Adjusted sparse tensor product spectral Galerkin method for solving pseudodifferential equations on the sphere with random input data

Duong Pham*† and Dinh Dũng‡

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Abstract

An adjusted sparse tensor product spectral Galerkin approximation method based on spherical harmonics is introduced and analyzed for solving pseudodifferential equations on the sphere with random input data. These equations arise from geodesy where the sphere is taken as a model of the earth. Numerical solutions to the corresponding k-th order statistical moment equations are found in adjusted sparse tensor approximation spaces which are accordingly designed to the regularity of the data and the equation. Established convergence theorem shows that the adjusted sparse tensor Galerkin discretization is superior not only to the full tensor product but also to the standard sparse tensor counterpart when the data's statistical moments are of mixed unequal regularity. Numerical experiments illustrate our theoretical results.

Keywords: Stochastic pseudodifferential equations, statistical moments, hyperbolic cross spectral methods

AMS Subject Classification: 35S05, 58J40, 65N15, 65N30

1 Introduction

Pseudodifferential operators have long been used [12, 17] as a modern and powerful tool to tackle linear boundary-value problems. Svensson [33] introduces this approach to geodesists who study [9, 11] these problems on the sphere which is taken as a model of the earth. Together with powerful computers, advanced numerical schemes are capable of producing highly accurate and efficient deterministic numerical simulations, provided that the problem data are known exactly. However, in reality the problem data are prone to uncertainty for many reasons. One is the unavoidable error due to imperfect measurement devices. Secondly, the error arises when estimating the problem parameters based on a large but finite number of system samples. Finally, parameters of the system originate from a mathematical model which is itself only an approximation of the actual process. Under such circumstances, highly accurate results of a single deterministic simulation for one particular set of problem parameters are of limited use. In this paper we suggest and analyze a natural and efficient numerical approach to solve pseudodifferential equations on the sphere, accounting for uncertainty in the input data.

We consider the following pseudifferential equation

$$Lu = f \quad \text{on } \mathbb{S}, \tag{1.1}$$

*Institute for Computational Science and Technology, SBI building, Quang Trung Software City, Tan Chanh Hiep ward, District 12, Ho Chi Minh city, Vietnam, ptduong01@gmail.com

†Vietnamese German University, Le Lai street, Binh Duong New City, Binh Duong Province, Vietnam

‡Information Technology Institute, Vietnam National University, Hanoi, 144 Xuan Thuy, Hanoi, Vietnam, dinhzung@gmail.com

where $\mathbb{S}$ is the unit sphere, i.e., $\mathbb{S} = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = 1\}$ and L is a pseudodifferential operator that assigns to any distribution v defined on $\mathbb{S}$ a distribution

$$Lv = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{L}(\ell)\widehat{v}_{\ell,m} Y_{\ell,m}. \tag{1.2}$$

Here, the sequence $\left\{\widehat{L}(\ell)\right\}_{\ell=0}^{\infty}$ is the *spherical symbol* of L, and $\widehat{v}_{\ell,m}$ is the Fourier coefficient of v with respect to the spherical harmonic $Y_{\ell,m}$ for all $\ell \in \mathbb{N}$ and $m = -\ell, \dots, \ell$. Precise definitions of pseudodifferential operators, Fourier coefficients and spherical harmonics are presented in Section 2.

Equation (1.1) with deterministic input data f has been solved by using different approximation methods. They could be continuous piecewise bilinear basis functions [14], spherical radial basis functions [19, 24, 25, 34] or spherical splines [26]. In this paper, spaces of spherical harmonics are used to solve (1.1) with random right-hand side. These functions are important in many theoretical and practical applications, particularly in the computation of atomic orbital electron configurations, representation of gravitational fields, geoids, and the magnetic fields of planetary bodies and stars, and characterization of the cosmic microwave background radiation, see e.g. [6, 10, 18, 27] and the references therein. The representation (1.2) results in diagonal Galerkin matrices of L with respect to bases of spherical harmonics. This property shows that subsets of spherical harmonics are very natural Galerkin bases to solve pseudodifferential equations on the sphere. In this paper, spherical harmonics will appear to be even more important for the case of the random data f when numerical solutions of the randomized problem will lead to a high-dimensional formulation, suffering from the *curse of dimensionality.* Under such circumstances, diagonality of the Galerkin matrix will make the solution of the linear system fast and simple.

In the present paper we assume that the input data (and therefore the solution) depends on a "random event" $\omega \in \Omega$ (assuming $(\Omega, \Sigma, \mathbb{P})$ is the underlying probability space) which allows to treat f as a *random field* being a measurable mapping

$$f : \begin{cases} \Omega & \to & H^{-\alpha}(\mathbb{S}), \\ \omega & \mapsto & f(\cdot, \omega). \end{cases} \tag{1.3}$$

This together with the continuity of the inverse of L implies that the solution u is a random field too. Since parametrization of Ω requires in general infinitely many variables, numerical solution to (1.1) leads to an infinite dimensional problem (see e.g. [7] for the case of a random coefficient). By a suitable truncation procedure [31] the number of stochastic dimensions can be reduced to a finite number, which, however, can still be large. This together with the tensor product discretization will lead to a prohibitive number of unknowns. Practical solution of this high-dimensional problem makes mandatory the use of sparse/adaptive discretization methods and specific solution procedures, see e.g. [15, 16, 28] and the references therein.

In many applications the solution $u(x, \omega)$ is not of interest as a function of ω. Frequently the mean field $\mathbb{E}_u(x) := \int_\Omega u(x, \omega) d\mathbb{P}(\omega)$, the covariance $C_u(x, y) := \mathbb{E}[(u(x, \omega) - \mathbb{E}_u(x))(u(y, \omega) - \mathbb{E}_u(y))]$, and higher order statistical moments $\mathcal{M}^k u(x_1, \dots, x_k)$ are the quantities of interest. Tensorization of (1.1) and integration over the set of elementary events Ω provides a formulation whose solution is the quantity of interest itself [29, 30, 35]:

$$\underbrace{(L \otimes \cdots \otimes L)}_{k \text{ times}} \mathcal{M}^k u = \mathcal{M}^k f, \tag{1.4}$$

Numerical solutions to the equation (1.4) have a huge challenge: the solution $\mathcal{M}^k u$ is a function on a $2k$-dimensional manifold $\mathbb{S} \times \cdots \times \mathbb{S}$ (k times) and this yields to an exponential in k growth of complexity for the full tensor product discretizations. An efficient discretization of (1.4) must be carefully adjusted to the regularity of $\mathcal{M}^k u$ and also to the nature of the equation itself.

Extending the approach in [2, 5] we propose in this paper an *adjusted* sparse tensor product spectral discretization of (1.4) based on suitable combination of bases of spherical harmonics on individual spheroids $\mathbb{S}$. We first consider the case when $\mathcal{M}^k f$ has a finite equal mixed regularity, i.e, $\mathcal{M}^k f \in H^s(\mathbb{S}) \otimes \cdots \otimes H^s(\mathbb{S})$ for some $s \in \mathbb{R}$. We prove that the Galerkin discretization by standard sparse tensor product spherical harmonics with degree from a standard hyperbolic cross achieves the same convergence rate as the discretization by the space of full tensor product spherical harmonics (see Theorem 3.5). This yields a significant gain in the convergence rate, expressed in terms of the total number of unknowns since the cardinality of the standard sparse space grows like $\frac{2^k}{(k-1)!}\mathcal{T}^2(\ln \mathcal{T} + k \ln 2)^{k-1}$ (see Lemma 3.6) compared to the cardinality $(\mathcal{T}+1)^{2k}$ of the full space. When the right hand side $\mathcal{M}^k f$ (and hence the $\mathcal{M}^k u$) has an *unequal mixed regularity*, i.e., $\mathcal{M}^k u \in H^{s_1}(\mathbb{S}) \otimes \cdots \otimes H^{s_k}(\mathbb{S})$ where $s_1 \le \ldots \le s_k$, we propose the use of an adjusted sparse tensor product spectral Galerkin approximation which follows an adjusted hyperbolic cross of degrees. This adjustment of degrees is based on not only the regularity of the input data but also on the order of the operator. This adjusted sparse tensor product approximation space produces the optimal convergence rate (see Theorem 4.3) with a minimal number of required unknowns. We prove an upper bound for the dimension of the adjusted sparse tensor product approximation space (see Theorem 4.1) which shows that to achieve the optimal convergence rate, the adjusted sparse approximation method requires

$$\frac{2^{\nu+1}\mathcal{T}^2}{\nu!}\left[\ln \mathcal{T} + (\nu+1)\ln 2\right]^{\nu} \times \exp\left(\sum_{\nu+2}^{k} \frac{s-\alpha}{s_j - s}\left(\frac{3}{2}\right)^{-\frac{2(s_j-s)}{s-\alpha}}\right)$$

unknowns if the smoothness satisfies $s = s_1 = \ldots = s_{\nu+1} < s_{\nu+2} \le \ldots \le s_k$, see Corollary 4.4.

The structure of the paper is as follows. In Section 2, we first review Sobolev spaces on the unit sphere and tensor products of these spaces. We then present pseudodifferential operators on the unit sphere and problems with random data. In Section 3, we investigate the use of standard sparse tensor product spectral Galerkin method in solving statistical moment equations which arises from pseudifferential equations with random input data. In Section 4, we introduce nonuniform (adjusted) hyperbolic cross sets and approximation spaces of spherical harmonics based on nonuniform hyperbolic scross sets. These adjusted sparse tensor product approximation spaces are then used to solve k-order statistical moment equations of pseudifferential equations on spheres. The final section (Section 5) presents our numerical experiments which illustrate our theoretical results.

Throughout, we adopt the following notation: for any positive real numbers x, y we write $x \lesssim y$ if there exists a constant C independent of any parameters which x and y might depend on, so that $x \le Cy$.

2 Preliminaries

2.1 Sobolev spaces

Throughout this paper, we denote by $\mathbb{S}$ the unit sphere in $\mathbb{R}^3$, i.e., $\mathbb{S} := \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = 1\}$ where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$. A spherical harmonic of order ℓ on $\mathbb{S}$ is the restriction to $\mathbb{S}$ of a homogeneous harmonic polynomial of degree ℓ in $\mathbb{R}^3$. The space of all spherical harmonics of order ℓ is the eigenspace of the Laplace–Beltrami operator $\Delta_{\mathbb{S}}$ corresponding to the eigenvalue $\lambda_\ell = -\ell(\ell+1)$. The dimension of this space being $2\ell+1$, see e.g. [20], one may choose for it an orthonormal basis $\{Y_{\ell,m}\}_{m=-\ell}^{\ell}$. The collection of all the spherical harmonics $Y_{\ell,m}$, $m = -\ell, \ldots, \ell$ and $\ell = 0, 1, \ldots$, forms an orthonormal basis for $L_2(\mathbb{S})$.

