LEVEL PROPERTY OF ORDINARY AND SYMBOLIC POWERS OF STANLEY-REISNER IDEALS

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ABSTRACT. In this paper, we prove that the t-th ordinary and/or symbolic power of a Stanley-Reisner ideal is level for some positive integer $t \geq 3$ if and only if I_Δ is a complete intersection and equi-generated. For $t = 2$, we give a characterzation of level property of the second symbolic power $I_\Delta^{(2)}$ when Δ is a matroid complex of dimension one.

1. INTRODUCTION

Let Δ be a simplicial complex on $[n] = \{1, \dots, n\}$ and $S = K[x_1, \dots, x_n]$ a polynomial over a field K. The Stanley-Reisner ideal I_Δ of Δ (over K) is the ideal in S which is generated by all square-free monomials $x_{i_1} \dots x_{i_p}$ such that $\{i_1, \dots, i_p\} \notin \Delta$. It is known that I_Δ has the primary decomposition $I_\Delta = \bigcap_{F:\text{ facet of }\Delta} P_F$, where $P_F = (x_i \mid i \in [n] \setminus F)$. Then for $t \geq 1$, the t-th symbolic power $I_\Delta^{(t)}$ of I_Δ is expressed as

$$I_\Delta^{(t)} = \bigcap_{F:\text{ facet of }\Delta} P_F^t.$$

The purpose of this paper is to study the following question:

Question. When is S/I_Δ^t or $S/I_\Delta^{(t)}$ a level ring for $t \geq 1$?

This question fits into an ongoing research program to characterize ring properties of S/I^t or $S/I^{(t)}$. The Cohen-Macaulayness, the Buchsbaumness, the generalized Cohen-Macaulayness , and the k-Buchsbaumness were studied, for example, in [MT1], [MT2], [TT], [RTY], [HMT], [TY] and [M]. For Cohen-Macaulay case it is known from [MT2] [V] [TT] that $I^{(t)}$ (resp. I^t) is Cohen-Macaulay for some $t \geq 3$ (and then for all $t \geq 1$) if and only if I is the Stanley-Reisner ideal of a matroid complex (resp. a complete intersection Stanley-Reisner ideal) for a squarefree monomial ideal I.

There are some equivalent ways to define a graded ring is level, but we shall use the following definition. The ring S/I is called a level ring (for shortly, I level) if S/I is Cohen-Macaulay and the last free module in the minimal graded free resolution of S-module S/I has a basis of the same degree. The concept of a level ring was firstly introduced by R. Stanley. The level property is weaker than the Gorenstein property. A level ring of type 1 is precisely a Gorenstein ring. Level rings have attracted a lot of

2010 *Mathematics Subject Classification.* 13D02, 05E40, 05E45, 05B35.

Key words and phrases. Matroid, Stanley-Reisner ideal, levelness, symbolic powers, ordinary powers.

attention as in the work of M. Boij ([B]), T. Hibi ([H]), A. Geramita et. al. ([GHMS]), but many fundamental questions about this class of rings are still open.

In this article we shall give a complete answer of the above question for $t \geq 3$. Namely, we prove the following theorem:

Theorem 1. Let $I = I_\Delta$ be the Stanley-Reisner ideal of a simplicial complex Δ. Then, the following conditions are equivalent:

(1) I^t is level for all $t \geq 1$;
(2) I^t is level for some $t \geq 3$;
(3) $I^{(t)}$ is level for all $t \geq 1$;
(4) $I^{(t)}$ is level for some $t \geq 3$;
(5) I is a complete intersection and equi-generated.

For $t \geq 3$, the level properties of the ordinary power I^t and the symbolic one $I^{(t)}$ are equivalent, that is different from Cohen-Macaylay case.

For $t = 2$, the situation is quite complicated. Hence we consider the case that a simplicial complex Δ has dimension one. The ordinary power I_Δ^2 is level if and only if Δ is one of the following simplicial complexes: a 2-vertex segment, a 3-vertex segment, a triangle, a quadrilateral, and a pentagon. It follows from the fact that I_Δ^2 is level if and only if Δ is one of the above simplicial complexes in [MT1].

For the symbolic power case, we only give an answer when I is the Stanley-Reisner ideal of a one-dimensional matroid complex Δ. In this case, we think of the facets of Δ as the edges of a simple graph on the vertex set $[n]$. In other words, I is the Stanley-Reisner ideal of a matroid graph. Note that there are non-matroid graphs of which the second symbolic power of the Stanley-Reisner ideals are level. See the last two examples of the paper.

Theorem 2. Let I be the Stanley-Reisner ideal of a matroid graph Δ. Then, $I^{(2)}$ is level if and only if Δ is either a complete graph or a complete bipartite graph.

Now we explain the organization of the paper. In Section 2, we recall some notations and basic facts about the Stanley-Reisner ideal and matroids. Section 3 contains results for non-vanishing reduced homology groups which are used later. Section 4 is devoted to the proof of Theorem 1. After, Theorem 2 is proved in the last section.

2. Preliminaries

We will use some notation on graphs according to [D]. We refer the reader to e. g. [BH], [S],[MS] for the detailed information about combinatorial and algebraic background.

Let Δ be a simplicial complex on $[n] = \{1, \ldots, n\}$ that is a collection of subsets of $[n]$ closed under taking subsets. We put $\dim F = |F| - 1$, where $|F|$ is the cardinality of F, and $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$, which is called the dimension of Δ. It is clear that Δ can be uniquely determinate by the set of its maximal elements under inclusion, called by facets. The set of all facets of Δ is denote by $\mathfrak{F}(\Delta)$. The complex Δ is said pure if all its facets have the same cardinality.

For fixed field K, the i-th reduced simplicial (co)homology group of Δ denoted by $\widetilde{H}_i(\Delta; K)$ (w. r. t $\widetilde{H}^i(\Delta; K)$). Note that $\widetilde{H}_i(\Delta; K) = 0$ for all $i \in \mathbb{Z}$ if Δ is a cone (i.e., there exists a vertex x such that $x \in F$ for any facet F of Δ).

