# GENERALIZED COHEN-MACAULAYNESS AND NON-COHEN-MACAULAY LOCUS OF CANONICAL MODULES

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**Abstract** [1]. Let $(R, \mathfrak{m})$ be a Noetherian local ring which is a quotient of a Gorenstein local ring. Let $M$ be a finitely generated $R$-module. Denote by $K_M$ the canonical module of $M$. In this paper, we study the generalized Cohen-Macaulayness and the non-Cohen-Macaulay locus of $K_M$. Firstly we introduce the notion of canonical system of parameters of $M$ in order to characterize the generalized Cohen-Macaulayness of $K_M$. We give two other parametric characterizations for $K_M$ to be generalized Cohen-Macaulay. Then we present the relation between the non-Cohen-Macaulay locus of $K_M$ and that of $M$.

## 1 Introduction

The depth and the Cohen-Macaulayness of canonical modules have attracted the interest of a number of researchers, see [A], [AG], [Sch1], [Nh], [BN]. Aoyama and Goto [AG] proved that if $R$ is a Noetherian local with the total quotient ring $Q(R)$ such that $R$ is unmixed and $R$ admits the canonical module $K_R$, then $K_R$ is a Cohen-Macaulay $R$-module if and only if there exists a Cohen-Macaulay intermediate ring $B$ between $R$ and $Q(R)$ such that $B$ is a finitely generated $R$-module with $\dim_R(B/R) \leq \dim R - 2$ and $\dim B_{\mathfrak{n}} = \dim R$ for any maximal ideal $\mathfrak{n}$ of $B$. However, the fact is not valid any more whenever $\dim_R(B/R) = \dim R - 1$, see Example 2.5.

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Let $(R, \mathfrak{m})$ be a Noetherian local ring, let $M$ be a finitely generated $R$-module with $\dim M = d$. For each system of parameters (s.o.p. for short) $(x_1, \ldots, x_d)$ of $M$, set

$$I(x_1, \ldots, x_d; M) = \ell_R(M/(x_1, \ldots, x_d)M) - e(x_1, \ldots, x_d; M).$$

It is well-known that $M$ is Cohen-Macaulay if and only if $I(x_1, \ldots, x_d; M) = 0$ for some (for all) s.o.p. $(x_1, \ldots, x_d)$ of $M$. A similar parametric characterization for the Cohen-Macaulayness of canonical module was given in [Nh] and [BN] as follows. Suppose that $R$ is a quotient of a Gorenstein local ring. Denote by $K_M$ the canonical module of $M$. For an Artinian $R$-module $A$, denote by $\mathrm{Rl}(A) := \ell_R(A/\mathfrak{m}^s A)$ the residual length of $A$ defined by Sharp and Hamieh [SH], where $s > 0$ is an integer such that $\mathfrak{m}^s A = \mathfrak{m}^n A$ for all $n \geq s$. Then $K_M$ is Cohen-Macaulay if and only if $\mathrm{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1, \ldots, x_{d-3})M)\right) = 0$ for some (for all) strict f-sequence $(x_1, \ldots, x_d)$ of $M$. Here, the notion of strict f-sequence was introduced in [CMN], and if $(x_1, \ldots, x_d)$ is a strict f-sequence of $M$, then it is a s.o.p. of $M$.

Set $I(M) := \sup I(x_1, \ldots, x_d; M)$, where the supremum runs over all s.o.p $(x_1, \ldots, x_d)$ of $M$. We say that $M$ is *generalized Cohen-Macaulay* if $I(M) < \infty$, see [CST].

**Theorem 1.1.** (See [CST], [Tr]). *The following statements are equivalent:*

(a) *$M$ is generalized Cohen-Macaulay;*

(b) *There exists a s.o.p. $(x_1, \ldots, x_d)$ of $M$ such that* $\sup\limits_{n_1, \ldots, n_d \in \mathbb{N}} I(x_1^{n_1}, \ldots, x_d^{n_d}; M) < \infty$*;*

(c) *$M$ has a standard s.o.p. $(x_1, \ldots, x_d)$, i.e. $I(x_1, \ldots, x_d; M) = I(x_1^2, \ldots, x_d^2; M)$.*

In this paper, firstly we establish an analogue for the canonical modules of the parametric characterizations in Theorem 1.1 for generalized Cohen-Macaulay modules, where the role of the number $\mathrm{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1, \ldots, x_{d-3})M)\right)$ in the study of $K_M$ is as useful as that of the number $I(x_1, \ldots, x_d; M)$ in the study of $M$, for strict f-sequences $(x_1, \ldots, x_d)$ of $M$. We introduce the notion of canonical system of parameters (canonical s.o.p. for short) as follows.

**Definition 1.2.** A strict f-sequence $\underline{x} = (x_1, \ldots, x_d)$ is said to be a *canonical s.o.p.* of $M$ if

$$\mathrm{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1, \ldots, x_{d-3})M)\right) = \mathrm{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1^2, \ldots, x_{d-3}^2)M)\right).$$

If $\underline{x}$ is at the same time an unconditioned strict f-sequence and a canonical s.o.p. of $M$, then $\underline{x}$ is said to be an *unconditioned canonical s.o.p.* of $M$.

The following theorem is the first main result of this paper.

**Theorem 1.3.** *Suppose that $R$ is a quotient of a Gorenstein local ring. The following four statements are equivalent:*

(a) *$K_M$ is generalized Cohen-Macaulay.*

(b) $c_M := \sup\limits_{\underline{x}} \mathrm{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1, \ldots, x_{d-3})M)\right) < \infty$ *where $\underline{x} = (x_1, \ldots, x_d)$ runs over all strict* f*-sequences of $M$.*

(c) *There exists a strict* f*-sequence* $(x_1, \ldots, x_d)$ *of* $M$ *such that*

$$\sup_{n_1,\ldots,n_{d-3}\in\mathbb{N}} \operatorname{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_{d-3}^{n_{d-3}})M)\right) < \infty.$$

(d) *There is an unconditioned canonical s.o.p. of* $M$*.*

*Futhermore, if* $(x_1, \ldots, x_d)$ *is an unconditioned canonical s.o.p. of* $M$*, then*

$$\operatorname{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1, \ldots, x_{d-3})M)\right) = c_M = \sum_{i=0}^{d-3} \binom{d-3}{i} \ell(H^{i+2}_{\mathfrak{m}}(K_M)).$$

It should be mentioned that the statements (b) and (c) of Theorem 1.3 improve the main result of [LN].

Secondly, Y. Aoyama [A, Theorem 1] studied the relation between the depth of $K_R$ and that of $R$ in case where $R$ is not Cohen-Macaulay. He proved that for given integers $0 \le r < n$ and $2 \le s \le n$, there exists a complete local ring $R$ such that $\dim R = n$, $\operatorname{depth} R = r$ and $\operatorname{depth} K_R = s$. This is the motivation for us to discuss about the relation between the non Cohen-Macaulay locus of $K_M$ and that of $M$.

Denote by $\operatorname{nCM}(M)$ the non-Cohen-Macaulay locus of $M$. If $R$ is a quotient of a Gorenstein local ring, then $\operatorname{nCM}(M)$ is closed under Zariski topology and $\dim \operatorname{nCM}(M) \le d-1$, see [C]. Moreover, if $M$ is unmixed, then $\dim \operatorname{nCM}(M) \le d-2$.

The following theorem is the second main result of this paper.

**Theorem 1.4.** *Suppose that* $R$ *is a quotient of a Gorenstein local ring. The following statements are true.*

(a) $\dim \operatorname{nCM}(K_M) \le \min\{d-3, \dim \operatorname{nCM}(M)\}$.

(b) *For given integers* $-1 \le s \le d-3$ *and* $s \le r \le d-2$*, there exists a complete unmixed Noetherian local ring* $(R, \mathfrak{m})$ *such that* $\dim \operatorname{nCM}(R) = r$ *and* $\dim \operatorname{nCM}(K_R) = s$*.*

In the next section, we present some preliminaries that will be used in the sequel. Section 3 and Section 4 are devoted to prove the main results of this paper (Theorems 1.3, 1.4).

## 2 Preliminaries

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring which is a quotient of an $n$-dimensional local Gorenstein ring $(R', \mathfrak{m}')$. Let $M$ be a finitely generated $R$-module with $\dim M = d$. For each integer $i \ge 0$, let $K^i_M := \operatorname{Ext}^{n-i}_{R'}(M, R')$ denote the $i$-th *deficiency module* of $M$. Then $K^i_M$ is a finitely generated $R$-module and the local duality (see [BS, 11.2.6]) gives the isomorphism $H^i_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R(K^i_M, E(R/\mathfrak{m}))$, where $E(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. Let $K_M$ be the *canonical module* $K^d_M$ of $M$. For an Artinian

$R$-module $A$, let $\mathrm{Rl}(A) := \ell_R(A/\mathfrak{m}^s A)$ be the *residual length* of $A$ defined by Sharp-Hamieh [SH], where $s > 0$ is an integer such that $\mathfrak{m}^n A = \mathfrak{m}^s A$ for all $n \geq s$.

