N} \delta_i x_i.$$

Hence, from Theorem 3, we obtain the following corollary.

Corollary 2. *Assume that condition (3) is satisfied,* $0 \le m_1(n) \le \ldots \le m_N(n) \le m_N,\ \forall n \in \mathbb{N}$, $0 < \lambda_* \le \lambda_n \le \lambda^* < \infty,\ 0 < \alpha_* \le \alpha_n \le \alpha^* < \infty,\ q_i(n) > 0\ \forall n \in \mathbb{N}$ *and* $\lambda^* + \sum\limits_{i=0}^{N} \alpha^* \delta_i < 1$. *Then every solution of (52) is ultimately bounded with the smallest ultimate upper bound* $q^* = \frac{d^*}{1-\lambda^*-\sum\limits_{i=0}^{N} \alpha^* \delta_i}$ *and the largest ultimate lower bound* $q_* = \frac{d_*}{1-\lambda_*}$.

Next, we consider the model (53). Clearly, this model is in the form of equation (1) with $F(n, x_0, x_1, \dots, x_N) = \sum\limits_{i=0}^{N} \frac{\zeta_i x_i}{1+x_i^p}$. It is not hard to check that for each $n \in \mathbb{N}$,

$$\Big|\frac{\partial F(n, x_0, x_1, \dots, x_N)}{\partial x_i}\Big| \le \zeta_i,\ \forall x_i \ge 0,\ i = 1, 2, \dots, N,\ (\text{in case } p = 1)$$

and

$$\Big|\frac{\partial F(n, x_0, x_1, \dots, x_N)}{\partial x_i}\Big| \le \frac{\zeta_i (p-1)^2}{4p},\ \forall x_i \ge 0,\ i = 1, 2, \dots, N,\ (\text{in case } p > 1).$$

Therefore $F(n, x_0, x_1, \dots, x_N)$ is L_n^i-locally Lipschitz in x_i with $L_n^i = 1$ when $p = 1$ and with $L_n^i = \frac{\zeta_i (p-1)^2}{4p}$ when $p > 1$. Hence, from Theorem 2, we obtain the following corollary.

Corollary 3. *Assume that condition (3) is satisfied and there exist* $b > 0,\ \beta \in (0, 1),\ n_1 \in \mathbb{N}_{n_0}$ *such that either* $p = 1$ *and*

$$\prod_{s=n_0}^{n-1} \lambda_s \le b,\ \Big[\sum_{t=n_0}^{n-1} \sum_{i=0}^{N} \zeta_i \alpha_t + d^*\Big] \prod_{s=t+1}^{n-1} \lambda_s \le \beta,\ n \ge n_1 \tag{55}$$

or $p > 1$ *and*

$$\prod_{s=n_0}^{n-1} \lambda_s \le b,\ \Big[\sum_{t=n_0}^{n-1} \sum_{i=0}^{N} \frac{\zeta_i (p-1)^2}{4p} \alpha_t + d^*\Big] \prod_{s=t+1}^{n-1} \lambda_s \le \beta,\ n \ge n_1. \tag{56}$$

Then the solutions of (53) are equi-bounded.

On the other hand, for $p > 0$, we have

$$0 \le F(n, x_0, x_1, \dots, x_N) = \sum_{i=0}^{N} \frac{\zeta_i x_i}{1 + x_i^p} \le \sum_{i=0}^{N} \zeta_i x_i.$$

Hence, from Theorem 3, we obtain the following corollary.

Corollary 4. *Assume that condition (3) is satisfied,* $0 \le m_1(n) \le \ldots \le m_N(n) \le m_N,\ \forall n \in \mathbb{N}$, $0 < \lambda_* \le \lambda_n \le \lambda^* < \infty,\ 0 < \alpha_* \le \alpha_n \le \alpha^* < \infty,\ \forall n \in \mathbb{N}$ *and* $\lambda^* + \sum\limits_{i=0}^{N} \alpha^* \zeta_i < 1$. *Then every solution of (53) is ultimately bounded with the smallest ultimate upper bound* $q^* = \frac{d^*}{1-\lambda^*-\sum\limits_{i=0}^{N} \alpha^* \zeta_i}$ *and the largest ultimate lower bound* $q_* = \frac{d_*}{1-\lambda_*}$.