A matroid M on the ground set $[n]$ is a collection $\mathfrak{F}$ of subsets of $[n]$, which are called independent sets, satisfying the following conditions:

(i) $\emptyset \in \mathfrak{F}$,
(ii) If $I \in \mathfrak{F}$ and $J \subseteq I$, then $J \in \mathfrak{F}$,
(iii) If $I, J \in \mathfrak{F}$ and $|J| < |I|$, then there exists an element $x \in I \setminus J$ such that $J \cup \{x\} \in \mathfrak{F}$.

Maximal independent sets of M are called bases. They have the same cardinality called the rank of M. Denote by $\mathfrak{B}(M)$ the set of all bases of M. A dependent set is a subset of E which is not in $\mathfrak{F}$. Minimal dependent sets are called circuits of M. Denote by $\mathfrak{C}(M)$ the set of all circuits of M. It is clear that $\mathfrak{C}(M)$ determines M: $\mathfrak{F}$ consists of subsets of E that do not contain any member of $\mathfrak{C}(M)$.

It is apparent from the definition that the collection of independent sets of a matroid M forms a simplicial complex, which is called the matroid complex (or the independence complex) of M. This one is a pure simplicial complex of dimension $r(M) - 1$. For simlicity, we also use $\mathfrak{C}(\Delta)$, $\mathfrak{B}(\Delta)$ as the set of circuits and the set of bases of a matroid Δ.

We will also need the following property of a matroid due to by Stanley.

Lemma 2.1 (S, Theorem 3.4)**.** *Let Δ be a matroid complex. Then, Δ is a cone if and only if Δ is acyclic (i.e., has vanishing reduced homology).*

Suppose $V_1 \cap V_2 = \emptyset$. Let Δ_1 (respectively Δ_2) be a simplicial complex on V_1 (respectively V_2). Then, the simplicial join of Δ_1 and Δ_2, denoted by $\Delta_1 * \Delta_2$, is defined by $\{F \cup G \mid F \in \Delta_1, G \in \Delta_2\}$. It is clear that it is a simplicial complex on $V_1 \cup V_2$. The following lemma is easy to check from the definition.

Lemma 2.2. *If Δ_1, Δ_2 be two matroid complexes, which are not cones, over disjoint ground sets V_1, V_2 then so is $\Delta_1 * \Delta_2$ with the ground set $V_1 \cup V_2$.*

For a face $F \in \Delta$, we define the link and the star of F in a simplicial complex Δ to be

$$\operatorname{lk}_\Delta F = \{G \in \Delta \mid F \cup G \in \Delta, F \cap G = \emptyset\};$$

$$\operatorname{st}_\Delta F = \{G \in \Delta \mid F \cup G \in \Delta\}.$$

The next lemma appeared in [MTr, Lemma 2.3], and we would like to sketch the proof just for completeness.

Lemma 2.3. *Let Δ be a matroid complex which it is not a cone. If $\operatorname{lk}_\Delta(F) \neq \emptyset$ for some F, then it is also a matroid complex and is not a cone.*

Proof. It suffices to prove the case $F = \{x\}$ for $x \in V$. It is well-known that $\mathrm{lk}_\Delta(x)$ is a matroid. Assume the contrary, that $\mathrm{lk}_\Delta(x) \neq \emptyset$ is a cone for some $x \in V$. Let y be a center of this cone. Obviously, $y \neq x$. Since Δ is not a cone, there exists $B \in \mathfrak{F}(\Delta)$ such that $y \notin B$ (i.e. $x \notin B$). Put $F \in \mathfrak{F}(\mathrm{lk}_\Delta(x))$, then $F \cup \{x\} \in \mathfrak{F}(\Delta), x \notin F$. Therefore, $F' = (F \cup \{x\}) \setminus \{y\} \in \Delta$ and $|(F \cup \{x\}) \setminus \{y\}| < |B|$. By the definition of matroids, there exists $z \in B \setminus F'$ such that $F' \cup \{z\} \in \mathfrak{F}(\Delta)$. Thus, $(F' \cup \{z\}) \setminus \{x\} \in \mathfrak{F}(\mathrm{lk}_\Delta(x))$ and $y \notin (F' \cup \{z\}) \setminus \{x\}$, which is a contradiction. □

Let
$$\mathrm{core}([n]) = \{i \in [n] \mid \mathrm{st}_\Delta(i) \neq \Delta\},$$
and $\mathrm{core}(\Delta) = \Delta[\mathrm{core}([n])]$. It is clear that $\Delta[[n] \setminus \mathrm{core}([n])]$ is a simplex and $\{x_i \mid i \in [n] \setminus \mathrm{core}([n])\}$ forms a linear regular sequence of $S/I^{(t)}$. Therefore, $I^{(t)}$ is level if and only if $I^{(t)}_{\mathrm{core}(\Delta)})$ is level. For simplicity of exposition, throughout the rest of this paper, we always assume $\Delta = \mathrm{core}(\Delta)$, i.e. Δ is not a cone.

3. Non-vanishing reduced homology groups

Let Δ be a matroid complex of dimension $(d-1) \geq 0$. We shall give some non-vanishing reduced homology groups of certain subcomplexes of Δ, which are used later. The first result is as follows.