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{S})$ is defined as usual by

$$H^s(\mathbb{S}) := \Big\{ v \in \mathcal{D}'(\mathbb{S}) : \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1)^{2s} |\widehat{v}_{\ell,m}|^2 < \infty \Big\},$$

where $\mathcal{D}'(\mathbb{S})$ is the space of distributions on $\mathbb{S}$ and

$$\widehat{v}_{\ell,m} := \int_{\mathbb{S}} v(\boldsymbol{x}) Y_{\ell,m}(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}}$$

are the Fourier coefficients of v. Here $d\sigma_{\boldsymbol{x}}$ is the element of surface area. The space $H^s(\mathbb{S})$ is equipped with the inner product

$$\langle v, w\rangle_{H^s(\mathbb{S})} := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1)^{2s} \widehat{v}_{\ell,m} \widehat{w}_{\ell,m}.$$

and the norm

$$\|v\|_{H^s(\mathbb{S})} := \sqrt{\langle v, v\rangle_{H^s(\mathbb{S})}} = \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1)^{2s} |\widehat{v}_{\ell,m}|^2 \right)^{1/2}. \tag{2.1}$$

When $s = 0$ we write $\langle \cdot, \cdot\rangle$ instead of $\langle \cdot, \cdot\rangle_{H^0(\mathbb{S})}$. This is in fact the $L_2(\mathbb{S})$-inner product. We note that

$$|\langle v, w\rangle_{H^s(\mathbb{S})}| \leq \|v\|_{H^s(\mathbb{S})} \|w\|_{H^s(\mathbb{S})} \quad \forall v, w \in H^s(\mathbb{S}),\ \forall s \in \mathbb{R}, \tag{2.2}$$

and

$$\|v\|_{H^{s_1}(\mathbb{S})} = \sup_{\substack{w \in H^{s_2}(\mathbb{S}) \\ w \neq 0}} \frac{\langle v, w\rangle_{H^{\frac{s_1+s_2}{2}}(\mathbb{S})}}{\|w\|_{H^{s_2}(\mathbb{S})}} \quad \forall v \in H^{s_1}(\mathbb{S}),\ \forall s_1, s_2 \in \mathbb{R}. \tag{2.3}$$

We now introduce tensor products of Sobolev spaces. Given $r, s \in \mathbb{R}$, the Sobolev space $H^{r,s}_{mix}(\mathbb{S}^2)$ is defined to be the space of all distributions v on $\mathbb{S}^2 := \mathbb{S} \times \mathbb{S}$ satisfying

$$\sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \sum_{m_1=-\ell_1}^{\ell_1} \sum_{m_2=-\ell_2}^{\ell_2} (1+\ell_1)^{2r} (1+\ell_2)^{2s} \left| \widehat{v}_{(\ell_1,\ell_2),(m_1,m_2)} \right|^2 < \infty, \tag{2.4}$$

with the Fourier coefficients

$$\widehat{v}_{(\ell_1,\ell_2),(m_1,m_2)} := \int_{\mathbb{S}} \int_{\mathbb{S}} v(\boldsymbol{x}, \boldsymbol{y})\, Y_{\ell_1,m_1}(\boldsymbol{x}) Y_{\ell_2,m_2}(\boldsymbol{y})\, d\sigma_{\boldsymbol{x}}\, d\sigma_{\boldsymbol{y}}. \tag{2.5}$$

The inner product on $H^{r,s}_{mix}(\mathbb{S}^2)$ is given by

$$\langle v, w\rangle_{H^{r,s}_{mix}(\mathbb{S}^2)} := \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \sum_{m_1=-\ell_1}^{\ell_1} \sum_{m_2=-\ell_2}^{\ell_2} (1+\ell_1)^{2r} (1+\ell_2)^{2s} \widehat{v}_{(\ell_1,\ell_2),(m_1,m_2)} \widehat{w}_{(\ell_1,\ell_2),(m_1,m_2)} \tag{2.6}$$

for any $v, w \in H^{r,s}_{mix}(\mathbb{S}^2)$ and $\|v\|_{H^{r,s}_{mix}(\mathbb{S}^2)} := \langle v, v\rangle^{1/2}_{H^{r,s}_{mix}(\mathbb{S}^2)}$.

We denote by $\bigotimes_{i=1}^k X_i$ the tensor product of separable Hilbert spaces X_i, for $i = 1, \ldots, k$. For the corresponding inner products there holds (see, e.g., [1, page 298])

$$\left\langle \bigotimes_{i=1}^k v_i, \bigotimes_{i=1}^k w_i \right\rangle_{\bigotimes_{i=1}^k X_i} = \prod_{i=1}^k \langle v_i, w_i\rangle_{X_i} \quad \forall v_i, w_i \in X_i,\ i = 1, \ldots, k, \tag{2.7}$$

and thus

$$\|v_1 \otimes \cdots \otimes v_k\|_{\bigotimes_{i=1}^k X_i} = \prod_{i=1}^k \|v_i\|_{X_i} \quad \forall v_i \in X_i,\ i = 1, \ldots, k. \tag{2.8}$$

In the case $X_i = X$ for $i = 1, \ldots, k$, we denote $X^{(k)} := \bigotimes_{i=1}^k X$. Noting (2.8), there holds

$$\|v_1 \otimes \cdots \otimes v_k\|_{X^{(k)}} = \|v_1\|_X \cdots \|v_k\|_X \tag{2.9}$$

for every $v_1, \ldots, v_k \in X$. Relation (2.7) and definition (2.6) provide the isometry $H^r(\mathbb{S}) \otimes H^s(\mathbb{S}) = H^{r,s}_{mix}(\mathbb{S}^2)$. In what follows, we identify the space $H^{r,s}_{mix}(\mathbb{S}^2)$ and the tensor product $H^r(\mathbb{S}) \otimes H^s(\mathbb{S})$. By this agreement, we define for any $(r_1, \ldots, r_k) \in \mathbb{R}^k$ and $k \geq 2$ the tensor product space on $\mathbb{S}^k := \mathbb{S} \times \ldots \times \mathbb{S}$ by

$$H^{r_1,\ldots,r_k}_{mix}(\mathbb{S}^k) := \bigotimes_{i=1}^k H^{r_i}(\mathbb{S}).$$

Noting (2.6), the corresponding inner product and norm in $H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k)$ are defined by

$$\langle v, w\rangle_{H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k)} := \sum_{\boldsymbol{\ell}=\boldsymbol{0}}^{\infty}\sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}}\prod_{i=1}^{k}(1+\ell_i)^{2r_i}\widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}}\widehat{w}_{\boldsymbol{\ell},\boldsymbol{m}}, \quad v, w \in H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k), \tag{2.10}$$

and

$$\|v\|_{H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k)} := \langle v, v\rangle^{1/2}_{H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k)}, \quad v \in H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k). \tag{2.11}$$

Here, we use the following notations

$$\sum_{\boldsymbol{\ell}=\boldsymbol{0}}^{\infty}\sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} := \sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_k=0}^{\infty}\sum_{m_1=-\ell_1}^{\ell_1}\cdots\sum_{m_k=-\ell_k}^{\ell_k},$$

and

$$\widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}} := \int_{\mathbb{S}^k} v(\boldsymbol{x}_1,\dots,\boldsymbol{x}_k)Y_{\ell_1,m_1}(\boldsymbol{x}_1)\cdots Y_{\ell_k,m_k}(\boldsymbol{x}_k)\,d\sigma_{\boldsymbol{x}_1}\dots d\sigma_{\boldsymbol{x}_k}, \tag{2.12}$$

where $\boldsymbol{\ell} = (\ell_1,\dots,\ell_k)$ and $\boldsymbol{m} = (m_1,\dots,m_k)$. In the case $r_1 = \cdots = r_k = r$, we denote $H^r_{mix}(\mathbb{S}^k)$ instead of $H^{r,\dots,r}_{mix}(\mathbb{S}^k)$. Inequality (2.2) and the identity (2.3) combined with above definitions imply

$$\langle v, w\rangle_{H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k)} \le \|v\|_{H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k)}\|w\|_{H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k)} \quad \forall v, w \in H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k) \tag{2.13}$$

for any $(r_1,\dots,r_k) \in \mathbb{R}^k$ and

$$\|v\|_{H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k)} = \sup_{\substack{w\in H^{t_1,\dots,t_k}_{mix}(\mathbb{S}^k)\\ w\neq 0}} \frac{\langle v, w\rangle_{H^{\frac{r_1+t_1}{2},\dots,\frac{r_k+t_k}{2}}_{mix}(\mathbb{S}^k)}}{\|w\|_{H^{t_1,\dots,t_k}_{mix}(\mathbb{S}^k)}} \quad \forall v \in H^{r_1,\dots,r_k}_{mix}(\mathbb{S}^k) \tag{2.14}$$

for any $(r_1,\dots,r_k)$ and $(t_1,\dots,t_k)$ in $\mathbb{R}^k$.

2.2 Pseudodifferential operators

Let $\{\widehat{L}(\ell)\}_{\ell\ge 0}$ be a sequence of real numbers. A pseudodifferential operator L is a linear operator that assigns to any $v \in \mathcal{D}'(\mathbb{S})$ a distribution

$$Lv := \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\widehat{L}(\ell)\widehat{v}_{\ell,m}Y_{\ell,m}. \tag{2.15}$$

The sequence $\{\widehat{L}(\ell)\}_{\ell\ge 0}$ is referred to as the *spherical symbol* of L. Let $\mathcal{K}(L) := \{\ell : \widehat{L}(\ell) = 0\}$. Then

$$\ker L = \operatorname{span}\{Y_{\ell,m} : \ell \in \mathcal{K}(L),\, m = -\ell,\dots,\ell\}.$$

Denoting $M := \dim\ker L$, we assume that $0 \le M < \infty$. In this paper, we assume that L is a *strongly elliptic* pseudodifferential operator of order 2α, i.e.,

$$C_1(\ell+1)^{2\alpha} \le \widehat{L}(\ell) \le C_2(\ell+1)^{2\alpha} \quad \text{for all } \ell \notin \mathcal{K}(L), \tag{2.16}$$

for some positive constants C_1 and C_2.

More general pseudodifferential operators can be defined via Fourier transforms by using local charts, see e.g., [13, 23]. It can be easily seen that if L is a pseudodifferential operator of order 2α then $L : H^{s+\alpha} \to H^{s-\alpha}$ is bounded for all $s \in \mathbb{R}$.

The following commonly seen pseudodifferential operators are strongly elliptic, see [33].

- The Laplace–Beltrami operator (with the minus sign) is an operator of order 2 and has as symbol $\widehat{L}(\ell) = \ell(\ell+1)$. This operator is the restriction of the Laplacian on the sphere.
- The hypersingular integral operator (with the minus sign) is an operator of order 1 and has as symbol $\widehat{L}(\ell) = \ell(\ell+1)/(2\ell+1)$. This operator arises from the boundary-integral reformulation of the Neumann problem with the Laplacian in the interior or exterior of the sphere.
- The weakly-singular integral operator is an operator of order -1 and has as symbol $\widehat{L}(\ell) = 1/(2\ell+1)$. This operator arises from the boundary-integral reformulation of the Dirichlet problem with the Laplacian in the interior or exterior of the sphere.

For a given function f, the pseudodifferential equation $Lu = f$ is not uniquely solvable if the set $\mathcal{K}(L)$ is nonempty. To assure the unique solvability of the equation, additional conditions must be included. These conditions could be the introduction of an unisolvent system of M additional equations, or the use of a new pseudodifferential equation $L^*u = f$ in which L^* is a strongly elliptic pseudodifferential operator defined by

$$\widehat{L^*}(\ell) = \begin{cases} \widehat{L}(\ell) & \text{if } \ell \notin \mathcal{K}(L) \\ (1+\ell)^{2\alpha} & \text{if } \ell \in \mathcal{K}(L), \end{cases}$$

see, e.g., [26]. Noting (2.16), there holds

$$C_1(1+\ell)^{2\alpha} \le \widehat{L^*}(\ell) \le C_2(1+\ell)^{2\alpha} \quad \forall \ell \ge 0.$$

For the sake of presentational simplicity, we consider in this paper the pseudodifferential equation

$$Lu = f \text{ on } \mathbb{S}, \tag{2.17}$$

where $f \in H^{-\alpha}(\mathbb{S})$ is a given function and the operator L satisfies

$$C_1(1+\ell)^{2\alpha} \le \widehat{L}(\ell) \le C_2(1+\ell)^{2\alpha} \quad \forall \ell \ge 0. \tag{2.18}$$

We define the bilinear form $a : H^\alpha(\mathbb{S}) \times H^\alpha(\mathbb{S}) \to \mathbb{R}$ by

$$a(v,w) := \langle Lv, w \rangle, \quad v, w \in H^\alpha(\mathbb{S}). \tag{2.19}$$

The corresponding variational formulation of (2.17) is: Find $u \in H^\alpha(\mathbb{S})$ satisfying

$$a(u,v) = \langle f, v \rangle \quad \forall v \in H^\alpha(\mathbb{S}). \tag{2.20}$$

Noting (2.1) and (2.18), the bilinear form $a(\cdot,\cdot)$ is bounded and coercive in $H^\alpha(\mathbb{S})$. The well-known Lax–Milgram theorem confirms the unique existence of the solution $u \in H^\alpha(\mathbb{S})$ of (2.20).

