BUBBLING OF THE PRESCRIBED Q-CURVATURE EQUATION ON 4-MANIFOLDS IN THE NULL CASE

QUỐC ANH NGÔ AND HONG ZHANG

ABSTRACT. Analog to the classical result of Kazdan–Warner for the existence of solutions to the prescribed Gaussian curvature equation on compact 2-manifolds without boundary, it is widely known that if (M, g_0) is a closed 4-manifold with zero Q-curvature and if f is any non-constant, smooth, sign-changing function with $\int_M f \, \mathrm{d}\mu_{g_0} < 0$, then there exists at least one solution u to the prescribed Q-curvature equation

$$\mathbf{P}_{g_0} u = f e^{4u},$$

where $\mathbf{P}_{g_0}$ is the Paneitz operator which is positive with kernel consisting of constant functions. In this paper, we fix a non-constant smooth function f_0 with

$$\max_{x \in M} f_0(x) = 0, \quad \int_M f_0 \, \mathrm{d}\mu_{g_0} < 0$$

and consider a family of prescribed Q-curvature equations

$$\mathbf{P}_{g_0} u = (f_0 + \lambda) e^{4u},$$

where $\lambda > 0$ is a suitably small constant. A solution to the equation above can be obtained from a minimizer u_λ of certain energy functional associated to the equation. Firstly, we prove that the minimizer u_λ exhibits bubbling phenomenon in a certain limit regime as $\lambda \searrow 0$. Then, we show that the analogous phenomenon occurs in the context of Q-curvature flow.

1. INTRODUCTION

The problem of describing the set of curvatures that a given manifold can possess is of importance in Riemannian geometry over the last 50 years starting from a seminal paper in 1960, or even before, due to Yamabe [Yam60] for the existence of conformal metrics of constant scalar curvature on closed manifolds of dimension $n \geqslant 3$. Without limiting to the case of constant scalar curvature, this problem is known as the prescribed scalar curvature problem and has been a main research topic in conformal geometry in recent decades. An analogue problem for manifolds of dimension 2, known as the prescribed Gaussian curvature problem, can be formulated in a similar way.

1.1. **The Kazdan–Warner result for the scalar curvature equation.**

Let (M, g_0) be a compact surface without boundary. Given a smooth function f on M, the prescribed Gaussian curvature problem asks if there exists a conformal metric g such that the Gaussian curvature of g is equal to f. By writing $g = e^{2u} g_0$, the Gaussian curvature of the metric g, denoted by K_g, satisfies the transformation law

$$K_g = e^{-2u}(-\Delta_{g_0} u + K_{g_0}).$$

This enables us to reduce the prescribed Gaussian curvature problem to the problem of solving the semilinear PDE

$$-\Delta_{g_0} u + K_{g_0} = f e^{2u} \tag{1.1}$$

Date: **29th March, 2019 at 00:38.**

2010 *Mathematics Subject Classification.* Primary 53C44; Secondary 35J60.

Key words and phrases. Q-curvature flow, prescribed Q-curvature, 4-manifolds, blow-up behaviour.

Since Eq.(1.1) is conformally covariant, we obtain that if v solves

$$-\Delta_{g_1} v + K_{g_1} = f e^{2v}$$

for some $g_1 = e^{2w} g_0$, then $u = v + w$ solves (1.1). This together with the uniformization theorem implies that we can freely choose the background metric g_0 in such a way that K_{g_0} is a constant whose sign depends on the Euler characteristic of M. In the case that M has genus one, namely, M is the torus, Eq. (1.1) becomes

$$-\Delta_{g_0} u = f e^{2u} \tag{1.2}$$

on M. In [KW74], Kazdan and Warner proved the following result:

Theorem 1.1 (see Kazdan–Warner [KW74]). *There is a solution u to (1.2) if, and only if, either $f \equiv 0$, or if the function f changes sign and satisfies*

$$\int_M f \, \mathrm{d}\mu_{g_0} < 0. \tag{1.3}$$

A solution u to (1.2) can be obtained by minimizing the Liouville energy

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 \, \mathrm{d}\mu_{g_0}$$

in the class

$$C_f = \Big\{ u \in H^1(M, g_0) : \int_M f e^{2u} \, \mathrm{d}\mu_{g_0} = 0 \Big\}.$$

We note that the constraint $\int_M f e^{2u} \, \mathrm{d}\mu_{g_0} = 0$ in the class C_f is quite natural in view of the Gauss–Bonnet theorem. Since the energy E and the constraint in C_f is left unchanged up to a constant addition, in order to show existence of a minimizer for E in the class C_f, one often restricts attention to those functions with vanishing mean. To be precise, we look for minimizer of E within the set

$$C'_f = \Big\{ u \in H^1(M, g_0) : \int_M f e^{2u} \, \mathrm{d}\mu_{g_0} = 0, \ \int_M u \, \mathrm{d}\mu_{g_0} = 0 \Big\}.$$

However, normalizing the volume will work equally well, that is, we can also look for minimizer of E within the set

$$C^*_f = \Big\{ u \in H^1(M, g_0) : \int_M f e^{2u} \, \mathrm{d}\mu_{g_0} = 0, \ \int_M e^{2u} \, \mathrm{d}\mu_{g_0} = \mathrm{vol}(M, g_0) \Big\}.$$

In [Gal15], Galimberti showed "bubbling" of the Kazdan–Warner metrics in a certain limit regime. To describe his result, we let f_0 be a non-constant and smooth function with $\max_M f_0 = 0$. Let $\lambda > 0$ be small such that $f_\lambda = f_0 + \lambda$ changes sign and satisfies (1.3). Therefore, by Theorem 1.1 there exists a solution $\widehat{u}_\lambda$ to (1.2), which can be obtained from a minizer u_λ of E in the set $C^*_{f_\lambda}$ with f replaced by f_λ. In fact, one can easily see that $\widehat{u}_\lambda$ and u_λ differ by a positive constant c_λ. With a delicate argument, he is able to control the total curvature of the conformal metrics $\widehat{g}_\lambda = e^{2\widehat{u}_\lambda} g_0$ for suibtable $\lambda \searrow 0$ and hence to show that after rescaling the metrics suitably near local maximum points of f, one or more "bubbles" may be extracted from $\widehat{g}_\lambda$; see [Gal15, Theorem 1.1].

Recently, Struwe [Str17] improves the result in [Gal15] by obtaining a more precise characterization of the bubbling. He shows that "slow blow-up" does not occur; see [Str17, Theorem 1.2]. This is achieved with the help of a new Liouville-type result; see [Str17, Theorem 1.3]. It is remarkable that the method developed in [Str17] is flexible enough to apply also in the presence of perturbation leading to a similar "bubbling" phenomenon for a family of prescribed curvature flows for f_λ with suitably chosen initial data in C_{f_λ}; see [Str17, Theorem 1.5].

In the last paragraph of Subsection 1.5 in [Str17], Struwe comments on future investigation of "bubbling" metrics of prescribing Q-curvature equation in arbitrary even dimensions $n \geqslant 4$. Inspired by his interesting work and comments, we aim to study the bubbling behavior of the prescribed Q-curvature equation in the null case. In fact, we have borrowed many ideas from [Str17] in the proof of the main theorems in the paper.

1.2. **A Kazdan–Warner type result for the Q-curvature equation.** Let (M, g_0) be a closed 4-dimensional Riemannian manifold endowed with a smooth background metric g_0. An analogue of the conformal Laplacian in dimension 2 is the Paneitz operator $\mathbf{P}_{g_0}$ discovered by [Pan82]. To be more precise, it is defined in terms of the Ricci tensor Ric_{g_0} and the scalar curvature scal_{g_0} as

$$\mathbf{P}_{g_0} = \Delta_{g_0}^2 - \mathrm{div}_{g_0}\Big((\frac{2}{3}\,\mathrm{scal}_{g_0}\, g_0 - 2\,\mathrm{Ric}_{g_0})d\Big).$$

Associated to the Paneitz operator $\mathbf{P}_{g_0}$, Branson [Bra85] found the Q-curvature which enjoys many similar properties as the Gaussian curvatue in dimension 2. It is also given, in terms of the Ricci tensor Ric_{g_0} and the scalar curvature R_{g_0}, by

$$Q_{g_0} = -\frac{1}{6}\Big(\Delta_{g_0}\,\mathrm{scal}_{g_0} - R_{g_0}^2 + 3|\,\mathrm{Ric}_{g_0}|^2\Big).$$

An important topic about the Q-curvature is the prescribed Q-curvature problem which is formulated as follows. Given a smooth function f on M, one may ask if there exists a conformal metric $g = e^{2u}g_0$ with Q-curvature $Q_g = f$. To solve the geometric problem is equivalent to finding the solution to the fourth order semilinear PDE.

$$\mathbf{P}_{g_0}u + Q_{g_0} = fe^{4u}. \tag{1.4}$$

There are many research works on the equation (1.4), see, for instance, [BFR06, Bre03, CY95, DM08, L LL12, MS06, WX98] and references therein.