Theorem 3.1. *For any circuit $C \in \mathfrak{C}(\Delta)$,*
$$\widetilde{H}_{d-1}(\bigcup_{i \in C} \mathrm{st}_\Delta(C \setminus \{i\}); K) \neq 0.$$

Proof. Since $C \in \mathfrak{C}(\Delta)$, $C \setminus \{i\} \in \Delta$ for any $i \in C$, i.e. $\mathrm{st}_\Delta(C \setminus \{i\})) \neq \emptyset$. It is well known that the sub-complex $\Delta[C]$ is also matroid complex with its facet set $\{C \setminus \{i\} \mid i \in C\}$. This implies that $\Delta[C]$ is always not a cone. Fix $i \in C$, take $B \in \mathrm{lk}_\Delta(C \setminus \{i\})$. By the third condition of a matroid, $B \cup (C \setminus \{j\}) \in \Delta$ for all $j \in C$. Thus,
$$\bigcup_{i \in C} \mathrm{st}_\Delta(C \setminus \{i\}) = \Delta[C] * \mathrm{lk}_\Delta(C \setminus \{i\}).$$
Combining Lemma 2.3 and Lemma 2.2, $\bigcup_{i \in C} \mathrm{st}_\Delta(C \setminus \{i\})$ is always a matroid complex and is not a cone. Then, our assertion comes from Lemma 2.1. □

Next, we obtain the second result that:

Theorem 3.2. *Assume every circuit of Δ has the same cardinality and there exist two circuits of Δ which have at least one common vertex. Choose $C \neq C' \in \mathfrak{C}(\Delta)$ such that $|C \cap C'|$ is as large as possible. Then,*
$$\widetilde{H}_{d-1}\big(\bigcup_{U \subseteq (C \cup C'), |U|=2} \mathrm{st}_\Delta(C \cup C' \setminus U); K\big) \neq 0.$$

Proof. Let $W = C \cap C'$, $V_0 = C \setminus W$ and $V_0' = C' \setminus W$. Then, $|W| \geq 1$ and $|V_0| = |V_0'| = \alpha \geq 1$. Now, we need to prepare the following claims.

Claim 1: For any $x \in W$, there exists $W_x \subseteq W$ such that $|W_x| = \alpha$, $x \in W_x$ and

$$C_x = (V_0 \cup V_0' \cup W) \setminus W_x \in \mathfrak{C}(\Delta).$$

By a basic property of a matroid (see [O, Proposition 1.4.11]), there exists $C'' \in \mathfrak{C}(\Delta)$ such that $C'' \subseteq (C \cup C') \setminus \{x\}$. Let $U_1 = W \cap C''$, $U_2 = (C \cap C'') \setminus U_1$ and $U_3 = (C' \cap C'') \setminus U_1$. It yields that $x \in W \setminus U_1$. It is noticed that

$$\begin{aligned} |C| &= |U_1| + |U_2| + |W \setminus U_1| + |C \setminus (C' \cup C'')| \\ |C'| &= |U_1| + |U_3| + |W \setminus U_1| + |C' \setminus (C \cup C'')| \\ |C''| &= |U_1| + |U_2| + |U_3|, \end{aligned}$$

and $|C'' \cap C| = |U_1| + |U_2|$, $|C'' \cap C'| = |U_1| + |U_3|$. By choosing of C, C', $|U_2| \leq |W \setminus U_1|$ and $|U_3| \leq |W \setminus U_1|$. From this and our assumption, one can see that $C \setminus (C' \cup C'') = C' \setminus (C \cup C'') = \emptyset$ and $|U_2| = |U_3| = |W \setminus U_1|$. Put $W_x = W \setminus U_1$ and $C_x = C''$, we will obtain the result as required of this Claim.

Claim 2: For any $x, y \in W$, then either $W_x = W_y$ or $W_x \cap W_y = \emptyset$.

Assume the contrary, that $W_x \cap W_y \neq \emptyset$ and $W_x \neq W_y$ for some $x, y \in W$. As in the above Claim,

$$\begin{aligned} C_x &= (V_0 \cup V_0' \cup W) \setminus W_x \in \mathfrak{C}(\Delta), \\ C_y &= (V_0 \cup V_0' \cup W) \setminus W_y \in \mathfrak{C}(\Delta). \end{aligned}$$

Therefore, $C_x \neq C_y$ and

$$|C_x \cap C_y| = |V_0| + |V_0'| + |W| - |W_x| - |W_y| + |W_x \cap W_y| > |W|,$$

which is a contradiction with choosing C and C'.

By Claim 2, we have a partition of W by W_i for $i = 1, \ldots, s$. For simplicity, we rewrite $W_0 = V_0$ and $W_{s+1} = V_0'$. Then, $C \cup C'$ is a disjoint union of W_i for $i = 0, \ldots, s+1$. And, for all i, $|W_i| = \alpha$ and

$$(C \cup C') \setminus W_i \in \mathfrak{C}(\Delta).$$

Claim 3: For any $U = \{x, y\} \subseteq C \cup C'$, then $(C \cup C') \setminus U \in \Delta$ if and only if x, y belong to two different subsets W_i for some $i = 0, \ldots, s+1$.

It is clear that if $x, y \in W_i$ for some $i = 0, \ldots, s+1$ then $(C \cup C') \setminus U \notin \Delta$ by $(C \cup C') \setminus W_i \in \mathfrak{C}(\Delta)$. Assume $x \in W_a$, $y \in W_b$ for some $0 \leq a \neq b \leq s+1$ and $(C \cup C') \setminus U \notin \Delta$. Therefore, there exists a circuit C'' of M such that $C'' \subseteq (C \cup C') \setminus U$. Let $\alpha_i = |W_i \setminus C''| \geq 0$ for all i. It is noted that $\alpha_a \geq 1$ and $\alpha_b \geq 1$. Then,

$$\sum_{i=0}^{s+1} \alpha_i = \alpha,$$

by C'' has the same cardinality with C, i.e. $|C''| = (s+1)\alpha$. Thus, $((C \cup C') \setminus W_a) \neq C''$, and we have

$$\begin{aligned} |((C \cup C') \setminus W_a) \cap C''| &= \sum_{i \neq a} |W_i \cap C''| \\ &= \sum_{i \neq a} (\alpha - \alpha_i) \\ &= (s+1)\alpha - \sum_{i \neq a} \alpha_i = s\alpha + \alpha_a > s\alpha = |C \cap C'|, \end{aligned}$$

a contradiction.