2.3 Problems with random data

In what follows we consider the equation (2.17) for random loading f, which leads to random solution u. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space consisting of the space of elementary events Ω, σ-algebra of its subsets Σ and the probability measure $\mathbb{P}$ on Σ. Suppose that f is a random field, i.e. a measurable mapping satisfying

$$f : \begin{cases} \Omega \to H^{-\alpha}(\mathbb{S}) \\ \omega \mapsto f(\cdot, \omega). \end{cases}$$

Since the solution operator $L^{-1} : H^{-\alpha}(\mathbb{S}) \to H^\alpha(\mathbb{S})$ is continuous, the solution $u := L^{-1}f$ is measurable and satisfies

$$u : \begin{cases} \Omega \to H^\alpha(\mathbb{S}) \\ \omega \mapsto u(\cdot, \omega). \end{cases}$$

In order to introduce the notion of a k-th statistical moment of a random field we define (following [30, 35]) for a positive integer k and a separable Hilbert space X a Bochner space

$$L^k(\Omega, X) := \{v : \Omega \to X : \int_\Omega \|v(\omega)\|_X^k \, d\mathbb{P}(\omega) < +\infty\}$$

equipped with the norm

$$\|v\|_{L^k(\Omega,X)} := \Big(\int_\Omega \|v(\omega)\|_X^k \, d\mathbb{P}(\omega)\Big)^{1/k}.$$

Definition 2.1. *Let $v \in L^k(\Omega, X)$. The k-th order moment $\mathcal{M}^k v \in X^{(k)}$ of v is given by*

$$\mathcal{M}^k v := \int_\Omega \Big(\underbrace{v(\omega) \otimes \cdots \otimes v(\omega)}_{k \ \textit{times}}\Big) \, d\mathbb{P}(\omega).$$

By (2.9), the k-th order moment $\mathcal{M}^k v$ is well defined for any $v \in L^k(\Omega, X)$. Let $u(\omega)$ be a random solution of (2.17) with a random right hand side $f(\omega)$. We consider the tensor product operator $L^{(k)} := L \otimes \cdots \otimes L$ (k times) which is a linear mapping

$$L^{(k)} : H^\alpha_{mix}(\mathbb{S}^k) \to H^{-\alpha}_{mix}(\mathbb{S}^k), \tag{2.21}$$

see [35, Proposition 2.4] for more details. Tensorization of $Lu(\omega) = f(\omega)$ yields for every fixed elementary event $\omega \in \Omega$

$$L^{(k)}\big(\otimes_{i=1}^k u(\omega)\big) = \otimes_{i=1}^k f(\omega) \quad \text{in} \quad H^{-\alpha}_{mix}(\mathbb{S}^k). \tag{2.22}$$

Taking the mean of (2.22) yields the deterministic k-th statistical moment problem: Given $\mathcal{M}^k f \in H^{-\alpha}_{mix}(\mathbb{S}^k)$, find $\mathcal{M}^k u \in H^\alpha_{mix}(\mathbb{S}^k)$ satisfying

$$L^{(k)} \mathcal{M}^k u = \mathcal{M}^k f. \tag{2.23}$$

Noting (2.21), the variational counterpart of (2.23) is: Given $\mathcal{M}^k f \in H^{-\alpha}_{mix}(\mathbb{S}^k)$, find $\mathcal{M}^k u \in H^\alpha_{mix}(\mathbb{S}^k)$ satisfying

$$\mathscr{A}(\mathcal{M}^k u, v) = \big\langle\big\langle \mathcal{M}^k f, v\big\rangle\big\rangle \quad \forall v \in H^\alpha_{mix}(\mathbb{S}^k), \tag{2.24}$$

where $\mathscr{A}(\cdot,\cdot) = \big\langle\big\langle L^{(k)}\cdot, \cdot\big\rangle\big\rangle$ is the bilinear form and $\langle\langle \cdot, \cdot\rangle\rangle$ is the $H^{-\alpha}_{mix}(\mathbb{S}^k) \times H^\alpha_{mix}(\mathbb{S}^k)$ duality pairing. Proposition 2.4 in [35] implies

Lemma 2.2. *The bilinear form $\mathscr{A}(\cdot,\cdot) : H^\alpha_{mix}(\mathbb{S}^k) \times H^\alpha_{mix}(\mathbb{S}^k) \to \mathbb{R}$ is bounded and $H^\alpha_{mix}(\mathbb{S}^k)$-elliptic, i.e.,*

$$\mathscr{A}(v, w) \le C_1^k \|v\|_{H^\alpha_{mix}(\mathbb{S}^k)} \|w\|_{H^\alpha_{mix}(\mathbb{S}^k)}, \tag{2.25}$$

and

$$C_2^k \|v\|^2_{H^\alpha_{mix}(\mathbb{S}^k)} \le \mathscr{A}(v, v) \tag{2.26}$$

for all $v, w \in H^\alpha_{mix}(\mathbb{S}^k)$.

The continuity and coercivity of the bilinear form $\mathscr{A}(\cdot,\cdot)$ assure the unique solvability of (2.24).

3 Standard sparse tensor product spectral Galerkin method

The choice of suitable finite dimensional approximation subspaces to solve the problem (2.23) numerically is one of the main ingredients deciding the efficiency of computation. The most naive option is to find Galerkin solutions in full tensor product approximation spaces. For every positive integer $\mathcal{T}$, we denote

$$\delta_{\mathcal{T}} := \Big\{\boldsymbol{\ell} = (\ell_1, \ldots, \ell_k) \in \mathbb{N}^k : \ell_i \le \mathcal{T} \quad \text{for } i = 1, \ldots, k\Big\}, \tag{3.1}$$

and

$$S_{\mathcal{T}} := \operatorname{span}\Big\{ \boldsymbol{Y}_{\boldsymbol{\ell},\boldsymbol{m}} : \boldsymbol{\ell} \in \delta_{\mathcal{T}},\ m_i = -\ell_i, \dots, \ell_i \quad \text{for } i = 1, \dots, k \Big\}, \tag{3.2}$$

where

$$\boldsymbol{Y}_{\boldsymbol{\ell},\boldsymbol{m}}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) := Y_{\ell_1,m_1}(\boldsymbol{x}_1) \cdots Y_{\ell_k,m_k}(\boldsymbol{x}_k), \quad (\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) \in \mathbb{S}^k.$$

It is obvious that

$$\dim(S_{\mathcal{T}}) = (\mathcal{T}+1)^{2k}. \tag{3.3}$$

Note that $\big\{ \boldsymbol{Y}_{\boldsymbol{\ell},\boldsymbol{m}} : \boldsymbol{\ell} \in \mathbb{N}^k, \boldsymbol{m} = -\boldsymbol{\ell}, \dots, \boldsymbol{\ell} \big\}$ forms an orthogonal basis of $H^0_{mix}(\mathbb{S}^k)$. For any distribution v defined on $\mathbb{S}$ of the form

$$v = \sum_{\boldsymbol{\ell}=\boldsymbol{0}}^{\boldsymbol{\infty}} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}} \boldsymbol{Y}_{\boldsymbol{\ell},\boldsymbol{m}}, \tag{3.4}$$

we denote by $P_{\mathcal{T}} v$ the truncated series

$$P_{\mathcal{T}} v := \sum_{\boldsymbol{\ell} \in \delta_{\mathcal{T}}} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}} \boldsymbol{Y}_{\boldsymbol{\ell},\boldsymbol{m}}. \tag{3.5}$$

This truncation is the orthogonal projection into the space $S_{\mathcal{T}}$ with respect to $\langle \cdot, \cdot \rangle_{H^0_{mix}(\mathbb{S}^k)}$.

The following lemma, which is included here for completeness, shows the projection error $\|v - P_{\mathcal{T}} v\|_{H^\alpha_{mix}(\mathbb{S}^k)}$ when

$$v \in \bigcap_{j=1}^{k} H_{mix}^{\alpha,\dots,\overset{jth}{\overset{\downarrow}{s}},\dots,\alpha}(\mathbb{S}^k).$$

Lemma 3.1. *For every $v \in \bigcap_{j=1}^{k} H_{mix}^{\alpha,\dots,\overset{jth}{\overset{\downarrow}{s}},\dots,\alpha}(\mathbb{S}^k)$, where $s \geq \alpha$, there holds*

$$\|v - P_{\mathcal{T}} v\|_{H^\alpha_{mix}(\mathbb{S}^k)} \leq \frac{1}{(\mathcal{T}+2)^{s-\alpha}} \left(\sum_{j=1}^{k} \|v\|^2_{H_{mix}^{\alpha,\dots,\overset{jth}{\overset{\downarrow}{s}},\dots,\alpha}(\mathbb{S}^k)} \right)^{1/2}. \tag{3.6}$$

Proof. Noting (3.5) and the definition of $\delta_{\mathcal{T}}$ (3.1), we have that

$$\begin{aligned} \|v - P_{\mathcal{T}} v\|^2_{H^\alpha_{mix}(\mathbb{S}^k)} &= \sum_{\boldsymbol{\ell} \notin \delta_{\mathcal{T}}} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \prod_{i=1}^{k} (1+\ell_i)^{2\alpha} \left| \widehat{v}_{\ell,m} \right|^2 \\ &= \sum_{j=1}^{k} \left(\sum_{\ell_1=0}^{\infty} \cdots \sum_{\ell_j=\mathcal{T}+1}^{\infty} \cdots \sum_{\ell_k=0}^{\infty} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \prod_{i=1}^{k} (1+\ell_i)^{2\alpha} \left| \widehat{v}_{\ell,m} \right|^2 \right). \end{aligned}$$

Since $v \in H_{mix}^{\alpha,\dots,\overset{jth}{\overset{\downarrow}{s}},\dots,\alpha}(\mathbb{S}^k)$ for every $j = 1, \dots, k$, there holds

$$\begin{aligned} \|v - P_{\mathcal{T}} v\|^2_{H^\alpha_{mix}(\mathbb{S}^k)} &= \sum_{j=1}^{k} \left(\sum_{\ell_1=0}^{\infty} \cdots \sum_{\ell_j=\mathcal{T}+1}^{\infty} \cdots \sum_{\ell_k=0}^{\infty} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \frac{(1+\ell_j)^{2s} \prod\limits_{\substack{i=1 \\ i \neq j}}^{k} (1+\ell_i)^{2\alpha}}{(1+\ell_j)^{2s-2\alpha}} \left| \widehat{v}_{\ell,m} \right|^2 \right) \\ &\leq \frac{1}{(\mathcal{T}+2)^{2(s-\alpha)}} \left(\sum_{j=1}^{k} \|v\|^2_{H_{mix}^{\alpha,\dots,\overset{jth}{\overset{\downarrow}{s}},\dots,\alpha}(\mathbb{S}^k)} \right). \end{aligned}$$

The inequality (3.6) is proved. $\square$

We consider the Galerkin formulation: Find $\mu_{\mathcal{T}} \in S_{\mathcal{T}}$ satisfying

$$\mathscr{A}(\mu_{\mathcal{T}}, v) = \left\langle\left\langle \mathcal{M}^k f, v \right\rangle\right\rangle \qquad \forall v \in S_{\mathcal{T}}. \tag{3.7}$$

The unique existence of the Galerkin solution $\mu_{\mathcal{T}} \in S_{\mathcal{T}}$ is guaranteed by the Lax–Milgram theorem, noting the boundedness and coercivity of the bilinear form $\mathscr{A}(\cdot,\cdot)$ (see Lemma 2.2). Céa's lemma and the approximation property of $S_{\mathcal{T}}$ (Lemma 3.1) imply the follwing lemma.

