A NOTE ON PSEUDOCONVEX HYPERSURFACES OF INFINITE TYPE IN $\mathbb{C}^n$

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Abstract. The purpose of this article is to prove that there exists a real smooth pseudoconvex hypersurface germ (M, p) of D'Angelo infinite type in $\mathbb{C}^{n+1}$ such that it does not admit any (singular) holomorphic curve in $\mathbb{C}^{n+1}$ tangent to M at p to infinite order.

1. Introduction

Let (M, p) be a smooth real hypersurface germ at p in $\mathbb{C}^{n+1}$ and let r be a local defining function for M near p. Suppose that (M, p) is of D'Angelo infinite type, i.e., there exists a sequence of nonconstant holomorphic curves $\gamma_m : (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, p)$ such that $\dfrac{\nu(r \circ \gamma_m)}{\nu(\gamma_m)} \to +\infty$ as $m \to \infty$, where $\nu(f)$ denotes the vanishing order of f at 0. It is natural to ask whether there exists a nonconstant holomorphic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, p)$ tangent to M at p to infinite order, i.e. $\nu(r \circ \gamma) = +\infty$.

This question plays a crucial role in the regularity of $\bar\partial$-Neumann problems over pseudoconvex domains (see [D'A82, Cat83, Cat84, Cat87, DK99], and the references therein). The main results around this question are due to T. Bloom and I. Graham [BG77], L. Lempert and J. P. D'Angelo [D'A93, Lem86], the first author and B. Stensønes [FS12], the first author, L. Lee and Y. Zhang [FLZ14], and K.-T. Kim and the second author [KN15].

If (M, p) is real-analytic, it was shown that M contains a nontrivial holomorphic curve γ_∞ passing through p (see [D'A93, Lem86, FS12]). For the case when (M, p) is a smooth real hypersurface in $\mathbb{C}^n$, the first author, L. Lee and Y. Zhang [FLZ14] proved that there exists a formal complex curve in the hypersurface M through p. Recently, in [KN15], K.-T. Kim and the second author proved that in general there is no such a regular holomorphic curve. However, the hypersurface constructed in [KN15] is not pseudoconvex.

In this paper, we ensure that this result still holds even for higher-dimensional pseudoconvex hypersurfaces and for singular holomorphic curves. More precisely, we prove the following theorem.

Theorem 1. *Let $n \geq 1$. There exists a smooth pseudoconvex real hypersurface germ $(M, 0)$ of D'Angelo infinite type in $\mathbb{C}^{n+1}$ that does not admit any nonconstant holomorphic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, 0)$ tangent to M at 0 to infinite order.*

Theorem 1 is a crucial consequence of the main result of this paper. In order to state our main result, let us recall the notion of points of Bloom-Graham type.

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A point $p \in M \subset \mathbb{C}^{n+1}$ is of Bloom-Graham type m (m is a positive integer or $+\infty$) if m is the supremum of the orders of tangency of M and codimension one complex submanifolds of $\mathbb{C}^{n+1}$ at p. It was proved in [BG77, Theorem 2.4] that a point $p \in M$ is of Bloom-Graham type m if and only if p is of type m in the sense of J. J. Kohn, defined in terms of iterated commutators of vector fields. We remark here that for smooth real hypersurfaces in $\mathbb{C}^2$, D'Angelo finite type, Bloom-Graham finite type, and Kohn finite type are equivalent. For various notions of points of finite type and their relationships to subelliptic estimates, we refer the reader to [DK99] and the references therein.

Let (M, p) be a smooth real hypersurface germ at p in $\mathbb{C}^{n+1}$. Suppose that (M, p) is of Bloom-Graham infinite type. Then, (M, p) is also of D'Angelo infinite type (see Lemma 5 in Section 3). As the above-mentioned notion of D'Angelo infinite type, it is natural to ask whether there exists a nonconstant holomorphic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, p)$ tangent to M at p to infinite order if (M, p) is of Bloom-Graham infinite type. In [BG77, Counterexamples 2.14], T. Bloom and I. Graham introduced a smooth real hypersurface germ $(M, 0)$ of infinite type in $\mathbb{C}^2$ that does not admit any complex submanifold tangent to M at the origin to infinite order.