We now return to prove our statement. Using Claim 3,

$$\bigcup_{U \subseteq (C \cup C'), |U|=2} \mathrm{st}_\Delta(C \cup C' \setminus U) = \bigcup_{x \in W_a, y \in W_b, a \neq b} \mathrm{st}_\Delta(C \cup C' \setminus \{x, y\}).$$

Also by this Claim, $\Delta[C \cup C']$ is a matroid complex with the facet set which consists of $C \cup C' \setminus \{x, y\}$ for x, y belong to two different subsets W_i. It implies that this complex is always neither emptyset nor a cone. Fix $x \in W_0$ and $y \in W_1$. Take any $B \in \mathrm{lk}_\Delta(C \cup C' \setminus \{x, y\})$. Then, by the third condition of a matroid, $B \in \mathrm{lk}_\Delta(C \cup C' \setminus \{x', y'\})$ for any x', y' belong to two different subsets W_i for some $i = 0, \ldots, s+1$. From this,

$$\bigcup_{U \subseteq (C \cup C'), |U|=2} \mathrm{st}_\Delta(C \cup C' \setminus U) = \Delta[C \cup C'] * \mathrm{lk}_\Delta(C \cup C' \setminus \{x, y\}).$$

Then, our statement comes from combining Lemmas 2.1, 2.2 and 2.3. □

4. Large symbolic powers

First, we need to recall a formula for computing the multigraded Betti numbers of a monomial ideal due to by Miller and Sturmfels throughout the (non)-vanishing of reduced homology groups of certain simplicial complexes. Let $\mathbf{e}_i$ be the i^{th}-unit vector for $i = 1, \ldots, n$. For each vector $\mathbf{a} \in \mathbb{N}^n$, define $\mathbf{e}_{\mathrm{supp}(\mathbf{a})} = \sum_{i \in \mathrm{supp}(\mathbf{a})} \mathbf{e}_i$, where $\mathrm{supp}(\mathbf{a}) = \{i \mid a_i \neq 0\}$. Given a monomial ideal J and a degree $\mathbf{a} \in \mathbb{N}^n$, the lower Koszul simplicial complex of S/J in degree $\mathbf{a}$ is

$$K_{\mathbf{a}}(J) = \{F \subseteq \mathrm{supp}(\mathbf{a}) \mid \mathbf{x}^{\mathbf{a} - \mathbf{e}_{\mathrm{supp}(\mathbf{a})}}.\mathbf{x}^F \notin J\},$$

where $\mathbf{x}^F = \prod_{i \in F} x_i$ and $\mathbf{x}^{\mathbf{a}} = \prod_{i \in \mathrm{supp}(\mathbf{a})} x_i^{a_i}$.

Theorem 4.1 (MS, Theorem 5.11). *Given a vector $\mathbf{a} \in \mathbb{N}^n$ with support $\mathrm{supp}(\mathbf{a})$ and a monomial ideal J in S, the Betti numbers of S/J in degree $\mathbf{a}$ can be expressed as*

$$\beta_{i,\mathbf{a}}(S/J) = \dim_K(\widetilde{H}^{|\mathrm{supp}(\mathbf{a})|-i-1}(K_{\mathbf{a}}(J); K)) = \dim_K(\widetilde{H}_{|\mathrm{supp}(\mathbf{a})|-i-1}(K_{\mathbf{a}}(J); K)),$$

for all i.

From the level property of a symbolic power for $t \geq 2$, we always obtain the condition that the original ideal is equi-generated as follows.

Theorem 4.2. *Let Δ be the matroid complex of dimension $(d-1) \geq 0$ and I be the Stanley-Reisner ideal of Δ. If $S/I^{(t)}$ is level for some $t \geq 2$, then I is equi-generated, i.e. every circuit of Δ has the same cardinality.*

Proof. For each circuit $C \in \mathfrak{C}(\Delta)$, let $\mathbf{a}_C = \sum_{i \in C} t\mathbf{e}_i + \sum_{i \notin C} \mathbf{e}_i$. Then,

$$K_{\mathbf{a}_C}(I^{(t)}) = \{F \subseteq [n] \mid f_C.\mathbf{x}^F \notin I^{(t)}\},$$

where $f_C = \prod_{i \in C} x_i^{t-1}$. For each $B \in \mathfrak{B}(\Delta)$, one can see that $|C \setminus B| \geq 1$. This implies that $f_C.\mathbf{x}^F \notin I^{(t)}$ if and only if $F \subseteq B$ for some $B \in \mathfrak{B}(\Delta)$ such that $|C \setminus B| = 1$. Therefore,

$$K_{\mathbf{a}_C}(I^{(t)}) = \bigcup_{i \in C} \mathrm{st}_\Delta(C \setminus \{i\}).$$

Using Theorem 4.1 and Theorem 3.1,

$$\beta_{n-d,\mathbf{a}_C}(S/I^{(t)}) = \dim_K(\widetilde{H}_{d-1}(\bigcup_{i \in C} \mathrm{st}_\Delta(C \setminus \{i\}); K)) \neq 0.$$

This yields $\beta_{n-d,(t-1)|C|+n}(S/I^{(t)}) \neq 0$ for each $C \in \mathfrak{C}(\Delta)$. By our assumption, every circuit of Δ has the same cardinality as required. □

We are now in a position to prove the first main result of this paper.

Theorem 4.3. *Let Δ be a simplicial complex of dimension $d-1 \geq 0$ and I be the Stanley-Reisner ideal of Δ. Then, the following conditions are equivalent:*

1. *S/I^t is level for all $t \geq 1$,*
2. *S/I^t is level for some $t \geq 3$,*
3. *$S/I^{(t)}$ is level for all $t \geq 1$,*
4. *$S/I^{(t)}$ is level for some $t \geq 3$,*
5. *I is equi-generated and a complete intersection.*

Proof. The implications (1) ⇒ (2) and (3) ⇒ (4) are clear. Note that for some $t \geq 1$ S/I^t is Cohen-Macaulay if and only if $S/I^{(t)}$ is Cohen-Macaulay and $I^t = I^{(t)}$. Hence S/I^t is level if and only if $S/I^{(t)}$ is level and $I^t = I^{(t)}$. Then the implications (1) ⇒ (3) and (2) ⇒ (4) are clear.

We consider the implication (5) ⇒ (1). The t-th power of the graded maximal ideal has a t-linear resolution. See, e.g., [BH, Exercises 4.1.17]. Hence if I is equi-generated and a complete intersection, then I^t has a pure resolution, since each pair of generators of I is coprime and has the same degree. Since S/I^t is Cohen-Macaulay, it is level.