In this paper, we consider the prescribed Q-curvature equation on 4-manifolds in the null case, that is, $\int_M Q_{g_0}\,\mathrm{d}\mu_{g_0} = 0$. Due to the resolution of the constant Q-curvature problem, we may assume, w.l.o.g., that the background metric g_0 has the null Q-curvature. Then the equation (1.4) becomes

$$\mathbf{P}_{g_0}u = fe^{4u}. \tag{1.5}$$

If $f \not\equiv 0$, then it is necessary that f changes sign for the existence of a solution to (1.5), since $\int_M fe^{4u}\,\mathrm{d}\mu_{g_0} = 0$. However, unlike the two-dimensional case, $\int_M f\,\mathrm{d}\mu_{g_0} < 0$ is not necessary anymore. The following result shows that $\int_M f\,\mathrm{d}\mu_{g_0} < 0$ is still sufficient.

Theorem 1.2 (see Ge-Xu [GX08])**.** *Let (M, g) be a compact, oriented four-dimensional Riemannian manifold. Assume that the Paneitz operator $\mathbf{P}_{g_0}$ is positive with kernel consisting of constant functions. If*

$$\sup_M f > 0 \ \textit{ and } \int_M f\,\mathrm{d}\mu_{g_0} < 0,$$

then there exists a smooth solution to (1.5).

In [GX08], Ge and Xu proved that a solution to (1.5) may be obtained by minimizing the energy

$$\mathscr{E}(u) = 2\langle \mathbf{P}_{g_0}u, u\rangle$$

under the constraint

$$F = \Big\{u \in H^2(M, g_0) : \int_M fe^{4u}\,\mathrm{d}\mu_{g_0} = 0 \text{ and } \int_M u\,\mathrm{d}\mu_{g_0} = 0\Big\}.$$

Here, for $u, v \in H^2(M, g_0)$, the inner product $\langle \mathbf{P}_{g_0}u, v\rangle$ is defined as follows

$$\langle \mathbf{P}_{g_0}u, v\rangle = \int_M \Big[\Delta_{g_0}u\Delta_{g_0}v + \frac{2}{3}R_{g_0}g_0(\nabla_{g_0}u, \nabla_{g_0}v) - 2Ric_{g_0}(\nabla_{g_0}u, \nabla_{g_0}v)\Big]\,\mathrm{d}\mu_{g_0}\,.$$

However, similar to the case of (1.2), the authors showed, in [NZ17, Theorem A.1], that the way of searching a solution is still successful if we minimize $\mathscr{E}(u)$ under the following constraint

$$X_f^* = \Big\{ u \in H^2(M, g_0) : \int_M f e^{4u} \, \mathrm{d}\mu_{g_0} = 0 \text{ and } \int_M e^{4u} \, \mathrm{d}\mu_{g_0} = 1 \Big\}.$$

2. Main results

We shall study "bubbling" of the prescribed Q-curvature equation on 4-manifolds in two different contexts: the static case and the flow case.

2.1. Bubbling metrics in the static case.

As in [Gal17, Str17] , we let f_0 be a smooth, non-constant function with $\max_{x \in M} f(x) = 0$, and let $f_\lambda = f_0 + \lambda$ for any $\lambda \in \mathbb{R}$. By assuming that $\mathrm{vol}(M, g_0) = 1$, we find that if

$$0 < \lambda < -\int_M f_0 \, \mathrm{d}\mu_{g_0} := \lambda_0, \tag{2.1}$$

then f_λ changes sign and $\int_M f_\lambda \, \mathrm{d}\mu_{g_0} < 0$. Hence, it follows from Theorem 1.2 that there exists a solution $\widetilde{u}_\lambda$ to (1.5) with f replaced by f_λ. In addition, [NZ17, Theorem A.1] implies that $\widetilde{u}_\lambda$ can be obtained as

$$\widetilde{u}_\lambda = u_\lambda + c_\lambda$$

from a minimizer u_λ of $\mathscr{E}$ in the set $X^*_{f_\lambda}$. Here u_λ satisfies

$$\mathbf{P}_{g_0} u_\lambda = \alpha_\lambda f_\lambda e^{4u_\lambda}, \tag{2.2}$$

with $\alpha_\lambda > 0$ and $c_\lambda = (\log \alpha_\lambda)/4$. Moreover, by setting

$$\widetilde{g}_\lambda = e^{2\widetilde{u}_\lambda} g_0,$$

we have

$$\alpha_\lambda = e^{4c_\lambda} = \int_M e^{4(u_\lambda + c_\lambda)} \, \mathrm{d}\mu_{g_0} = \mathrm{vol}(M, \widetilde{g}_\lambda). \tag{2.3}$$

Also, set

$$\beta_\lambda := \mathscr{E}(u_\lambda) = \min \big\{ \mathscr{E}(u) : u \in X^*_{f_\lambda} \big\}. \tag{2.4}$$

Then one will see from Lemma 4.1 below that $\beta_\lambda \to +\infty$ as $\lambda \searrow 0$; Thus, one should expect the bubbling phenomenon associated with the family of metrics $\widetilde{g}_\lambda$ to occur.

The purpose of this part of the paper is to characterize the bubbling behavior of $\widetilde{g}_\lambda$. First, when the function f_0 has only non-degenerate maxima, we have the following result:

Theorem 2.1. *Assume that the Paneitz operator $\mathbf{P}_{g_0}$ is positive with kernel consisting of constant functions. Let $f_0 \leqslant 0$ be a smooth, non-constant function with $\max_M f_0 = 0$ having only non-degenerate maximum points. Then for suitable $\lambda_k \searrow 0$, for $u_k = u_{\lambda_k}$ as above and suitable $I \in \mathbb{N}$, $r_k^{(i)} \searrow 0$, $x_k^{(i)} \to x_\infty^{(i)} \in M$ with $f_0(x_\infty^{(i)}) = 0$, $i \leqslant i \leqslant I$, as $k \to +\infty$ the following hold:*

(i) *$u_k \to -\infty$ locally uniformly on $M_\infty = M \backslash \{x_\infty^{(i)} : 1 \leqslant i \leqslant I\}$.*

(ii) *In normal coordinates around $x_\infty^{(i)}$, set*

$$z_k^{(i)} = \exp^{-1}_{x_\infty^{(i)}}(x_k^{(i)}), \quad \widetilde{u}_k = u_k \circ \exp_{x_\infty^{(i)}}.$$

Then for each $1 \leqslant i \leqslant I$, either

(a) $\limsup_{k\to\infty} r_k^{(i)}/\sqrt{\lambda_k} = 0$ *and*

$$\widehat{u}_k(z) := \widetilde{u}_k\big(z_k^{(i)} + r_k^{(i)} z\big) + \log r_k^{(i)} \to \widehat{u}_\infty(z)$$

strongly in $H^4_{\rm loc}(\mathbf{R}^4)$*, where* $\widehat{u}_\infty$*, up to a translation and a scaling, is given by*

$$\widehat{u}_\infty(z) = \log\left(\frac{4\sqrt{6}}{4\sqrt{6}+|z|^2}\right)$$

and it induces a spherical metric

$$\widehat{g}_\infty = e^{4\widehat{u}_\infty} g_{\mathbf{R}^4}$$

of Q*-curvature*

$$Q_{\widehat{g}_\infty} \equiv 1$$

on $\mathbf{R}^4$ *and* $1 \leqslant I \leqslant 4$*, or*

(b) $\limsup_{k\to\infty} r_k^{(i)}/\sqrt{\lambda_k} > 0$ *and*

$$\widehat{u}_k(z) := \widetilde{u}_k\big(z_k^{(i)} + r_k^{(i)} z\big) + \log r_k^{(i)} \to \widehat{u}_\infty(z)$$

strongly in $H^4_{\rm loc}(\mathbf{R}^4)$*, where* $\widehat{u}_\infty$*, up to a translation and a scaling, solves*

$$\Delta_z^2 \widehat{u}_\infty(z) = \Big(1 + \frac{1}{2}\mathrm{Hess}_{f_0}\big(x_\infty^{(i)}\big)\big[z,z\big]\Big) e^{4\widehat{u}_\infty(z)}, \tag{2.5}$$