Lemma 3.2. *Let s be a real number satisfying $s \geq \alpha$. Assume that $\mathcal{M}^k u$ is the solution of* (2.24) *and $\mu_{\mathcal{T}} \in S_{\mathcal{T}}$ is the full tensor Galerkin solution of* (3.7). *If $\mathcal{M}^k u \in \bigcap_{j=1}^k H_{mix}^{\alpha,\ldots,\overset{jth}{\overset{\downarrow}{s}},\ldots,\alpha}(\mathbb{S}^k)$ for some $s \geq \alpha$, then*

$$\left\|\mathcal{M}^k u - \mu_{\mathcal{T}}\right\|_{H^\alpha_{mix}(\mathbb{S}^k)} \leq \frac{1}{(\mathcal{T}+2)^{s-\alpha}} \left(\sum_{j=1}^k \|\mathcal{M}^k u\|^2_{H_{mix}^{\alpha,\ldots,\overset{jth}{\overset{\downarrow}{s}},\ldots,\alpha}(\mathbb{S}^k)} \right)^{1/2}. \tag{3.8}$$

The error estimation (3.8) shows that if $s > \alpha$, the approximate solution $\mu_{\mathcal{T}}$ converges to the weak solution $\mathcal{M}^k u$ when $\mathcal{T}$ goes to infinity. However, the size of the discretized problem (3.7) grows exponentially when the order k increases, see (3.3). The error estimation (3.8) can also be written as

$$\left\|\mathcal{M}^k u - \mu_{\mathcal{T}}\right\|_{H^\alpha_{mix}(\mathbb{S}^k)} \leq \left(\frac{1}{\dim(S_{\mathcal{T}})}\right)^{\frac{s-\alpha}{2k}} \left(\sum_{j=1}^k \|\mathcal{M}^k u\|^2_{H_{mix}^{\alpha,\ldots,\overset{jth}{\overset{\downarrow}{s}},\ldots,\alpha}(\mathbb{S}^k)} \right)^{1/2}, \tag{3.9}$$

noting (3.3). We observe that the convergence rate in (3.9) depends heavily on the number k which can lead to a prohibitive number of unknowns in the corresponding discretized problems. This effect is known as the *curse of dimensionality*. In practical applications, approximation spaces must be adapted to the solution of the problem in order to avoid taking unnecessarily many degrees of freedom which are responsible for consuming most of computational resources but contribute insignificantly to approximation quality [22].

In the remainder of this section, we present the use of a standard hyperbolic cross approximation in solving pseudifferential equations on the sphere with random input data. This approximation method has been used to solve the Dirichlet-to-Neumann equation arising when solving the Neumann problem exterior to a spheroid with random boundary condition, see [4]. Let $\mathcal{T}$ be a positive real number. We introduce the uniform hyperbolic cross index set

$$\delta^e_{\mathcal{T}} := \left\{ \boldsymbol{\ell} = (\ell_1,\ldots,\ell_k) \in \mathbb{N}^k : \prod_{i=1}^k (1+\ell_i) \leq \mathcal{T} \right\}, \tag{3.10}$$

and the associated finite element space

$$S^e_{\mathcal{T}} := \operatorname{span}\left\{ \boldsymbol{Y}_{\boldsymbol{\ell},\boldsymbol{m}} : \boldsymbol{\ell} \in \delta^e_{\mathcal{T}},\ m_i = -\ell_i,\ldots,\ell_i \text{ for } i = 1,\ldots,k \right\}. \tag{3.11}$$

For any

$$v = \sum_{\boldsymbol{\ell}=\boldsymbol{0}}^{\boldsymbol{\infty}} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}} \boldsymbol{Y}_{\boldsymbol{\ell},\boldsymbol{m}},$$

we define by $P^e_{\mathcal{T}} v$ the truncated series

$$P^e_{\mathcal{T}} v := \sum_{\boldsymbol{\ell}\in\delta^e_{\mathcal{T}}} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}} \boldsymbol{Y}_{\boldsymbol{\ell},\boldsymbol{m}} \tag{3.12}$$

which is the orthogonal projection onto $S^e_{\mathcal{T}}$ w.r.t. the inner product $\langle\cdot,\cdot\rangle_{H^0_{mix}(\mathbb{S}^k)}$. The following lemma, which is proved in [4], allows to quantify the projection error.

Lemma 3.3. *Let $\mathcal{T}$ be a positive integer. Let s and t be real numbers such that $t \le s$. For any $v \in H^s_{mix}(\mathbb{S}^k)$, there holds*

$$\|v - P^e_{\mathcal{T}} v\|_{H^t_{mix}(\mathbb{S}^k)} \le \frac{1}{(\mathcal{T}+1)^{s-t}} \|v\|_{H^s_{mix}(\mathbb{S}^k)}. \tag{3.13}$$

Lemma 3.3 gives the approximation property of the projection $P^e_{\mathcal{T}}$ in the case that $\mathcal{T}$ is a positive integer. For technical reasons, we also need the aproximation error when $\mathcal{T}$ is a positive real number which is presented in the next lemma.

Lemma 3.4. *Let $\mathcal{T}$ be a positive real number. Let s and t be real numbers such that $t \le s$. For any $v \in H^s_{mix}(\mathbb{S}^k)$, there holds*

$$\|v - P^e_{\mathcal{T}} v\|_{H^t_{mix}(\mathbb{S}^k)} \le \frac{1}{\mathcal{T}^{s-t}} \|v\|_{H^s_{mix}(\mathbb{S}^k)}. \tag{3.14}$$

Proof. Noting (3.10) and (3.11), there hold

$$\delta^e_{\lfloor \mathcal{T} \rfloor} \subset \delta^e_{\mathcal{T}} \quad \text{and} \quad S^e_{\lfloor \mathcal{T} \rfloor} \subset S^e_{\mathcal{T}},$$

where $\lfloor \mathcal{T} \rfloor := \max\{n \in \mathbb{Z} : n \le \mathcal{T}\}$. Since $P^e_{\mathcal{T}} v$ is the best approximation of v in the space $S^e_{\mathcal{T}}$ w.r.t. the $\|\cdot\|_{H^t_{mix}(\mathbb{S}^k)}$, we then have

$$\|v - P^e_{\mathcal{T}} v\|_{H^t_{mix}(\mathbb{S}^k)} \le \|v - \eta\|_{H^t_{mix}(\mathbb{S}^k)} \quad \forall \eta \in S^e_{\mathcal{T}}. \tag{3.15}$$

Noting that $P^e_{\lfloor \mathcal{T} \rfloor} v \in S^e_{\lfloor \mathcal{T} \rfloor} \subset S^e_{\mathcal{T}}$, it follows from (3.15) that

$$\|v - P^e_{\mathcal{T}} v\|_{H^t_{mix}(\mathbb{S}^k)} \le \left\|v - P^e_{\lfloor \mathcal{T} \rfloor} v\right\|_{H^t_{mix}(\mathbb{S}^k)}. \tag{3.16}$$

Applying Lemma 3.3 for $\lfloor \mathcal{T} \rfloor$ and using (3.16), we obtain

$$\begin{aligned}\|v - P^e_{\mathcal{T}} v\|_{H^t_{mix}(\mathbb{S}^k)} &\le \left\|v - P^e_{\lfloor \mathcal{T} \rfloor} v\right\|_{H^t_{mix}(\mathbb{S}^k)} \\ &\le \frac{1}{(\lfloor \mathcal{T} \rfloor + 1)^{s-t}} \|v\|_{H^s_{mix}(\mathbb{S}^k)} \\ &\le \frac{1}{\mathcal{T}^{s-t}} \|v\|_{H^s_{mix}(\mathbb{S}^k)},\end{aligned} \tag{3.17}$$

finishing the proof of the lemma.

□

Lemma 3.3 suggests that the sparse tensor product space $S^e_{\mathcal{T}}$ can be used to solve the k-th problem (2.23) approximately. We consider the following Galerkin formulation: Given $\mathcal{M}^k f \in H^{-\alpha}_{mix}(\mathbb{S}^k)$, find $\mu^e_{\mathcal{T}} \in S^e_{\mathcal{T}}$ satisfying

$$\mathscr{A}(\mu^e_{\mathcal{T}}, v) = \left\langle\!\left\langle \mathcal{M}^k f, v \right\rangle\!\right\rangle \quad \forall v \in S^e_{\mathcal{T}}. \tag{3.18}$$

The unique existence of the Galerkin solution $\mu^e_{\mathcal{T}}$ of (3.18) is assured due to the boundedness and coercivity of the bilinear form $\mathscr{A}(\cdot,\cdot)$ (see Lemma 2.2). Recalling (2.24) and (3.18), we have

$$\mathscr{A}(\mathcal{M}^k u - \mu^e_{\mathcal{T}}, v) = 0 \quad \forall v \in S^e_{\mathcal{T}}. \tag{3.19}$$

We have the following *a priori* convergence estimate.

Lemma 3.5. *Let $\mathcal{T}$ be a positive integer. Assume that $\mathcal{M}^k u$ and $\mu^e_{\mathcal{T}}$ are the weak and approximate solutions of the k-th problem (2.23) defined by (2.24) and (3.18), respectively. Let s and t be real numbers satisfying $t \le \alpha \le s$. If $\mathcal{M}^k f \in H^{s-2\alpha}_{mix}(\mathbb{S}^k)$, then*

$$\left\|\mathcal{M}^k u - \mu^e_{\mathcal{T}}\right\|_{H^t_{mix}(\mathbb{S}^k)} \lesssim \frac{1}{(\mathcal{T}+1)^{s-t}} \left\|\mathcal{M}^k f\right\|_{H^{s-2\alpha}_{mix}(\mathbb{S}^k)} \tag{3.20}$$

with a constant independent of $\mathcal{T}$.

Proof. Noting the continuity and $H^\alpha_{mix}(\mathbb{S}^k)$-ellipticity of $\mathscr{A}$ (Lemma 2.2), Céa's Lemma gives

$$\left\|\mathcal{M}^k u - \mu^e_{\mathcal{T}}\right\|_{H^\alpha_{mix}(\mathbb{S}^k)} \lesssim \inf_{v\in S^e_{\mathcal{T}}} \left\|\mathcal{M}^k u - v\right\|_{H^\alpha_{mix}(\mathbb{S}^k)}.$$

Combining this and the result in Lemma 3.3 we derive

$$\left\|\mathcal{M}^k u - \mu^e_{\mathcal{T}}\right\|_{H^\alpha_{mix}(\mathbb{S}^k)} \lesssim \frac{1}{(\mathcal{T}+1)^{s-\alpha}} \left\|\mathcal{M}^k u\right\|_{H^s_{mix}(\mathbb{S}^k)}. \tag{3.21}$$

Assume that $t < \alpha$. The duality (2.14), (2.10) and (2.18) yield

$$\begin{aligned}\left\|\mathcal{M}^k u - \mu^e_{\mathcal{T}}\right\|_{H^t_{mix}(\mathbb{S}^k)} &= \sup_{\substack{v\in H^{2\alpha-t}_{mix}(\mathbb{S}^k)\\ v\neq 0}} \frac{\left\langle \mathcal{M}^k u - \mu^e_{\mathcal{T}}, v\right\rangle_{H^\alpha_{mix}(\mathbb{S}^k)}}{\|v\|_{H^{2\alpha-t}_{mix}(\mathbb{S}^k)}}\\ &\lesssim \sup_{\substack{v\in H^{2\alpha-t}_{mix}(\mathbb{S}^k)\\ v\neq 0}} \frac{\mathscr{A}(\mathcal{M}^k u - \mu^e_{\mathcal{T}}, v)}{\|v\|_{H^{2\alpha-t}_{mix}(\mathbb{S}^k)}}.\end{aligned} \tag{3.22}$$

This together with the Galerkin orthogonality (3.19) and Lemma 2.2 implies

$$\begin{aligned}\left\|\mathcal{M}^k u - \mu^e_{\mathcal{T}}\right\|_{H^t_{mix}(\mathbb{S}^k)} &\lesssim \sup_{\substack{v\in H^{2\alpha-t}_{mix}(\mathbb{S}^k)\\ v\neq 0}} \frac{\mathscr{A}(\mathcal{M}^k u - \mu^e_{\mathcal{T}}, v - P^e_{\mathcal{T}} v)}{\|v\|_{H^{2\alpha-t}_{mix}(\mathbb{S}^k)}}\\ &\lesssim \left\|\mathcal{M}^k u - \mu^e_{\mathcal{T}}\right\|_{H^\alpha_{mix}(\mathbb{S}^k)} \sup_{\substack{v\in H^{2\alpha-t}_{mix}(\mathbb{S}^k)\\ v\neq 0}} \frac{\|v - P^e_{\mathcal{T}} v\|_{H^\alpha_{mix}(\mathbb{S}^k)}}{\|v\|_{H^{2\alpha-t}_{mix}(\mathbb{S}^k)}}.\end{aligned} \tag{3.23}$$

Recalling (3.13) and (3.21) we obtain

$$\begin{aligned}\left\|\mathcal{M}^k u - \mu^e_{\mathcal{T}}\right\|_{H^t_{mix}(\mathbb{S}^k)} &\lesssim \frac{1}{(\mathcal{T}+1)^{s-t}} \left\|\mathcal{M}^k u\right\|_{H^s_{mix}(\mathbb{S}^k)}\\ &= \frac{1}{(\mathcal{T}+1)^{s-t}} \left\|\left(L^{-1}\right)^{(k)} \mathcal{M}^k f\right\|_{H^s_{mix}(\mathbb{S}^k)}.\end{aligned}$$

The inequality (3.20) follows then by the continuity of the operator $(L^{-1})^{(k)} : H^{s-2\alpha}_{mix}(\mathbb{S}^k) \to H^s_{mix}(\mathbb{S}^k)$. $\square$

The results in Lemmas 3.5 and 3.2 show that the standard sparse hyperbolic cross approximation space $S^e_{\mathcal{T}}$ produces the same convergence rate as the full tensor product approximation space $S_{\mathcal{T}}$ when the right hand side $\mathcal{M}^k f$ is of equal mixed regularity. In the following lemma, we prove an upper bound for the dimension of $S^e_{\mathcal{T}}$ which shows a significant advantage of the standard sparse tensor product approximation method over the naive full tensor product one.