We note that the hypersurface given in [BG77] is again not pseudoconvex. In this paper, we show that there exists a smooth pseudoconvex real hypersurface germ (M, p) of Bloom-Graham infinite type in $\mathbb{C}^{n+1}$ such that there is no any nonconstant holomorphic curve in $\mathbb{C}^{n+1}$ tangent to M at p to infinite order. Namely, we prove the following theorem as our main result.

Theorem 2. *Let $n \geq 1$. There exists a smooth pseudoconvex real hypersurface germ $(M, 0)$ of Bloom-Graham infinite type in $\mathbb{C}^{n+1}$ that does not admit any nonconstant holomorphic curve in $\mathbb{C}^{n+1}$ tangent to M at 0 to infinite order.*

The proof of Theorem 2 is split into several step. First, for an increasing sequence of positive real numbers $\{a_m\}_{m=1}^{\infty}$, we construct a $\mathcal{C}^\infty$-smooth subharmonic function f on $\mathbb{C}$ such that its Taylor series at the origin is exactly $\sum_{m=1}^{\infty} \mathrm{Re}(a_m z^m)$ (cf. Proposition 1 in Section 2). Next, choose n suitable sequences of positive real numbers $\{a_m^1\}_{m=1}^{\infty}, \dots, \{a_m^n\}_{m=1}^{\infty}$ and let $f_1, \dots, f_n$ be $\mathcal{C}^\infty$-smooth subharmonic functions $\mathbb{C}$ constructed as in Proposition 1 with respect to these sequences. Therefore, the desired hypersurface M is defined by

$$M = \left\{(z, w) \in \mathbb{C}^{n+1} \colon \mathrm{Re}(w) + f_1(z_1) + \cdots + f_n(z_n) = 0\right\}.$$

As a consequence of Theorem 2, we obtain the following corollary.

Corollary 1. *Let $n \geq 1$. There exists a smooth pseudoconvex real hypersurface germ $(M, 0)$ of Bloom-Graham infinite type in $\mathbb{C}^{n+1}$ that does not admit any n-dimensional complex submanifold tangent to M at 0 to infinite order.*

2. Construction of a $\mathcal{C}^\infty$-smooth subharmonic function

This section is devoted to proving Theorem 1. To do this, we need the following proposition.

Proposition 1. *Let $\{a_m\}_{m=1}^{\infty}$ be an increasing sequence of positive real numbers. Then, there exists a $\mathcal{C}^\infty$-smooth subharmonic function f on $\mathbb{C}$ satisfying that its Taylor series at the origin is exactly $\sum_{m=1}^{\infty} Re(a_m z^m)$.*

In order to give a proof of Proposition 1, we need following lemmas. First of all, denote by χ a nonnegative $\mathcal{C}^\infty$-smooth cut-off function on $\mathbb{R}$ such that

$$\chi(t) = \begin{cases} 1 & \text{if } t < 1/4 \\ 0 & \text{if } t > 1. \end{cases}$$

Let $\{a_m\}_{m=1}^\infty$be a given increasing sequence of positive real numbers. Denote by $\{\epsilon_m\}_{m=1}^\infty$ an increasing sequence of positive real numbers such that $\epsilon_m \geq \max\{m, a_m^{2/m}\}$ for every $m = 1, 2, \ldots$. Then, for each $m = 1, 2, \ldots$, denote $u_m(z)$ by setting

$$u_m(z) := \chi\left(\epsilon_m^2|z|^2\right) \mathrm{Re}\left(a_m z^m\right).$$

