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Besov-Morrey spaces associated to Hermite operators and applications to fractional Hermite equations

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Abstract
The purpose of this paper is to establish the molecular decomposition of the homogeneous Besov-Morrey spaces associated to the Hermite operator $\mathbb{H} = -\Delta + |x|^2$ on the Euclidean space $\mathbb{R}^n$. Particularly, we obtain some estimates for the operator $\mathbb{H}$ on the Hermite-Besov-Morrey spaces and the regularity results to the fractional Hermite equations:

$$(-\Delta + |x|^2)^s u = f,$$

and

$$(-\Delta + |x|^2 + I)^s u = f.$$

Our results generalize some results of Anh and Thinh, [1].

Keywords
Fractional Hermite equations — Hermite-Besov-Morrey space — Molecular decomposition

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1. Introduction

In this article, we want to study the Besov and Triebel-Lizorkin spaces associated to the Hermite operator $\mathbb{H} = -\Delta + |x|^2$ on $\mathbb{R}^n$, $n \geq 1$. It is known that the classical theory of the Besov and Triebel-Lizorkin spaces plays a crucial role not only in the theory

of function spaces, but aslo in the theory of partial differential equations and harmonic analysis, see e.g., [3, 5, 6, 7, 8, 10, 11], and the references therein.
Recently, the theory of the Besov and Triebel-Lizorkin spaces associated to the operators has been developed by many authors when one observed that the classical Besov and Triebel-Lizorkin spaces are not always the most suitable to investigate a number of operators, see [1, 4, 9, 14, 6, 7, 15], and their references. For example, Petrusev and Xu, [9] studied the characterization of the inhomogeneous Besov and Triebel-Lizorkin spaces in terms of Littlewood-Paley decomposition in the context of Hermite expansions that the frame elements have almost exponential localization. Note that these frame elements can be viewed as an analogue of the φ-transform of Frazier and Jawerth, [3]. Another approach introduced by Anh and Thinh, [1] is of defining the Besov and Triebel-Lizorkin spaces in terms of the heat kernels via square functions. Their approach adapted to the study of the theory of both homogeneous and inhomogeneous Besov and Triebel-Lizorkin spaces. This allows them to extend the range of indices $1 \le p,q \le \infty$ of the homogeneous Besov space $\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q}$ (resp. Triebel-Lizorkin spaces $\mathbf{FM}^{\alpha,\mathbb{H}}_{p,q}$) to $0 < p,q \le \infty$, compare to the results in [4].

One of the most interesting studys of the theory of Besov spaces is the Besov-Morrey spaces, introduced first by Kozono and Yamazaki [5] in order to investigate time-local solutions of the Navier-Stokes equations with the initial data in the spaces of this type. As a matter of fact, the Besov-Morrey spaces share several features of Besov and Morrey spaces. They represent the local oscillations and singularities of functions more precisely than the classical Besov spaces. Thus, they are behaved better in many aspects, particularly under the action of singular integrals and pseudo-differential operators. In addition, Mazzucato [7, 8] established the wavelet decompositions to characterize the homogeneous and inhomogeneous Besov-Morrey spaces. For more results on the Besov-Morrey spaces, we refer the reader to [5, 6, 7, 8, 10, 11, 13, 15] and the references therein.

Inspired by the above results, we would like to generalize the theory of the homogeneous Besov spaces associated to the Hermite operator $\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q}$ to the one of the homogeneous Besov-Morrey spaces associated to the Hermite operator $\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ in this paper. To study $\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$, we use the results in [1], specifically, the estimates on the heat kernels via the square functions. Beside, we also establish the molecular decompositions for $\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$. As applications, we obtain the regularity of solutions to the fractional Hermite equations:

$$\mathbb{H}^s u = f,$$

and

$$(\mathbb{H}+I)^s u = f.$$

Then, we organize this paper as follows: Section 2 contains some preliminary results and definitions of functional spaces. Section 3 is devoted to the study of the molecular decomposition for the Hermite-Besov-Morrey space. Finally, we investigate the regularity of solutions on Hermite-Besov-Morrey spaces to the fractional Hermite equations in Section 4.
Throughout this paper, we always use C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We write $A \lesssim B$ if there is a universal constant C such that $A \le CB$; and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We also use the following notations:

$$\mathbb{N} = \{0,1,2,...\}, \mathbb{N}_+ = \{1,2,3,...\}$$

$$\mathbb{Z}^- = \{-1,-2,...\}, \mathbb{Z}_0^- = \{0,-1,-2,...\}$$

$$a \wedge b = \min\{a,b\}, a \vee b = \max\{a,b\}.$$

Finally, $\mathrm{int}[a]$ is the integer part of a.

2. Preliminaries

2.1 Dyadic cube

The set of all dyadic cubes $\mathscr{D}$ in $\mathbb{R}^n$ is defined by

$$\mathscr{D} = \left\{ \prod_{j=1}^{n} \left[m_j 2^k, (m_j+1)2^k \right) : m_1, m_2, ..., m_n, k \in \mathbb{Z} \right\}.$$

For a dyadic cube $Q := \prod_{j=1}^{n} \left[m_j 2^k, (m_j+1)2^k \right)$, for some $m_1, m_2, ..., m_n, k \in \mathbb{Z}$ we denote by $\ell(Q)$ and x_Q the length and the center of the dyadic cube Q. In this case, $\ell(Q) = 2^k$ and $x_Q = \left((m_j + 1/2)2^k\right)_{j=1}^n$.
Moreover, for every $\nu \in \mathbb{Z}$, we set

$$\mathscr{D}_\nu = \{Q \in \mathscr{D} : \ell(Q) = 2^\nu\}.$$

2.2 Morrey space

Let us first recall the definition of the Morrey spaces.

Definition 2.1. For every $0 < p \le r < \infty$, the Morrey space $\mathbf{M}_p^r$ is defined by

$$\mathbf{M}_p^r \equiv \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{\mathbf{M}_p^r} = \sup_{x_0 \in \mathbb{R}^n} \sup_{R>0} R^{n\left(\frac{1}{r}-\frac{1}{p}\right)} \|f\|_{L^p(B(x_0,R))} < \infty \right\}.$$

Next, we point out some known results about the Morrey norms.