In addition, the metric

$$\widehat{g}_\infty = e^{4\widehat{u}_\infty} g_{\mathbf{R}^4}$$

on $\mathbf{R}^4$ *has finite volume and finite total* Q*-curvature*

$$Q_{\widehat{g}_\infty}(z) = 1 + \frac{1}{2}\mathrm{Hess}_{f_0}\big(x_\infty^{(i)}\big)\big[z,z\big]$$

and $1 \leqslant I \leqslant 8$*.*

Remark 2.2. Unlike the Struwe's result in [Str17], the "slow blow-up" case (b) is unable to be ruled out here. In fact, the limiting equation (2.5) associated with blow-up points $x_\infty^{(i)}$ with $1 \leqslant i \leqslant I$ may have a solution with finite energy and finite total curvature. To see this, one may apply a general existence result due to Chang and Chen [CC01] to obtain that there is a solution to

$$\Delta_z^2 \widehat{u}_\infty = \Big(1 + \frac{1}{2}\,\mathrm{Hess}_{f_0}\,\big(x_\infty^{(i)}\big)\big[z,z\big]\Big) e^{4\widehat{u}_\infty}$$

with

$$\int_{\mathbf{R}^4} e^{4\widehat{u}_\infty(z)}\,\mathrm{d}z < +\infty.$$

and

$$\int_{\mathbf{R}^4} \Big(1 + \frac{1}{2}\,\mathrm{Hess}_{f_0}\,\big(x_\infty^{(i)}\big)[z,z]\Big) e^{4\widehat{u}_\infty(z)}\,\mathrm{d}z < +\infty.$$

Since $x_\infty^{(i)}$ is a non-degenerate maxima of f_0, the matrix $\mathrm{Hess}_{f_0}(x_\infty^{(i)})$ is negative definite. Consequently, we also have

$$\int_{\mathbf{R}^4} \Big|1 + \frac{1}{2}\,\mathrm{Hess}_{f_0}\,\big(x_\infty^{(i)}\big)[z,z]\Big| e^{4\widehat{u}_\infty(z)}\,\mathrm{d}z < +\infty.$$

Now, we consider the case that the function f_0 may have a degenerate maxima. To describe our next result, motivated by [Str17], we propose the following condition on f_0 analog to Condition A in [Str17].

Condition A: Let $M_0 = \{x \in M : f_0(x) = 0\}$ and $d(x) = \text{dist}(x, M_0)$ for $x \in M$. There exist $d_0 > 0$ and $A_0 > 0$ such that, letting

$$K_0 = \Big\{ z = (z^1, z^2, z^3, z^4) \in \mathbf{R}^4 : \sqrt{\sum_{i=1}^3 (z^i)^2} < z^4, |z| < d_0 \Big\}$$

for any $x \in M$ with $0 < d(x) < d_0$ there is a rotated copy $K_x \subset \mathbf{R}^4$ of K_0 with vertex at x such that in Euclidean coordinates z around $x = 0$ there holds

$$A_0 \inf_{z \in K_x} |f_0(\exp_x(z))| \geqslant |f_0(x)|.$$

Since any function on a closed manifold with only non-degenerate maxima admits finitely many maximum points, it is then clear to see that Condition A is automatically satisfied by such functions. Let us take one example of a function f_0 satisfying Condition A. We use $(r, \theta_1, \theta_2, \theta_3)$ to denote the polar coordinates in the Euclidean space $\mathbf{R}^4$. Let f_0 be as follows

$$f_0(r, \theta_1, \theta_2, \theta_3) = \begin{cases} 0 & \text{if } r \leqslant 1, \\ -e^{-1/(r-1)} \Big(\sum_{i=1}^3 \sin\big(\frac{1}{r-1} + \theta_i\big) + 4 \Big) & \text{if } r > 1. \end{cases}$$

Then it is straightforward to verify that the function f_0 above satisfies Condition A with $A_0 = 7$. Furthermore, f_0 has degenerate maximum points.

Return to characterizing the bubbling behavior of $\widetilde{g}_\lambda$ in the degenerate situation, our second result reads as follows.

Theorem 2.3. *Assume all the conditions, expcept for the assumption of the non-degeneracy of the function f_0 at a maxima, in Theorem 2.1 above. If, in addition, (M, g_0) is locally conformally flat and f_0 satisfies the* **Condition A** *with $d_0, A_0 > 0$, then for u_k defined as in the Theorem 2.1 there exist suitable $I \in \mathbb{N}$ with $I \leqslant 8$, $r_k^{(i)} \searrow 0$ and $x_k^{(i)} \to x_\infty^{(i)} \in M$ with $f_0(x_\infty^{(i)}) = 0$, $1 \leqslant i \leqslant I$ such that the following hold*

(i) *$u_k \to -\infty$ locally uniformly on $M_\infty = M \backslash \{x_\infty^{(i)} : 1 \leqslant i \leqslant I\}$.*
(ii) *For each $1 \leqslant i \leqslant I$, we have*

$$\widehat{u}_k(z) := \widetilde{u}_k\big(z_k^{(i)} + r_k^{(i)} z\big) + \log r_k^{(i)} \to \widehat{u}_\infty(z)$$

strongly in $H^4_{\text{loc}}(\mathbf{R}^4)$, where $z_k^{(i)} = \exp^{-1}_{x_\infty^{(i)}}(x_k^{(i)})$ and $\widehat{u}_\infty$ induces a metric

$$\widehat{g}_\infty = e^{4\widehat{u}_\infty} g_{\mathbf{R}^4}$$

on $\mathbf{R}^4$ of locally bounded curvature and of volume less than or equal 1.

Remark 2.4. By comparing Theorems 2.1 and 2.3, one can easily notice that in the degenerate case we made an extra assumption on the manifold (M, g_0) except for the Condition A, that is, we require the manifold (M, g_0) to be locally conformally flat. It would be interesting to investigate the bubbling phenomenon in the degenerate situation without assuming the locally conformal flatness.

2.2. **Bubbling metrics along the prescribed curvature flow.** In contrast to the statics case, our second goal is to obtain an analogous bubbling behavior described in Theorem 2.1 for a family of prescribed Q-curvature flows for f_λ with suitably chosen initial data in X_{f_λ}, where

$$X_{f_\lambda} = \Big\{ u \in H^2(M) : \int_M f_\lambda e^{4u} \, \mathrm{d}\mu_{g_0} = 0 \Big\}.$$

To describe our second result precisely, let us briefly recall the prescribed Q-curvature flow introduced in [NZ17]. Let $g_\lambda(t) = e^{2u_\lambda(t)}g_0$ be a family of time-dependent conformal metrics satisfying

$$\frac{\partial g_\lambda}{\partial t} = -2(Q_{g_\lambda} - \alpha_\lambda(t)f_\lambda)g_\lambda$$

with the initial conformal metric $g_\lambda(0) = e^{2u_{0\lambda}}g_0$. In terms of $u_\lambda(t)$, the evolution equation above becomes

$$\frac{\partial u_\lambda}{\partial t} = \alpha_\lambda(t)f_\lambda - Q_{g_\lambda} \tag{2.6}$$

with the initial data

$$u_\lambda(0) = u_{0\lambda} \in X_{f_\lambda}.$$

The function $\alpha_\lambda = \alpha_\lambda(t)$ is chosen in such a way that $\int_M f_\lambda \, d\mu_{g_\lambda}$ remains constant, namely,

$$\frac{d}{dt}\int_M f_\lambda \, d\mu_{g_\lambda} = 4\int_M u_{\lambda t} f_\lambda \, d\mu_{g_\lambda} = 4\int_M (\alpha_\lambda f_\lambda - Q_{g_\lambda})f_\lambda \, d\mu_{g_\lambda} = 0. \tag{2.7}$$

Solving (2.7) for α_λ gives

$$\alpha_\lambda = \frac{\int_M f_\lambda Q_{g_\lambda} \, d\mu_{g_\lambda}}{\int_M f_\lambda^2 \, d\mu_{g_\lambda}}.$$

It is easy to verify that

$$u_\lambda(t) \in X_{f_\lambda}$$

for all $t \geqslant 0$. We thus have by conformal invariant of Q-curvature that

$$\frac{1}{4}\frac{d}{dt}\mathrm{vol}(M, g_\lambda(t)) = \int_M u_{\lambda t} \, d\mu_{g_\lambda} = \alpha_\lambda \int_M f_\lambda \, d\mu_{g_\lambda} - \int_M Q_{g_\lambda} \, d\mu_{g_\lambda} = 0.$$

Normalizing the initial metric $g_\lambda(0)$ to satisfy $\mathrm{vol}(M, g_\lambda(0)) = 1$, we then get

$$\mathrm{vol}(M, g_\lambda(t)) = \int_M d\mu_{g_\lambda} = \int_M d\mu_{g_\lambda(0)} = 1 \tag{2.8}$$

for all $t > 0$. This implies that

$$u_\lambda(t) \in X^*_{f_\lambda} \tag{2.9}$$

for all $t \geqslant 0$.