Then, we have the following estimates

$$\begin{aligned}\frac{\partial^2 u_m}{\partial z \partial \bar{z}}(z) &= \epsilon_m^2 \chi'\left(\epsilon_m^2|z|^2\right) \mathrm{Re}\left(a_m z^m\right) + \epsilon_m^4|z|^2\chi''\left(\epsilon_m^2|z|^2\right) \mathrm{Re}\left(a_n z^n\right) + m\epsilon_m^2\chi'\left(\epsilon_m^2|z|^2\right) \mathrm{Re}\left(a_m z^m\right) \\ &= \epsilon_m^2(m+1)\chi'\left(\epsilon_m^2|z|^2\right) \mathrm{Re}\left(a_m z^m\right) + \epsilon_m^4|z|^2\chi''\left(\epsilon_m^2|z|^2\right) \mathrm{Re}\left(a_m z^m\right)\end{aligned}$$

for all $z \in \mathbb{C}^*$. Thus, one has the following lemma

Lemma 1. *For each $m = 1, 2, \ldots$, the following assertions hold:*

i) $\Delta u_m(z) = 0$ *for all $z \in \mathbb{C}$ with $|z| \leq \dfrac{1}{2\epsilon_m}$ or $|z| \geq \dfrac{1}{\epsilon_m}$,*

ii) $|\Delta u_m(z)| \lesssim \dfrac{m a_m \epsilon_m^2}{\epsilon_m^m}$ *for all $z \in \mathbb{C}$ with $\dfrac{1}{2\epsilon_m} < |z| < \dfrac{1}{\epsilon_m}$, where the constant is independent of m.*

Next, let us denote by Λ a $\mathcal{C}^\infty$-smooth convex function on $\mathbb{R}$ such that

a) $\Lambda(x) = 0$ if $x \leq -2$,
b) $\Lambda''(x) > 0$ if $-2 < x < 2$,
c) $\Lambda'(x)$ is constant if $x \geq 2$.

Define $v_m(z) := C\,\dfrac{m a_m}{\epsilon_m^m}\Lambda\left(\log|z|^2 + 2\log\epsilon_m\right)$ for every $m = 1, 2, \ldots$, where $C > 0$ will be chosen later. Then, the function v_m is subharmonic on $\mathbb{C}$ for every $m = 1, 2, \ldots$. Moreover, we obtain the following lemma.

Lemma 2. *There exists a positive constant $C' > 0$ such that*

$$\Delta v_m(z) \geq C' \frac{m a_m \epsilon_m^2}{\epsilon_m^m}$$

for every $m = 1, 2, \ldots$ and for all $z \in \mathbb{C}$ with $\dfrac{1}{2\epsilon_m} < |z| < \dfrac{1}{\epsilon_m}$.

Proof. A direct computation shows that

$$\begin{aligned}\frac{\partial^2 v_m}{\partial z \partial \bar{z}}(z) &= C\,\frac{m a_m}{\epsilon_m^m}\Lambda''\left(\log|z|^2 + 2\log\epsilon_m\right)\left|\frac{\partial}{\partial z}\left(\log|z|^2\right)\right|^2 \\ &\quad + C\,\frac{m a_m}{\epsilon_m^m}\Lambda'\left(\log|z|^2 + 2\log\epsilon_m\right)\frac{\partial^2}{\partial z \partial \bar{z}}\left(\log|z|^2\right) \\ &= C\,\frac{m a_m}{\epsilon_m^m}\Lambda''\left(\log|z|^2 + 2\log\epsilon_m\right)\frac{1}{|z|^2} \\ &\geq C\,\frac{m a_m \epsilon_m^2}{\epsilon_m^m}\Lambda''\left(\log|z|^2 + 2\log\epsilon_m\right)\end{aligned}$$

$$\gtrsim \frac{na_n\epsilon_m^2}{\epsilon_m^m}$$

for all $z \in \mathbb{C}$ with $\dfrac{1}{2\epsilon_m} < |z| < \dfrac{1}{\epsilon_m}$, where the positive constant is independent of m. The proof is complete. □

It follows from Lemmas 1 and 2 that if C is chosen fixed and large enough, then $u_m + v_m$ are all subharmonic. Furthermore, we have the following lemmas.

Lemma 3. *If* $|z| < \dfrac{1}{e\epsilon_m}$, *then* $u_m(z) + v_m(z) = Re\,(a_m z^m)$ *for every* $m = 1, 2, \ldots$.