Proposition 2.2. *Let $0 < p \le r < \infty$. Then, we have*

$$\|f\|_{\mathbf{M}_p^r} \sim \sup_{Q \in \mathscr{D}} |Q|^{\frac{1}{r}-\frac{1}{p}} \|f\|_{L^p(Q)}. \tag{2.1}$$

$$\|f^\theta\|_{\mathbf{M}_p^r} = \|f\|^\theta_{\mathbf{M}^{r\theta}_{p\theta}}, \quad \forall \theta > 0. \tag{2.2}$$

$$\left\| \left(\int_a^b |F(\cdot,t)|^q \frac{dt}{t} \right)^{1/q} \right\|_{\mathbf{M}_p^r} \le \left(\int_a^b \|F(\cdot,t)\|^q_{\mathbf{M}_p^r} \frac{dt}{t} \right)^{1/q}, \quad \text{for } 0 < q \le p. \tag{2.3}$$

Proof. (2.1) and (2.2) just follow from the definition of the Morrey spaces. While, (2.3) can be obtained by using Minkowski integral inequality, see also (2.20) in [4]. □

Next, for any $\theta > 0$, we denote by $\mathbb{M}_\theta$ the Hardy-Littlewood maximal function:

$$\mathbb{M}_\theta f(x) = \sup_{x \in B} \left(\frac{1}{|B|} \int_B |f(y)|^\theta dy \right)^{1/\theta}, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x.
Then, we have a version of the Fefferman-Stein vector-valued maximal inequality for the Morrey spaces, see Proposition 2.1, [12].

Proposition 2.3. *Let $0 < q \le \infty$, $0 < p \le r < \infty$, and $0 < \theta < \min\{p,q\}$. Then, we have*

$$\left\| \left(\sum_{k \in \mathbb{Z}} |\mathbb{M}_\theta f_k|^q \right)^{1/q} \right\|_{\mathbf{M}_p^r} \lesssim \left\| \left(\sum_{k \in \mathbb{Z}} |f_k|^q \right)^{1/q} \right\|_{\mathbf{M}_p^r}.$$

Remark 2.4. As a consequence of Proposition 2.3, the Hardy-Littlewood maximal operator $\mathbb{M}_\theta$ is bounded on $\mathbf{M}_p^r$.

Next, let us put

$$A_\nu = \left(\sup_{J \in \mathscr{D}, \ell(J) \geqslant 2^\nu} \left(\frac{1}{|J|} \right)^{1-p/r} \sum_{Q \in \mathscr{D}_\nu, Q \subset J} |Q|^{1-p/r} |s_Q|^p \right)^{1/p}.$$

We borrow a result of Wang [15, p.779] involving the characterization of A_ν in the Morrey norms.

Lemma 2.5. *Let $0 < p \le r < \infty$, and $\nu \in \mathbb{Z}$. Assume that the sequence $\{s_Q : Q \in \mathscr{D}_\nu\}$ satisfy*

$$\left\| \sum_{Q\in\mathscr{D}_\nu} |Q|^{-1/r}|s_Q|\chi_Q \right\|_{\mathbf{M}_p^r} < \infty.$$

Then, we have

$$\left\| \sum_{Q\in\mathscr{D}_\nu} |Q|^{-1/r}|s_Q|\chi_Q \right\|_{\mathbf{M}_p^r} \sim A_\nu.$$

2.3 Kernel estimates on Hermite operators

For any $k \ge 0$ and for $t > 0$, we denote the kernel associated to $(t\sqrt{\mathbb{H}})^k e^{-t\sqrt{\mathbb{H}}}$ by $p_{t,k}(x,y)$. We recall here the results of Lemma 2.1 and Propisition 2.2 in [1].

Proposition 2.6. *For $k \in \mathbb{N}$, there exist $C > 0$ and $\delta > 0$ so that*

1. $|p_{t,k}(x,y)| \le C\dfrac{t^k}{\Big(t+|x-y|\Big)^{n+k}}$, *for $x,y \in \mathbb{R}^n$.*
2. *For any $|h| < t$, we have*

$$|p_{t,k}(x+h,y) - p_{t,k}(x,y)| \le C\Big(\frac{|h|}{t}\Big)^{\delta}\frac{t^k}{\Big(t+|x-y|\Big)^{n+k}}, \quad \text{for } x,y \in \mathbb{R}^n.$$

Proposition 2.7. *For every $y \in \mathbb{R}^n$, we have $p_{t,k}(\cdot,y) \in \mathscr{S}$.*

2.4 Calderón reproducing formulas

In this part, we recall the following Calderón reproducing formula in [1]. It is useful to study the homogeneous Besov-Morrey spaces.

Proposition 2.8. *Let $m_1, m_2 \in \mathbb{N}^+$ and $f \in \mathscr{S}'$. Then we have*

$$f = -\frac{1}{2^{m-1}(m-1)!}\int_0^\infty (t\sqrt{\mathbb{H}})^{m_1}e^{-t\sqrt{\mathbb{H}}}(t\sqrt{\mathbb{H}})^{m_2}e^{-t\sqrt{\mathbb{H}}}f\,\frac{dt}{t} \text{ in } \mathscr{S}',$$

where $m = m_1 + m_2$, and $\mathscr{S}'$ is the dual space of the Schwartz functions $\mathscr{S}$ as usual.

3. Besov-Morrey Spaces associated to the Hermite operators

It is convenient for us to introduce first the homogeneous Besov-Morrey spaces corresponding to the Hermite operator $\mathbb{H}$.

Definition 3.1. Let $\alpha \in \mathbb{R}$, $0 < p,q \le \infty$, $p \le r \le \infty$, and for every positive integer $m > n + \max\{\alpha, 0\} + \text{int}\left[n\left(\frac{1}{\theta_0} - 1\right)\right] + 1$, with $\theta_0 = \min\{1,p,q\}$. Then, we define the homogeneous Hermite-Besov-Morrey space $\mathbf{BM}_{p,q,r}^{\alpha,\mathbb{H},m}$ as follows:

$$\mathbf{BM}_{p,q,r}^{\alpha,\mathbb{H},m} := \left\{ f \in \mathscr{S}' : \|f\|_{\mathbf{BM}_{p,q,r}^{\alpha,\mathbb{H},m}} = \left(\int_0^\infty \left(t^{-\alpha}\left\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}}f\right\|_{\mathbf{M}_p^r}\right)^q \frac{dt}{t}\right)^{1/q} < \infty \right\}.$$

Remark 3.2. If $r = p$, then the space $\mathbf{BM}_{p,q,r}^{\alpha,\mathbb{H},m}$ is exactly the space $\mathbf{BM}_{p,q}^{\alpha,\mathbb{H},m}$ in [1].