Now it is enough to prove that (4) implies (5). By Theorem 4.2, we only need to show that two different circuits of Δ must be disjoint. Assume the contrary, that there exist two circuits of Δ which have at least a common vertex. Choose

$C \neq C' \in \mathfrak{C}(\Delta)$ such that cardinality of $\emptyset \neq W = C \cap C'$ is as large as possible. Let $\mathbf{a}_{(C,C')} = \sum_{i\in C}(t-1)\mathbf{e}_i + 2\sum_{i\in C'\setminus C}\mathbf{e}_i + \sum_{i\notin C\cup C'}\mathbf{e}_i$. Then,

$$K_{\mathbf{a}_{(C,C')}}(I^{(t)}) = \{F \subseteq [n] \mid f_{(C,C')}.\mathbf{x}^F \notin I^{(t)}\},$$

where $f_{(C,C')} = \prod_{i\in C} x_i^{t-2}\prod_{i\in C'\setminus C} x_i$. For each $B \in \mathfrak{B}(\Delta)$, one can see that $|C \setminus B| \geq 1$ and $|C' \setminus B| \geq 1$.

If $|(C \cup C') \setminus B| = 1$, assume $x \in (C \cup C') \setminus B$, then x must belong to W and $(C \cup C') \setminus \{x\} \subseteq B$. Since Claim 1 in the Theorem 3.2, there exists $x \in W_x \subseteq W$ such that $C_x = (V_0 \cup V_0' \cup W) \setminus W_x \in \mathfrak{C}(\Delta)$, which is a contradiction by $C_x \subsetneq B \in \Delta$.

If $|(C \cup C') \setminus B| \geq 3$, then $f_{(C,C')} \in P_B^t$ by $t \geq 3$. Therefore, $f_{(C,C')}.\mathbf{x}^F \notin I^{(t)}$ if and only if $F \subseteq B$ for some $B \in \mathfrak{B}(\Delta)$ such that either $|(C \cup C') \setminus B| = 2$ if $t = 3$ or $(C \cup C') \setminus B = \{x, y\}$ for $x \in C, y \in C' \setminus C$ if $t \geq 4$.

We consider two cases as follows.

Case 1: $t = 3$. Then, as in the above,

$$K_{\mathbf{a}_{(C,C')}}(I^{(t)}) = \bigcup_{U\subseteq(C\cup C'),|U|=2} \mathrm{st}_\Delta(C \cup C' \setminus U).$$

Using Theorem 3.2, $\widetilde{H}_{d-1}(K_{\mathbf{a}_{(C,C')}}(I^{(t)}); K) \neq 0$.

Case 2: $t \geq 4$. We can see that

$$K_{\mathbf{a}_{(C,C')}}(I^{(t)}) = \bigcup_{x\in C,y\in(C'\setminus C)} \mathrm{st}_\Delta(C \cup C' \setminus \{x, y\}).$$

Similarly as in the proof of Theorem 3.2, fixed $x \in C, y \in C' \setminus C$, one can check that

$$\bigcup_{x\in C,y\in(C'\setminus C)} \mathrm{st}_\Delta(C \cup C' \setminus \{x, y\}) = \Delta[C] * \Gamma * \mathrm{lk}_\Delta(C \cup C' \setminus \{x, y\})$$

where Γ is the matroid complex which consists of all subsets $(C' \setminus C) \setminus \{z\}$ for $z \in C' \setminus C$. Using again Lemma 2.1, Lemma 2.3 and Lemma 2.2, $\widetilde{H}_{d-1}(K_{\mathbf{a}_{(C,C')}}(I^{(t)}); K) \neq 0$.

From both of cases and Theorem 4.1, one can see that $\beta_{n-d,(t-1)|C|+n-|W|}(S/I^{(t)}) \neq 0$. Combining it and Theorem 4.2, we will obtain a contradiction with the levelness of $S/I^{(t)}$. □

It can be noted that there is a Stanley-Reisner ideal I such that $S/I^{(2)}$ is level but S/I^2 is not (see the last example of next section). So, $t = 3$ is the best value for this theorem.

Corollary 4.4. *Let Δ be a simplicial complex and I be the Stanley-Reisner ideal of Δ. Then, the following conditions are equivalent:*

(1) *S/I^t is Gorenstein for all $t \geq 1$,*
(2) *S/I^t is Gorenstein for some $t \geq 3$,*
(3) *$S/I^{(t)}$ is Gorenstein for all $t \geq 1$,*
(4) *$S/I^{(t)}$ is Gorenstein for some $t \geq 3$,*

(5) *I is a principal ideal.*

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (4), (1) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (1) are clear. Hence it is enough to prove that (4) implies (5). Assume the condition (4). By Theorem 4.3, I is equi-generated and a complete intersection. Suppose I is not principal. Suppose I is minimally generated by p monomials for $p \geq 2$. Set $J = (x_1, x_2, \dots, x_p)$. Then for $t \geq 3$, J^t is not Gorenstein, since the coefficient of the highest degree of the numerator of Hilbert series of S/J^t is $\binom{p+t-2}{t-1} \neq 1$. Hence I^t is not Gorenstein, which is a constradiction with the condition (4). $\square$

5. The second symbolic power

In this section we only consider the second symbolic power of Stanley-Reisner ideal of a one-dimensional matroid complex. For simplicity of exposition, in this section, we assume that Δ is a matroid complex of dimension one. Then, $S/I_\Delta^{(2)}$ is Cohen-Macaulay of dimension two. It is clear that Δ can be viewed as a simple graph on $[n]$ for $n \geq 2$. It can be noted that if $n = 2, 3$ then Δ is a complete graph and I_Δ is a principal ideal, so $I_\Delta^{(2)}$ is always level. So, we may assume that $n \geq 4$.

For the proof of the main theorem, some more preparations are needed.