Proof. Fix a positive integer m. Then, $u_m(z) = \mathrm{Re}\,(a_m z^m)$ and $v_m(z) = 0$ for all $z \in \mathbb{C}$ with $|z| < \dfrac{1}{e\epsilon_m}$. Therefore, the proof follows. □

Lemma 4. *The sums* $\sum_{m=1}^{\infty} u_m + v_m$ *are uniformly convergent on compact sets in any* $\mathcal{C}^k$ *norm.*

Proof. Let K be a fixed compact subset in $\mathbb{C}$ and fix a positive integer m. Since $\mathrm{supp}(u_m) \subset \{z \in \mathbb{C} \colon |z| \le 1/\epsilon_m\}$ for every $m = 1, 2, \ldots$, a computation shows that

$$\sup_{z\in K} \left| \frac{\partial^k u_m(z)}{\partial z^k \partial \bar{z}^{k-j}} \right| = \sup_{|z|\le 1/\epsilon_m} \left| \frac{\partial^k u_m(z)}{\partial z^k \partial \bar{z}^{k-j}} \right| \le C_k \frac{a_m \epsilon_m^{2k}}{\epsilon_m^m}$$

for all $0 \le k \le m$ and $0 \le j \le k$, where C_k is a positive constant depending on k. Notice that $a_m \le \epsilon_m^{m/2}$ and $\epsilon_m \ge m$ for every $m = 1, 2, \ldots$. Therefore, by Weierstrass M-test the following series $\sum_{m=1}^{\infty} \frac{\partial^k u_n(z)}{\partial z^k \partial \bar{z}^{k-j}}$ are uniformly convergent on any compact subsets of $\mathbb{C}$ for any nonnegative integers k, j.

On the other hand, since $\Lambda(x) \le C_1(|x| + 1)$ for all $x \in \mathbb{R}$, where $C_1 > 0$ is a constant and $\mathrm{supp}(v_m) \subset \{z \in \mathbb{C} \colon |z| \ge \dfrac{1}{e\epsilon_n}\}$, it follows that

$$\sup_{z\in K} \left| \frac{\partial^k v_m(z)}{\partial z^k \partial \bar{z}^{k-j}} \right| = \sup_{|z|\ge 1/(e\epsilon_m)} \left| \frac{\partial^k v_m(z)}{\partial z^k \partial \bar{z}^{k-j}} \right| \le \widetilde{C}_k \frac{m a_m \epsilon_m^{k}}{\epsilon_m^m}$$

for all $0 \le k \le m$ and $0 \le j \le k$, where $\widetilde{C}_k$ is a positive constant depending on k and K. Note that $a_m \le \epsilon_m^{m/2}$ and $\epsilon_m \ge m$ for every $m = 1, 2, \ldots$. Hence, by Weierstrass M-test the following series $\sum_{m=1}^{\infty} \frac{\partial^k v_m(z)}{\partial z^k \partial \bar{z}^{k-j}}$ are also uniformly convergent on any compact subsets of $\mathbb{C}$ for any nonnegative integers k, j.

Altogether, the proof is now complete. □

The following corollary immediately follows from Lemma 4.

Corollary 2. $\sum_{m=1}^{\infty} u_m + v_m$ *is a* $\mathcal{C}^\infty$ *function and any derivative at* 0 *is the sum of the corresponding derivatives of the* $u_m + v_m$.

Proof of Proposition 1. Define $f(z) = \sum_{m=1}^{\infty} u_m(z) + v_m(z)$. Then, f is a $\mathcal{C}^\infty$-smooth subharmonic function on $\mathbb{C}$. Moreover, by Lemma 3 and Corollary 2, the Taylor series of f at the origin is exactly $\sum_{m=1}^{\infty} \mathrm{Re}(a_m z^m)$. □

3. Pseudoconvex hypersurface of Bloom-Graham infinite type

In this section, we shall give proofs of Theorem 2, Theorem 1, and Corollary 1.

Proof of Theorem 2. Let $\{a_m^1\}_{m=1}^\infty, \ldots, \{a_m^n\}_{m=1}^\infty$ be n sequences of positive real numbers satisfying:

(i) $a_m^1 \geq 2m^m$ for $m = 1, 2, \ldots$;

(ii) $a_m^2/(\sum_{j=1}^{m-1} a_j^1) \geq e^{m^2}, \ldots, a_m^n/(\sum_{j=1}^{m-1} a_j^n + \sum_{k=1}^{n-1}\sum_{j=1}^{m} a_j^k) \geq e^{m^2}$ for $m = 1, 2, \ldots$.