The notion of filter regular sequence (f-sequence for short) introduced in [CST] can be considered as a generalization of the known concept of regular sequence. An element $x \in \mathfrak{m}$ is said to be a *filter regular element* (f-element for short) of $M$ if $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R M \setminus \{\mathfrak{m}\}$. A sequence $(x_1, \ldots, x_t)$ of elements in $\mathfrak{m}$ is said to be an f-*sequence* of $M$ if $x_i$ is an f-element of $M/(x_1, \ldots, x_{i-1})M$ for all $i \leq t$.

**Remark 2.1.** *An element $x \in \mathfrak{m}$ is an* f*-element of $M$ if and only if $\ell_R(0 :_M x) < \infty$. Moreover, each* f*-sequence of length $d$ is a s.o.p. of $M$.*

A special kind of f-sequences is the class of strict f-sequences introduced in [CMN]. In the original definition of strict f-sequence, the set of attached primes $\mathrm{Att}_R H^i_{\mathfrak{m}}(M)$ defined by I. G. Macdonald [Mac] was used. However, we note that $\mathrm{Att}_R H^i_{\mathfrak{m}}(M) = \mathrm{Ass}_R K^i_M$ by [S, Theorem 2.3], therefore we can recall the notion of strict f-sequence as follows.

**Definition 2.2.** An element $x \in \mathfrak{m}$ is said to be a *strict* f*-element* of $M$ if $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \Big(\bigcup_{i=1}^{d} \mathrm{Ass}_R K^i_M\Big) \setminus \{\mathfrak{m}\}$. A sequence $(x_1, \ldots, x_t)$ of elements in $\mathfrak{m}$ is said to be a *strict* f*-sequence* of $M$ if $x_{j+1}$ is a strict f-element of $M/(x_1, \ldots, x_j)M$ for all $j = 0, \ldots, t-1$. A sequence $(x_1, \ldots, x_t)$ of elemets in $\mathfrak{m}$ is said to be an *unconditioned strict* f*-sequence* of $M$ if it is a strict f-sequence in any order.

Note that $\mathrm{Ass}_R M \subseteq \bigcup_{i=0}^{d} \mathrm{Ass}_R K^i_M$, see [Sch2, Proposition 2.3(c)]. Hence, each strict f-sequence is an f-sequence of $M$. In particular, if $x \in \mathfrak{m}$ is a strict f-element of $M$, then $\ell_R(0 :_M x) < \infty$. Moreover, if $(x_1, \ldots, x_d)$ is a strict f-sequence, then it is a s.o.p. of $M$.

Here are some properties of strict f-sequence that we need in the proof of the main results.

**Lemma 2.3.** ([CMN, Lemmas 3.4, 4.2 ], [GN, Theorem 3.5])

(a) *A sequence $(x_1, \ldots, x_t)$ of elements in $\mathfrak{m}$ is a strict* f*-sequence of $M$ if and only if it is an* f*-sequence of $K^i_M$ for all integers $i \geq 0$.*

(b) *If $(x_1, \ldots, x_t) \in \mathfrak{m}$ is a strict* f*-sequence of $M$, then so is $(x_1^{n_1}, \ldots, x_t^{n_t})$ for all positive integers $n_1, \ldots, n_t$.*

(c) *For each integer $t > 0$, there exists an unconditioned strict* f*-sequence of $M$ of length $t$.*

**Lemma 2.4.** ([LN, Lemmas 2.5, 2.7]) *Let $x \in \mathfrak{m}$ be a strict* f*-element of $M$. The following statements are true.*

(a) *For each integer $i \geq 0$, there exists an integer $n_0$ such that for all $n \geq n_0$ we have*

$$\mathrm{Rl}(H^i_{\mathfrak{m}}(M)) = \ell_R(H^0_{\mathfrak{m}}(K^i_M)) = \ell_R(0 :_{K^i_M} x^n).$$

(b) *For each integer $i \geq 1$, there is an exact sequence*

$$0 \to K_M^{i+1}/xK_M^{i+1} \to K_{M/xM}^i \to (0 :_{K_M^i} x) \to 0.$$

*In particular,* $H_{\mathfrak{m}}^i\big(K_M/xK_M\big) \cong H_{\mathfrak{m}}^i\big(K_{M/xM}\big)$ *for any $i \geq 2$.*

Next, we discuss about the Cohen-Macaulayness and generalized Cohen-Macaulayness of the canonical module. Following P. Schenzel [Sch2, Definition 5.1], $R$ is said to have a *birational Macaulayfication* if there is an intermediate ring $B$ between $R$ and $Q(R)$ such that $B$ is a finitely generated Cohen-Macaulay $R$-module. As we mentioned in the introduction, Aoyama and Goto [AG] proved that if $R$ is unmixed, then $K_R$ is Cohen-Macaulay if and only if there exists a birational Macaulayfication $B$ of $R$ such that $\dim_R(B/R) \leq \dim R - 2$. When this is the case, $B$ is uniquely determined and $B \cong \mathrm{End}_R(K_R)$ as an $R$-algebra. Note that the condition $\dim_R(B/R) \leq \dim R - 2$ can not be removed. The following example given by S. Goto shows that the result does not valid any more if $\dim_R(B/R) = \dim R - 1$.

**Example 2.5.** Let $A = F[X, Y]$ be the polynomial ring over an infinite field $F$ and $J = (X^3, V)(X^3, XV, Y^3)$, where $V = X^2 + XY + Y^2$. Let $\mathfrak{M} = (X, Y)$. Then $\sqrt{J} = \mathfrak{M}$. We set $I = JA_{\mathfrak{M}}$. Then the Rees algebra $\mathcal{R} = \mathcal{R}(I)$ of $I$ is a Buchsbaum ring with $\operatorname{depth} \mathcal{R}(I) = 2$. Since $A_{\mathfrak{M}}$ is a regular local ring of dimension 2, $\overline{\mathcal{R}} = \mathcal{R}(\overline{I})$ is a Cohen-Macaulay ring, but $K_{\mathcal{R}}$ is not a Cohen-Macaulay $\mathcal{R}$-module. Therefore $K_{\mathcal{R}_{\mathfrak{n}}}$ is not a Cohen-Macaulay $\mathcal{R}_{\mathfrak{n}}$-module where $\mathfrak{n}$ denotes the graded maximal ideal of $\mathcal{R}$, although the Noetherian local domain $\mathcal{R}_{\mathfrak{n}}$ possesses a birational Cohen-Macaulayfication.

It is well-known that $M$ is Cohen-Macaulay if and only if $I(x_1, \ldots, x_d; M) = 0$ for some (for all) s.o.p. $(x_1, \ldots, x_d)$ of $M$, where

$$I(x_1, \ldots, x_d; M) := \ell(M/(x_1, \ldots, x_d)M) - e(x_1, \ldots, x_d; M)$$

and $e(x_1, \ldots, x_d; M)$ is the multiplicity of $M$ with respect to $(x_1, \ldots, x_d)$. Moreover, if $x \in \mathfrak{m}$ is an $M$-regular element, then $M$ is Cohen-Macaulay if and only if so is $M/xM$.

It is clear that if $M$ is Cohen-Macaulay, then so is $K_M$. The converse statement is not true, see Theorem 1.4(b). Note that $K_M$ satisfies the condition Serre $(S_2)$. Therefore $K_M$ is Cohen-Macaulay whenever $d \leq 2$. In case where $d \geq 3$, we have the following characterizations for the canonical module to be Cohen-Macaulay.

**Lemma 2.6.** ([Nh, Theorem 4.2], [BN, Theorem 2.5]). *The following statements are true.*

(a) *$K_M$ is Cohen-Macaulay if and only if* $\mathrm{Rl}\left(H_{\mathfrak{m}}^{d-k-1}(M/(x_1, \ldots, x_k)M)\right) = 0$ *for a (and for all) strict* f*-sequence $(x_1, \ldots, x_d)$ of $M$ and all $k = 0, \ldots, d-3$.*

(b) *If $d \geq 4$, then $K_M$ is Cohen-Macaulay if and only if $K_{M/xM}$ is Cohen-Macaulay for every strict* f*-element $x$ of $M$.*

Following Cuong, Schenzel and Trung [CST], $M$ is said to be *generalized Cohen-Macaulay* if $I(M) := \sup I(x_1, \ldots, x_d; M) < \infty$, where $(x_1, \ldots, x_d)$ runs over all s.o.p. of $M$. Note that

$M$ is generalized Cohen-Macaulay if and only if $\ell_R(H^i_{\mathfrak{m}}(M)) < \infty$ for all $i < d$. Note that $K_M$ satisfies the condition Serre $(S_2)$, therefore $K_M$ is generalized Cohen-Macaulay whenever $d \le 3$. In case where $d \ge 4$, we have the following characterizations for the canonical module to be generalized Cohen-Macaulay.