We will show that $\mathbf{BM}_{p,q,r}^{\alpha,\mathbb{H},m}$ is independent of the choice of m when m is large enough. Precisely, we have the following result.

Theorem 3.3. *Let $\alpha \in \mathbb{R}$, $0 < p,q \le \infty$, and $p \le r \le \infty$. Let m_1, m_2 be the positive integers such that*

$$m_1, m_2 > n + \max\{\alpha, 0\} + \operatorname{int}\left[n\left(\frac{1}{\theta_0} - 1\right)\right] + 1,$$

with $\theta_0 = \min\{1, p, q\}$. Then, the spaces $\mathbf{BM}^{\alpha,\mathbb{H},m_1}_{p,q,r}$ and $\mathbf{BM}^{\alpha,\mathbb{H},m_2}_{p,q,r}$ coincide with equivalent norms.

As a consequence of Theorem 3.3, we can define the Besov space $\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ as any space $\mathbf{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$, for any positive integer $m > n + \max\{\alpha, 0\} + \operatorname{int}\left[n\left(\frac{1}{\theta_0} - 1\right)\right] + 1$.

We now recall the definition of the molecules associated to the Hermite operator in [1].

Definition 3.4. Let $0 < r \le \infty, \alpha \in \mathbb{R}$, and $N, M \in \mathbb{N}_+$. A function u is said to be an $(\mathbb{H}, M, N, \alpha, r)$ molecule if there exist a function b from the domain $(\sqrt{\mathbb{H}})^M$ and a dyadic cube $Q \in \mathscr{D}$ so that

i) $u = (\sqrt{\mathbb{H}})^M b$,

ii) $\left|(\sqrt{\mathbb{H}})^k b(x)\right| \le \ell(Q)^{M-k} |Q|^{\alpha/n - 1/r} \left(1 + \frac{|x - x_Q|}{\ell(Q)}\right)^{-n-N}, \quad$ for $k = 0, \dots, 2M$.

Briefly, we denote $u = m_Q$, for every dyadic cube $Q \in \mathscr{D}$.

Next, we have some elementary estimates.

Lemma 3.5. *Let $N \in \mathbb{N}_+$ and $t, a > 0$. For any $x, z \in \mathbb{R}^n$, then we have*

i) $\displaystyle\int_{\mathbb{R}^n} \left(1 + \frac{|x-y|}{t}\right)^{-n-N} \left(1 + \frac{|z-y|}{a}\right)^{-n-N} dy \lesssim t^n \left(\frac{a}{t}\right)^n \left(1 + \frac{|x-z|}{a}\right)^{-n-N}, \quad$ *if $t \le a$.*

ii) $\displaystyle\int_{\mathbb{R}^n} \left(1 + \frac{|x-y|}{t}\right)^{-n-N} \left(1 + \frac{|z-y|}{a}\right)^{-n-N} dy \lesssim t^n \left(1 + \frac{|x-z|}{t}\right)^{-n-N}, \quad$ *if $t \ge a$.*

Proof. We split its proof and refer to Lemma 3.6, [1]. □

Next, we have a result of the molecular decomposition for $\mathbf{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$.

Theorem 3.6. *Let $\alpha \in \mathbb{R}$, $0 < p,q \le \infty$, $p \le r \le \infty$, and $\theta_0 = \min\{1, p, q\}$.*

i) For every $M, N \in \mathbb{N}_+$ and $m > n + \max\{\alpha, 0\} + \operatorname{int}\left[n\left(\frac{1}{\theta_0} - 1\right)\right] + 1$, if $f \in \mathbf{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$, then there exist a sequence of $(\mathbb{H}, M, N, \alpha, r)$ molecules $\{m_Q\}_{Q \in \mathscr{D}_\nu, \nu \in \mathbb{Z}}$ and a sequence of coefficients $\{s_Q\}_{Q \in \mathscr{D}_\nu, \nu \in \mathbb{Z}}$ so that

$$f = \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathscr{D}_\nu} s_Q m_Q, \quad \text{in } \mathscr{S}'.$$

Moreover, we have

$$\left(\sum_{\nu \in \mathbb{Z}} A_\nu^q\right)^{\frac{1}{q}} \lesssim \|f\|_{\mathbf{BM}^{\alpha,\mathbb{H},m}_{p,q,r}}. \tag{3.1}$$

ii) Conversely, if

$$f = \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathscr{D}_\nu} s_Q m_Q, \quad \text{in } \mathscr{S}',$$

where $\{m_Q\}_{Q \in \mathscr{D}_\nu, \nu \in \mathbb{Z}}$ is a sequence of $(\mathbb{H}, M, N, \alpha, r)$ molecules and $\{s_Q\}_{Q \in \mathscr{D}_\nu, \nu \in \mathbb{Z}}$ is a sequence of coefficients verifying $\left(\sum_{\nu \in \mathbb{Z}} A_\nu^q\right)^{\frac{1}{q}} < \infty$, then $f \in \mathbf{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$, and

$$\|f\|_{\mathbf{BM}^{\alpha,\mathbb{H},m}_{p,q,r}} \lesssim \left(\sum_{\nu \in \mathbb{Z}} A_\nu^q\right)^{\frac{1}{q}}, \tag{3.2}$$

provided that $N, M \in \mathbb{N}_+$ such that $\frac{n}{n+N} < \theta_0$, $M > \max\{\frac{n}{\theta_0} - \alpha, m\}$, with $m > \max\{\alpha, 0\} + N + n$.