Denote by $\{\epsilon_m^k\}_{m=1}^\infty, 1 \leq k \leq n$, n sequences of positive real numbers satisfying $\epsilon_m^k \geq (a_m^k)^{2/n}, 1 \leq k \leq n$, and $\epsilon_m^1 \leq \cdots \leq \epsilon_m^n$.

For each $k = 1, \ldots, n$, let f_k be a $\mathcal{C}^\infty$-smooth subharmonic function on $\mathbb{C}$ constructed in the proof of Proposition 1 for a pair of sequences $\{a_m^k\}_{m=1}^\infty$ and $\{\epsilon_m^k\}_{m=1}^\infty$. That is, the Taylor series at the origin of f_k is $\sum_{m=1}^{\infty} \mathrm{Re}\left(a_m^k z^m\right)$, $k = 1, 2, \ldots, n$.

We now define a hypersurface germ M at $p = 0$ by setting

$$M = \left\{(z, w) \in \mathbb{C}^{n+1} \colon \tilde{\rho}(z, w) := \mathrm{Re}(w) + f_1(z_1) + \cdots + f_n(z_n) = 0\right\}.$$

Since $f_k, 1 \leq k \leq n$, are subharmonic on $\mathbb{C}$, M is pseudoconvex. Moreover, $(M, 0)$ is of infinite type in the sense of Bloom-Graham.

Indeed, for each $m = 1, 2, \ldots$, consider an n-dimensional complex submanifold X_m in $\mathbb{C}^{n+1}$ defined by

$$X_m = \left\{(z, w) \in \mathbb{C}^{n+1} \colon w = -\sum_{k=1}^{m} a_k^1 z_1^k - \cdots - \sum_{k=1}^{m} a_k^n z_n^k, |z_j| < \frac{1}{e\epsilon_m^j}, 1 \leq j \leq n\right\}.$$

Then $\rho\mid_{X_m}(z, w) = o(|z_1|^m) + \cdots + o(|z_n|^m)$ vanishes to order $\geq m$ at $p = 0$. Consequently, X_m is tangent to M at 0 to order $\geq m$. This yields that M is of infinite type in the sense of Bloom-Graham at $p = 0$.

We now prove that there does not exist a nonconstant holomorphic curve $\gamma_\infty := (h_1, \ldots, h_n; g) : (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, 0)$, where $g, h_j, 1 \leq j \leq n$, are holomorphic functions on a neighborhood of the origin in $\mathbb{C}$, such that $\nu(\rho \circ \gamma_\infty) = +\infty$, that is,

$$\rho \circ \gamma_\infty(t) = \mathrm{Re}(g(t)) + f_1(h_1(t)) + \cdots + f_n(h_n(t))) = o(t^\infty). \tag{1}$$

Suppose otherwise that there exists such a holomorphic curve. Without loss of generality, we may assume that $g, h_j, 1 \leq j \leq n$, are all holomorphic on the unit disk $\Delta := \{z \in \mathbb{C} \colon |z| < 1\}$.

We now consider the following cases

Case 1. $h_j \equiv 0$ for $1 \leq j \leq n$. In this case we have $f_j(h_j(t)) \equiv 0$ for $1 \leq j \leq n$. Therefore, it follows from (1) that

$$\mathrm{Re}(g(t)) = o(t^\infty).$$

This implies that $g \equiv 0$, which is impossible.