**Lemma 2.7.** ([LN, Main theorem]) *The following statements are equivalent:*

(a) *$K_M$ is generalized Cohen-Macaulay.*

(b) *There exists a number $c(M)$ such that $\mathrm{Rl}\left(H^{d-k-1}_{\mathfrak{m}}(M/(x_1,\ldots,x_k)M)\right) \le c(M)$ for all strict* f*-sequences $\underline{x} = (x_1,\ldots,x_d)$ of $M$ and all $k = 1,\ldots,d-3$.*

(c) *There exist a strict* f*-sequence $\underline{x} = (x_1,\ldots,x_d)$ of $M$ and a number $c(\underline{x}, M)$ such that $\mathrm{Rl}\left(H^{d-k-1}_{\mathfrak{m}}(M/(x_1^{n_1},\ldots,x_k^{n_k})M)\right) \le c(\underline{x}, M)$ for all $k = 1,\ldots,d-3$ and all positive integers $n_1,\ldots,n_{d-3}$.*

*Furthermore, if the conditions (a), (b), (c) satisfy, then*

$$\mathrm{Rl}\left(H^{d-k-1}_{\mathfrak{m}}(M/(x_1,\ldots,x_k)M)\right) \le \sum_{i=0}^{k} \binom{k}{i} \ell(H^{i+2}_{\mathfrak{m}}(K_M))$$

*for any $k = 1,\ldots,d-3$. The equality holds true when $x_1,\ldots,x_k \in \mathfrak{m}^{2^{k-1}q}$, where*

$$q = \min\{t \in \mathbb{N} \mid \mathfrak{m}^t H^i_{\mathfrak{m}}(K_M) = 0 \text{ for all } i < d\}.$$

The notion of standard system of parameters (standard s.o.p. for short) defined in [Tr] (see also [Sch1]) is very important in the study of generalized Cohen-Macaulay modules. A s.o.p. $(x_1,\ldots,x_d)$ of $M$ is said to be a *standard s.o.p.* if

$$\ell_R(M/(x_1,\ldots,x_d)M) - e(x_1,\ldots,x_d; M) = \ell_R(M/(x_1^2,\ldots,x_d^2)M) - e(x_1^2,\ldots,x_d^2; M).$$

Then $M$ is generalized Cohen-Macaulay if and only if there exists a standard s.o.p. of $M$. Note that if $(x_1,\ldots,x_d)$ is a standard s.o.p. of $M$, then

$$I(x_1,\ldots,x_d; M) = I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(M)).$$

In the introduction, we introduce the notion of canonical s.o.p. (see Definition 1.2), which will be used in the next section to characterize the generalized Cohen-Macaulayness of the canonical module. The following lemma gives a relation between standard s.o.p. of $M$ and canonical s.o.p. of $M$.

**Lemma 2.8.** *If $(x_1,\ldots,x_d)$ be a standard s.o.p. of $M$, then it is a canonical s.o.p of $M$.*

*Proof.* If $d \le 2$, there is nothing to prove. Let $d \ge 3$. Suppose that $(x_1,\ldots,x_d)$ is a standard s.o.p. of $M$. Then $M$ is generalized Cohen-Macaulay, cf. [Tr]. Hence $\ell_R(K^i_M) < \infty$

for all $i < d$. So, each s.o.p. of $M$ is an f-sequence of $K_M^i$ for all $i$. It follows by Lemma 2.3(a) that each s.o.p. of $M$ is a strict f-sequence. Since $(x_1, \ldots, x_d)$ is a standard s.o.p. of $M$, so is $(x_1^2, \ldots, x_d^2)$. Note that $M/(x_1, \ldots, x_{d-3})M$ is generalized Cohen-Macaulay. Hence $\ell_R\big(H_{\mathfrak{m}}^2(M/(x_1, \ldots, x_{d-3})M)\big) < \infty$. Similarly, $\ell_R\big(H_{\mathfrak{m}}^2(M/(x_1^2, \ldots, x_{d-3}^2)M)\big) < \infty$. Therefore, we get by [Tr, Proposition 2.9] that

$$\begin{aligned}\mathrm{Rl}\left(H_{\mathfrak{m}}^2(M/(x_1, \ldots, x_{d-3})M)\right) &= \ell(H_{\mathfrak{m}}^2(M/(x_1, \ldots, x_{d-3})M)) \\ &= \sum_{i=2}^{d-1} \binom{d-3}{i-1} \ell(H_{\mathfrak{m}}^i(M)) \\ &= \ell_R\big(H_{\mathfrak{m}}^2(M/(x_1^2, \ldots, x_{d-3}^2)M)\big) \\ &= \mathrm{Rl}\left(H_{\mathfrak{m}}^2(M/(x_1^2, \ldots, x_{d-3}^2)M)\right).\end{aligned}$$

□

The converse statement of Lemma 2.8 is not true. In fact, by Theorem 1.4, there is an unmixed complete local ring $R$ such that $R$ is not generalized Cohen-Macaulay, but $K_R$ is generalized Cohen-Macaulay. By Theorem 1.3, there is a canonical s.o.p. of $R$, but $R$ does not admit a standard s.o.p.

## 3 Proof of Theorem 1.3

Before proving Theorem 1.3, we need some auxiliary lemmas.

For an Artinian $R$-module $A$, set $\dim_R A = \dim(R/\operatorname{Ann}_R A)$. Note that $A$ has a natural structure as an Artinian $\widehat{R}$-module and $\dim_R A \geq \dim_{\widehat{R}} A$, see [CN, Proposition 2.4(ii), Corollary 4.7]. Moreover, $\ell_R(A) < \infty$ if and only if $\dim_R A = \dim_{\widehat{R}} A \leq 0$.

Since $K_M$ satisfies the condition Serre $(S_2)$, we have $\dim_R H_{\mathfrak{m}}^i(K_M) \leq i - 2$ for all $i < d$ (see [Sch2, Propositions 2.2(c), 2.3(d)]). In particular, if $d \geq 3$, then $H_{\mathfrak{m}}^i(K_M) = 0$ for $i \leq 1$ and $\ell_R(H_{\mathfrak{m}}^2(K_M)) < \infty$.

**Lemma 3.1.** *Let $d \geq 4$, let $x \in \mathfrak{m}$ be a strict* f*-element of $M$. Then $\ell_R(0 :_{H_{\mathfrak{m}}^3(K_M)} x) < \infty$ and*

$$\mathrm{Rl}(H_{\mathfrak{m}}^{d-2}(M/xM)) = \ell_R(H_{\mathfrak{m}}^2(K_M)/xH_{\mathfrak{m}}^2(K_M)) + \ell_R(0 :_{H_{\mathfrak{m}}^3(K_M)} x).$$

*Proof.* Set $N = M/xM$. Let $y$ be a strict f-element of $N$. Then by Lemma 2.4(a) that

$$\mathrm{Rl}(H_{\mathfrak{m}}^{d-2}(N)) = \ell_R(0 :_{K_N^{d-2}} y^n) < \infty$$

for all large enough integers $n$. Note that $y^n$ is a strict f-element of $N$. Therefore, we have by Lemma 2.4(b) the exact sequence

$$0 \to K_N/y^nK_N \to K_{N/y^nN} \to (0 :_{K_N^{d-2}} y^n) \to 0.$$

Since $d \geq 4$ and $K_{N/y^nN}$ satisfies the condition Serre $(S_2)$, we have depth $K_{N/y^nN} \geq 2$. Since $\ell_R(0 :_{K_N^{d-2}} y^n) < \infty$, it follows by the above exact sequence that

$$(0 :_{K_N^{d-2}} y^n) = H^0_{\mathfrak{m}}(0 :_{K_N^{d-2}} y^n) \cong H^1_{\mathfrak{m}}(K_N/y^nK_N).$$

Since $y^n$ is $K_N$-regular, we have the exact sequence

$$0 \to K_N \to K_N \to K_N/y^nK_N \to 0.$$

As $\dim K_N \geq 3$ and $K_N$ satisfies the condition Serre $(S_2)$, we get depth $K_N \geq 2$. So,

$$H^1_{\mathfrak{m}}(K_N/y^nK_N) \cong (0 :_{H^2_{\mathfrak{m}}(K_N)} y^n).$$

Since $\ell_R(H^2_{\mathfrak{m}}(K_N)) < \infty$, it follows by Lemma 2.4(b) that

$$(0 :_{H^2_{\mathfrak{m}}(K_N)} y^n) = H^2_{\mathfrak{m}}(K_N) \cong H^2_{\mathfrak{m}}(K_M/xK_M)$$

for all large enough integers $n$. Therefore we get by all the aboves facts that

$$\mathrm{Rl}\left(H^{d-2}_{\mathfrak{m}}(M/xM)\right) = \ell_R\left(H^2_{\mathfrak{m}}(K_M/xK_M)\right).$$