Proof. We first prove i). For every $f \in \mathbf{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$, it follows from Proposition 2.8 that

$$f = c_{m,M,N} \int_0^\infty (t\sqrt{\mathbb{H}})^{M+N} e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \frac{dt}{t}, \quad \text{in } \mathscr{S}',$$

with $c_{m,M,N} = -\dfrac{1}{2^{m+M+N-1}(m+M+N-1)!}$.
Thus,

$$\begin{aligned} f &= c_{m,M,N} \sum_{v\in\mathbb{Z}} \int_{2^v}^{2^{v+1}} (t\sqrt{\mathbb{H}})^{M+N} e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \frac{dt}{t} \\ &= c_{m,M,N} \sum_{v\in\mathbb{Z}} \sum_{Q\in\mathscr{D}_v} \int_{2^v}^{2^{v+1}} (t\sqrt{\mathbb{H}})^{M+N} e^{-t\sqrt{\mathbb{H}}} \left[(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f.\chi_Q \right] \frac{dt}{t}. \end{aligned}$$

For any $v \in \mathbb{Z}$ and $Q \in \mathscr{D}_v$, we set

$$s_Q = 2^{-v(\alpha - n/r)} \sup_{(y,t)\in Q\times[2^v,2^{v+1})} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y) \right|, \quad \text{and } m_Q = \mathbb{H}^{M/2} b_Q, \tag{3.3}$$

with

$$b_Q = \frac{1}{s_Q} \int_{2^v}^{2^{v+1}} t^M (t\sqrt{\mathbb{H}})^N e^{-t\sqrt{\mathbb{H}}} \left[(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f.\chi_Q \right] \frac{dt}{t}.$$

Obviously, we have

$$f = \sum_{v\in\mathbb{Z}} \sum_{Q\in\mathscr{D}_v} s_Q m_Q, \quad \text{in } \mathscr{S}'.$$

Thus, it remains to show that m_Q is an $(\mathbb{H}, M, N, \alpha, r)$ molecule.
Indeed, for $k = 0, ..., 2M$, and for any $x \in \mathbb{R}^n$, we have from Proposition 2.6.

$$\begin{aligned} \left| \mathbb{H}^{k/2} b_Q(x) \right| &= \left| \frac{1}{s_Q} \int_{2^v}^{2^{v+1}} t^{M-k} (t\sqrt{\mathbb{H}})^{N+k} e^{-t\sqrt{\mathbb{H}}} \left[(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f.\chi_Q \right] \frac{dt}{t} \right| \\ &\le \frac{1}{s_Q} \int_{2^v}^{2^{v+1}} t^{M-k} \int_Q |p_{t,N+k}(x,y)| \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y) \right| dy \frac{dt}{t} \\ &\lesssim \frac{1}{s_Q} \sup_{(z,t)\in Q\times[2^v,2^{v+1})} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(z) \right| \int_{2^v}^{2^{v+1}} t^{M-k} \int_Q \frac{t^N}{(t+|x-y|)^{n+N}} dy \frac{dt}{t}. \end{aligned} \tag{3.4}$$

On the other hand, it is not difficult to verify that

$$\int_Q \frac{t^N}{(t+|x-y|)^{n+N}} dy \le C(n,N) \left(1 + \frac{|x-x_Q|}{2^v} \right)^{-n-N}, \quad \forall t \in [2^v, 2^{v+1}). \tag{3.5}$$

A combination of (3.3), (3.4) and (3.5) yields

$$\left| \mathbb{H}^{k/2} b_Q(x) \right| \lesssim 2^{v(\alpha + M - k - n/r)} \left(1 + \frac{|x-x_Q|}{2^v} \right)^{-n-N}.$$

This implies that m_Q is an $(\mathbb{H}, M, N, \alpha, r)$ molecule.

Next, we prove (3.1). We observe that $w(x,t) \equiv \mathbb{H}^{m/2} e^{-t\sqrt{\mathbb{H}}} f(x)$ is a solution of the equation

$$-\left(\Delta_{x,t} + |x|^2\right) w = 0, \quad \text{with } \Delta_{x,t} w = w_{tt} + \Delta w.$$

So, w is a subharmonic function. Thanks to Lemma 5.2 in [2], for every $\theta \in (0,\infty)$ we obtain

$$\sup_{(y,t)\in\widetilde{Q}} \left|\mathbb{H}^{m/2} e^{-t\sqrt{\mathbb{H}}} f(y)\right| \lesssim \left(\frac{1}{|\widetilde{Q}|} \int_{\frac{3}{2}\widetilde{Q}} \left|\mathbb{H}^{m/2} e^{-t\sqrt{\mathbb{H}}} f(y)\right|^{\theta} dydt \right)^{1/\theta},$$

where $\widetilde{Q} = Q \times [2^{\nu}, 2^{\nu+1})$ is a cube in $\mathbb{R}^{n+1}$.
Note that $|\widetilde{Q}| \sim 2^{\nu}|Q|$ and $t \sim 2^{\nu}$, for any $(y,t) \in \widetilde{Q}$. Hence, it follows from the last inequality that

$$\begin{aligned} \sup_{(y,t)\in\widetilde{Q}} \left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y)\right| &\lesssim \left(\frac{1}{|Q|} \int_{\frac{3}{4}2^{\nu}}^{\frac{9}{8}2^{\nu+1}} \int_{\frac{3}{2}Q} \left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y)\right|^{\theta} dy \frac{dt}{t} \right)^{1/\theta} \\ &\lesssim \left(\int_{\frac{3}{4}2^{\nu}}^{\frac{9}{8}2^{\nu+1}} \left[\mathbb{M}_{\theta}\left(\left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right|\right)(x)\right]^{\theta} \frac{dt}{t} \right)^{1/\theta}, \end{aligned} \tag{3.6}$$

for any $x \in Q$. From (3.3) and (3.6), we get

$$|s_Q|\chi_Q(x) \lesssim 2^{-\nu(\alpha - n/r)} \left(\int_{\frac{3}{4}2^{\nu}}^{\frac{9}{8}2^{\nu+1}} \left[\mathbb{M}_{\theta}\left(\left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right|\right)(x)\right]^{\theta} \frac{dt}{t} \right)^{1/\theta} \chi_Q(x).$$

Or,

$$\sum_{Q\in\mathscr{D}_{\nu}} |Q|^{-1/r} |s_Q|\chi_Q(x) \lesssim 2^{-\nu\alpha} \left(\int_{\frac{3}{4}2^{\nu}}^{\frac{9}{8}2^{\nu+1}} \left[\mathbb{M}_{\theta}\left(\left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right|\right)(x)\right]^{\theta} \frac{dt}{t} \right)^{1/\theta}.$$