Lemma 3.6. *The dimension of the standard sparse hyperbolic cross approximation space defined by* (3.11) *is bounded above by*

$$\dim(S^e_{\mathcal{T}}) \le \frac{2^k}{(k-1)!} \mathcal{T}^2 (\ln \mathcal{T} + k \ln 2)^{k-1} \quad \forall \mathcal{T} \ge \mathcal{T}^*, \tag{3.24}$$

for some positive integer $\mathcal{T}^*$ which depends only k.

Proof. Recalling the definition of $S^e_{\mathcal{T}}$ in (3.11), the dimension of $S^e_{\mathcal{T}}$ is estimated by

$$\begin{aligned}\dim(S^e_{\mathcal{T}}) &= \sum_{\boldsymbol{\ell}\in\delta^e_{\mathcal{T}}} \prod_{i=1}^k (1+2\ell_i) \le 2^k \sum_{\boldsymbol{\ell}\in\delta^e_{\mathcal{T}}} \prod_{i=1}^k (1+\ell_i)\\ &\le 2^k \mathcal{T} \operatorname{card}(\delta^e_{\mathcal{T}}).\end{aligned} \tag{3.25}$$

Applying Theorem 3.5 in [3], there exists a $\mathcal{T}^*$ depending only on k such that for every $\mathcal{T} \geq \mathcal{T}^*$, there holds

$$\operatorname{card}(\delta^e_{\mathcal{T}}) < \frac{\mathcal{T}(\ln \mathcal{T} + k \ln 2)^k}{(k-1)!\,(\ln \mathcal{T} + k \ln \mathcal{T} + k - 1)} \leq \frac{\mathcal{T}(\ln \mathcal{T} + k \ln 2)^{k-1}}{(k-1)!}. \tag{3.26}$$

It follows from (3.25) and (3.26) that

$$\dim(S^e_{\mathcal{T}}) \leq \frac{2^k}{(k-1)!}\mathcal{T}^2 (\ln \mathcal{T} + k \ln 2)^{k-1},$$

finishing the proof of the lemma. □

4 Adjusted sparse tensor product spectral Galerkin method

The uniform hyperbolic cross sets $\delta^e_{\mathcal{T}}$ defined in (3.10) is designed according to the level sets of the Fourier coefficients of functions v from the $H^s_{mix}(\mathbb{S}^k)$ of equal mixed regularity. In this case, the standard sparse tensor product spectral Galerkin solution $\mu^e_{\mathcal{T}} \in S^e_{\mathcal{T}}$ behaves asymptotically as the best N-term approximation. If a function v possesses a nonequal mixed regularity, the level sets of the Fourier coefficients of v do not agree with the uniform hyperbolic cross set $\delta^e_{\mathcal{T}}$ and therefore, the standard sparse tensor product spectral Galerkin method will not be robust and other adjusted sparse tensor product spectral methods must be employed, accordingly to the level sets of the Fourier coefficients. In this section, we consider the case when the solution $\mathcal{M}^k u$ is of unequal mixed smooth regularity, i.e., when $\mathcal{M}^k u \in H^{s_1}(\mathbb{S}) \otimes \cdots \otimes H^{s_k}(\mathbb{S})$ and $s_1 \leq \cdots \leq s_k$. The spectral approximation spaces must be asymptotically adjusted to the level sets of the Fourier coefficients. We first introduce nonuniform hyperbolic cross set which will be used to define the approximation spaces. Let $r_1, \ldots,$ r_k be k real numbers satisfying

$$0 < r = r_1 = \ldots = r_{\nu+1} < r_{\nu+2} \leq \ldots \leq r_k \quad (0 \leq \nu \leq k-2). \tag{4.1}$$

We denote by $\gamma_{\mathcal{T}}(r_1, \ldots, r_k)$ the nonuniform hyperbolic cross index set

$$\gamma_{\mathcal{T}}(r_1, \ldots, r_k) := \left\{ \boldsymbol{\ell} = (\ell_1, \ldots, \ell_k) \in \mathbb{N}^k : \prod_{i=1}^{k} (1 + \ell_i)^{r_i} \leq \mathcal{T} \right\}. \tag{4.2}$$

The space $S^\gamma_{\mathcal{T}}(r_1, \ldots, r_k)$ is then defined by

$$S^\gamma_{\mathcal{T}}(r_1, \ldots, r_k) := \operatorname{span}\left\{ Y_{\boldsymbol{\ell},\boldsymbol{m}} : \boldsymbol{\ell} \in \gamma_{\mathcal{T}}(r_1, \ldots, r_k), m_i = -\ell_i, \ldots, \ell_i \text{ for } i = 1, \ldots, k \right\}. \tag{4.3}$$

The following theorem presents one of the main results of this paper in which we prove an upper bound for the dimension of the space $S^\gamma_{\mathcal{T}}(r_1, \ldots, r_k)$.

Theorem 4.1. *Let $r_1, \ldots, r_k$ be positive real numbers satisfying* (4.1)*. The dimension of the space $S^\gamma_{\mathcal{T}}(r_1, \ldots, r_k)$ satisfies*

$$\begin{aligned} \dim(S^\gamma_{\mathcal{T}}(r_1, \ldots, r_k)) \;\leq\; & \frac{2^{\nu+1}\mathcal{T}^{2/r}}{\nu!} \left[\ln\left(\mathcal{T}^{1/r}\right) + (\nu+1)\ln 2 \right]^{\nu} \\ & \times \exp\left(\sum_{\nu+2}^{k} \frac{1}{r_j/r - 1} \left(\frac{3}{2}\right)^{-2(r_j/r-1)} \right) \quad \forall \mathcal{T} \geq \mathcal{T}^*, \end{aligned}$$

for some positive integer $\mathcal{T}^*$ which depends only on k.

Proof. Noting (4.2) and (4.3), the dimension of $S^{\gamma}_{\mathcal{T}}(r_1,\dots,r_k)$ is equal to

$$\dim\left(S^{\gamma}_{\mathcal{T}}(r_1,\dots,r_k)\right)=\sum_{\boldsymbol{\ell}\in\gamma_{\mathcal{T}}(r_1,\dots,r_k)}\prod_{i=1}^{k}(1+2\ell_i)=\sum_{\substack{\prod_{i=1}^{\nu+1}(1+\ell_i)^r\prod_{j=\nu+2}^{k}(1+\ell_j)^{r_j}\\ \le\mathcal{T}}}\prod_{i=1}^{k}(1+2\ell_i). \tag{4.4}$$

Noting that if $(\ell_1,\dots\ell_k)\in\mathbb{N}^k$, then there holds

$$\prod_{i=1}^{\nu+1}(1+\ell_i)^r\prod_{j=\nu+2}^{k}(1+\ell_j)^{r_j}\le\mathcal{T}\iff\begin{cases}\prod_{j=\nu+2}^{k}(1+\ell_j)^{r_j}\le\mathcal{T},\\[2ex] \prod_{i=1}^{\nu+1}(1+\ell_i)\le\mathcal{T}^{1/r}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r}.\end{cases}$$

This together with (4.4) yields

$$\dim\left(S^{\gamma}_{\mathcal{T}}(r_1,\dots,r_k)\right)=\sum_{\prod_{j=\nu+2}^{k}(1+\ell_j)^{r_j}\le\mathcal{T}}\prod_{j=\nu+2}^{k}(1+2\ell_j)\sum_{\substack{\prod_{i=1}^{\nu+1}(1+\ell_i)\\ \le\mathcal{T}^{1/r}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r}}}\prod_{i=1}^{\nu+1}(1+2\ell_i). \tag{4.5}$$

We have

$$\begin{aligned}\sum_{\substack{\prod_{i=1}^{\nu+1}(1+\ell_i)\\ \le\mathcal{T}^{1/r}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r}}}\prod_{i=1}^{\nu+1}(1+2\ell_i)\ &\le\ 2^{\nu+1}\sum_{\substack{\prod_{i=1}^{\nu+1}(1+\ell_i)\\ \le\mathcal{T}^{1/r}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r}}}\prod_{i=1}^{\nu+1}(1+\ell_i)\\ \le 2^{\nu+1}\mathcal{T}^{1/r}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r}&\mathrm{card}\left\{(\ell_1,\dots,\ell_{\nu+1}):\prod_{i=1}^{\nu+1}(1+\ell_i)\le\mathcal{T}^{1/r}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r}\right\}.\end{aligned}$$

This together with (4.5) gives

$$\begin{aligned}\dim\left(S^{\gamma}_{\mathcal{T}}(r_1,\dots,r_k)\right)\le 2^{\nu+1}\mathcal{T}^{1/r}&\sum_{\prod_{j=\nu+2}^{k}(1+\ell_j)^{r_j}\le\mathcal{T}}\left(\prod_{j=\nu+2}^{k}(1+2\ell_j)(1+\ell_j)^{-r_j/r}\right)\\ \times\,\mathrm{card}&\left\{(\ell_1,\dots,\ell_{\nu+1}):\prod_{i=1}^{\nu+1}(1+\ell_i)\le\mathcal{T}^{1/r}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r}\right\}.\end{aligned} \tag{4.6}$$

Applying (3.26), there exists $\mathcal{T}^*>0$ depending only k such that for any $\mathcal{T}>\mathcal{T}^*$ there holds

$$\begin{aligned}&\mathrm{card}\left\{(\ell_1,\dots,\ell_{\nu+1}):\prod_{i=1}^{\nu+1}(1+\ell_i)\le\mathcal{T}^{1/r}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r}\right\}\\ &\le\frac{1}{\nu!}\mathcal{T}^{1/r}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r}\left(\ln\left[\mathcal{T}^{1/r}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r}\right]+(\nu+1)\ln 2\right)^{\nu}\\ &\le\frac{1}{\nu!}\mathcal{T}^{1/r}\left[\ln\left(\mathcal{T}^{1/r}\right)+(\nu+1)\ln 2\right]^{\nu}\prod_{j=\nu+2}^{k}(1+\ell_j)^{-r_j/r},\end{aligned}$$

noting that r and r_j are positive numbers. This together (4.6) yields

$$\begin{aligned}\dim\left(S^{\gamma}_{\mathcal{T}}(r_1,\dots,r_k)\right)\le&\frac{2^{\nu+1}\mathcal{T}^{2/r}}{\nu!}\left[\ln\left(\mathcal{T}^{1/r}\right)+(\nu+1)\ln 2\right]^{\nu}\\ &\times\sum_{\prod_{j=\nu+2}^{k}(1+\ell_j)^{r_j}\le\mathcal{T}}\prod_{j=\nu+2}^{k}(1+2\ell_j)(1+\ell_j)^{-2r_j/r}.\end{aligned} \tag{4.7}$$

We also have

$$\sum_{\prod_{j=\nu+2}^k (1+\ell_j)^{r_j}\le\mathcal{T}} \prod_{j=\nu+2}^k (1+2\ell_j)(1+\ell_j)^{-2r_j/r} \le \prod_{j=\nu+2}^k \sum_{(1+\ell_j)^{r_j}\le\mathcal{T}} (1+2\ell_j)(1+\ell_j)^{-2r_j/r}$$
$$\le \prod_{j=\nu+2}^k \sum_{\substack{n_j^{r_j}\le\mathcal{T},\\ n_j\ge 1}} (2n_j-1)n_j^{-2r_j/r}$$
$$\le \prod_{j=\nu+2}^k \sum_{n_j=1}^\infty (2n_j-1)n_j^{-2r_j/r}. \tag{4.8}$$

We can write

$$\sum_{n_j=1}^\infty (2n_j-1)n_j^{-2r_j/r} = 1+\sum_{n_j=2}^\infty (2n_j-1)n_j^{-2r_j/r} \le 1+2\sum_{n_j=2}^\infty n_j^{1-2r_j/r}$$
$$\le 1+\frac{1}{r_j/r-1}\left(\frac{3}{2}\right)^{-2(r_j/r-1)} < \exp\left(\frac{1}{r_j/r-1}\left(\frac{3}{2}\right)^{-2(r_j/r-1)}\right),$$

where in the second inequality we use the result in [8, Lemma 2.2] and in the last inequality we employ the well-known inequality

$$1+x<e^x \quad \forall x>0.$$

Hence, there holds

$$\prod_{j=\nu+2}^k \sum_{n_j=1}^\infty (2n_j-1)n_j^{-2r_j/r} \le \exp\left(\sum_{\nu+2}^k \frac{1}{r_j/r-1}\left(\frac{3}{2}\right)^{-2(r_j/r-1)}\right). \tag{4.9}$$

From (4.7)–(4.9) follows the theorem.