Case 2. $g \equiv 0$. Then, (1) becomes

$$f_1(h_1(t)) + \cdots + f_n(h_n(t))) = o(t^\infty). \tag{2}$$

Expanding $h_1, \dots, h_n$ into the Taylor series at $t=0$, we obtain

$$h_j(t) = \sum_{m=1}^{\infty} \alpha_m^j t^m,\ \alpha_m^j \in \mathbb{C},\ 1 \le j \le n.$$

Hence, by (2) one must have

$$\begin{aligned} a_1^1\alpha_1^1 + \cdots + a_1^n\alpha_1^n &= 0;\\ a_1^1\alpha_2^1 + \cdots + a_1^n\alpha_2^n + a_2^1(\alpha_1^1)^2 + \cdots + a_2^n(\alpha_1^n)^2 &= 0;\\ \cdots&\\ \sum_{m=1}^{k} a_m^1 \sum_{\substack{n_1+\cdots+n_m=k\\ n_1,\dots,n_m\ge 1}} \alpha_{n_1}^1\cdots\alpha_{n_m}^1 + \cdots +&\\ + \sum_{m=1}^{k} a_m^n \sum_{\substack{n_1+\cdots+n_m=k\\ n_1,\dots,n_m\ge 1}} \alpha_{n_1}^n\cdots\alpha_{n_m}^n &= 0;\\ \cdots& \end{aligned} \tag{3}$$

Since $h_j, 1 \le j \le n$, are all holomorphic on the unit disk Δ, without loss of generality we can assume that $|\alpha_m^j| \le 1$ for every $m \ge 1$ and $1 \le j \le n$. Moreover, without loss of generality, we may assume that $h_n \not\equiv 0$ and $\alpha_1^n = 1$. Therefore, one has

$$\left| \sum_{\substack{n_1+\cdots+n_m=k\\ n_1,\dots,n_m\ge 1}} \alpha_{n_1}^j\cdots\alpha_{n_m}^j \right| \le k^k$$

for every $k \ge 1$ and $1 \le j \le n$. Hence, (3) yields that

$$|a_k^n| \le k^k \left(\sum_{m=1}^{k-1} a_m^n + \sum_{j=1}^{n-1}\sum_{m=1}^{k} a_m^j \right),\ k \ge 1.$$

This contradicts the condition (ii).

Case 3. $g \not\equiv 0, h_j \not\equiv 0$ for some $j \in \{1, 2, \dots, n\}$. Then, (1) becomes

$$\mathrm{Re}(g(t)) + f_1(h_1(t)) + \cdots + f_n(h_n(t))) = o(t^\infty).$$

Expanding $g, h_1, \dots, h_n$ into the Taylor series at $t=0$, we have

$$h_j(t) = \sum_{m=1}^{\infty} \alpha_m^j t^m,\ \alpha_m^j \in \mathbb{C},\ 1 \le j \le n;$$
$$g(t) = \sum_{m=1}^{\infty} \gamma_m t^m,\ \gamma_m \in \mathbb{C}.$$

Hence, by (1) one must have

$$\begin{aligned} \gamma_1 + a_1^1\alpha_1^1 + \cdots + a_1^n\alpha_1^n &= 0;\\ \gamma_2 + a_1^1\alpha_2^1 + \cdots + a_1^n\alpha_2^n + a_2^1(\alpha_1^1)^2 + \cdots + a_2^n(\alpha_1^n)^2 &= 0;\\ \cdots& \end{aligned}$$

$$\gamma_k + \sum_{m=1}^{k} a_m^1 \sum_{\substack{n_1+\cdots+n_m=k\\ n_1,\ldots,n_m\geq 1}} \alpha_{n_1}^1 \cdots \alpha_{n_m}^1 + \cdots + \tag{4}$$

$$+ \sum_{m=1}^{k} a_m^n \sum_{\substack{n_1+\cdots+n_m=k\\ n_1,\ldots,n_m\geq 1}} \alpha_{n_1}^n \cdots \alpha_{n_m}^n = 0;$$

$$\cdots$$

Since $g, h_j, 1 \leq j \leq n$, are all holomorphic on the unit disk Δ, without loss of generality we can assume that $|\alpha_m^j| \leq 1, |\gamma_m| \leq 1$ for every $m \geq 1$ and $1 \leq j \leq n$. Moreover, without loss of generality, we may assume that $h_n \not\equiv 0$ and $\alpha_1^n = 1$. Therefore, one has

$$\left| \sum_{\substack{n_1+\cdots+n_m=k\\ n_1,\ldots,n_m\geq 1}} \alpha_{n_1}^j \cdots \alpha_{n_m}^j \right| \leq k^k$$

for every $k \geq 1$ and $1 \leq j \leq n$. Hence, (4) yields that

$$|a_k^n| \leq k^k \left(\sum_{m=1}^{k-1} a_m^n + \sum_{j=1}^{n-1} \sum_{m=1}^{k} a_m^j \right) + 1, \ k \geq 1.$$

This again contradicts the condition (i) and (ii).