Thanks to Lemma 2.5, we have

$$A_{\nu} \lesssim 2^{-\nu\alpha} \left\| \left(\int_{\frac{3}{4}2^{\nu}}^{\frac{9}{8}2^{\nu+1}} \left[\mathbb{M}_{\theta}\left(\left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right|\right)\right]^{\theta} \frac{dt}{t} \right)^{1/\theta} \right\|_{\mathbf{M}_p^r}.$$

Next, using Minkowski integral inequality (see (2.3)) yields

$$A_{\nu} \lesssim 2^{-\nu\alpha} \left[\int_{\frac{3}{4}2^{\nu}}^{\frac{9}{8}2^{\nu+1}} \left\|\mathbb{M}_{\theta}\left(\left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right|\right)\right\|_{\mathbf{M}_p^r}^{\theta} \frac{dt}{t} \right]^{1/\theta}.$$

At the moment, for a fixed $\theta \in (0,\theta_0)$, then $\mathbb{M}_\theta$ is a bounded operator on $\mathbf{M}_p^r$, likewise

$$\begin{aligned}
A_v &\lesssim 2^{-v\alpha}\left[\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right\|_{\mathbf{M}_p^r}^{\theta} \frac{dt}{t}\right]^{1/\theta} \\
&\lesssim \left[\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(t^{-\alpha}\left\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right\|_{\mathbf{M}_p^r}\right)^{\theta} \frac{dt}{t}\right]^{1/\theta} \\
&\lesssim \left[\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(t^{-\alpha}\left\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right\|_{\mathbf{M}_p^r}\right)^{q} \frac{dt}{t}\right]^{1/q},
\end{aligned}$$

where the last inequality is obtained by using Hölder's inequality.
Therefore,

$$\left(\sum_{v\in\mathbb{Z}} A_v^q\right)^{1/q} \lesssim \left[\sum_{v\in\mathbb{Z}} \int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(t^{-\alpha}\left\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right\|_{\mathbf{M}_p^r}\right)^{q} \frac{dt}{t}\right]^{1/q}.$$

By noting that $\sum_{v\in\mathbb{Z}} \chi_{(\frac{3}{4}2^v, \frac{9}{8}2^{v+1})} \leq 2$, we obtain

$$\sum_{v\in\mathbb{Z}} \int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \left(t^{-\alpha}\left\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right\|_{\mathbf{M}_p^r}\right)^{q} \frac{dt}{t} \leq 2\int_0^\infty \left(t^{-\alpha}\left\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right\|_{\mathbf{M}_p^r}\right)^{q} \frac{dt}{t},$$

which implies

$$\left(\sum_{v\in\mathbb{Z}} A_v^q\right)^{1/q} \lesssim \left[\int_0^\infty \left(t^{-\alpha}\left\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\right\|_{\mathbf{M}_p^r}\right)^{q} \frac{dt}{t}\right]^{1/q} = \|f\|_{\mathbf{BM}^{\alpha,\mathbb{H},m}_{p,q,r}}.$$

This puts an end to the proof of i) Theorem 3.6.

In order to prove ii), we need the following auxiliary lemmas.

Lemma 3.7. *Let $N>0$, and let $\eta, v \in \mathbb{Z}$ be such that $v \leq \eta$. Let $\{f_Q\}_{Q\in\mathscr{D}_v}$ be a sequence of functions satisfying*

$$|f_Q(x)| \lesssim \left(1+2^{-\eta}|x-x_Q|\right)^{-n-N}.$$

Then, for any $\theta \in (\frac{n}{n+N},\infty)$ and for a sequence of numbers $\{s_Q\}_{Q\in\mathscr{D}_v}$, we have

$$\sum_{Q\in\mathscr{D}_v} |s_Q||f_Q(x)| \lesssim 2^{\frac{(\eta-v)n}{\theta}} \mathbb{M}_\theta\left(\sum_{Q\in\mathscr{D}_v} |s_Q|\chi_Q\right)(x).$$

Proof. We refer to [3, p.147] for the proof of this lemma. □

Next, we recall [1, Lemma 3.6] here for a convenience.

Lemma 3.8. *Under the assumptions as in ii) of Theorem 3.6, we have*

$$\left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x)\right| \lesssim |Q|^{\frac{\alpha}{n}-\frac{1}{r}} \left(\frac{t}{2^v}\right)^{m-N-n} \left(1+\frac{|x-x_Q|}{2^v}\right)^{-n-N}, \quad \forall t<2^v,$$

$$\left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x)\right| \lesssim |Q|^{\frac{\alpha}{n}-\frac{1}{r}} \left(\frac{2^v}{t}\right)^{M} \left(1+\frac{|x-x_Q|}{t}\right)^{-n-N}, \quad \forall t \geq 2^v.$$

We are now ready to give the proof of ii) Theorem 3.6. At the beginning, we write

$$\begin{aligned}\|f\|^q_{\mathbf{BM}^{\alpha,\mathbb{H},m}_{p,q,r}} &= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(t^{-\alpha} \left\| \sum_{v\in\mathbb{Z}} \sum_{Q\in\mathscr{D}_v} s_Q (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q \right\|_{\mathbf{M}^r_p} \right)^q \frac{dt}{t} \\ &\lesssim \sum_{k\in\mathbb{Z}} \left(2^{-k\alpha} \left\| \sum_{v>k} \sum_{Q\in\mathscr{D}_v} |s_Q| \sup_{t\in[2^k,2^{k+1})} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q \right| \right\|_{\mathbf{M}^r_p} \right)^q \\ &+ \sum_{k\in\mathbb{Z}} \left(2^{-k\alpha} \left\| \sum_{v\le k} \sum_{Q\in\mathscr{D}_v} |s_Q| \sup_{t\in[2^k,2^{k+1})} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q \right| \right\|_{\mathbf{M}^r_p} \right)^q \\ &:= I_1 + I_2. \end{aligned} \tag{3.7}$$