□

In the remainder of this section, approximation spaces of tensors of spherical harmonics based on nonuniform hyperbolic crosses will be used to find approximate solutions to the statistical moment equation (2.23). Suppose that the solution $\mathcal{M}^k u$ of (2.23) has an unequal mixed smooth regularity, i.e.,

$$\mathcal{M}^k u \in H_{mix}^{s_1,\dots,s_k}(\mathbb{S}^k),$$

where

$$\alpha < s = s_1 = \ldots = s_{\nu+1} < s_{\nu+2} \le \cdots \le s_k \quad (0\le\nu\le k-2). \tag{4.10}$$

Noting (2.18), this happens when

$$\mathcal{M}^k f \in H_{mix}^{s-2\alpha,\dots,s-2\alpha,s_{\nu+2}-2\alpha,\dots,s_k-2\alpha}(\mathbb{S}^k).$$

Noting (4.2) and (4.3), we denote

$$\Gamma_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k) := \gamma_{\mathcal{T}}\left(1,\dots,1,\frac{s_{\nu+2}-\alpha}{s-\alpha},\dots,\frac{s_k-\alpha}{s-\alpha}\right)$$

and

$$\mathscr{S}_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k) := S_{\mathcal{T}}^{\gamma}\left(1,\dots,1,\frac{s_{\nu+2}-\alpha}{s-\alpha},\dots,\frac{s_k-\alpha}{s-\alpha}\right). \tag{4.11}$$

For any

$$v=\sum_{\boldsymbol{\ell}=\boldsymbol{0}}^\infty \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}} Y_{\boldsymbol{\ell},\boldsymbol{m}},$$

we define by $P^{s,\dots,s,s_{\nu+2},\dots,s_k}_{\mathcal{T},\alpha} v$ the truncated series

$$P^{s,\dots,s,s_{\nu+2},\dots,s_k}_{\mathcal{T},\alpha} v := \sum_{\boldsymbol{\ell}\in\Gamma_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k)} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}} Y_{\boldsymbol{\ell},\boldsymbol{m}}.$$

The operator $P^{s,\dots,s,s_{\nu+2},\dots,s_k}_{\mathcal{T},\alpha}$ is the orthogonal projection into $\mathscr{S}_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k)$ with respect to the inner product $\langle\cdot,\cdot\rangle_{H^0_{mix}(\mathbb{S}^k)}$. The following lemma presents approximation error of the projection $P^{s,\dots,s,s_{\nu+2},\dots,s_k}_{\mathcal{T},\alpha}$.

Lemma 4.2. *Let s, $s_{\nu+2}$, $\dots$, s_k be real numbers satisfying*

$$\alpha \le s < s_{\nu+2} \le \cdots \le s_k.$$

For any $v \in H^{s,\dots,s,s_{\nu+2},\dots,s_k}_{mix}(\mathbb{S}^k)$, there holds

$$\left\| v - P^{s,\dots,s,s_{\nu+2},\dots,s_k}_{\mathcal{T},\alpha} v \right\|_{H^\alpha_{mix}(\mathbb{S}^k)} \le \frac{1}{\mathcal{T}^{s-\alpha}} \|v\|_{H^{s,\dots,s,s_{\nu+2},\dots,s_k}_{mix}(\mathbb{S}^k)}.$$

Proof. We have

$$\begin{aligned}
\left\| v - P^{s,\dots,s,s_{\nu+2},\dots,s_k}_{\mathcal{T},\alpha} v \right\|^2_{H^\alpha_{mix}(\mathbb{S}^k)} &= \sum_{\boldsymbol{\ell}\notin\Gamma_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k)} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \prod_{i=1}^{k} (1+\ell_i)^{2\alpha} |\widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}}|^2 \\
= \sum_{\boldsymbol{\ell}\notin\Gamma_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k)} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} & \frac{\prod_{i=1}^{\nu+1}(1+\ell_i)^{2s} \prod_{j=\nu+2}^{k} (1+\ell_j)^{2s_j}}{\left[\prod_{i=1}^{\nu+1}(1+\ell_i) \prod_{j=\nu+2}^{k} (1+\ell_j)^{(s_j-\alpha)/(s-\alpha)}\right]^{2(s-\alpha)}} |\widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}}|^2 \\
\le \frac{1}{\mathcal{T}^{2(s-\alpha)}} \sum_{\boldsymbol{\ell}\notin\Gamma_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k)} & \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \prod_{i=1}^{\nu+1}(1+\ell_i)^{2s} \prod_{j=\nu+2}^{k} (1+\ell_j)^{2s_j} |\widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}}|^2 \\
\le \frac{1}{\mathcal{T}^{2(s-\alpha)}} \|v\|^2_{H^{s,\dots,s,s_{\nu+2},\dots,s_k}_{mix}(\mathbb{S}^k)}. &
\end{aligned}$$

The desired inequality is obtained by taking the square root both sides of the above inequality. $\square$

The results in Lemma 4.2 suggests that the space $\mathscr{S}_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k)$ defined by (4.11) can be used to solve the problem (2.23) approximately. We consider the following Galerkin formulation: Given $\mathcal{M}^k f \in H^{s-2\alpha,\dots,s-2\alpha,s_{\nu+2}-2\alpha,\dots,s_k-2\alpha}_{mix}(\mathbb{S}^k)$, find $\widetilde{\mu}_{\mathcal{T}} \in \mathscr{S}_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k)$ satisfying

$$\mathscr{A}(\widetilde{\mu}_{\mathcal{T}}, v) = \left\langle\!\left\langle \mathcal{M}^k f, v \right\rangle\!\right\rangle \quad \forall v \in \mathscr{S}_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k). \tag{4.12}$$

By Lemma 2.2, Galerkin formulation (4.12) is well posed.

The next theorem presents an error estimate when solving the statistical moment equation by using the adjusted sparse tensor product approximation spaces, see (4.12).

Theorem 4.3. *Let $s_1,\dots$, s_k be real numbers satisfying* (4.10). *Assume that $\mathcal{M}^k u$ is the solution of* (2.24) *and $\widetilde{\mu}_{\mathcal{T}} \in \mathscr{S}_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k)$ is the adjusted sparse tensor Galerkin solution of* (4.12) *for the data $\mathcal{M}^k f \in H^{s-2\alpha,\dots,s-2\alpha,s_{\nu+2}-2\alpha,\dots,s_k-2\alpha}_{mix}(\mathbb{S}^k)$. Then for every $t \le \alpha$, there holds*

$$\left\| \mathcal{M}^k u - \widetilde{\mu}_{\mathcal{T}} \right\|_{H^t_{mix}(\mathbb{S}^k)} \lesssim \frac{1}{\mathcal{T}^{\frac{(s-\alpha)(s_k-t)}{s_k-\alpha}}} \left\| \mathcal{M}^k f \right\|_{H^{s-2\alpha,\dots,s-2\alpha,s_{\nu+2}-2\alpha,\dots,s_k-2\alpha}_{mix}(\mathbb{S}^k)}.$$

In particular, there holds

$$\left\| \mathcal{M}^k u - \widetilde{\mu}_{\mathcal{T}} \right\|_{H^\alpha_{mix}(\mathbb{S}^k)} \lesssim \frac{1}{\mathcal{T}^{s-\alpha}} \left\| \mathcal{M}^k f \right\|_{H^{s-2\alpha,\dots,s-2\alpha,s_{\nu+2}-2\alpha,\dots,s_k-2\alpha}_{mix}(\mathbb{S}^k)}.$$

Proof. Recalling (2.24) and (4.12), we have

$$\mathscr{A}(\mathcal{M}^k u - \widetilde{\mu}_{\mathcal{T}}, v) = 0 \quad \forall v \in \mathscr{S}_{\mathcal{T},\alpha}(s, \dots, s, s_{\nu+2}, \dots, s_k). \tag{4.13}$$

The Céa's lemma gives

$$\left\| \mathcal{M}^k u - \widetilde{\mu}_{\mathcal{T}} \right\|_{H^{\alpha}_{mix}(\mathbb{S}^k)} \lesssim \inf_{v \in \mathscr{S}_{\mathcal{T},\alpha}(s,\dots,s,s_{\nu+2},\dots,s_k)} \left\| \mathcal{M}^k u - v \right\|_{H^{\alpha}_{mix}(\mathbb{S}^k)}.$$

This together with the result in Lemma 4.2 yields

$$\left\| \mathcal{M}^k u - \widetilde{\mu}_{\mathcal{T}} \right\|_{H^{\alpha}_{mix}(\mathbb{S}^k)} \lesssim \frac{1}{\mathcal{T}^{s-\alpha}} \left\| \mathcal{M}^k u \right\|_{H^{s,\dots,s,s_{\nu+2},\dots,s_k}_{mix}(\mathbb{S}^k)}. \tag{4.14}$$

Noting (3.10), we have

$$(\ell_1, \dots, \ell_k) \in \delta^e_{\mathcal{T}^{(s-\alpha)/(s_k-\alpha)}} \iff \prod_{i=1}^k (1+\ell_i) \le \mathcal{T}^{\frac{s-\alpha}{s_k-\alpha}} \qquad \iff \prod_{i=1}^k (1+\ell_i)^{\frac{s_k-\alpha}{s-\alpha}} \le \mathcal{T}.$$

By (4.10), there holds

$$1 < \frac{s_{\nu+2}-\alpha}{s-\alpha} \le \dots \le \frac{s_k-\alpha}{s-\alpha}.$$

This suggests

$$\begin{aligned} \prod_{i=1}^k (1+\ell_i)^{\frac{s_k-\alpha}{s-\alpha}} \le \mathcal{T} &\implies \prod_{i=1}^{\nu+1} (1+\ell_i) \prod_{i=\nu+2}^{k} (1+\ell_i)^{\frac{s_i-\alpha}{s-\alpha}} \le \mathcal{T} \\ &\implies (\ell_1, \dots, \ell_k) \in \gamma_{\mathcal{T}} \left(1, \dots, 1, \frac{s_{\nu+2}-\alpha}{s-\alpha}, \dots, \frac{s_k-\alpha}{s-\alpha}\right). \end{aligned}$$