Hence $\ell_R(H^2_{\mathfrak{m}}(K_M/xK_M)) < \infty$. From the exact sequence

$$0 \to K_M \to K_M \to K_M/xK_M \to 0,$$

we have the exact sequence

$$0 \to H^2_{\mathfrak{m}}(K_M)/xH^2_{\mathfrak{m}}(K_M) \to H^2_{\mathfrak{m}}(K_M/xK_M) \to (0 :_{H^3_{\mathfrak{m}}(K_M)} x) \to 0.$$

Now, the result follows. □

**Lemma 3.2.** *Let $d \geq 4$, let $(x_1, \ldots, x_d)$ be an unconditioned strict* f*-sequence of $M$. Then*

$$\mathrm{Rl}(H^{d-k-1}_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_k^{n_k})M)) \leq \mathrm{Rl}\left(H^{d-k-1}_{\mathfrak{m}}(M/(x_1^{m_1}, \ldots, x_k^{m_k})M)\right)$$

*for all integers $1 \leq k \leq d-3$ and all positive integers $n_i \leq m_i$ for $i = 1, \ldots, k$.*

*Proof.* We prove the lemma by induction on $d$.

Let $d = 4$. Then $k = 1$. We have by Lemma 3.1 that

$$\begin{aligned}\mathrm{Rl}\left(H^{d-2}_{\mathfrak{m}}(M/x^nM)\right) &= \ell_R(H^2_{\mathfrak{m}}(K_M)/x^nH^2_{\mathfrak{m}}(K_M)) + \ell_R(0 :_{H^3_{\mathfrak{m}}(K_M)} x^n) \\ &\leq \ell_R(H^2_{\mathfrak{m}}(K_M)/x^mH^2_{\mathfrak{m}}(K_M)) + \ell_R(0 :_{H^3_{\mathfrak{m}}(K_M)} x^m) \\ &= \mathrm{Rl}\left(H^{d-2}_{\mathfrak{m}}(M/x^mM)\right).\end{aligned}$$

Assume that $d > 4$. Set $N = M/(x_2^{n_2}, \ldots, x_k^{n_k})M$ and $L = M/x_1^{m_1}M$. Then $\dim L \geq 4$ and $\dim N = d - k + 1 \geq 4$. Since $(x_1, \ldots, x_d)$ is an unconditioned strict f-sequence of $M$, it follows by Lemma 2.3(b) that $(x_1, x_2^{n_2}, \ldots, x_k^{n_k})$ is also an unconditioned strict f-sequence of $M$. Hence, $x_1$ is a strict f-element of $N$. Therefore we get

$$\begin{aligned}\mathrm{Rl}\left(H_{\mathfrak{m}}^{d-k-1}(M/(x_1^{n_1}, \ldots, x_k^{n_k})M)\right) &= \mathrm{Rl}\left(H_{\mathfrak{m}}^{(d-k+1)-2}(N/x_1^{n_1}N)\right)\\ &\leq \mathrm{Rl}\left(H_{\mathfrak{m}}^{(d-k+1)-2}(N/x_1^{m_1}N)\right)\\ &= \mathrm{Rl}\left(H_{\mathfrak{m}}^{d-k-1}(M/(x_1^{m_1}, x_2^{n_2}, \ldots, x_k^{n_k})M)\right)\\ &= \mathrm{Rl}\left(H_{\mathfrak{m}}^{d-k-1}(L/(x_2^{n_2}, \ldots, x_k^{n_k})L)\right).\end{aligned}$$

It is clear that $(x_2, \ldots, x_k)$ is an unconditioned strict f-sequence of $L$ and $\dim L = d-1$. So, we get by induction hypothesis that

$$\begin{aligned}\mathrm{Rl}\left(H_{\mathfrak{m}}^{d-k-1}(L/(x_2^{n_2}, \ldots, x_k^{n_k})L)\right) &= \mathrm{Rl}\left(H_{\mathfrak{m}}^{(d-1)-(k-1)-1}(L/(x_2^{n_2}, \ldots, x_k^{n_k})L)\right)\\ &\leq \mathrm{Rl}\left(H_{\mathfrak{m}}^{(d-1)-(k-1)-1}(L/(x_2^{m_2}, \ldots, x_k^{m_k})L)\right)\\ &= \mathrm{Rl}\left(H_{\mathfrak{m}}^{d-k-1}(M/(x_1^{m_1}, x_2^{m_2}, \ldots, x_k^{m_k})M)\right).\end{aligned}$$

□

The following property of Artinian module is useful in the proof of Theorem 1.3. Let $A$ be an Artinian $R$-module. It follows by [Ro, Theorem 6] and [CN, Corollary 4.7] that

$$\dim_{\widehat{R}} A = \inf\{t \in \mathbb{N} \mid \exists x_1, \ldots, x_t \in \mathfrak{m} \text{ such that } \ell_R(0 :_A (x_1, \ldots, x_t)) < \infty\}.$$

A system $(x_1, \ldots, x_t)$ of elements in $\mathfrak{m}$ (where $t = \dim_{\widehat{R}} A$) is said to be a *system of parameters* of $A$ if $\ell_R(0 :_A (x_1, \ldots, x_t)) < \infty$. It is clear that if $(x_1, \ldots, x_t)$ is a system of parameters of $A$, then $\dim_{\widehat{R}}(0 :_A (x_1, \ldots, x_n)) = t - n$ for all $n \leq t$. If $\dim_{\widehat{R}} A > 0$ and $x \in \mathfrak{m}$ be such that $\dim_{\widehat{R}}(0 :_A x) = \dim_{\widehat{R}} A - 1$, then $x$ is said to be a *parameter* of $A$.

**Lemma 3.3.** *Let $A$ be an Artinian $R$-module. If $\dim_{\widehat{R}} A > 0$ and $x$ is a parameter of $A$, then for all positive integers $n$ we have*

$$(0 :_A x^n) \neq (0 :_A x^{n+1}).$$

*Proof.* Assume in contrary that $(0 :_A x^n) = (0 :_A x^{n+1})$ for some integer $n > 0$. We claim that $A = (0 :_A x^n)$. In fact, let $a \in A$. Since $A$ is $\mathfrak{m}$-torsion, we have $\mathfrak{m}^s a = 0$ for some integer $s > 0$. Hence $x^s a = 0$. If $s \leq n$, then $a \in (0 :_A x^n)$. So, we assume that $s \geq n+1$. Then we have $x^{n+1}(x^{s-n-1}a) = 0$. Hence $x^{s-n-1}a \in (0 :_A x^{n+1}) = (0 :_A x^n)$. Therefore $x^{s-1}a = 0$. Continue this process, after some steps we have $x^{n+1}a = 0$. Hence $a \in (0 :_A x^{n+1}) = (0 :_A x^n)$. Therefore, $A = (0 :_A x^n)$ and the claim is proved. Note that $x^n$ is also a parameter of $A$. Since $\dim_{\widehat{R}} A > 0$, we have by the claim that

$$\dim_{\widehat{R}} A = \dim_{\widehat{R}}(0 :_A x^n) = \dim_{\widehat{R}} A - 1.$$

This gives a contradiction. □

**Corollary 3.4.** *Let $d \geq 4$ and let $x \in \mathfrak{m}$ be a strict* f*-element of $M$ such that*

$$\mathrm{Rl}\left(H_{\mathfrak{m}}^{d-2}(M/xM)\right) = \mathrm{Rl}\left(H_{\mathfrak{m}}^{d-2}(M/x^2M)\right).$$

*Then* $\ell_R(H^3_{\mathfrak{m}}(K_M)) < \infty$, $xH^i_{\mathfrak{m}}(K_M) = 0$ *for all* $i \leq 3$, *and*

$$\mathrm{Rl}(H^{d-2}_{\mathfrak{m}}(M/xM)) = \mathrm{Rl}(H^{d-2}_{\mathfrak{m}}(M/x^nM)) = \ell_R(H^2_{\mathfrak{m}}(K_M)) + \ell_R(H^3_{\mathfrak{m}}(K_M))$$

*for all* $n > 0$. *In particular, if* $d = 4$, *then* $M$ *is generalized Cohen-Macaulay canonical.*