Thus, Theorem 3.6 is done if we can demonstrate that

$$I_1, I_2 \lesssim \sum_{v\in\mathbb{Z}} A^q_v. \tag{3.8}$$

We first prove (3.8) for I_1. Keep in mind that $v \ge k+1$ in this case.
Since $\theta_0 > \dfrac{n}{n+N}$ and $M > \max\{\dfrac{n}{\theta_0} - \alpha, m\}$, we can choose a real number $\theta \in (\dfrac{n}{n+N}, \theta_0)$ such that $M > \dfrac{n}{\theta} - \alpha$. By noting that $2^v \ge 2^{k+1} > t$, Lemma 3.8 deduces

$$\sup_{t\in[2^k,2^{k+1})} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x) \right| \lesssim |Q|^{\frac{\alpha}{n} - \frac{1}{r}} 2^{(k-v)(m-N-n)} \left(1 + 2^{-v}|x - x_Q|\right)^{-n-N}.$$

Thus,

$$\begin{aligned}\sum_{Q\in\mathscr{D}_v} |s_Q| \sup_{t\in[2^k,2^{k+1})} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x) \right| &\lesssim \sum_{Q\in\mathscr{D}_v} |Q|^{\frac{\alpha}{n} - \frac{1}{r}} 2^{(k-v)(m-N-n)} |s_Q| \left(1 + 2^{-v}|x - x_Q|\right)^{-n-N} \\ &\lesssim 2^{v\alpha} 2^{(k-v)(m-N-n)} \sum_{Q\in\mathscr{D}_v} |Q|^{-\frac{1}{r}} |s_Q| \left(1 + 2^{-v}|x - x_Q|\right)^{-n-N}. \end{aligned} \tag{3.9}$$

Now, we apply Lemma 3.7 with $\eta = v$ and $f_Q(x) = (1 + 2^{-v}|x - x_Q|)^{-n-N}$ to get

$$\sum_{Q\in\mathscr{D}_v} |Q|^{-\frac{1}{r}} |s_Q| \left(1 + 2^{-v}|x - x_Q|\right)^{-n-N} \lesssim \mathbb{M}_\theta \left(\sum_{Q\in\mathscr{D}_v} |Q|^{-\frac{1}{r}} |s_Q| \chi_Q \right)(x), \quad \text{for } \theta \in (\frac{n}{n+N}, \theta_0). \tag{3.10}$$

Inserting (3.10) into (3.9) yields

$$\sum_{Q\in\mathscr{D}_v} |s_Q| \sup_{t\in[2^k,2^{k+1})} \left| (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x) \right| \lesssim 2^{v\alpha} 2^{(k-v)(m-N-n)} \mathbb{M}_\theta \left(\sum_{Q\in\mathscr{D}_v} |Q|^{-\frac{1}{r}} |s_Q| \chi_Q \right)(x).$$

Then,

$$\begin{aligned} I_1 &\lesssim \sum_{k\in\mathbb{Z}} \left[2^{-k\alpha} \left\| \sum_{v>k} 2^{\alpha v} 2^{(k-v)(m-N-n)} \mathbb{M}_\theta \left(\sum_{Q\in\mathscr{D}_v} |Q|^{-1/r} |s_Q| \chi_Q \right) \right\|_{\mathbf{M}^r_p} \right]^q \\ &= \sum_{k\in\mathbb{Z}} \left\| \sum_{v>k} 2^{(k-v)(m-N-n-\alpha)} \mathbb{M}_\theta \left(\sum_{Q\in\mathscr{D}_v} |Q|^{-1/r} |s_Q| \chi_Q \right) \right\|^q_{\mathbf{M}^r_p} \\ &\lesssim \sum_{k\in\mathbb{Z}} \left[\sum_{v>k} 2^{(k-v)(m-N-n-\alpha)} \left\| \mathbb{M}_\theta \left(\sum_{Q\in\mathscr{D}_v} |Q|^{-1/r} |s_Q| \chi_Q \right) \right\|_{\mathbf{M}^r_p} \right]^q. \end{aligned} \tag{3.11}$$

Again the fact that $\mathbb{M}_\theta$ is bounded on $\mathbf{M}_p^r$ deduces

$$\left\|\mathbb{M}_\theta\left(\sum_{Q\in\mathscr{D}_v}|Q|^{-1/r}|s_Q|\chi_Q\right)\right\|_{\mathbf{M}_p^r}\lesssim\left\|\sum_{Q\in\mathscr{D}_v}|Q|^{-1/r}|s_Q|\chi_Q\right\|_{\mathbf{M}_p^r}\sim A_v. \tag{3.12}$$

A combination of (3.11) and (3.12) deduces

$$I_1\lesssim\sum_{k\in\mathbb{Z}}\left[\sum_{v>k}2^{(k-v)(m-N-n-\alpha)}A_v\right]^q.$$

Apllying Young's inequality yields

$$\sum_{v>k}2^{(k-v)(m-N-n-\alpha)}A_v\le\left(\sum_{v>k}2^{\frac{(k-v)(m-N-n-\alpha)q}{2(q-1)}}\right)^{\frac{q-1}{q}}\left(\sum_{v>k}2^{\frac{(k-v)(m-N-n-\alpha)q}{2}}A_v^q\right)^{\frac{1}{q}}.$$

Since $m>N+n+\alpha$, $\sum_{v>k}2^{\frac{(k-v)(m-N-n-\alpha)q}{2(q-1)}}$ is then bounded by a constant not depending on k,v. Thus,

$$I_1\lesssim\sum_{k\in\mathbb{Z}}\sum_{v>k}2^{\dfrac{(k-v)(m-N-n-\alpha)q}{2}}A_v^q=\sum_{v\in\mathbb{Z}}\left(\sum_{k<v}2^{\dfrac{(k-v)(m-N-n-\alpha)q}{2}}\right)A_v^q\lesssim\sum_{v\in\mathbb{Z}}A_v^q.$$

It remains to show that estimate (3.8) holds for I_2. Actually, the proof for I_2 is most likely to the one for I_1, with only one different point that we use Lemma 3.8 for $v\le k$, i.e:

$$\sup_{t\in[2^k,2^{k+1})}\left|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}}m_Q(x)\right|\lesssim|Q|^{\frac{\alpha}{n}-\frac{1}{r}}2^{(v-k)M}\left(1+\frac{|x-x_Q|}{2^v}\right)^{-n-N}.$$

Proceed similarly to the proof (from (3.9) to (3.12)) above, we obtain

$$I_2\lesssim\sum_{k\in\mathbb{Z}}\left[\sum_{v\le k}2^{(v-k)(M+\alpha)}A_v\right]^q.$$

By noting that $M+\alpha>0$, aplly Young's inequality yields the result.
This puts an end to the proof of Theorem 3.6. □

Next, we provide the proof of Theorem 3.3.