By applying [NZ17, Theorem 1.1] to f_λ, we obtain the sequential convergence of the flow (2.6).

Theorem 2.5 (see Ngô–Zhang [NZ17])**.** *The flow (2.6) has a smooth solution $u_\lambda(t)$ on $[0, +\infty)$. Moreover, there exists a suitable time sequence $(t_j)_j$ with $t_j \to +\infty$ as $j \to +\infty$ and a suitable non-zero constant $\alpha_{\infty\lambda} \in \mathbb{R}$ such that $u_\lambda(t_j) \to u_{\infty\lambda}$ in $C^\infty(M, g_0)$, $|\alpha_\lambda(t_j) - \alpha_{\infty\lambda}| \to 0$ and $\|Q_{g_\lambda}(t_j) - \alpha_{\infty\lambda}f_\lambda\|_{C^\infty(M,g_0)} \to 0$ as $j \to +\infty$. Finally, $u_{\infty\lambda}$ satisfies*

$$\mathbf{P}_{g_0}u_{\infty\lambda} = \alpha_{\infty\lambda}f_\lambda e^{4u_{\infty\lambda}}.$$

For any $0 < \lambda < \lambda_0$ and any $\sigma \in (-\sigma_0, 0)$, with the number $\sigma_0 = \sigma_0(\lambda)$ to be determined in Lemma 5.2 below, we choose $u^\sigma_{0\lambda} \in X^*_{f_\lambda}$ such that

$$\mathscr{E}(u^\sigma_{0\lambda}) \leqslant \beta_\lambda + \sigma^2.$$

For such an initial data $u^\sigma_{0\lambda}$, it follows from Theorem 2.5 that the flow (2.6) possesses the smooth solution $u^\sigma_\lambda = u^\sigma_\lambda(t)$ with $\alpha^\sigma_\lambda = \alpha^\sigma_\lambda(t)$. Unlike the case of prescribed Gaussian curvature flow in the dimension two, the sign of $\alpha_{\infty\lambda}$ in the Q-curvature flow is unable to be determined. So, we have to assume that there exist a sequence $(\lambda_k)_k, k \in \mathbb{N}$ with $\lambda_k \searrow 0$ as $k \to +\infty$ such that $\alpha_{\infty\lambda_k} > 0$ for all k large. With σ_k and T_k defined by (5.6) below, we let, for a suitable time sequence $(t_k)_k$ with $t_k \geqslant T_k$,

$$u_k = u^{\sigma_k}_{\lambda_k}(t_k), \alpha_k = \alpha^{\sigma_k}_{\lambda_k}(t_k). \tag{2.10}$$

Now, our second result reads as

Theorem 2.6. *Let f_0 be, respectively, as in the Theorems 2.1 and 2.3 above. Then for $\lambda_k \searrow 0$ with $\alpha_{\infty\lambda_k} > 0$, suitable $u_{0\lambda_k} \in X^*_{f_{\lambda_k}}$ with $\mathscr{E}(u_{0\lambda_k}) - \beta_{\lambda_k} \leqslant \sigma_k^2 \searrow 0$, and sufficiently large $t_k \geqslant T_k \to +\infty$ as $k \to +\infty$, the conclusions of Theorems 2.1 and 2.3 hold for u_k defined by* (2.10).

Our paper is organized as the table of contents below.

CONTENTS

1. Introduction 1
1.1. The Kazdan–Warner result for the scalar curvature equation 1
1.2. A Kazdan–Warner type result for the Q-curvature equation 3
2. Main results 4
2.1. Bubbling metrics in the static case 4
2.2. Bubbling metrics along the prescribed curvature flow 6
3. Notations and preliminaries 8
4. Bubbling in the static case 10
4.1. Bounds for total curvature 10
4.2. Concentration of curvature 14
4.3. Blow-up analysis 20
5. Bubbling along the flow 39
5.1. Bounds for total curvature along the flow 39
5.2. Bubbling of the prescribed curvature flow 42
Acknowledgments 43
References 43

3. NOTATIONS AND PRELIMINARIES

In this brief section, we collect some useful facts frequently used throughout the paper. First, given a function w on M, let us denote by $\overline{w}$ the average of w over (M, g_0), namely,

$$\overline{w} = \int_M w \, d\mu_{g_0} .$$

(Keep in mind that $\text{vol}(M, g_0) = 1$.) We shall use a double bar for w, namely $\overline{\overline{w}}$, if we want to emphasize that the average of w is taking over M with any other conformal metric.

Recalling that the higher order Moser–Trudinger inequality for Paneitz operator $\mathbf{P}_{g_0}$, known as Adam's inequality; see [Ada88, Theorem 2] states that if $\mathbf{P}_{g_0}$ is self-adjoint and positive with kernel consisting of constant functions, then there is some constant $\mathscr{C}_A > 0$ such that

$$\int_M \exp\Big(32\pi^2 \frac{(u - \overline{u})^2}{\langle \mathbf{P}_{g_0} u, u \rangle}\Big) \, d\mu_{g_0} \leqslant \mathscr{C}_A \tag{3.1}$$

for every $u \in H^2(M, g_0)$. As a consequence of (3.1) and Young's inequality, we obtain the following inequality

$$\int_M \exp\left(\alpha(u - \overline{u})\right) \, \mathrm{d}\mu_{g_0} \leqslant \mathscr{C}_A \exp\left(\frac{\alpha^2}{128\pi^2} \langle \mathbf{P}_{g_0} u, u \rangle\right) \tag{3.2}$$

for all real number α.

Now we collect some information of Green's function, denoted by $\mathbb{G}$, of the Paneitz operator $\mathbf{P}_{g_0}$. By the results in [CY95], Green's function $\mathbb{G}$ is symmetric and fulfills the following properties:

(P1) $\mathbb{G}$ is smooth on $M \times M \backslash$diagonal;

(P2) there exists a positive constant $\mathscr{C}_{\mathbb{G}}$ depending only on (M, g_0) such that

$$\left| \mathbb{G}(x,y) - \frac{1}{8\pi^2} \log \frac{1}{d(x,y)} \right| \leqslant \mathscr{C}_{\mathbb{G}}$$

for any $x, y \in M$ with $x \neq y$; while for its derivatives and for $1 \leqslant j \leqslant 3$ there holds

$$\left| \nabla^j \mathbb{G}(x,y) \right| \leqslant \frac{\mathscr{C}_{\mathbb{G}}}{d(x,y)^j}$$

for any $x, y \in M$ with $x \neq y$.

As clearly described in [Mal06, page 145], the higher order estimates in (P2) are not shown in [CY95] but they can be derived with the same approach, by an expansion of $\mathbb{G}$ at higher order using the parametrix.

It is well known that if $\varphi \in L^1(M, g_0)$ with $\overline{\varphi} = 0$, then w solves

$$\mathbf{P}_{g_0} w = \varphi,$$

if and only if

$$w(x) = \overline{w} + \int_M \mathbb{G}(x,y) \varphi(y) \, \mathrm{d}\mu_{g_0} . \tag{3.3}$$

For convenience, we cite the following lemma proved in [Mal06, Lemma 2.3].

Lemma 3.1. *Let $(w_k)_k$ and $(\varphi_k)_k$ be two sequences of functions on (M, g_0) satisfying*

$$\mathbf{P}_{g_0} w_k = \varphi_k$$

with $\|\varphi_k\|_{L^1(M,g_0)} \leqslant \alpha_0$ for some positive constant α_0 independent of k. Then for any $x \in M$, any small $r > 0$, and any $s \in [1, 4/j)$ with $j = 1, 2, 3$, there holds

$$\int_{B_r(x)} |\nabla^j w_k|^s \, \mathrm{d}\mu_{g_0} \leqslant C r^{4-js},$$

where C, independent of k, is a positive constant depending only on α_0, M, and s.