Lemma 5.1. *If Δ does not contain any triangles then Δ is a complete bipartite graph.*

Proof. By the connectedness of Δ, one may assume that $12, 13 \in \Delta$. Let

$$X = \{i \in [n] \mid i \neq 2, 2i \in \Delta\}$$

and

$$Y = \{j \in [n] \mid j \neq 1, \text{ there exists a vertex } i_j \in X \text{ such that } ji_j \in \Delta\}.$$

It is clear that $1 \in X$ and both of $2, 3$ are in Y. Firstly, for all $a \neq b \in X$, then $ab \notin \Delta$ by the triangle-free property of Δ. Take $a \neq b \in Y$, then there exist $i_a, i_b \in X$ such that $ai_a, bi_b \in \Delta$. If $i_a = i_b$ then $ab \notin \Delta$ as above. If $i_a \neq i_b$ then $i_a i_b \notin \Delta$. Therefore, $i_a b \in \Delta$ by the matroid condition. Thus, $ab \notin \Delta$. Secondly, take any vertex $u \in [n] \setminus \{1, 2, 3\}$, one may see that either $1u$ or $2u$ is in Δ by the matroid property. Therefore, $X \cup Y = [n]$ and it can check that $X \cap Y = \emptyset$. Take any $u \in X, v \in Y$. If $v = 2$ then $uv \in \Delta$. If $v \neq 2$ then there exists $i(v) \in X$ such that $vi_v \in \Delta$. If $i_v = u$ then $uv \in \Delta$, otherwise $i_v \neq u$ then $uv \in \Delta$ by its matroid property. Thus, uv always belongs to Δ which implies that Δ is the complete bipartite graph over X and Y as required.

$\square$

Proposition 5.2. *If Δ be a complete graph then $I_\Delta^{(2)}$ is level.*

Proof. Let $\mathbf{a} = 2(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + \sum_{i=4}^n \mathbf{e}_i$. Then, $\operatorname{supp}(\mathbf{a}) = [n]$ and by definition,

$$K_{\mathbf{a}}(I_\Delta^{(t)}) = \{F \subseteq [n] \mid x_1 x_2 x_3.\mathbf{x}^F \notin I_\Delta^{(2)}.\}$$

Note that, if $|F \setminus \{1,2,3\}| \geq 1$ then $x_1x_2x_3.\mathbf{x}^F \in I_\Delta^{(2)}$. If $F \subseteq \{1,2,3\}$ then one can see that the facets of $K_{\mathbf{a}}(I_\Delta^{(t)})$ are $12, 23, 31$. Therefore, by Theorem 4.1, $\beta_{n-2,\mathbf{a}}(S/I_\Delta^{(t)}) = \dim(\widetilde{H}_1(K_{\mathbf{a}}(I_\Delta^{(t)}); K)) = \dim(\widetilde{H}_1(\mathbb{S}^1; K)) \neq 0$. It is enough to show that $\widetilde{H}_{|\operatorname{supp}(\mathbf{b})|-n+1}(K_{\mathbf{b}}(I_\Delta^{(2)}); K) = 0$ for all $\mathbf{b} \in \mathbb{N}^n$ and $|\mathbf{b}| \neq n+3$. Fix a vector $\mathbf{b} \in \mathbb{N}^n$ with $|\mathbf{b}| \neq n+3$, let $W = \operatorname{supp}(\mathbf{b}), \mathbf{u} = \mathbf{b} - \mathbf{e}_{\operatorname{supp}(\mathbf{b})}$. Let

$$\Delta_{\mathbf{u}} = \{F \subseteq [n] \mid \mathbf{x}^{\mathbf{u}}.\mathbf{x}^F \notin I_\Delta^{(2)}\},$$

then $K_{\mathbf{b}}(I_\Delta^{(2)}) = \Delta_{\mathbf{u}}[W]$. It is clear that $\operatorname{supp}(\mathbf{u}) \subseteq W$. We distinguish some types of $\Delta_{\mathbf{u}}$.

Type 1: $|\operatorname{supp}(\mathbf{u})| \geq 4$. It is clear that $\mathbf{x}^{\mathbf{u}} \in I_\Delta^{(2)}$. Therefore, $\Delta_{\mathbf{u}} = \emptyset$.

Type 2: $|\operatorname{supp}(\mathbf{u})| = 3$. Write $1, 2, 3 \in \operatorname{supp}(\mathbf{u})$.

(i) If $u_1 = u_2 = u_3 = 1$ then the facets of $\Delta_{\mathbf{u}}$ are $12, 13, 23$;
(ii) If $u_1 \geq 2, u_2 = u_3 = 1$ then the facets of $\Delta_{\mathbf{u}}$ are $12, 13$;
(iii) If $u_1 \geq 2, u_2 \geq 2, u_3 = 1$ then the facets of $\Delta_{\mathbf{u}}$ are 12;
(iv) If $u_1 \geq 2, u_2 \geq 2, u_3 \geq 2$ then $\Delta_{\mathbf{u}} = \emptyset$ by $x_1^2x_2^2x_3^2 \in I_\Delta^{(2)}$.

Type 3: $|\operatorname{supp}(\mathbf{u})| = 2$. Write $1, 2 \in \operatorname{supp}(\mathbf{u})$. If $|F \setminus \{1,2\}| \geq 2$ then $\mathbf{x}^{\mathbf{u}}.\mathbf{x}^F \in I_\Delta^{(2)}$. Note that $\mathbf{x}^{\mathbf{u}}.x_i \notin P_{1,2}^2$ for all i. Therefore, the facets of $\Delta_{\mathbf{u}}$ are $\{12i \mid i = 3, \ldots, n\}$.

Type 4: $|\operatorname{supp}(\mathbf{u})| = 1$. Write $1 \in \operatorname{supp}(\mathbf{u})$. If $|F \setminus \{1\}| \geq 3$ then $\mathbf{x}^{\mathbf{u}}.\mathbf{x}^F \in I_\Delta^{(2)}$. From $\mathbf{x}^{\mathbf{u}}.x_ix_j \notin P_{1,i}^2$ for all $i \neq j$, the facets of $\Delta_{\mathbf{u}}$ are $\{1ij \mid 2 \leq i < j \leq n\}$.

Type 5: $|\operatorname{supp}(\mathbf{u})| = 0$. One can see that the facets of $\Delta_{\mathbf{u}}$ are $\{ijh \mid 1 \leq i < j < h \leq n\}$.