*Proof.* Since $K_M$ satisfies the condition Serre $(S_2)$, it follows that $\ell_R(H^2_{\mathfrak{m}}(K_M) < \infty$ and $\dim_R H^3_{\mathfrak{m}}(K_M) \leq 1$. By Lemma 3.1 and by our assumption, $xH^2_{\mathfrak{m}}(K_M) = x^2H^2_{\mathfrak{m}}(K_M)$ and $(0 :_{H^3_{\mathfrak{m}}(K_M)} x) = (0 :_{H^3_{\mathfrak{m}}(K_M)} x^2)$. It follows by Nakayama Lemma that $xH^2_{\mathfrak{m}}(K_M) = 0$. Next, we claim that $\ell_R(H^3_{\mathfrak{m}}(K_M)) < \infty$. In fact, suppose $\ell_R(H^3_{\mathfrak{m}}(K_M)) = \infty$. Then $\dim_R H^3_{\mathfrak{m}}(K_M) = 1$. Hence $\dim_{\widehat{R}} H^3_{\mathfrak{m}}(K_M) = 1$ by [CNN, Proposition 2.4, Corollary 4.2(iii)]. Since $\ell_R(0 :_{H^3_{\mathfrak{m}}(K_M)} x) < \infty$ by Lemma 3.1, it follows that $x$ is a parameter of $H^3_{\mathfrak{m}}(K_M)$. Hence $(0 :_{H^3_{\mathfrak{m}}(K_M)} x) \neq (0 :_{H^3_{\mathfrak{m}}(K_M)} x^2)$ by Lemma 3.3. This gives a contradiction, and the claim is proved. Since $(0 :_{H^3_{\mathfrak{m}}(K_M)} x) = (0 :_{H^3_{\mathfrak{m}}(K_M)} x^2)$, we get by the same arguments as in the proof of Lemma 3.3 that $H^3_{\mathfrak{m}}(K_M) = (0 :_{H^3_{\mathfrak{m}}(K_M)} x)$. So, $xH^3_{\mathfrak{m}}(K_M) = 0$. Now, the rest statement follows by Lemma 3.1. □

**Lemma 3.5.** *Suppose that* $d \geq 4$. *Let* $\underline{x} = (x_1, \ldots, x_k)$ *be a strict* f*-sequence of* $M$, *where* $1 \leq k \leq d-3$ *is an integer. Then, there exists a positive integer* $m(\underline{x})$ *such that*

$$\mathrm{Rl}\big(H^{d-k}_{\mathfrak{m}}(M/(x_1, \ldots, x_{k-1})M)\big) \leq \mathrm{Rl}\big(H^{d-k-1}_{\mathfrak{m}}(M/(x_1, \ldots, x_{k-1}, x_k^{m(\underline{x})})M)\big).$$

*Proof.* Set $N := M/(x_1, \ldots, x_{k-1})M)$. We can choose a positive integer $m(\underline{x})$ such that $x_k^{m(\underline{x})}H^0_{\mathfrak{m}}(K_N^{d-k}) = 0$. Note that $x_k^{m(\underline{x})}$ is a filter regular element of $K_N^{d-k}$ by Lemma 2.3(a), i.e. it is $K_N^{d-k}$-regular in dimension $> 0$ in sense of [BN1]. Therefore, we have by [DN, Lemma 2.3] the following exact sequence

$$0 \to H^0_{\mathfrak{m}}(K_N^{d-k}) \to H^0_{\mathfrak{m}}(K_N^{d-k}/x_k^{m(\underline{x})}K_N^{d-k}) \to (0 :_{H^1_{\mathfrak{m}}(K_N^{d-k})} x_k^{m(\underline{x})}) \to 0.$$

Since $x_k^{m(\underline{x})}$ is a strict f-element of $N$, we have by Lemma 2.4(b) the following exact sequence

$$0 \to K_N^{d-k}/x_k^{m(\underline{x})}K_N^{d-k} \to K^{d-k-1}_{N/x_k^{m(\underline{x})}N} \to (0 :_{K_N^{d-k-1}} x_k^{m(\underline{x})}) \to 0.$$

Therefore, it follows by Lemma 2.4(a) that

$$\begin{aligned}\mathrm{Rl}\big(H^{d-k}_{\mathfrak{m}}(M/(x_1, \ldots, x_{k-1})M)\big) &= \ell_R(H^0_{\mathfrak{m}}(K_N^{d-k})) \\ &\leq \ell_R\big(H^0_{\mathfrak{m}}(K_N^{d-k}/x_k^{m(\underline{x})}K_N^{d-k})\big) \\ &\leq \ell_R\big(H^0_{\mathfrak{m}}(K^{d-k-1}_{N/x_k^{m(\underline{x})}N})\big) \\ &= \mathrm{Rl}\big(H^{d-k-1}_{\mathfrak{m}}(M/(x_1, \ldots, x_{k-1}, x_k^{m(\underline{x})})M)\big).\end{aligned}$$

□

Now, we are ready to prove the first main result of this paper.

***Proof of Theorem 1.3***. (a) $\Rightarrow$ (d). By our assumption (a), $\ell_R(H^i_{\mathfrak{m}}(K_M)) < \infty$ for all $i < d$. Set

$$q = \min\{t \in \mathbb{N} \mid \mathfrak{m}^t H^i_{\mathfrak{m}}(K_M) = 0 \text{ for all } i < d\}.$$

Then there exists by Lemma 2.3 an unconditioned f-sequence $(x_1, \ldots, x_d)$ of $M$ contained in $\mathfrak{m}^{2^{d-4}q}$. We have by Lemma 2.7 that

$$\begin{aligned}\mathrm{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1, \ldots, x_{d-3})M)\right) &= \sum_{i=0}^{d-3} \binom{d-3}{i} \ell_R(H^{i+2}_{\mathfrak{m}} K_M)) \\ &= \mathrm{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1^2, \ldots, x_{d-3}^2)M)\right).\end{aligned}$$

Therefore, $(x_1, \ldots, x_d)$ is an unconditioned canonical s.o.p. of $M$.

(d) $\Rightarrow$ (c). Suppose that $(x_1, \ldots, x_d)$ is an unconditioned canonical s.o.p. of $M$. It is enough to prove the following equalities

$$\mathrm{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1, \ldots, x_{d-3})M)\right) = \mathrm{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_{d-3}^{n_{d-3}})M)\right)$$

for all positive integers $n_1, \ldots, n_{d-3}$. We prove this by induction on $d$. If $d \leq 3$, there is nothing to prove. The case where $d = 4$ follows by Corollary 3.4.

Let $d > 4$ and assume that the result is valid for $d-1$. Let $n_1, \ldots, n_k$ be positive integers. Set $N = M/x_1M$ and $N' = M/x_1^2M$. Then $\dim N = d - 1 = \dim N'$. Since $(x_1, \ldots, x_d)$ is an unconditioned canonical s.o.p. of $M$, we have by Lemma 3.2 that

$$\begin{aligned}\mathrm{Rl}\left(H^2_{\mathfrak{m}}(N/(x_2, \ldots, x_{d-3})N)\right) &\leq \mathrm{Rl}\left(H^2_{\mathfrak{m}}(N/(x_2^2, \ldots, x_{d-3}^2)N)\right) \\ &\leq \mathrm{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1^2, \ldots, x_{d-3}^2)M)\right) \\ &= \mathrm{Rl}(H^2_{\mathfrak{m}}(N/(x_2, \ldots, x_{d-3})N)).\end{aligned}$$

Hence $\mathrm{Rl}\left(H^2_{\mathfrak{m}}(N/(x_2, \ldots, x_{d-3})N)\right) = \mathrm{Rl}\left(H^2_{\mathfrak{m}}(N/(x_2^2, \ldots, x_{d-3}^2)N)\right)$. It follows that $(x_2, \ldots, x_d)$ is an unconditioned canonical s.o.p. of $N$. Therefore, we get by induction hypothesis that

$$\mathrm{Rl}\left(H^2_{\mathfrak{m}}(N/(x_2, \ldots, x_{d-3})N)\right) = \mathrm{Rl}\left(H^2_{\mathfrak{m}}(N/(x_2^{n_2}, \ldots, x_{d-3}^{n_{d-3}})N)\right)$$

for all positive integers $n_2, \ldots, n_{d-3}$. Similarly, since $(x_1, \ldots, x_d)$ is an unconditioned canonical s.o.p. of $M$, we have by Lemma 3.2 that

$$\mathrm{Rl}\left(H^2_{\mathfrak{m}}(N'/(x_2, \ldots, x_{d-3})N')\right) = \mathrm{Rl}\left(H^2_{\mathfrak{m}}(N'/(x_2^2, \ldots, x_{d-3}^2)N')\right).$$

Hence $(x_2, \ldots, x_d)$ is an unconditioned canonical s.o.p. of $N'$. Therefore, we get by induction that

$$\mathrm{Rl}\left(H^2_{\mathfrak{m}}(N'/(x_2, \ldots, x_{d-3})N')\right) = \mathrm{Rl}\left(H^2_{\mathfrak{m}}(N'/(x_2^{n_2}, \ldots, x_{d-3}^{n_{d-3}})N')\right),$$

for any positive integers $n_2, \ldots, n_{d-3}$. As $(x_1, \ldots, x_d)$ is an unconditioned canonical s.o.p. of $M$, it follows by Lemma 3.2 that

$$\mathrm{Rl}\left(H^2_{\mathfrak{m}}(N/(x_2, \ldots, x_{d-3})N)\right) = \mathrm{Rl}\left(H^2_{\mathfrak{m}}(N'/(x_2, \ldots, x_{d-3})N')\right).$$

For given positive integers $n_2, \ldots, n_{d-3}$, we have $\operatorname{Rl}\left(H^2_{\mathfrak{m}}(L/x_1L)\right) = \operatorname{Rl}\left(H^2_{\mathfrak{m}}(L/x_1^2L)\right)$ by the above equalities, where $L = M/(x_2^{n_2}, \ldots, x_k^{n_k})M$. Since $x_1$ is a strict f-element of $L$, we have by Corollary 3.4 that

$$\operatorname{Rl}\left(H^2_{\mathfrak{m}}(L/x_1L)\right) = \operatorname{Rl}\left(H^2_{\mathfrak{m}}(L/x_1^{n_1}L)\right),$$

the equalities are proved.