Altogether, the proof is complete. □

Proof of Corollary 1. Let M be the smooth pseudoconvex real hypersurface given in the proof of Theorem 2. Then M is of Bloom-Graham infinite type at $p = 0$ and moreover it does not admit any nonconstant holomorphic curve tangent to M at $p = 0$ to infinite order. We shall show that M does also not admit any complex submanifold X_∞ of codimension one in $\mathbb{C}^{n+1}$ tangent to M at 0 to infinite order. That is, $\tilde{\rho}\,|_{X_\infty}$ vanishes to infinite order at $p = 0$.

Indeed, suppose otherwise. Then, $\tilde{\rho}\,|_{X_\infty \cap \{z_2=\cdots=z_n=0\}}$ vanishes to infinite order at $p = 0$. Note that $X_\infty \cap \{z_2 = \cdots = z_n = 0\}$ is locally represented as the graph of a holomorphic curve. Therefore, there exists a nonconstant holomorphic curve $\gamma_\infty : (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, 0)$ tangent to M at 0 to infinite order, which is a contradiction. Hence, this completes the proof. □

In order to give a proof of Theorem 1, we need the following lemma.

Lemma 5. *Let $n \geq 1$. If (M, p) is a smooth real hypersurface germ of Bloom-Graham infinite type in $\mathbb{C}^{n+1}$, then (M, p) is of D'Angelo infinite type.*

Proof. Suppose that (M, p) is a smooth real hypersurface germ of Bloom-Graham infinite type in $\mathbb{C}^{n+1}$ and let r be a local defining function for M near p. By definition, it follows that there exists a sequence $\{X_m\}_{m=1}^\infty$ of codimension-one complex submanifolds of $\mathbb{C}^{n+1}$ such that $p \in X_m$ and

$$ord_p(r\,|_{X_m}) \geq m$$

for every $m = 1, 2, \ldots$.

For each $m = 1, 2, \ldots$, choose a regular holomorphic curve $\gamma_m : \Delta := \{z \in \mathbb{C} \colon |z| < 1\} \to X_m \subset \mathbb{C}^{n+1}$ such that $p = \gamma_m(0)$ and $\nu_0(\gamma_m) = 1$. Then, $\nu_0(r \circ \gamma_m) \geq$

$ord_p(r\mid_{X_m}) \geq m$. This yields that the D'Angelo type of M at p equals $+\infty$. Therefore, the proof is complete. □

Proof of Theorem 1. Let M be the smooth pseudoconvex real hypersurface given in the proof of Theorem 2. Then M is of Bloom-Graham infinite type at $p = 0$. By Lemma 5, M is of D'Angelo infinite type at $p = 0$. Hence, the proof follows from Theorem 2. □

Remark 1. Let X be a variety in $\mathbb{C}^n$ defined by $f_1(z_1) + \cdots + f_n(z_n) = 0$, where $f_j, 1 \leq j \leq n$, given in the proof of Theorem 2. Following the proof of Theorem 2, we conclude that X does not admit any nonconstant holomorphic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ with convergent Taylor series tangent to X at 0 to infinite order. Moreover, let $\widetilde{X}$ be a formal variety in $\mathbb{C}^n$ defined by $\tilde{f}_1(z_1)+\cdots+\tilde{f}_n(z_n) = 0$, where $\tilde{f}_j, 1 \leq j \leq n$, respectively, are the (divergent) Taylor series of smooth harmonic functions $f_j, 1 \leq j \leq n$, given in the proof of Theorem 2. Following the proof of Theorem 2, we also conclude that $\widetilde{X}$ does not contain any nonconstant holomorphic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ with convergent Taylor series.

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