From these types and $\operatorname{supp}(\mathbf{u}) \subseteq W$, we always obtain $\widetilde{H}_{|W|-n+1}(\Delta_{\mathbf{u}}[W]; K)) = 0$ except the case type 2 (i) occurs and $|W| = n$, i.e. $|\mathbf{b}| = n+3$. From this, we obtain as required. □

Proposition 5.3. *If Δ is a complete bipartite graph then $I_\Delta^{(2)}$ is level.*

Proof. Assume that Δ is a complete bipartite graph $K_{|X|,|Y|}$ for $X \cup Y = [n], X \cap Y = \emptyset, X, Y \neq \emptyset$. Fix a vector $\mathbf{b} \in \mathbb{N}^n$, let $W = \operatorname{supp}(\mathbf{b}), \mathbf{u} = \mathbf{b} - \mathbf{e}_{\operatorname{supp}(\mathbf{b})}$. Let

$$\Delta_{\mathbf{u}} = \{F \subseteq [n] \mid \mathbf{x}^{\mathbf{u}}.\mathbf{x}^F \notin I_\Delta^{(2)}\},$$

then $K_{\mathbf{b}}(I_\Delta^{(2)}) = \Delta_{\mathbf{u}}[W]$. Similarly as in the above proof, we have some types of $\Delta_{\mathbf{u}}$.

Type 1: $|\operatorname{supp}(\mathbf{u})| \geq 4$. It is clear that $\mathbf{x}^{\mathbf{u}} \in I_\Delta^{(2)}$. Therefore, $\Delta_{\mathbf{u}} = \emptyset$.

Type 2: $|\operatorname{supp}(\mathbf{u})| = 3$. Write $1, 2, 3 \in \operatorname{supp}(\mathbf{u})$.

(i) If $1, 2, 3 \in X$ or $1, 2, 3 \in Y$ then $\Delta_{\mathbf{u}} = \emptyset$ by $x_1x_2x_3 \in I_\Delta^{(2)}$;
(ii) If $1, 2 \in X$ and $3 \in Y$ then the facets of $\Delta_{\mathbf{u}}$ are $23, 13$ if $u_1 = u_2 = u_3 = 1$, or 13 if $u_1 \geq 2, u_2 = u_3 = 1$, or 23 if $u_1 = 1, u_2 \geq 2, u_3 = 1$, or $\emptyset$ otherwise.

Type 3: $|\operatorname{supp}(\mathbf{u})| = 2$. Write $1, 2 \in \operatorname{supp}(\mathbf{u})$.

(i) If $1, 2 \in X$ or $1, 2 \in Y$ then $\Delta_{\mathbf{u}}$ is $\operatorname{st}_\Delta(1) \cup \operatorname{st}_\Delta(2)$ if $u_1 = u_2 = 1$, or $\operatorname{st}_\Delta(1)$ if $u_1 \geq 2, u_2 = 1$, or $\operatorname{st}_\Delta(2)$ if $u_1 = 1, u_2 \geq 2$, or $\emptyset$ otherwise.

(ii) If $1 \in X$ and $2 \in Y$ then the facets of $\Delta_{\mathbf{u}}$ are $\{12i \mid i = 3, \ldots, n\}$ if $u_1 = u_2 = 1$, or $\{1i \mid i = 3, \ldots, n\}$ if $u_1 \geq 2, u_2 = 1$, or $\{2i \mid i = 3, \ldots, n\}$ if $u_1 = 1, u_2 \geq 2$, or $\emptyset$ otherwise.

Type 4: $|\operatorname{supp}(\mathbf{u})| = 1$. Write $1 \in \operatorname{supp}(\mathbf{u})$. Assume $1 \in X$, then the facets of $\Delta_{\mathbf{u}}$ are $\{1ij \mid i \in Y \text{ or } j \in Y\}$.

Type 5: $|\operatorname{supp}(\mathbf{u})| = 0$. One can see that the facets of $\Delta_{\mathbf{u}}$ are

$$\{ijh \mid \text{ except in the case of } i,j,h \in X \text{ or in the case of } i,j,h \in Y\}.$$

One can see that $\widetilde{H}_{|W|-n+1}(\Delta_{\mathbf{u}}[W]; K)) = 0$ if form of $\Delta_{\mathbf{u}}$ likes as type 1, type 2, type 3 (ii) and type 4 by $\operatorname{supp}(\mathbf{u}) \subseteq W$ and the acyclic property of a cone. We distinguish some cases as follows.

Case 1: $|X| = 1$ or $|Y| = 1$. Assume $|X| = 1$ and $t \in X$. Therefore, if $\Delta_{\mathbf{u}}$ has form as type 3 (i) $\widetilde{H}_{|W|-n+1}(\Delta_{\mathbf{u}}[W]; K)) \neq 0$ when $W = [n] \setminus \{t\}$ and $u_1 = u_2 = 1$ for $1, 2 \in Y$. In this case, $\Delta_{\mathbf{u}}[W]$ consists of two points $1, 2$. One can see that $\widetilde{H}_{|W|-n+1}(\Delta_{\mathbf{u}}[W]; K)) = 0$ if $\Delta_{\mathbf{u}}$ has form as type 5 because it is a cone over t.

Case 2: $|X| = 2$ and $|Y| = 2$. Then, I_Δ is a complete intersection which implies the level property of $I_\Delta^{(2)}$.

Case 3: $|X| \geq 2$ and $|Y| \geq 3$ or $|X| \geq 3$ and $|Y| \geq 2$. Assume $|X| \geq 2$ and $|Y| \geq 3$. If $\Delta_{\mathbf{u}}$ has form as type 3 (i) then $\widetilde{H}^{|W|-n+1}(\Delta_{\mathbf{u}}[W]; K)) \neq 0$ when $\mathbf{b}$ has a form $2(\mathbf{e}_1 + \mathbf{e}_2) + \sum_{i \geq 3} \mathbf{e}_n$ (for $1, 2 \in X$ or $1, 2 \in Y$). In this case $W = [n], \mathbf{u} = \mathbf{e}_1 + \mathbf{e}_2$ and the reduced cohomology groups are not vanishing by there exists a "empty" circle in $\Delta_{\mathbf{u}}[W]$.