(c) $\Rightarrow$ (a). Assume that there exists a strict f-sequence $\underline{x} = (x_1, \ldots, x_d)$ of $M$ such that

$$c_{\underline{x},M} := \sup_{n_1,\ldots,n_{d-3}\in\mathbb{N}} \operatorname{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_{d-3}^{n_{d-3}})M)\right) < \infty.$$

Let $\underline{n} = (n_1, \ldots, n_{d-3})$ be a tuple of $d-3$ positive integers. Note that $(x_1^{n_1}, \ldots, x_{d-3}^{n_{d-3}})$ is a strict f-sequence of $M$ by Lemma 2.3(b). Therefore, there exists by Lemma 3.5 a positive integer $m(\underline{x}, \underline{n})$ such that

$$\operatorname{Rl}\left(H^3_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_{d-4}^{n_{d-4}})M)\right) \le \operatorname{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_{d-4}^{n_{d-4}}, x_{d-3}^{n_{d-3}m(\underline{x},\underline{n})})M)\right).$$

It follows by our assumption that

$$\begin{aligned}\sup_{n_1,\ldots,n_{d-4}} \operatorname{Rl}\left(H^3_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_{d-4}^{n_{d-4}})M)\right) &\le \sup_{n_1,\ldots,n_{d-3}} \operatorname{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_{d-4}^{n_{d-4}}, x_{d-3}^{n_{d-3}m(\underline{x},\underline{n})})M)\right)\\ &\le \sup_{m_1,\ldots,m_{d-3}\in\mathbb{N}} \operatorname{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1^{m_1}, \ldots, x_{d-3}^{m_{d-3}})M)\right) < \infty.\end{aligned}$$

By the same arguments, we get

$$\sup_{n_1,\ldots,n_k} \operatorname{Rl}\left(H^{d-k-1}_{\mathfrak{m}}(M/(x_1^{n_1}, \ldots, x_k^{n_k})M)\right) < \infty$$

for all $k = 1, \ldots, d-3$. Therefore, $K_M$ is generalized Cohen-Macaulay by Lemma 2.7(c)$\Rightarrow$(a).

(a) $\Rightarrow$ (b) follows by Lemma 2.7 (a)$\Rightarrow$(b).

(b) $\Rightarrow$ (c) is trivial.

Finally, let $(x_1, \ldots, x_d)$ is an unconditioned canonical s.o.p. of $M$. Let $n > 2^{d-4}q$ be an integer, where $q$ is the number defined from the begining. Then, we get by Lemma 2.3 and by the proof of (d) $\Rightarrow$ (c) that

$$\begin{aligned}\operatorname{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1, \ldots, x_{d-3})M)\right) &= \operatorname{Rl}\left(H^2_{\mathfrak{m}}(M/(x_1^n, \ldots, x_{d-3}^n)M)\right)\\ &= \sum_{i=0}^{d-3} \binom{d-3}{i} \ell_R(H^{i+2}_{\mathfrak{m}}(K_M)).\end{aligned}$$

□

## 4 Proof of Theorem 1.4

In this section, we keep the assumption that $(R, \mathfrak{m})$ is a Noetherian local ring which is a quotient of a Gorenstein local ring, $M$ is a finitely generated $R$-module with $\dim M = d$. Let

$K_M$ be the canonical module of $M$. For each integer $i \geq 0$, let $K_M^i$ be the $i$-th deficiency module of $M$. *The non Cohen-Macaulay locus* of $M$, denoted by $\mathrm{nCM}(M)$, is defined by

$$\mathrm{nCM}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_\mathfrak{p} \text{ is not Cohen-Macaulay}\}.$$

Because $R$ is a quotient of a Gorenstein local ring, $\mathrm{nCM}(M)$ is closed under Zariski topology, see [CNN, Corollary 4.2](iv). Therefore, we can define its dimension $\dim \mathrm{nCM}(M)$. If we stipulate that $\dim \emptyset = -1$, then $M$ is Cohen-Macaulay if and only if $\dim \mathrm{nCM}(M) = -1$. Moreover, $M$ is generalized Cohen-Macaulay if and only if $\dim \mathrm{nCM}(M) \leq 0$. In general, we have the following result.

**Lemma 4.1.** ([C, Theorems 3.1, 3.3]). $\dim \mathrm{nCM}(M) \leq \max\limits_{i<d} \dim_R H^i_\mathfrak{m}(M)$. *The equality holds true if $M$ is equidimensional.*

It is clear that $\dim \mathrm{nCM}(M) \leq d-1$. Moreover, $\dim \mathrm{nCM}(M) = d-1$ if and only if $M$ has an embedded prime of dimension $d-1$. Following M. Nagata [Na], $M$ is said to be *unmixed* if $\dim(\widehat{R}/\mathfrak{P}) = \dim_{\widehat{R}} \widehat{M}$ for all $\mathfrak{P} \in \mathrm{Ass}_{\widehat{R}} \widehat{M}$. Since $R$ is a quotient of a Gorenstein local ring, it follows by [Mat, Theorem 23.2] that $M$ is unmixed if and only if $\dim(R/\mathfrak{p}) = d$ for all $\mathfrak{p} \in \mathrm{Ass}_R M$. Note that if $M$ is unmixed, then $\dim \mathrm{nCM}(M) \leq d-2$. For each integer $k \geq 1$, it should be noticed that if $M$ satisfies the condition Serre $(S_k)$, then $\dim_R H^i_\mathfrak{m}(M) \leq i-k$ for all $i < d$, see [Sch2, Proposition 2.2](c). Therefore, by Lemma 4.1, we have the following consequence.

**Corollary 4.2.** *If $M$ satisfies the condition Serre $(S_k)$, then* $\dim \mathrm{nCM}(M) \leq d-k-1$. *In particular,* $\dim \mathrm{nCM}(K_M) \leq d-3$.

Next, we study the non Cohen-Macaulay locus under a flat extension.

**Lemma 4.3.** *Let $(S, \mathfrak{n})$ be a Noetherian local ring and $\varphi : R \to S$ a flat local homomorphism such that $S/\mathfrak{m}S$ is Cohen-Macaulay of dimension $t$. If $M$ is not Cohen-Macaulay, then*

$$\max_{i<d+t} \dim_S H^i_\mathfrak{n}(M \otimes_R S) = \dim(S/\mathfrak{m}S) + \max_{i<d} \dim_R H^i_\mathfrak{m}(M).$$

*In addition, if $M$ and $M \otimes_R S$ are equidimensional, then*

$$\dim \mathrm{nCM}(M \otimes_R S) = \dim(S/\mathfrak{m}S) + \dim \mathrm{nCM}(M).$$

*Proof.* Set $\dim(S/\mathfrak{m}S) = t$. We have $\dim(M \otimes_R S) = d+t$ by [Mat, Theorem 15.1]. Since $S/\mathfrak{m}S$ is Cohen-Macaulay, it follows by [BS1, Theorem 2.1] that

$$H^i_\mathfrak{n}(M \otimes_R S) \cong H^t_\mathfrak{n}\big(H^{i-t}_\mathfrak{m}(M) \otimes_R S\big)$$

for all $i \geq t$. Moreover, according to [Mat, Theorem 23.3], we have

$$\mathrm{depth}(M \otimes_R S) = \mathrm{depth}\, M + \mathrm{depth}(S/\mathfrak{m}S) \geq t.$$