To end the section, we provide the following concentration-compactness result proved in [Mal06, Proposition 3.1].

Proposition 3.2. *Let $(w_k)_k$ and $(\varphi_k)_k$ be two sequences of functions on (M, g_0) satisfying*

$$\mathbf{P}_{g_0} w_k = \varphi_k$$

with $\|\varphi_k\|_{L^1(M,g_0)} \leqslant \alpha_0$ for some positive constant α_0 independent of k. Then, up to a subsequence, we have one of the following alternatives:

(i) either there exist some constant $s > 1$ and some positive constant C independent of k such that

$$\int_M e^{4s(w_k - \overline{w}_k)} \, \mathrm{d}\mu_{g_0} \leqslant C,$$

(ii) or there exist points $x_1, x_2, \dots, x_L \in M$ such that for any $r > 0$ and any $i \in \{1, \dots, L\}$ one has

$$\liminf_{k \to +\infty} \int_{B_r(x_i)} |\varphi_k| \, \mathrm{d}\mu_{g_0} \geqslant 8\pi^2.$$

Remark 3.3. As clearly stated in [Mal06], Proposition 3.2 remains valid if the metric g_0 is replaced by a sequence of metrics $(g_k)_k$ which is uniformly bounded in $C^N(M, g_0)$ for any $N \in \mathbb{N}$. It also holds true if one replaces M by any bounded open ball in $\mathbf{R}^4$, in which all the functions are compactly supported.

4. Bubbling in the static case

In this section, we are going to prove the "bubbling" phenomena in the static case, namely, Theorems 2.1 and 2.3.

4.1. Bounds for total curvature.

We derive, in this subsection, the bounds for the total Q-curvature. As an initial step, we show the unboundedness of the minimum energy β_λ defined in (2.4).

Lemma 4.1. *As $\lambda \searrow 0$, there holds $\beta_\lambda \to +\infty$.*

Proof. Assume by contradiction that $\beta_\lambda \leqslant C_1$ for some constant C_1. Thanks to $u_\lambda \in X^*_{f_\lambda}$, we can use $\mathscr{E}(u_\lambda)$ to bound $\exp(-4\overline{u}_\lambda)$ from above by applying Adams' inequality (3.2) as follows

$$e^{-4\overline{u}_\lambda} = \int_M \exp\big(4(u_\lambda - \overline{u}_\lambda)\big) \, \mathrm{d}\mu_{g_0} \leqslant \mathscr{C}_A \exp\Big(\frac{\mathscr{E}(u_\lambda)}{16\pi^2}\Big).$$

Keep in mind that $|f_0| = -f_0 = \lambda - f_\lambda$. From this together with Hölder's inequality we can estimate

$$\begin{aligned}
0 &< \Big(\int_M |f_0| \, \mathrm{d}\mu_{g_0}\Big)^2 \\
&\leqslant \int_M |f_0| e^{-4u_\lambda} \, \mathrm{d}\mu_{g_0} \int_M |f_0| e^{4u_\lambda} \, \mathrm{d}\mu_{g_0} \\
&\leqslant \|f_0\|_{L^\infty(M,g_0)} e^{-4\overline{u}_\lambda} \int_M e^{-4(u_\lambda - \overline{u}_\lambda)} \, \mathrm{d}\mu_{g_0} \int_M (\lambda - f_\lambda) e^{4u_\lambda} \, \mathrm{d}\mu_{g_0} \\
&\leqslant \lambda \|f_0\|_{L^\infty(M,g_0)} \mathscr{C}_A^2 \exp\Big(\frac{\beta_\lambda}{8\pi^2}\Big),
\end{aligned}$$

which is obviously a contradiction if λ is sufficiently small. □

The following monotonicity property result is a key gradient for the uniform bound of the total Q-curvature of the metric $\widetilde{g}_\lambda = e^{4\widetilde{u}_\lambda} g_0$.

Lemma 4.2. *The function $\lambda \mapsto \beta_\lambda$ is non-increasing in λ for small $0 < \lambda < \lambda_0$ and*

$$\limsup_{\mu \searrow \lambda} \frac{\beta_\mu - \beta_\lambda}{\mu - \lambda} \leqslant -\alpha_\lambda,$$

where λ_0 is given in (2.1).

Proof. Fix $\lambda \in (0, \lambda_0)$. As always, let $u_\lambda \in X^*_{f_\lambda}$ be a minimizer of $\mathscr{E}$ as above, namely $\int_M f_\lambda e^{4u_\lambda} \, \mathrm{d}\mu_{g_0} = 0$ and $\int_M e^{4u_\lambda} \, \mathrm{d}\mu_{g_0} = 1$. Then for small $\sigma \in \mathbb{R}$ we have, by Taylor's expansion, that

$$\int_M f_\lambda e^{4(u_\lambda + \sigma f_\lambda)} \, \mathrm{d}\mu_{g_0} = \int_M f_\lambda \Big[e^{4(u_\lambda + \sigma f_\lambda)} - e^{4u_\lambda}\Big] \, \mathrm{d}\mu_{g_0}$$

$$
\begin{aligned}
&= \int_M f_\lambda e^{4u_\lambda}\left[e^{4\sigma f_\lambda}-1\right] \mathrm{d}\mu_{g_0} \\
&= 4\sigma \int_M f_\lambda^2 e^{4u_\lambda} \,\mathrm{d}\mu_{g_0} + O(\sigma^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_M e^{4(u_\lambda+\sigma f_\lambda)} \,\mathrm{d}\mu_{g_0} &= 1 + \int_M e^{4u_\lambda}\left[e^{4\sigma f_\lambda}-1\right] \mathrm{d}\mu_{g_0} \\
&= 1 + 4\sigma \int_M f_\lambda e^{4u_\lambda} \,\mathrm{d}\mu_{g_0} + O(\sigma^2), \\
&= 1 + O(\sigma^2).
\end{aligned}
$$

So, for $0 < |\sigma| \ll 1$ sufficiently small, if we let

$$
\mu = \lambda - 4\sigma \int_M f_\lambda^2 e^{4u_\lambda} \,\mathrm{d}\mu_{g_0} + O(\sigma^2),
$$

then we can find some constant c such that

$$
u_\lambda + \sigma f_\lambda + c \in X^*_{f_\mu}. \tag{4.1}
$$

In particular, for $\sigma < 0$ sufficiently close to zero, we have $\mu > \lambda$ and $\sigma = O(\mu - \lambda)$. Notice that it follows from (2.2) that

$$
\begin{aligned}
\mathscr{E}(u_\lambda + \sigma f_\lambda + c) &= 2\langle u_\lambda + \sigma f_\lambda + c, \mathbf{P}_{g_0} u_\lambda + \sigma \mathbf{P}_{g_0} f_\lambda\rangle \\
&= \mathscr{E}(u_\lambda) + 4\sigma \int_M f_\lambda \mathbf{P}_{g_0} u_\lambda \,\mathrm{d}\mu_{g_0} + 2\sigma^2 \langle f_\lambda, \mathbf{P}_{g_0} f_\lambda\rangle \\
&= \mathscr{E}(u_\lambda) + 4\sigma\alpha_\lambda \int_M f_\lambda^2 e^{4u_\lambda} \,\mathrm{d}\mu_{g_0} + O(\sigma^2).
\end{aligned}
$$

Now, by (4.1), we get that

$$
\begin{aligned}
\beta_\mu &\leqslant \mathscr{E}(u_\lambda + \sigma f_\lambda + c) \\
&\leqslant \mathscr{E}(u_\lambda) + 4\sigma\alpha_\lambda \int_M f_\lambda^2 e^{4u_\lambda} \,\mathrm{d}\mu_{g_0} + O(\sigma^2) \\
&= \beta_\lambda - \alpha_\lambda(\mu - \lambda) + O((\mu-\lambda)^2) < \beta_\lambda,
\end{aligned}
$$

for $\sigma < 0$ sufficiently close to zero. Hence the map $\lambda \mapsto \beta_\lambda$ is non-increasing and

$$
\limsup_{\mu \searrow \lambda} \frac{\beta_\mu - \beta_\lambda}{\mu - \lambda} \leqslant -\alpha_\lambda
$$

as claimed. □

We can find the following bound on β_λ.