Proof of Theorem 3.3. Let us take $N=\operatorname{int}\left[n\left(\frac{1}{\theta_0}-1\right)\right]+1$, and $M>\max\left\{m_1,m_2,\dfrac{n}{\theta_0}-\alpha\right\}$. Because m_1 and m_2 play the same role, it then suffices to prove that $\mathbf{BM}_{p,q}^{\alpha,\mathbb{H},m_1}\hookrightarrow\mathbf{BM}_{p,q}^{\alpha,\mathbb{H},m_2}$.

In fact, for $f\in\mathbf{BM}_{p,q,r}^{\alpha,\mathbb{H},m_1}$, thanks to i) of Theorem 3.6, there exist a sequence of $(\mathbb{H},M,N,\alpha,r)$ molecules $\left\{m_Q:Q\in\mathscr{D}_v,v\in\mathbb{Z}\right\}$, and a sequence of coefficients $\left\{s_Q:Q\in\mathscr{D}_v,v\in\mathbb{Z}\right\}$ so that

$$f=\sum_{v\in\mathbb{Z}}\sum_{Q\in\mathscr{D}_v}s_Qm_Q,\quad\text{in }\mathscr{S}',$$

and

$$\left(\sum_{v\in\mathbb{Z}}A_v^q\right)^{1/q}\lesssim\|f\|_{\mathbf{BM}_{p,q,r}^{\alpha,\mathbb{H},m_1}}.$$

In other words, $\left(\sum_{v\in\mathbb{Z}}A_v^q\right)^{1/q}$ is finite.

By ii) of Theorem 3.6, we obtain $f\in\mathbf{BM}_{p,q,r}^{\alpha,\mathbb{H},m_2}$. Furthermore, f fulfills

$$\|f\|_{\mathbf{BM}_{p,q,r}^{\alpha,\mathbb{H},m_2}}\lesssim\left(\sum_{v\in\mathbb{Z}}A_v^q\right)^{1/q}.$$

Or, we get the result. □

4. Regularity on Besov-Morrey spaces for fractional Hermite equations

In this part, we study the regularity results of solutions of the two fractional Hermite equations:

$$\mathbb{H}^s u = f, \quad \text{and } (I+\mathbb{H})^s = f, \quad \text{on } \mathbb{R}^n,$$

for any $s>0$, and for $f \in \mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$.

To solve the indicated equations, it is necessary to investigate the operators $\mathbb{H}^{-s}$ and $(I+\mathbb{H})^{-s}$, named by Riesz potential of the Hermite operator and Bessel potential of Hermite operator respectively.

In fact, by following Proposition 2.5 in [1], we can define the operators $\mathbb{H}^{-s}: \mathscr{S}' \to \mathscr{S}'$ and $(I+\mathbb{H})^{-s}: \mathscr{S}' \to \mathscr{S}'$ by setting

$$\langle \mathbb{H}^{-s} f, \phi \rangle = \langle f, \mathbb{H}^{-s}\phi \rangle, \quad \text{and } \langle (I+\mathbb{H})^{-s} f, \phi \rangle = \langle f, (I+\mathbb{H})^{-s}\phi \rangle,$$

for any $f \in \mathscr{S}'$, and for $\phi \in \mathscr{S}$. Note that $\langle \cdot, \cdot \rangle$ is the pair between a linear function in $\mathscr{S}'$ and a function in $\mathscr{S}$. And $\mathscr{S}'$ is the dual space of the Schwartz space $\mathscr{S}$ as usual.
Moreover, we have for any $\phi \in \mathscr{S}$,

$$\mathbb{H}^{-s}\phi = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-t\mathbb{H}} \phi \frac{dt}{t} \in \mathscr{S}, \tag{4.1}$$

$$(I+\mathbb{H})^{-s}\phi = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-t} e^{-t\mathbb{H}} \phi \frac{dt}{t} \in \mathscr{S}. \tag{4.2}$$

Then, our regularity results are as follows:

Theorem 4.1. *Let $\alpha \in \mathbb{R}$, $0<q\le\infty$, $0<p\le r\le\infty$, and $f \in \mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$. Assume that u is a solution of equation $\mathbb{H}^s u = f$, Then, there exists a constant $C>0$ such that*

$$\|u\|_{\mathbf{BM}^{\alpha+s,\mathbb{H}}_{p,q,r}} \le C \|f\|_{\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}}.$$

Theorem 4.2. *Let $\alpha \in \mathbb{R}$, $0<q\le\infty$, $0<p\le r\le\infty$, and $f \in \mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$. Assume that u is a solution of equation $(\mathbb{H}+I)^s u = f$. Then, there exists a constant $C>0$ such that*

$$\|u\|_{\mathbf{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}} \le C \|f\|_{\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}}.$$

Theorem 4.1 and Theorem4.2 are just a consequence of the theorem below.

Theorem 4.3. *Let $\alpha \in \mathbb{R}$, $0<p\le r<\infty$, and $0<q\le\infty$. For any $s>0$, the operator $\mathbb{H}^{-s}$ (resp. $(I+\mathbb{H})^{-s}$) is bounded from $\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ to $\mathbf{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}$.*

Proof of Theorem 4.3. Before giving the proof, we emphasize that our approach is not similar to the one in [1]. Here, we give a direct proof, based on the estimates for the Hermite operator $\mathbb{H}^{-s}$ (resp. $(I+\mathbb{H})^{-s}$) instead of proving that $H^{-s}m_Q$ is a $(H,M,N,\alpha+2s,r)$ molecular, see Theorem 5.1, [1].