Hence $H^i_\mathfrak{n}(M \otimes_R S) = 0$ for all integers $i < t$. We set $\mathfrak{a}(M) = \mathfrak{a}_0(M).\mathfrak{a}_1(M)\ldots\mathfrak{a}_{d-1}(M)$ and $\mathfrak{a}(M \otimes_R S) = \mathfrak{a}_0(M \otimes_R S).\mathfrak{a}_1(M \otimes_R S)\ldots\mathfrak{a}_{d+t-1}(M \otimes_R S)$, where $\mathfrak{a}_i(M) = \mathrm{Ann}_R H^i_\mathfrak{m}(M)$

and $\mathfrak{a}_i(M \otimes_R S) = \mathrm{Ann}_S H^i_{\mathfrak{n}}(M \otimes_R S)$ for all $i$. Then $\mathfrak{a}(M)S \subseteq \mathfrak{a}(M \otimes_R S)$ by the above isomorphism. Since $M$ is not Cohen-Macaulay, it follows by [C, Theorem 3.1(i)] and by the same arguments as in the proof of [C, Theorem 5.1] that

$$\max_{i<d+t} \dim_S H^i_{\mathfrak{n}}(M \otimes_R S) = \dim(S/\mathfrak{m}S) + \max_{i<d} \dim_R H^i_{\mathfrak{m}}(M).$$

The rest statement follows by this equality and by Lemma 4.1. □

Let $t > 0$ be an integer, let $S = R[[x_1, \ldots, x_t]]$ be the formal power series ring of $t$ variables over $R$. Then the natural map $R \to S$ is flat local and the fiber ring $S/\mathfrak{m}S$ is Cohen-Macaulay. So, $R$ is Cohen-Macaulay if and only if so is $S$. The following lemma shows some relations between the canonical modules and the deficiency modules of $R$ and that of $S$. The proof of this lemma given below was suggested by P. Schenzel.

**Lemma 4.4.** *Let $S = R[[x_1, \ldots, x_t]]$ be the formal power series ring over $R$. Then*

(a) *$K_R$ is Cohen-Macaulay if and only if so is $K_S$. If $K_R$ is not Cohen-Macaulay, then $\dim \mathrm{nCM}(K_S) = t + \dim \mathrm{nCM}(K_R)$.*

(b) *$K^i_S \cong K^{i-t}_R \otimes_R S$ for all $i \geq t$ and $K^i_S = 0$ for all $i < t$. In particular, if $K^{i-t}_R \neq 0$, then $\dim_S K^i_S = t + \dim_R K^{i-t}_R$.*

*Proof.* (a) Since the ring $S/\mathfrak{m}S$ is Gorenstein, $K_S \cong K_R \otimes_R S$ by [AG, Theorem 4.1]. It is clear that the natural injection $R \to S$ is a local flat homomorphism. Since $\dim S/\mathfrak{m}S = t$, we have $\dim K_S = t + \dim K_R$. Because $\mathrm{depth}(S/\mathfrak{m}S) = t$, it follows that $\mathrm{depth}\, K_S = t + \mathrm{depth}\, K_R$. Therefore, $K_R$ is Cohen-Macaulay if and only if so is $K_S$. Suppose that $K_R$ is not Cohen-Macaulay. Note that $K_R$ and $K_S$ are equidimensional. So, we get by Lemma 4.3 that

$$\dim \mathrm{nCM}(K_S) = \dim \mathrm{nCM}(K_R \otimes_R S) = t + \dim \mathrm{nCM}(K_R).$$

(b) Let $(R', \mathfrak{m}')$ be a Gorenstein local ring such that $R$ is a factor ring of $R'$. Set $R = R'/J$ for some ideal $J$ of $R'$. Suppose that $\dim R' = n$. The Local Duality Theorem (see [BS, 11.2.6]) provides the natural isomorphisms

$$H^i_{\mathfrak{m}}(R) \cong \mathrm{Hom}_R(K^i_R, E(R/\mathfrak{m})) = \mathrm{Hom}_R(\mathrm{Ext}^{n-i}_{R'}(R, R'), E(R/\mathfrak{m}))$$

for all $i \in \mathbb{N}$. Let $S' = R'[[x_1, \ldots, x_t]]$ be the formal power series ring of $t$ variables over $R'$. For each integer $i \geq 0$, we have the isomorphism

$$\mathrm{Ext}^i_{R'}(R, R') \otimes_{R'} S' \cong \mathrm{Ext}^i_{S'}(R \otimes_{R'} S', S').$$

Note that $S'$ is a Gorenstein ring with $\dim S' = n+t$. This implies the following isomorphisms

$$K^i_R \otimes_{R'} S' \cong K^{i+t}_{R \otimes_{R'} S'}$$

for all integers $i \geq 0$. Now $K^i_R$ and $R$ have the structure of an $R'$-module. So,

$$R \otimes_{R'} S' \cong R \otimes_R R'/J \otimes_{R'} S' \cong R \otimes_R S \cong S;$$
$$K^i_R \otimes_{R'} S' \cong K^i_R \otimes_R R'/J \otimes_{R'} S' \cong K^i_R \otimes_R S.$$

Thus, $K_S^i \cong K_R^{i-t} \otimes_R S$ for all $i \geq t$. It is clear that $K_S^i = 0$ for all $i < t$. Therefore, if $K_R^{i-t} \neq 0$, then $\dim_S K_S^i = t + \dim_R K_R^{i-t}$. $\square$

In order to prove Theorem 1.4, we need recall the notion of idealization introduced by M. Nagata [Na]. We can make Cartesian product $R \times M$ into a ring with respect to the componentwise addition and the multiplication defined by

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1).$$

This ring is called the *idealization* of $M$ over $R$, and denoted by $R \ltimes M$. Note that $R \ltimes M$ is a commutative Noetherian local ring with the identity $(1, 0)$. The unique maximal ideal of $R \ltimes M$ is $\mathfrak{m} \times M$. Note that $K_R, K_M, K_{R\ltimes M}$ are equidimensional. Therefore the following lemma can be verified by Lemma 4.1, [C, Theorem 3.1] and [L].

**Lemma 4.5.** *The following statements are true*

(a) *If* $\dim M < \dim R$*, then* $\dim \mathrm{nCM}(K_{R\ltimes M}) = \dim \mathrm{nCM}(K_R)$.

(b) *If* $\dim M = \dim R$*, then* $\dim \mathrm{nCM}(K_{R\ltimes M}) = \max\{\dim \mathrm{nCM}(K_R), \dim \mathrm{nCM}(K_M)\}$.

***Proof of Theorem 1.4.*** (a) Note that $\dim \mathrm{nCM}(K_M) \leq d - 3$ by Corollary 4.2. So, it is enough to prove that $\dim \mathrm{nCM}(K_M) \leq \dim \mathrm{nCM}(M)$. If $K_M$ is Cohen-Macaulay, then there is nothing to prove. Assume that $K_M$ is not Cohen-Macaulay. Set $s = \dim \mathrm{nCM}(K_M)$. Then there exists $\mathfrak{p} \in \mathrm{nCM}(K_M)$ such that $\dim R/\mathfrak{p} = s$. Hence $(K_M)_\mathfrak{p}$ is not Cohen-Macaulay. It follows that $\mathfrak{p} \in \mathrm{Supp}(K_M)$. Note that $\mathrm{Ass}(K_M) = \{\mathfrak{p} \in \mathrm{Ass}(M) \mid \dim R/\mathfrak{p} = d\}$. Therefore $\dim R/\mathfrak{p} + \dim M_\mathfrak{p} = d$. Hence $(K_M)_\mathfrak{p} \cong K_{M_\mathfrak{p}}$, see [Sch2, Proposition 2.2](b). So $K_{M_\mathfrak{p}}$ is not Cohen-Macaulay. It follows that $M_\mathfrak{p}$ is not Cohen-Macaulay. Hence $\dim \mathrm{nCM}(M) \geq s$.

(b) Let $d \geq 3$ be an integer. Let $r, s$ be integers such that $-1 \leq s \leq d-3$ and $s \leq r \leq d-2$. We consider the following two cases.