Lemma 4.3. *There holds*

$$
\limsup_{\lambda \searrow 0} \frac{\beta_\lambda}{\log(1/\lambda)} \leqslant 64\pi^2.
$$

Proof. Let $p_0 \in M$ be such that $f_0(p_0) = 0$ and assume that $\lambda \in (0, \lambda_0)$. By fixing a natural number $N \geqslant 5$, we can find a smooth conformal metric $g_N = e^{2\varphi_N} g_0$ such that

$$
\det(g_N) = 1 + O(r^N)_{r \searrow 0}, \tag{4.2}
$$

where $r = |x|$ and x are g_N-normal coordinates around p_0 which is identified as 0 in this new coordinate system. Now, letting

$$
A = \frac{1}{2}\mathrm{Hess}_{f_0}(p_0).
$$

Since p_0 is an isolated maxima of f_0, for a suitable constant $L > 0$ we have

$$f_0(x) = (Ax, x) + O(|x|^3) \geqslant -\frac{\lambda}{2}$$

on $B_{\sqrt{\lambda}/L}(0)$, and therefore $f_\lambda \geqslant \lambda/2$ on $B_{\sqrt{\lambda}/L}(0)$ for all $\lambda \in (0, \lambda_0)$. Fix a cut-off function $\tau \in C_c^\infty([0,\infty))$ with $0 \leqslant \tau \leqslant 1$ and

$$\tau(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant 1/2, \\ 0 & \text{if } t \geqslant 1. \end{cases}$$

For any $A_0 > 1$, we can find a smooth function $\xi \in C^\infty([0,\infty))$ such that $1 \leqslant \xi \leqslant 2$, $\xi' \geqslant 0$, $\sup_{t \geqslant 0} \xi'(t) \leqslant A_0$, and

$$\xi(t) = \begin{cases} t & \text{if } 0 \leqslant t \leqslant 1, \\ 2 & \text{if } t \geqslant 2. \end{cases}$$

Then we define

$$z_\lambda(x) = \begin{cases} \log(1/\lambda) & \text{if } |x| \leqslant \lambda, \\ \frac{1}{2}\log(1/\lambda)\, \xi\Big(\frac{2\log|x|}{\log(\lambda)}\Big)\tau(|x|) & \text{if } \lambda \leqslant |x| \leqslant 1. \end{cases}$$

It is easy to see that $z_\lambda \in C^\infty(B_1(0))$ with $\operatorname{supp}(z_\lambda) \subset \overline{B_1(0)}$. Finally, we define for

$$w_\lambda(x) = \begin{cases} z_\lambda\big(\frac{Lx}{\sqrt{\lambda}}\big) & \text{if } x \in B_{\sqrt{\lambda}/L}(0), \\ 0 & \text{if } x \in M \setminus B_{\sqrt{\lambda}/L}(0). \end{cases}$$

Then, $w_\lambda \in C^\infty(M)$ with $\operatorname{supp}(w_\lambda) \subset \overline{B_{\sqrt{\lambda}/L}(0)}$. Consider the continuous function $\eta : [0,\infty) \mapsto \mathbb{R}$ defined by

$$\eta(s) = \int_M f_\lambda e^{4sw_\lambda}\, \mathrm{d}\mu_{g_0}.$$

It follows from (2.1) that $\eta(0) < 0$. On the other hand, by the definition of w_λ and the fact that $f_\lambda \geqslant \lambda/2$ on $B_{\sqrt{\lambda}/L}(0)$, we conclude that

$$\begin{aligned} \eta(s) &= \int_M f_\lambda\, \mathrm{d}\mu_{g_0} + \int_{B_{\sqrt{\lambda}/L}(0)} f_\lambda \big(e^{4sw_\lambda} - 1\big)\, \mathrm{d}\mu_{g_0} \\ &\geqslant \int_M f_\lambda\, \mathrm{d}\mu_{g_0} + \frac{\lambda}{2}\int_{B_{\sqrt{\lambda}/L}(0)} \big(e^{4sw_\lambda} - 1\big)\, \mathrm{d}\mu_{g_0}. \end{aligned}$$

This implies that $\eta(s) \to +\infty$ as $s \nearrow +\infty$. Hence, there exists some $s(\lambda) \in (0, +\infty)$ depending on λ such that

$$0 = \eta(s) = \int_M f_\lambda e^{4s(\lambda)w_\lambda}\, \mathrm{d}\mu_{g_0},$$

that is $s(\lambda)w_\lambda \in X_{f_\lambda}$. In addition, we may find a constant $c(\lambda)$ such that

$$s(\lambda)w_\lambda + c(\lambda) \in X^*_{f_\lambda}.$$

Now, we provide a more precise estimate of $s(\lambda)$. Since $\operatorname{vol}(M, g_0) = 1$, $\operatorname{supp}(w_\lambda) \subset \overline{B_{\sqrt{\lambda}/L}(0)}$, and $\mathrm{d}\mu_{g_N} = e^{4\varphi_N}\, \mathrm{d}\mu_{g_0}$ we get that

$$\begin{aligned} 0 &= \int_M f_\lambda e^{4s(\lambda)w_\lambda}\, \mathrm{d}\mu_{g_0} \\ &= \int_{B_{\sqrt{\lambda}/L}(0)} f_\lambda e^{4s(\lambda)w_\lambda}\, \mathrm{d}\mu_{g_0} + \int_{M \setminus B_{\sqrt{\lambda}/L}(0)} f_\lambda\, \mathrm{d}\mu_{g_0} \\ &\geqslant \frac{\lambda}{2}\int_{B_{\sqrt{\lambda}/L}(0)} e^{4[s(\lambda)w_\lambda - \varphi_N]}\, \mathrm{d}\mu_{g_N} - \|f_0\|_\infty. \end{aligned}$$

By (4.2), we have $\mathrm{d}\mu_{g_N} = \sqrt{1+O(r^N)}dx$. Thus, for any $\varepsilon \in (0,1)$, we can find $\lambda_\varepsilon \in (0,\lambda_0)$, independent of s, such that for any $\lambda \in (0,\lambda_\varepsilon)$

$$\begin{aligned}\int_{B_{\sqrt{\lambda}/L}(0)} e^{4[s(\lambda)w_\lambda - \varphi_N]}\,\mathrm{d}\mu_{g_N} \geqslant& (\min_M e^{-4\varphi_N})\int_{B_{\sqrt{\lambda}/L}(0)} e^{4s(\lambda)w_\lambda}\sqrt{1+O(r^N)}\,dx\\ \geqslant& (\min_M e^{-4\varphi_N})(1-\varepsilon)\int_{B_{\sqrt{\lambda}/L}(0)} e^{4s(\lambda)w_\lambda}\,dx.\end{aligned}$$

It follows from the definition of z_λ and after substituting $y = Lx/\sqrt{\lambda}$ that

$$\begin{aligned}\lambda\int_{B_{\sqrt{\lambda}/L}(0)} e^{4s(\lambda)w_\lambda}\,dx =& \frac{\lambda^3}{L^4}\int_{B_1(0)} e^{4s(\lambda)z_\lambda(y)}\,dy\\ \geqslant& \frac{\lambda^{3-4s(\lambda)}}{L^4}\int_{B_{\lambda^{5/4}}(0)} dy = \frac{\pi^2\lambda^{8-4s(\lambda)}}{2L^4}.\end{aligned}$$

By combining all estimates above, we obtain

$$\frac{1}{4L^4}(\min_M e^{-4\varphi_N})(1-\varepsilon)\pi^2\,\lambda^{8-4s(\lambda)} \leqslant \|f_0\|_\infty.$$

Solving the preceding inequality for s gives

$$0 < s(\lambda) \leqslant 2 + \frac{1}{4\log(1/\lambda)}\log\Big(\frac{4L^4\|f_0\|_\infty \max_M e^{4\varphi_N}}{(1-\varepsilon)\pi^2}\Big) := 2 + O(1/\log(1/\lambda)). \quad (4.3)$$

Next, following the proof of [Gal17, Lemma 3.6], we obtain that given any $A_0 > 1$, there exists $\lambda^\varepsilon \in (0,\lambda_0)$, independent of A_0, such that for any $0<\lambda<\lambda^\varepsilon$ there holds

$$\langle \mathbf{P}_{g_0} w_\lambda, w_\lambda\rangle \leqslant 4\pi^2(1+\varepsilon)(A_0^2+1)\log(1/\lambda) + C_0,$$

where C_0 does not depend on neither λ nor ε. Keep in mind that $s(\lambda)w_\lambda + c(\lambda) \in X_f^*$ with $s(\lambda)$ satisfying (4.3). Hence, we have

$$\begin{aligned}\beta_\lambda \leqslant& 2\langle \mathbf{P}_{g_0}(s(\lambda)w_\lambda + c), s(\lambda)w_\lambda + c\rangle\\ =& 2s(\lambda)^2\langle \mathbf{P}_{g_0} w_\lambda, w_\lambda\rangle\\ \leqslant& 32\pi^2(1+\varepsilon)(A_0^2+1)\log(1/\lambda) + O(1).\end{aligned}$$