For any $f \in \mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$, thanks to i) of Theorem 3.6, there exist a sequence $\left\{m_Q : Q \in \mathscr{D}_\nu, \nu \in \mathbb{Z}\right\}$ of $(\mathbb{H},M,N,\alpha,r)$ molecules and a sequence of coefficients $\left\{s_Q : Q \in \mathscr{D}_\nu, \nu \in \mathbb{Z}\right\}$, such that

$$f = \sum_{\nu\in\mathbb{Z}} \sum_{Q\in\mathscr{D}_\nu} s_Q m_Q,$$

and

$$\sum_{\nu\in\mathbb{Z}} A_\nu^q \lesssim \|f\|^q_{\mathbf{BM}^{\alpha,\mathbb{H},m'}_{p,q,r}},$$

with $M,N \in \mathbb{N}_+$, such that $\dfrac{n}{n+N} < \theta_0$, $M > \max\left\{\dfrac{n}{\theta_0} - \alpha - 2s, m'\right\}$, and $m' > n + \max\{\alpha+2s, 2s\} + \operatorname{int}\left[n\left(\frac{1}{\theta_0} - 1\right)\right] + 1$. It is of course that we want to choose m' large enough in order to be sure that $\mathbf{BM}^{\alpha,\mathbb{H},m'}_{p,q,r} = \mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ due to Theorem 3.3.

To prove that H^{-s} is a bounded operator from $\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ to $\mathbf{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}$, it suffices to show that

$$\left\|H^{-s}f\right\|^q_{\mathbf{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}} \lesssim \sum_{\nu\in\mathbb{Z}} A^q_\nu. \tag{4.3}$$

It follows from Theorem 3.3 that

$$\begin{aligned}\left\|H^{-s}f\right\|^q_{\mathbf{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}} \sim \left\|H^{-s}f\right\|^q_{\mathbf{BM}^{\alpha+2s,\mathbb{H},m'}_{p,q,r}} &= \int_0^\infty \left(t^{-(\alpha+2s)}\left\|(t\sqrt{H})^{m'}e^{-t\sqrt{H}}H^{-s}f\right\|_{\mathbf{M}^r_p}\right)^q \frac{dt}{t}\\
&= \int_0^\infty \left(t^{-\alpha}\left\|(t\sqrt{H})^{m'-2s}e^{-t\sqrt{H}}f\right\|_{\mathbf{M}^r_p}\right)^q \frac{dt}{t}\\
&= \sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}} \left(t^{-\alpha}\left\|\sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathscr{D}_\nu} s_Q(t\sqrt{H})^{m'-2s}e^{-t\sqrt{H}}m_Q\right\|_{\mathbf{M}^r_p}\right)^q \frac{dt}{t}\\
&\lesssim \sum_{k\in\mathbb{Z}}\left(2^{-k\alpha}\left\|\sum_{\nu>k}\sum_{Q\in\mathscr{D}_\nu}|s_Q|\sup_{t\in[2^k,2^{k+1})}\left|(t\sqrt{H})^{m'-2s}e^{-t\sqrt{H}}m_Q\right|\right\|_{\mathbf{M}^r_p}\right)^q\\
&+\sum_{k\in\mathbb{Z}}\left(2^{-k\alpha}\left\|\sum_{\nu\le k}\sum_{Q\in\mathscr{D}_\nu}|s_Q|\sup_{t\in[2^k,2^{k+1})}\left|(t\sqrt{H})^{m'-2s}e^{-t\sqrt{H}}m_Q\right|\right\|_{\mathbf{M}^r_p}\right)^q.\end{aligned}$$

Obviously the last inequality is just a version of (3.7), in that m is replaced by $m'-2s$. Note that $m'-2s > n+\max\{\alpha,0\}+\operatorname{int}\left[n\left(\frac{1}{\theta_0}-1\right)\right]+1$. Therefore, (4.3) follows by apllying ii) of Theorem 3.6 to $m=m'-2s$. Or, we get the conclusion. Similarly, we can establish the boundedness of the Bessel potential $(I+H)^{-s}$ from $\mathbf{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ to $\mathbf{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}$. Then, we leave the proof to the reader. □

Remark 4.4. We emphasize that our proofs in this paper, can be applied to study the homogeneous and inhomogeneous Hermite-Triebel-Lizorkin-Morrey spaces. The ones will appear in our forthcoming papers.

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References

[1] B. T. Anh, D. X. Thinh. Besov and Triebel-Lizorkin Spaces Associated to Hermite Operators. J. Fourier. Anal. Appl **21** (2015) 405–448.

[2] P. Auscher, B. Ben Ali. Maximal inequalities and Riesz transform estimates on L^p spaces for Schrödinger operators with non-negative potentials. Annales de I'Institut Fourier. **57**(6) (2007) 1975–2013.

[3] M. Frazier, B. Jawerth. A discrete transform and decomposition of distribution spaces. J. Funct. Anal. 93 (10990) 34–170.

[4] G. Kerkyacharian, P. Petrushev. Heat Kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces. Trans. Amer. Math. Soc. **367 (1)** (2015) 121–189.

[5] H. Kozono, M. Yamazaki. Semilinear heat equations and the Navier-Stokes equation with distributions in the new function spaces as initial data. Comm. Partial Differential Equations **19** (5–6) (1994) 959–1014.

[6] C. C. Lin, Q. Yang. Semigroup characterization of Besov type Morrey spaces and well-posedness of generalized Navier–Stokes equations. J. Differential Equations **254** (2013) 804–846.

[7] Anna L. Mazzucato. Besov-Morrey spaces: Function space theory and applications to non-linear PDE. Trans. Amer. Math. Soc. **355** (2003) 1297–1364.

[8] Anna L. Mazzucato. Decomposition of Besov-Morrey spaces, Proceedings of the Conference on Harmonic Analysis. Contemp. Math. **320**, Amer. Math. Soc., Providence, RI, (2003) 279–294.

[9] P. Petrushev, Y. Xu. Decomposition of spaces of distributions induced by Hermite expansion. J. Fourier Anal. Appl. **14**(3) (2008) 372–414.

[10] Y. Sawano, H. Tanaka. Decompositions of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces. Math. Z. **257** (4) (2007) 871—905.

[11] Y. Sawano, H. Tanaka.Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces for nondoubling measures. Math. Nachr. **282** (12). (2009) 1788–1810.

[12] Y. Sawano. Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces on domains. Math. Nachr. **283**(10) (2010) 1456–1487.

[13] L. Tang, J. Xu. Some properties of Morrey type Besov-Triebel spaces. Math. Nachr. **278** (2005) 904—917.

[14] B. H. Qui, D. X. Thinh, L. Yan. Calderón reproducing formulas and new Besov spaces associated with operators. Adv. Math. **229** (4) (2012) 2449—2502.

[15] H. Wang. Decomposition for Morrey type Besov-Triebel spaces. Math. Nachr. **282** (5) (2009) 774–787.