It follows that

$$\delta^e_{\mathcal{T}^{(s-\alpha)/(s_k-\alpha)}} \subset \gamma_{\mathcal{T}} \left(1, \dots, 1, \frac{s_{\nu+2}-\alpha}{s-\alpha}, \dots, \frac{s_k-\alpha}{s-\alpha}\right)$$

and thus

$$S^e_{\mathcal{T}^{(s-\alpha)/(s_k-\alpha)}} \subset \mathscr{S}_{\mathcal{T},\alpha}\left(s, \dots, s, s_{\nu+2}, \dots, s_k\right). \tag{4.15}$$

Noting (3.12) and (4.15), and applying Lemma 3.4, for every $v \in H^{2\alpha-t}_{mix}(\mathbb{S}^k)$ we have

$$P^e_{\mathcal{T}^{(s-\alpha)/(s_k-\alpha)}} v \in S^e_{\mathcal{T}^{(s-\alpha)/(s_k-\alpha)}} \subset \mathscr{S}_{\mathcal{T}^{(s-\alpha)/(s_k-\alpha)},\alpha}\left(s, \dots, s, s_{\nu+2}, \dots, s_k\right), \tag{4.16}$$

and

$$\left\| v - P^e_{\mathcal{T}^{(s-\alpha)/(s_k-\alpha)}} v \right\|_{H^{\alpha}_{mix}(\mathbb{S}^k)} \le \frac{1}{\left(\mathcal{T}^{\frac{s-\alpha}{s_k-\alpha}}\right)^{\alpha-t}} \left\| v \right\|_{H^{2\alpha-t}_{mix}}. \tag{4.17}$$

Similar arguments as used to obtain (3.22) yield

$$\begin{aligned} \left\| \mathcal{M}^k u - \widetilde{\mu}_{\mathcal{T}} \right\|_{H^{t}_{mix}(\mathbb{S}^k)} &= \sup_{\substack{v \in H^{2\alpha-t}_{mix}(\mathbb{S}^k) \\ v \neq 0}} \frac{\left\langle \mathcal{M}^k u - \widetilde{\mu}_{\mathcal{T}}, v \right\rangle_{H^{\alpha}_{mix}(\mathbb{S}^k)}}{\left\| v \right\|_{H^{2\alpha-t}_{mix}(\mathbb{S}^k)}} \\ &\lesssim \sup_{\substack{v \in H^{2\alpha-t}_{mix}(\mathbb{S}^k) \\ v \neq 0}} \frac{\mathscr{A}\left(\mathcal{M}^k u - \widetilde{\mu}_{\mathcal{T}}, v\right)}{\left\| v \right\|_{H^{2\alpha-t}_{mix}(\mathbb{S}^k)}}. \end{aligned} \tag{4.18}$$

It follows from (4.18), (4.13) and (2.25) that

$$\left\|\mathcal{M}^k u-\widetilde{\mu}_{\mathcal{T}}\right\|_{H^t_{mix}(\mathbb{S}^k)} \lesssim \sup_{\substack{v\in H^{2\alpha-t}_{mix}(\mathbb{S}^k)\\ v\neq 0}} \frac{\mathscr{A}\left(\mathcal{M}^k u-\widetilde{\mu}_{\mathcal{T}}, v-P^e_{\mathcal{T}^{(s-\alpha)/(s_k-\alpha)}}v\right)}{\|v\|_{H^{2\alpha-t}_{mix}(\mathbb{S}^k)}}$$

$$\lesssim \left\|\mathcal{M}^k u-\widetilde{\mu}_{\mathcal{T}}\right\|_{H^\alpha_{mix}(\mathbb{S}^k)} \sup_{\substack{v\in H^{2\alpha-t}_{mix}(\mathbb{S}^k)\\ v\neq 0}} \frac{\left\|v-P^e_{\mathcal{T}^{(s-\alpha)/(s_k-\alpha)}}v\right\|_{H^\alpha_{mix}(\mathbb{S}^k)}}{\|v\|_{H^{2\alpha-t}_{mix}(\mathbb{S}^k)}}.$$

This together with (4.14) and (4.17) implies

$$\left\|\mathcal{M}^k u-\widetilde{\mu}_{\mathcal{T}}\right\|_{H^t_{mix}(\mathbb{S}^k)} \lesssim \frac{1}{\mathcal{T}^{s-\alpha}\left(\mathcal{T}^{\frac{s-\alpha}{s_k-\alpha}}\right)^{\alpha-t}} \left\|\mathcal{M}^k u\right\|_{H^{s,\ldots,s,s_{\nu+2},\ldots,s_k}_{mix}(\mathbb{S}^k)}.$$

The above inequality together with the continuity of

$$\left(L^{(k)}\right)^{-1}: H^{s-\alpha,\ldots,s-\alpha,s_{\nu+2}-\alpha,\ldots,s_k-\alpha}_{mix}(\mathbb{S}^k) \to H^{s,\ldots,s,s_{\nu+2},\ldots,s_k}_{mix}(\mathbb{S}^k)$$

yields the desired inequality.

□

We next present an estimate for the dimension of the space $\mathscr{S}_{\mathcal{T},\alpha}(s,\ldots,s,s_{\nu+2},\ldots,s_k)$ in which the Galerkin solution defined by (4.12) is found.

Corollary 4.4. *There exists a positive number $\mathcal{T}^*$ depending only on k such that for every $\mathcal{T}\geq\mathcal{T}^*$, there holds*

$$\dim\left(\mathscr{S}_{\mathcal{T},\alpha}(s,\ldots,s,s_{\nu+2},\ldots,s_k)\right) \leq \frac{2^{\nu+1}\mathcal{T}^2}{\nu!}\left[\ln\mathcal{T}+(\nu+1)\ln 2\right]^\nu \times \exp\left(\sum_{\nu+2}^k \frac{s-\alpha}{s_j-s}\left(\frac{3}{2}\right)^{-\frac{2(s_j-s)}{s-\alpha}}\right). \tag{4.19}$$

Proof. The inequality (4.19) is obtained by applying Theorem 4.1 for the space $\mathscr{S}_{\mathcal{T},\alpha}(s,\ldots,s,s_{\nu+2},\ldots,s_k)$ and noting (4.11). □

The estimation (4.19) gives an upper bound for the dimension of the approximation space. In our numerical experiments (Section 5), the space $\mathscr{S}_{\mathcal{T},1/2}(3,5)$ will be used to solve the second order statistical moment equation of the hypersingular integral equation on the unit sphere. We observe the dimensions of $\mathscr{S}_{\mathcal{T},1/2}(3,5)$ with respect to different values of $\mathcal{T}$. Corollary 4.4 shows that

$$\dim\left(\mathscr{S}_{\mathcal{T},1/2}(3,5)\right) \leq 2\mathcal{T}^2\exp\left(\frac{5}{4}\left(\frac{3}{2}\right)^{-8/5}\right).$$

The numbers in Table 1 show that $\dim\left(\mathscr{S}_{\mathcal{T},1/2}(3,5)\right)$ behaves like $\mathcal{O}\left(\mathcal{T}^\eta\right)$ where η appears to be close to number 2 as suggested by our theoretical results.

In Theorem 4.3, the approximate solution $\widetilde{\mu}_{\mathcal{T}}$ is sought in the space

$$\mathscr{S}_{\mathcal{T},\alpha}(s,\ldots,s,s_{\nu+2},\ldots,s_k)$$

which gives the optimal convergence rate when the error is estimated in the enery norm ($H^\alpha_{mix}(\mathbb{S}^k)$-norm). In the case that we want to find an approximate solution to (2.23) and evaluate the error in a $H^t_{mix}(\mathbb{S})$-norm for some $t<\alpha$, an optimal convergence rate can be obtained by finding the approximate solution $\mu^t_{\mathcal{T}}$ in the space

$$\mathscr{S}_{\mathcal{T},t}(s,\ldots,s,s_{\nu+2},\ldots,s_k) = S^\gamma_{\mathcal{T}}\left(1,\ldots,1,\frac{s_{\nu+2}-t}{s-t},\ldots,\frac{s_k-t}{s-t}\right),$$

$\mathcal{T}$	$\dim\left(\mathscr{S}_{\mathcal{T},1/2}(3,5)\right)$	Expected order of $\mathcal{T}$ (≈ 2)
1	4	
4	28	1.4037
8	98	1.8074
12	208	1.8561
16	364	1.9453
20	585	2.1262
30	1281	1.9331
40	2292	2.0223
50	3654	2.0901
60	5253	1.9909
70	7049	1.9078
80	9372	2.1331
90	11888	2.0190
100	14675	1.9990

Table 1: Sizes of the adjusted sparse tensor product approximation spaces $\mathscr{S}_{\mathcal{T},1/2}(3,5)$.

see (4.11). The corresponding Galerkin equation is

$$\mathscr{A}\left(\mu^t_{\mathcal{T}}, v\right) = \left\langle\left\langle \mathcal{M}^k f, v\right\rangle\right\rangle \quad \forall v \in \mathscr{S}_{\mathcal{T},t}(s,\dots,s,s_{\nu+2},\dots,s_k).$$

Employing a similar argument as in the proof of Theorem 4.3 yields

$$\left\|\mathcal{M}^k - \mu^t_{\mathcal{T}}\right\|_{H^t_{mix}(\mathbb{S}^k)} \lesssim \frac{1}{(\mathcal{T}+1)^{s-t}} \left\|\mathcal{M}^k f\right\|_{H^{s-2\alpha,\dots,s-2\alpha,s_{\nu+2}-2\alpha,\dots,s_k-2\alpha}_{mix}(\mathbb{S}^k)},$$

where the positive constant in the above inequality is independent of $\mathcal{T}$.

5 Numerical experiments

In this section, we consider the hypersingular integral equation on the sphere

$$\mathcal{N}u(\omega) = f(\omega) \quad \text{on } \mathbb{S}, \tag{5.1}$$

where $\mathcal{N}$ is the hypersingular integral operator (with a minus) given by

$$\mathcal{N}v(\boldsymbol{x}) = -\frac{1}{4\pi}\frac{\partial}{\partial\nu_{\boldsymbol{x}}}\int_{\mathbb{S}} v(\boldsymbol{y})\frac{\partial}{\partial\nu_{\boldsymbol{y}}}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}d\sigma_{\boldsymbol{y}} + \int_{\mathbb{S}} v\, d\sigma. \tag{5.2}$$

The equation arises from the boundary-integral reformulation of the Neumann problem with the Laplacian in the interior or exterior of the sphere; see e.g. [21, 32, 36].

We present numerical results illustrating convergence of the standard and adjusted sparse approximations for the k-order statistical moment equation which arising from (5.1). For sake of simplicity, we restrict to the second moment problem,

$$\mathcal{N}^{(2)}\mathcal{M}^2 u = \mathcal{M}^2 f \quad \text{on } \mathbb{S}^2. \tag{5.3}$$

We assume that second moments of the data $\mathcal{M}^2 f$ are known from elsewhere and report on convergence behavior of the standard and adjusted spectral approximations to second moment of the solution to the exact function $\mathcal{M}^2 u$. In our two numerical experiments below we choose two different functions $\mathcal{M}^2 f$ with equal and unequal mixed smoothnesses. We compare performance of standard sparse, adjusted sparse and full tensor product approximations analyzed in Sections 3 and 4. We emphasize that the data $\mathcal{M}^2 f$ in both numerical tests are artificial. They do not come from a mathematical model and are chosen to illustrate convergence of the numerical schemes. An application of the suggested method to real models and data, e.g. in Earth observation, is outside the scope of this paper.