This implies that

$$\limsup_{\lambda\searrow 0}\frac{\beta_\lambda}{\log(1/\lambda)} \leqslant 32\pi^2(1+\varepsilon)(A_0^2+1).$$

Letting $\varepsilon \searrow 0$ and $A_0 \searrow 1$ gives the assertion. □

Lemma 4.4. *There holds*

$$\liminf_{\lambda\searrow 0}(\lambda\alpha_\lambda) \leqslant \liminf_{\lambda\searrow 0}|\lambda\beta'_\lambda| \leqslant 64\pi^2.$$

Proof. Notice that the monotone function β_λ is differentiable almost everywhere. Then by Lemma 4.2 we can easily get that

$$\liminf_{\lambda\searrow 0}(\lambda\alpha_\lambda) \leqslant \liminf_{\lambda\searrow 0}|\lambda\beta'_\lambda|.$$

So, it remains to show that

$$\liminf_{\lambda\searrow 0}|\lambda\beta'_\lambda| \leqslant 64\pi^2.$$

Indeed, if we assume that for some $0<\lambda_*<\lambda_0$, some $c_0 > 64\pi^2$ and almost all $0<\lambda<\lambda_*$ the absolutely continuous part of the differential of β_λ satisfies $|\beta'_\lambda| \geqslant c_0/\lambda$, then for $K = 32\pi^2 + c_0/2 > 64\pi^2$ and any sufficiently small $0<\lambda<\lambda_*$ we have, by Lebesgue's theorem, that

$$\beta_\lambda - \beta_{\lambda_*} \geqslant \int_\lambda^{\lambda_*}|\beta'_\lambda|\,d\lambda$$

$$\begin{aligned}&\geqslant K\log(1/\lambda)+(c_0-K)\log(1/\lambda)+c_0\log\lambda_*\\&>K\log(1/\lambda).\end{aligned}$$

This contradicts the bound in Lemma 4.3. □

With help of this lemma, we can now obtain a bound for the total Q-curvatures of the metric $\widetilde{g}_\lambda$ and the normalized metric g_λ. Recall that $\widetilde{g}_\lambda = e^{2\widetilde{u}_\lambda} g_0$ with $Q_{\widetilde{g}_\lambda} = f_\lambda$ and $g_\lambda = e^{2u_\lambda} g_0$ with $Q_{g_\lambda} = \alpha_\lambda f_\lambda$.

Lemma 4.5. *There holds*

$$\liminf_{\lambda\searrow 0}\int_M |Q_{g_\lambda}|\,\mathrm{d}\mu_{g_\lambda} = \liminf_{\lambda\searrow 0}\int_M |Q_{\widetilde{g}_\lambda}|\,\mathrm{d}\mu_{\widetilde{g}_\lambda} \leqslant 128\pi^2.$$

Proof. Notice that we can estimate

$$|Q_{\widetilde{g}_\lambda}| = |f_0+\lambda| \leqslant -f_0+\lambda = -f_\lambda + 2\lambda. \tag{4.4}$$

Keep in mind that $\alpha_\lambda = \mathrm{vol}(M,\widetilde{g}_\lambda)$ and that $\int_M f_\lambda\,\mathrm{d}\mu_{\widetilde{g}_\lambda} = 0$. Then by (2.3), Lemma 4.4, and the fact that $u_\lambda \in X^*_{f_\lambda}$, we get that

$$\begin{aligned}\liminf_{\lambda\searrow 0}\int_M |Q_{\widetilde{g}_\lambda}|\,\mathrm{d}\mu_{\widetilde{g}_\lambda} \leqslant&\liminf_{\lambda\searrow 0}\Big[\int_M (-f_\lambda)\,\mathrm{d}\mu_{\widetilde{g}_\lambda} + 2\lambda\alpha_\lambda\Big]\\ =&2\liminf_{\lambda\searrow 0}(\lambda\alpha_\lambda) \leqslant 128\pi^2.\end{aligned}$$

Since $|Q_{g_\lambda}| = e^{4c_\lambda}|Q_{\widetilde{g}_\lambda}|$ and $\mathrm{d}\mu_{g_\lambda} = e^{4u_\lambda}\,\mathrm{d}\mu_{g_0} = e^{-4c_\lambda}\,\mathrm{d}\mu_{\widetilde{g}_\lambda}$, we deduce that

$$\int_M |Q_{g_\lambda}|\,\mathrm{d}\mu_{g_\lambda} = \int_M |Q_{\widetilde{g}_\lambda}|\,\mathrm{d}\mu_{\widetilde{g}_\lambda},$$

we thus complete the proof. □

4.2. **Concentration of curvature.** In the following, we consider the prescribed Q-curvature equation with an error term. To be precise, for a suitable sequence $\lambda_k \searrow 0$ and suitable $\alpha_k > 0$ we let functions $w_k \in X^*_{f_{\lambda_k}}$ with corresponding metrics $g_k = e^{2w_k} g_0$ solve

$$\mathbf{P}_{g_0} w_k = \alpha_k f_{\lambda_k} e^{4w_k} + h_k e^{4w_k} \tag{4.5}$$

with $Q_{g_k} = \alpha_k f_{\lambda_k} + h_k$. Then

$$\int_M Q_{g_k} e^{4w_k}\,\mathrm{d}\mu_{g_0} = 0.$$

In view of Lemma 4.4, we further assume that α_k satisfies

$$\limsup_{k\to+\infty}(\lambda_k\alpha_k) \leqslant 64\pi^2. \tag{4.6}$$

Moreover, we let functions h_k on M be such that

$$\|h_k\|_{L^2(M,g_k)} =: \varepsilon_k \to 0 \tag{4.7}$$

as $k\to+\infty$. Denote

$$\overline{\overline{h}}_k = \int_M h_k e^{4w_k}\,\mathrm{d}\mu_{g_0}. \tag{4.8}$$

Then the assumption (4.7) implies that

$$|\overline{\overline{h}}_k| \leqslant \varepsilon_k \quad \text{and} \quad \|h_k - \overline{\overline{h}}_k\|_{L^1(M,g_k)} \leqslant 2\varepsilon_k. \tag{4.9}$$

With all these assumptions, we then have the same conclusion as Lemma 4.5. To see this, we set

$$s^\pm = \pm\max\{\pm s, 0\}$$

for any $s \in \mathbb{R}$. Upon writing $|Q_{g_k}| = -Q_{g_k} + 2Q_{g_k}^+$, estimating $Q_{g_k}^+ \leqslant \alpha_k\lambda_k + |h_k|$, and integrating (4.5) we obtain, by Hölder's inequality and the assumption (4.7), that

$$\begin{aligned}\limsup_{k\to+\infty}\int_M |Q_{g_k}|\,\mathrm{d}\mu_{g_k} =& 2\limsup_{k\to+\infty}\int_M Q_{g_k}^+ e^{4w_k}\,\mathrm{d}\mu_{g_0}\\ \leqslant& 2\limsup_{k\to+\infty}\big(\alpha_k\lambda_k + \|h_k\|_{L^2(M,g_k)}\big) \leqslant 128\pi^2.\end{aligned} \tag{4.10}$$

It is worth emphasizing that by allowing the "error term" h_k in the perturbed equation (4.5), we will also be able to apply Theorems 4.7 and 4.8 below in the flow context, where $w_k = u(t_k)$ for a solution $u = u(t)$ to (2.6) and $h_t = u_t(t_k)$ for a sequence of times $t_k \to +\infty$. On the other hand, by choosing $w_k = u_k \in X^*_{f_{\lambda_k}}$, satisfying (4.5) with $h_k = 0$ for all $k \in \mathbb{N}$, Theorems 2.1 and 2.3 will become the special cases of Theorems 4.7 and 4.8 below respectively.

It is worth noting that we are not interested in the existence of solutions to (4.5) in $X^*_{f_{\lambda_k}}$ under the conditions (4.6) and (4.7). What we are interested in is the concentration behavior of any sequence of solutions to (4.5) in $X^*_{f_{\lambda_k}}$, if exists. To be more precise, we prove the following concentration result.