In fact, if $|W| = n-2$ then $\Delta_{\mathbf{u}}[W] \neq \{\emptyset\}$ by it contains some points; if $|W| = n-1$ then $\Delta_{\mathbf{u}}[W]$ is always connected; if $|W| = n$ and either $u_1 \geq 2$ or $u_2 \geq 2$ then $\widetilde{H}_1(\Delta_{\mathbf{u}}[W]; K)) = 0$. If $\Delta_{\mathbf{u}}$ has form as type 5, then $\Delta_{\mathbf{u}}[W] \neq \{\emptyset\}$ if $|W| = n-2$ and $\Delta_{\mathbf{u}}[W]$ is connected if $|W| = n-1$. When $|W| = n$, by induction on $|X| \geq 1$ and the Mayer-Vietoris sequence, one can check that $\widetilde{H}_1(\Delta_{\mathbf{u}}; K) = 0$.

From these cases, $\beta_{n-2}((S/I_\Delta^{(2)}))$ only concentrated at degree $n+2$, which implies the conclusion as required. □

Proposition 5.4. *If Δ is neither a complete graph nor a complete bipartite graph then $I_\Delta^{(2)}$ is not level.*

Proof. By Lemma 5.1, Δ must contain at least a triangle, say $12, 23, 31 \in \Delta$. Put $\mathbf{a} = 2(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + \sum_{i=4}^n \mathbf{e}_i$. Arguing as in the proof of Proposition 5.2, $\beta_{n-2,\mathbf{a}}(S/I_\Delta^{(2)}) \neq 0$. Because Δ is not a complete graph, we assume $14 \notin \Delta$. From the matroid property of Δ, $24, 34 \in \Delta$. Let $\mathbf{b} = 2(\mathbf{e}_1 + \mathbf{e}_4) + \mathbf{e}_2 + \mathbf{e}_3 + \sum_{i>4}^n \mathbf{e}_i$ then $\operatorname{supp}(\mathbf{b}) = [n]$ and $|b| = n+2$. Then,

$$K_{\mathbf{b}}(I_\Delta^{(2)}) = \{F \subseteq [n] \mid x_1 x_4 . \mathbf{x}^F \notin I_\Delta^{(2)}\} = \operatorname{st}_\Delta(1) \cup \operatorname{st}_\Delta(4).$$

We can rewrite $\operatorname{st}_\Delta(1) \cup \operatorname{st}_\Delta(4) = \Delta_1 \cup \Delta_2$, where the facets of Δ_1 are $12, 13, 24, 34$ and the facets of Δ_2 are the other facets of $\operatorname{st}_\Delta(1) \cup \operatorname{st}_\Delta(4)$. Therefore, $\dim(\Delta_1 \cap \Delta_2) \leq 0$.

Then, $\widetilde{H}_1(\Delta_1 \cap \Delta_2; K) = 0$. And, it is clear that $\widetilde{H}_1(\Delta_1; K) \neq 0$. By using the Mayer-Vietoris sequence, $\cdots \to \widetilde{H}_1(\Delta_1 \cap \Delta_2; K) \to \widetilde{H}_1(\Delta_1; K) \oplus \widetilde{H}_1(\Delta_2; K) \to \widetilde{H}_1(\Delta_1 \cup \Delta_2; K) \to \widetilde{H}_0(\Delta_1 \cap \Delta_2; K) \to \cdots$, we have $\widetilde{H}_1(\Delta_1 \cup \Delta_2; K) \neq 0$. Thus, by Theorem 4.1,

$$\beta_{n-2,\mathbf{b}}(S/I_\Delta^{(2)}) = \dim_K(\widetilde{H}_1(K_{\mathbf{b}}(I_\Delta^{(2)}); K)) \neq 0.$$

This proves our assertion. □

Combining Proposition 5.2, Proposition 5.3 and Proposition5.4 yields the result as follows.

Theorem 5.5. *Let Δ be a matroid graph over $[n]$ for $n \geq 2$. Then, $I_\Delta^{(2)}$ is level if and only if Δ is either a complete graph or a complete bipartite graph.*

In the end of this section, we shall give two examples of non-matroid graphs of which the second symbolic power of the Stanley-Reisner ideals are level. These examples are inspired by computations of the computer algebra system as CoCoA [Co]. For the second example, it can be noted that the second ordinary power of its Stanley-Reisner ideal is not Cohen-Macaulay by [MT1, Corollary 3.4], so it is not also level.

Example 5.6. (1) *Let $n = 5$ and Δ be a pentagon such that its facet set is $\{12, 23, 34, 45, 15\}$. Then, $I_\Delta^{(2)}$ is level. This induced from the minimal graded resolution of $S/I_\Delta^{(2)}$ as follows:*

$$0 \to S(-6)^{10} \longrightarrow S(-5)^{24} \longrightarrow S(-4)^{15} \longrightarrow S \to 0.$$

(2) *Let $n = 10$ and Δ be the Petersen graph such that its facet set is*

$$\{12, 23, 34, 45, 15, 16, 27, 38, 49, 510, 68, 69, 79, 710, 810\}.$$

Then, $I_\Delta^{(2)}$ is level but I_Δ^2 is not level. In fact that, $S/I_\Delta^{(2)}$ has a minimal graded resolution that

$$0 \to S(-11)^{90} \longrightarrow S(-10)^{684} \longrightarrow S(-9)^{2240} \longrightarrow S(-8)^{4095} \longrightarrow S(-6)^5 \oplus S(-7)^{4500}$$
$$\longrightarrow S(-5)^{60} \oplus S(-6)^{2945} \longrightarrow S(-4)^{75} \oplus S(-5)^{1068} \longrightarrow S(-3)^{30} \oplus S(-4)^{165} \longrightarrow S \to 0.$$

Acknowledgment. A part of this paper was done while the first author was visiting Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the VIASM for hospitality and financial support.

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