We first solve the equation (5.3) when

$$\mathcal{M}^2 f(\boldsymbol{x}, \boldsymbol{y}) = \sin\left(|\boldsymbol{x} - \boldsymbol{n}|\,|\boldsymbol{y} - \boldsymbol{n}|\right), \quad \boldsymbol{n} = (0, 0, 1). \tag{5.4}$$

The function $\mathcal{M}^2 f$ belongs to $H_{mix}^{2-\epsilon,2-\epsilon}(\mathbb{S}^2)$ for any $\epsilon > 0$. Thus, the solution $\mathcal{M}^2 u$ of the equation (5.3) belongs to $H_{mix}^{3-\epsilon,3-\epsilon}(\mathbb{S}^2)$. We compute approximate solutions by using full and standard sparse tensor products of polynomials for different discretization levels $\mathcal{T}$. In Figure 1 we plot the $H_{mix}^{1/2}(\mathbb{S}^2)$-norm of the Galerkin errors and observe that the convergence rate with respect to the polynomial degree is almost the same for both methods: the slope of the convergence curve in the logarithmic plot is around 2.5. This agrees with the statement of Lemma 3.5. On the other hand, the standard sparse

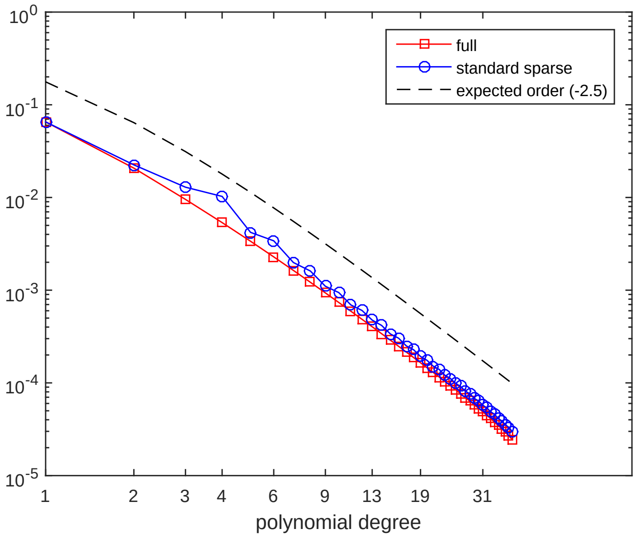

Figure 1: Relative error in energy norm for the right-hand side $\mathcal{M}^2 f$ from (5.4) vs. the maximal polynomial degree.

tensor discretization requires only $\mathcal{O}\left(\mathcal{T}^2(\ln \mathcal{T} + 2\ln 2)\right)$ unknowns, which is significantly smaller than $\mathcal{O}\left(\mathcal{T}^4\right)$ unknowns when using the full tensor discretization, see Table 5 and Figure 2. The coefficient distribution of the solution is also observed in Figure 3, in which we present

$$c_{\ell_1,\ell_2} = \sqrt{\sum_{m_1=-\ell_1}^{\ell_1} \sum_{m_2=-\ell_2}^{\ell_2} \widehat{(\mathcal{M}^2 u)}_{\ell_1,\ell_2,m_1,m_2}^{2}}, \quad \ell_1, \ell_2 = 0, \ldots, 40.$$

In Figure 3 we also sketch two hyperbolic curves $(\ell_1+1)(\ell_2+1) = L+1$ for $L = 19$ and 39 to illustrate the hyperbolic decay pattern of the coefficients c_{ℓ_1,ℓ_2}.

We then solve the equation (5.3) with unequal mixed regularity right hand side given by

$$\mathcal{M}^2 f(\boldsymbol{x}, \boldsymbol{y}) = e^{|\boldsymbol{x}-\boldsymbol{n}||\boldsymbol{y}-\boldsymbol{n}|^3}. \tag{5.5}$$

This right hand side belongs to the space $H_{mix}^{2-\epsilon,4-\epsilon}(\mathbb{S}^2)$ for any $\epsilon > 0$ and the solution is an element of $H_{mix}^{3-\epsilon,5-\epsilon}(\mathbb{S}^2)$. The equation (5.3) with the right hand side (5.5) is then solved by using the Galerkin method with full, standard sparse and adjusted sparse tensor product approximation spaces. Here, the adjusted sparse tensor product approximation spaces are the spaces $\mathscr{S}_{\mathcal{T},1/2}(3,5)$, (see (4.11)), which

Degree	full	standard sparse	adjusted sparse
1	16	7	4
4	625	58	28
8	6561	267	98
12	28561	633	208
16	83521	1180	364
20	194481	2066	585
30	923521	4981	1281

Table 1: Sizes of full tensor product approximation space $S_{\mathcal{T}}$, standard sparse tensor product approximation space $S^{\epsilon}_{\mathcal{T}}$ and adjusted sparse tensor product approximation space $\mathscr{S}_{\mathcal{T},1/2}(3,5)$.

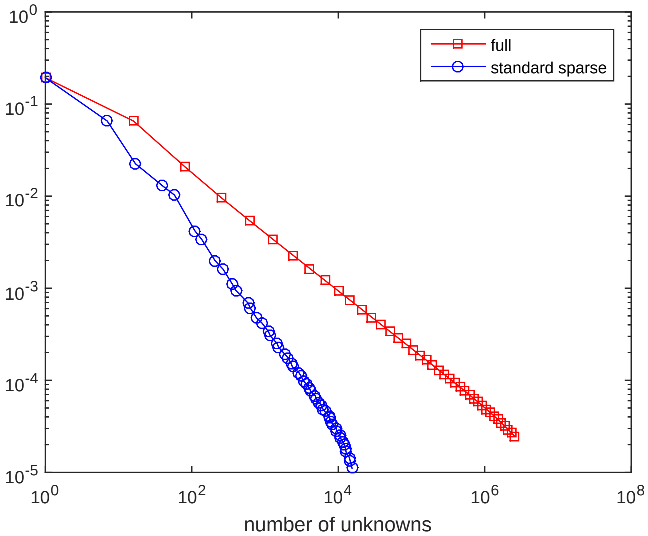

Figure 2: Relative error in energy norm for the right-hand side $\mathcal{M}^2 f$ from (5.4) vs. the number of unknowns.

are accordingly chosen for the pseudodifferential equation (5.3) of order 1 with the right hand side $\mathcal{M}^2 f$ belonging to the space $H^{2-\epsilon,4-\epsilon}_{mix}(\mathbb{S}^2)$. We compute approximate solutions by using these three approximation methods.

In Figure 4, we plot the $H^{1/2}_{mix}(\mathbb{S})$-error and observe the convergence rates. It appears that the convergence rates with respect to polynomials degrees are almost the same for all methods (around 2.5). This agrees with our theoretical results in Theorem 4.3. Nevertheless, the adjusted sparse tensor product approximation approach requires only $\mathcal{O}\left(\mathcal{T}^2\right)$ unknowns in comparison to $\mathcal{O}\left(\mathcal{T}^2(\ln\mathcal{T} + 2\ln 2)\right)$ unknowns of the standard sparse and to $\mathcal{O}(\mathcal{T}^4)$ unknowns of the full tensor product approximation approaches, see Table 5. We also plot the $H^{1/2}_{mix}(\mathbb{S})$-error with respect to the number of unknowns, see Figure 5. In this figure, our adjusted sparse tensor product approximation method is superior not only to the full tensor but also to the standard sparse tensor approximation methods. This superiority is due to the fact that the adjusted sparse tensor product approximation spaces are based on the hyperbolic cross set $\Gamma_{\mathcal{T},1/2}(3,5)$ which is asymptotically adjusted to the level sets of the Fourier coefficients of the solution $\mathcal{M}^2 u$. Figure 6 illustrates the adjusted hyperbolic decay pattern of the coefficients c_{ℓ_1,ℓ_2} of the solution. In this figure, we also sketch two adjusted hyperbolic curves $(1+\ell_1)(1+\ell_2)^{(5-1/2)/(3-1/2)} = L$ for $L = 20$ and $L = 126$.

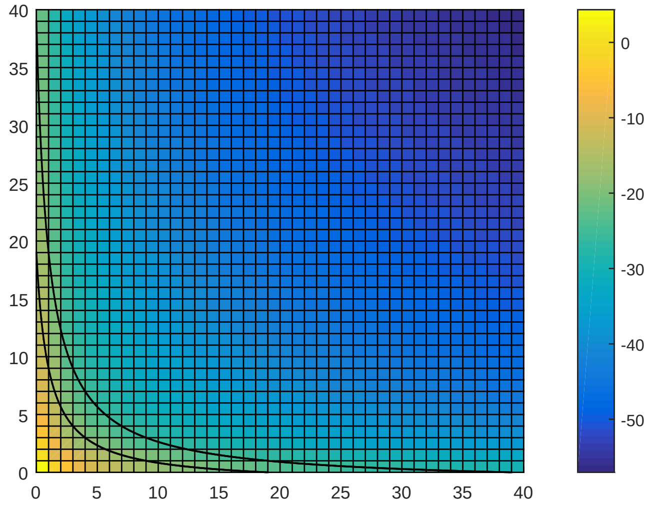

Figure 3: Coefficient distribution of $\mathcal{M}^2 u$ when $\mathcal{M}^2 f$ is defined by (5.4).

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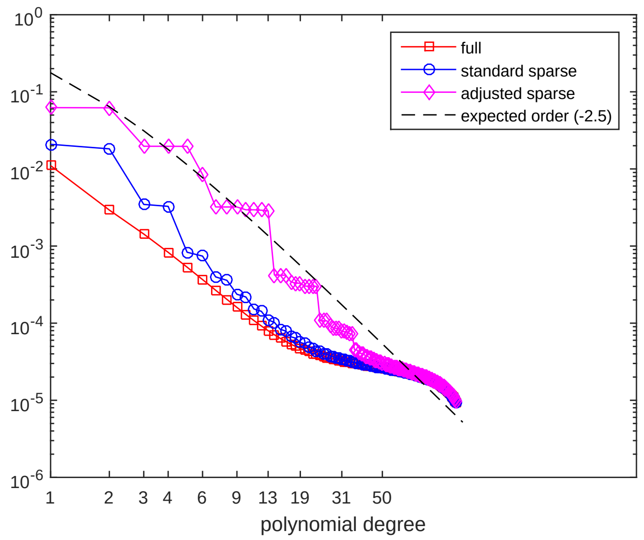

Figure 4: Relative error in energy norm for the right-hand side $\mathcal{M}^2 f$ from (5.5) vs. the maximal polynomial degree.

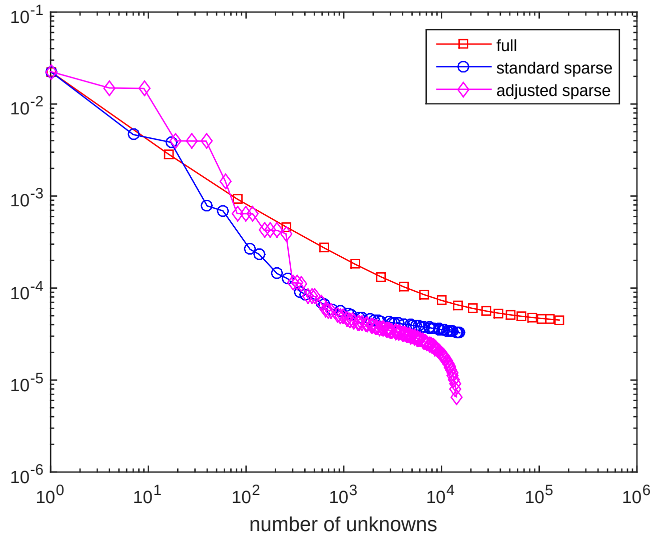

Figure 5: Relative error in energy norm for the right-hand side $\mathcal{M}^2 f$ from (5.5) vs. the number of unknowns.

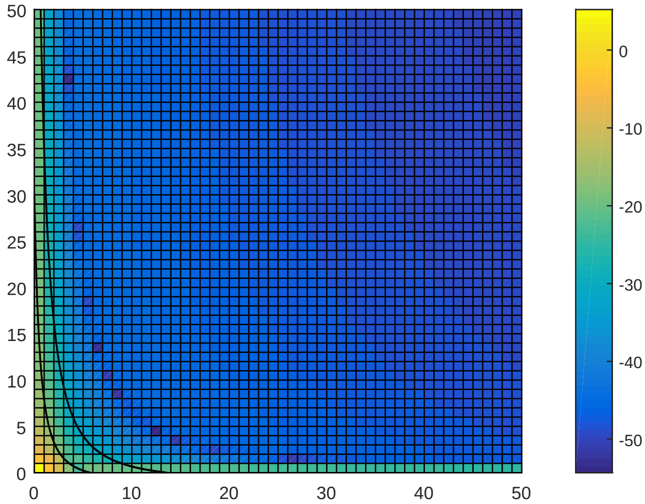

Figure 6: Coefficient distribution of $\mathcal{M}^2 u$ when $\mathcal{M}^2 f$ is defined by (5.5).

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