• The case where $s = -1$. If $r = -1$, then any Cohen-Macaulay complete local ring of dimension $d$ satisfies the requirement. Assume that $r \geq 0$. Let $(R_1, \mathfrak{m}_1)$ be a Buchsbaum complete local ring such that $\dim R_1 = d - r \geq 2$, $H^1_{\mathfrak{m}_1}(R_1) \neq 0$ and $H^i_{\mathfrak{m}_1}(R_1) = 0$ for $i \neq d$ and $i \neq 1$ (such a local ring $R_1$ exists by the construction of S. Goto [Go]). Then $R_1$ is not Cohen-Macaulay. Hence $\dim \mathrm{nCM}(R_1) = 0$. Note that $R_1$ is generalized Cohen-Macaulay. Therefore, it follows by [BN, Corollary 2.7] that $K_{R_1}$ is Cohen-Macaulay. Hence $\dim \mathrm{nCM}(K_{R_1}) = -1$. Let $R = R_1[[x_1, \ldots, x_r]]$ be the formal power series ring of $r$ variables over $R_1$. Then, $R$ is a Noetherian complete local ring and $\dim R = d$. Because $K_{R_1}$ is Cohen-Macaulay, it follows by Lemma 4.4 that $K_R$ is Cohen-Macaulay, i.e. $\dim \mathrm{nCM}(K_R) = -1 = s$. Since $R_1$ is Buchsbaum and $H^0_{\mathfrak{m}_1}(R_1) = 0$, it follows that $R_1$ is unmixed, i.e. $\dim(R_1/\mathfrak{p}) = \dim R_1$ for all $\mathfrak{p} \in \mathrm{Ass}\, R_1$. Since $R_1$ is not Cohen-Macaulay, we have by Lemma 4.4 that $\dim \mathrm{nCM}(R) = r + \dim \mathrm{nCM}(R_1) = r$. For each $\mathfrak{p} \in \mathrm{Spec}(R_1)$, since $R/\mathfrak{p}R \cong (R_1/\mathfrak{p})[[x_1, \ldots, x_r]]$ is a domain, it follows that $\mathfrak{p}R \in \mathrm{Spec}(R)$ and $\dim(R/\mathfrak{p}R) = r + \dim(R_1/\mathfrak{p})$. Therefore, we have by the flatness of the natural injection $R_1 \to R$ and by

[Mat, Theorem 23.2] that

$$\operatorname{Ass} R = \bigcup_{\mathfrak{p}\in \operatorname{Ass} R_1} \operatorname{Ass}_R(R/\mathfrak{p}R) = \{\mathfrak{p}R \mid \mathfrak{p} \in \operatorname{Ass} R_1\}.$$

It follows that for each $\mathfrak{P} \in \operatorname{Ass} R$, there exists $\mathfrak{p} \in \operatorname{Ass} R_1$ such that $\mathfrak{P} = \mathfrak{p}R$. Hence $\dim(R/\mathfrak{P}) = r + \dim(R_1/\mathfrak{p}) = r + \dim R_1 = \dim R = d$. Therefore, $R$ is a unmixed complete local ring which satisfies the requirement.

- The left case where $0 \le s \le r$. Let $(R_2, \mathfrak{m}_2)$ be a Buchsbaum complete local ring such that $\dim R_2 = d - s \ge 3$, $H^0_{\mathfrak{m}_2}(R_2) = 0$ and $H^{d-s-1}_{\mathfrak{m}_2}(R_2) \neq 0$ (such a local ring $R_2$ exists by the construction of S. Goto [Go]). It is clear that $\dim \mathrm{nCM}(R_2) = 0$. Moreover, $K_{R_2}$ is generalized Cohen-Macaulay. Note that $\mathrm{Rl}(H^{d-s-1}_{\mathfrak{m}_2}(R_2)) = \ell_{R_2}(H^{d-s-1}_{\mathfrak{m}_2}(R_2)) \neq 0$. Therefore, it follows by Lemma 2.6(a) that $K_{R_2}$ is not Cohen-Macaulay. Hence $\dim \mathrm{nCM}(K_{R_2}) = 0$. Let $R_3 = R_2[[x_1, \ldots, x_s]]$ be the formal power series ring of $s$ variables over $R_2$. By the same arguments in the above, we can show that $R_3$ is a Noetherian unmixed complete local ring with the unique maximal ideal $\mathfrak{n} = (\mathfrak{m}_2, x_1, \ldots, x_s)R_3$ and $\dim R_3 = d$. Since $R_2$ and $K_{R_2}$ are not Cohen-Macaulay, we get by Lemma 4.4 that

$$\begin{aligned}\dim \mathrm{nCM}(K_{R_3}) &= s + \dim \mathrm{nCM}(K_{R_2}) = s;\\ \dim \mathrm{nCM}(R_3) &= s + \dim \mathrm{nCM}(R_2) = s.\end{aligned}$$

Therefore, if $s = r$, then we set $R = R_3$ and the ring $R$ satisfies the requirement. Now, we can assume that $s < r$. It is clear that $H^i_{\mathfrak{n}}(R_3) = 0$ for all $i < s$. For each integer $i \ge s$, we have $K^i_{R_3} \cong K^{i-s}_{R_2} \otimes_{R_2} R_3$ by Lemma 4.4(b). Note that the natural map $R_2 \to R_3$ is flat. Therefore, for any integer $i < d$, we get by Local Duality Theorem (see [BS, 11.2.6]) that if $H^i_{\mathfrak{n}}(R_3) \neq 0$, then $H^{i-s}_{\mathfrak{m}_2}(R_2) \neq 0$ and

$$\dim_{R_3} H^i_{\mathfrak{n}}(R_3) = \dim_{R_2} H^{i-s}_{\mathfrak{m}_2}(R_2) + s = s.$$

Therefore, $\dim_{R_3} H^i_{\mathfrak{n}}(R_3) \le s$, for all $i < d$. Let $a_1, \ldots, a_{d-r}$ be a part of a system of parameters of $R_3$. Set $P = (a_1, \ldots, a_{d-r})R_3$ and $Q = R_3/(a_1, \ldots, a_{d-r})R_3$. Then we have the following exact sequence of $R_3$-modules

$$0 \to P \to R_3 \to Q \to 0.$$

So, we have the following sequences

$$H^i_{\mathfrak{n}}(R_3) \to H^i_{\mathfrak{n}}(Q) \to H^{i+1}_{\mathfrak{n}}(P) \to H^{i+1}_{\mathfrak{n}}(R_3),$$

for all $i$. Note that $\dim Q = r \le d - 2$ and $\dim P = \dim R_3 = d$. Hence $\dim_{R_3} H^r_{\mathfrak{n}}(Q) = r$ and $\dim_{R_3} H^i_{\mathfrak{n}}(Q) < r$ for all $i \neq r$. Since $r \le d - 2$ and $\dim_{R_3} H^i_{\mathfrak{n}}(R_3) \le s$ for all $i < d$, we have $\dim_{R_3} H^r_{\mathfrak{n}}(R_3) \le s < r$ and $\dim_{R_3} H^{r+1}_{\mathfrak{n}}(R_3) \le s < r$. Hence $\dim_{R_3} H^{r+1}_{\mathfrak{n}}(P) = r$ and $\dim_{R_3} H^i_{\mathfrak{n}}(P) < r$ for all $i \neq r + 1$ and $i \neq d$. Since $R_3$ is unmixed, $P$ is unmixed. Therefore, we get by Lemma 4.1 that

$$\dim \mathrm{nCM}(P) = \max_{i<d}\ \dim_{R_3} H^i_{\mathfrak{n}}(P) = r.$$

Let $R = R_3 \ltimes P$ be the idealization of the $R_3$-module $P$. Then $R$ is a Noetherian local ring with the unique maximal ideal $\mathfrak{m} = \mathfrak{n} \times P$. Since $R_3$ is complete under $\mathfrak{n}$-adic topology, $R$ is complete under $\mathfrak{m}$-adic topology, see [AW, Theorem 4.11]. Since $R_3$ and $P$ are unmixed of dimension $d$, we can check that $R$ is unmixed of dimension $d$. Since $R$ is completed and unmixed, we have by Lemma 4.1 that

$$\dim \mathrm{nCM}(R) = \max_{i<d} \dim_R H^i_{\mathfrak{m}}(R).$$

Consider the following exact sequence $0 \to P \xrightarrow{\epsilon} R \xrightarrow{\rho} R_3 \to 0$, where $\epsilon(x) = (0, x)$ for all $x \in P$ and $\rho(a, x) = a$ for all $(a, x) \in R$. From the induced long exact sequence of local cohomology modules, we can check that

$$\dim_R H^{r+1}_{\mathfrak{m}}(R) = \dim_R H^{r+1}_{\mathfrak{m}}(P) = \dim_{R_3} H^{r+1}_{\mathfrak{n}}(P) = r$$

and $\dim_R H^i_{\mathfrak{m}}(R) \leq s < r$ for all $i \neq r+1$ and $i \neq d$. Thus, $\dim \mathrm{nCM}(R) = r$. Since $r \leq d-2$ and $\dim_{R_3} Q = r$, we have by the exact sequence $0 \to P \to R_3 \to Q \to 0$ that $K_P \cong K_{R_3}$. Moreover, since $K_{R_2}$ is not Cohen-Macaulay, we have by Lemma 4.4(a) that $\dim \mathrm{nCM}(K_{R_3}) = s + \dim \mathrm{nCM}(K_{R_2}) = s$. Hence, by Lemma 4.1 and Lemma 4.5, we have

$$\dim \mathrm{nCM}(K_R) = \max\{\dim \mathrm{nCM}(K_{R_3}), \dim \mathrm{nCM}(K_P)\} = \dim \mathrm{nCM}(K_{R_3}) = s.$$

□

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