4. Numerical examples

Example 1. Consider the model (52) with $\lambda_n = \frac{2017n+6}{20170(n+2)}$, $\alpha_n = \frac{n+1}{5n+2}$, $q_0(n) = q_1(n) = \frac{1}{n+1}$ $\forall n \in \mathbb{N}$, $d_n = 0.25|\sin n| + 0.15$, $\delta_0 = 0.25$, $\delta_1 = 0.2$, $m_1(n) = \left[\frac{n}{2}\right]$, where $\left[\cdot\right]$ as the integer function. Clearly, there exists $b = 1$ such that $\prod_{s=n_0}^{n-1} \lambda_s \leq b$. Moreover, we have for $n_0 = 0$,

$$\begin{aligned}
&\Big[\sum_{t=n_0}^{n-1}\sum_{i=0}^{N}(1+q_i(t))\delta_i\alpha_t + d^*\Big]\prod_{s=t+1}^{n-1}\lambda_s \\
= \;& \sum_{t=0}^{n-1}\Big(0.45(\frac{t+2}{5t+2}) + 0.25\Big)\prod_{s=t+1}^{n-1}\Big(\frac{2017s+6}{20170(s+2)}\Big) \\
\leq \;& 0.9444\Big(1 - \Big(\frac{1}{10}\Big)^n\Big) \leq 0.9444, \; n \in \mathbb{N}.
\end{aligned}$$

Hence, according to the Corollary 1, the solutions of (52) are equi-bounded.

Next, we consider the model (52) with $\lambda_n = \frac{5n+1}{10n+4}$, $\alpha_n = \frac{n+1}{n+2}$, $q_0(n) = q_1(n) = \frac{1}{n+1}$ $\forall n \in \mathbb{N}$, $m_1(n) = 50 - \frac{1}{n}$, $d_n = |\sin n| + 1$, $\delta_0 = 0.25$, $\delta_1 = 0.2$. We have $\lambda_* = 0.25$, $\lambda^* = 0.5$, $\alpha_* = 0.5$, $\alpha^* = 1$, $\lambda^* + \alpha^*(\delta_0 + \delta_1) = 0.95 < 1$, $d^* = 2$, $d_* = 1$. According to the Corollary 2, every positive solution of the model (52) ultimately bounded with the smallest ultimate upper bound $q^* = 40$ and the largest ultimate lower bound $q_* = 1.3333$.

Example 2. *Consider the model (53) with* $\lambda_n = \frac{n+1}{9(n+2)}$, $\alpha_n = \frac{5n+9}{5n+2}$, $q_0(n) = q_1(n) = \frac{1}{n+1}$ $\forall n \in \mathbb{N}$, $d_n = 0.2|\sin n| + 0.3$, $\zeta_0 = 0.1667$, $\zeta_1 = 0.1429$, $p = 0.2$, $m_1(n) = \left[\frac{n}{2}\right]$, *where* $\left[\cdot\right]$ *as the integer function. It is easy to check that all conditions of Corollary 3 are satisfied. Hence, the solutions of (53) are equi-bounded.*

Next, we consider the model (53) with $\lambda_n = \frac{5n+1}{10n+4}$, $\alpha_n = \frac{5n+9}{5n+18}$, $q_0(n) = q_1(n) = \frac{1}{n+1}$ $\forall n \in \mathbb{N}$, $p = 0.2$, $m_1(n) = 50 - \frac{1}{n}$, $d_n = |\sin n| + 1$, $\zeta_0 = 0.1667$, $\zeta_1 = 0.1429$. *We have* $\lambda_* = 0.25$, $\lambda^* = 0.5$, $\alpha_* = 0.5$, $\alpha^* = 1$, $\lambda^* + \alpha^*(\zeta_0 + \zeta_1) = 0.8095 < 1$, $d^* = 2$, $d_* = 1$. *According to the Corollary 4, every positive solution of the model (53) ultimately bounded with the smallest ultimate upper bound* $q^* = 10.5$ *and the largest ultimate lower bound* $q_* = 1.3333$.

5. Conclusion

In this paper, we have provided some new results on the equi-boundedness and the ultimate boundedness for a general class of non-autonomous nonlinear time-varying delay difference equations subject to external bounded disturbances. By using the fixed point theory together with some analytical techniques, we derive some new results on the existence of positive solutions, the equi-boundedness and the ultimate boundedness of solutions of the above class of difference equations. We have applied the obtained results to analyze the equi-boundedness and the ultimate boundedness of the Nicholson's blowflies model and the bobwhite quail population model. Numerical examples have also been given to illustrate the effectiveness of the proposed theoretical results.

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