Lemma 4.6. *Given (w_k) a sequence of solutions to (4.5) as above we have $\alpha_k \to +\infty$ as $k \to +\infty$. Moreover, there exist a suitable positive integer I with $I \leqslant 8$ and finitely many points $x_\infty^{(i)} \in M$ with $1 \leqslant i \leqslant I$ such that, for any $r > 0$ and each $1 \leqslant i \leqslant I$, there hold*

$$f_0(x_\infty^{(i)}) = 0 \tag{4.11}$$

and

$$\liminf_{k\to+\infty}\int_{B_r(x_\infty^{(i)})} Q_{g_k}^+\,\mathrm{d}\mu_{g_k} \geqslant 8\pi^2. \tag{4.12}$$

Proof. Our proof consists of two parts.

PART 1. We prove (4.12) for $1 \leqslant i \leqslant I$ and $\alpha_k \to +\infty$ as $k \to +\infty$.

By way of contradiction, we assume that for every $x \in M$ there exists some $r_x > 0$ such that

$$\int_{B_{r_x}(x)} Q_{g_k}^+\,\mathrm{d}\mu_{g_k} \leqslant 8\pi^2 - \delta_x, \tag{4.13}$$

for some $\delta_x > 0$ and for k large enough. Since the proof presented here is rather long, we split it into several steps for clarity.

Step 1. In this step, from the contradiction assumption (4.13), we shall establish the key estimate (4.22) below. Since M is compact, we can cover M by N balls $B_i = B_{r_{x^i}/2}(x^i)$ with $1 \leqslant i \leqslant N$. By the property (P2) of Green's function, we conclude that $\mathbb{G}(x,y) > 0$ for any $x \in M$, $y \in B_{r_x}(x)$ with r_x suitably small. So, in the following, we choose the radius r_{x^i} small enough such that $\mathbb{G}(x,y) > 0$ for any $x, y \in \widetilde{B}_i = B_{r_{x^i}}(x^i)$. Let

$$\mu_k = \int_M Q_k^+ e^{4w_k}\,\mathrm{d}\mu_{g_0}\,.$$

Then the fact that $\int_M Q_{g_k} e^{4w_k}\,\mathrm{d}\mu_{g_0} = 0$ implies that $\int_M Q_k^- e^{4w_k}\,\mathrm{d}\mu_{g_0} = -\mu_k$. Moreover, by (4.10) we have

$$0 \leqslant \mu_k \leqslant 128\pi^2 + o(1).$$

Now we let $w_k^{(\pm)}$ solve the equations

$$\mathbf{P}_{g_0} w_k^{(\pm)} = Q_{g_k}^{\pm} e^{4w_k} \mp \mu_k \tag{4.14}$$

on M. Since $\ker \mathbf{P}_{g_0} = \{\text{constants}\}$ and $\mathbf{P}_{g_0} w_k = Q_{g_k} e^{4w_k}$, we can decompose w_k as the following

$$w_k = w_k^{(+)} + w_k^{(-)} + d_k, \tag{4.15}$$

where d_k is some constant. Integrating (4.15) with respect to the metric g_0 yields

$$d_k = \overline{w}_k - \overline{w_k^{(+)}} - \overline{w_k^{(-)}}. \tag{4.16}$$

Recall that M is covered by all balls B_i with $1 \leqslant i \leqslant N$. Hence, for each $x \in M$ we can find some $1 \leqslant i \leqslant N$ such that $x \in B_i$. By applying the formula (3.3) to the equations (4.14) we obtain

$$\begin{aligned} w_k^{(\pm)}(x) =& \overline{w_k^{(\pm)}} + \int_M \mathbb{G}(x,y)\big(Q_{g_k}^{\pm} e^{4w_k} \mp \mu_k\big)(y)\, \mathrm{d}\mu_{g_0}(y) \\ =& \overline{w_k^{(\pm)}} + \int_{\widetilde{B}_i} \mathbb{G}(x,y) Q_{g_k}^{\pm}\, \mathrm{d}\mu_{g_k}(y) \mp \mu_k \int_M \mathbb{G}(x,y)\, \mathrm{d}\mu_{g_0}(y) \\ &+ \int_{M\setminus \widetilde{B}_i} \mathbb{G}(x,y) Q_{g_k}^{\pm}\, \mathrm{d}\mu_{g_k}(y). \end{aligned} \tag{4.17}$$

It follows from the property (P1) of Green's function and (4.10) that there exists positive constants C_i independent of k such that

$$\Big|\mu_k \int_M \mathbb{G}(x,y)\, \mathrm{d}\mu_{g_0}(y)\Big| + \Big| \int_{M\setminus \widetilde{B}_i} \mathbb{G}(x,y) Q_{g_k}^{\pm}\, \mathrm{d}\mu_{g_k}(y)\Big| \leqslant C_i. \tag{4.18}$$

If we set

$$c_* = \max \big\{ C_i : 1 \leqslant i \leqslant N \big\},$$

then the positivity of G on each $\widetilde{B}_i$ and (4.17) imply that for any $x \in M$, we have

$$\begin{cases} w_k^{(+)}(x) \geqslant \overline{w_k^{(+)}} - c_*, \\ w_k^{(-)}(x) \leqslant \overline{w_k^{(-)}} + c_*. \end{cases} \tag{4.19}$$

Now, we define

$$\begin{cases} v_k = w_k - \overline{w}_k, \\ v_k^{(+)} = w_k^{(+)} - \overline{w_k^{(+)}} + c_*, \\ v_k^{(-)} = w_k^{(-)} - \overline{w_k^{(-)}} - c_*. \end{cases} \tag{4.20}$$

Then the relations in (4.19) imply that

$$v_k^{(+)} \geqslant 0 \geqslant v_k^{(-)}.$$

Furthermore, it follows from (4.15) and (4.16) that

$$w_k - \overline{w}_k = w_k^{(+)} + w_k^{(-)} - \overline{w_k^{(+)}} - \overline{w_k^{(-)}},$$

which then yields

$$v_k = v_k^{(+)} + v_k^{(-)}.$$

In view of (4.13), we may choose some real number $s_0 > 4$ so that

$$s_0 \int_{\widetilde{B}_i} Q_{g_k}^{+}\, \mathrm{d}\mu_{g_k} < 32\pi^2 \tag{4.21}$$

for any $1 \leqslant i \leqslant N$. By using (4.17) and (4.18), we can bound

$$v_k^{(+)} - c_* = w_k^{(+)} - \overline{w_k^{(+)}} \leqslant c_* + \int_{\widetilde{B}_i} \mathbb{G}(x,y) Q_{g_k}^{+}\, \mathrm{d}\mu_{g_k}(y),$$

which then gives

$$\exp\big[s_0 v_k^{(+)}\big] \leqslant e^{2s_0 c_*} \exp\Big[s_0 \int_{\widetilde{B}_i} \mathbb{G}(x,y) Q_{g_k}^{+}\, \mathrm{d}\mu_{g_k}(y)\Big]$$

$$\begin{aligned}&\leqslant e^{2s_0c_*}\exp\Big[s_0\int_M \|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}|\mathbb{G}(x,y)|\frac{Q^+_{g_k}\chi_{\widetilde B_i}(y)}{\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}}\,\mathrm{d}\mu_{g_k}(y)\Big]\\&\leqslant e^{2s_0c_*}\int_M \exp\Big[s_0\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}|\mathbb{G}(x,y)|\Big]\frac{Q^+_{g_k}\chi_{\widetilde B_i}(y)}{\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}}\,\mathrm{d}\mu_{g_k}(y),\end{aligned}$$

thanks to the 'weighted' Jensen inequality; see [BM91, page 1227]. By integrating the inequality above and using Fubini's theorem, we obtain

$$\begin{aligned}\int_{B_i} e^{s_0 v_k^{(+)}}\,\mathrm{d}\mu_{g_0}(x) \leqslant e^{2s_0c_*}&\int_M\Big(\int_{B_i}\exp\Big[s_0\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}|\mathbb{G}(x,y)|\Big]\,\mathrm{d}\mu_{g_0}(x)\Big)\\&\times\frac{Q^+_{g_k}\chi_{\widetilde B_i}(y)}{\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}}\,\mathrm{d}\mu_{g_k}(y)\\&\leqslant e^{2s_0c_*}\sup_{y\in M}\int_M\exp\Big[s_0\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}|\mathbb{G}(x,y)|\Big]\,\mathrm{d}\mu_{g_0}(x).\end{aligned}$$

By the property (P2), we know that $|\mathbb{G}(x,y)|\leqslant(1/(8\pi^2))\log(1/d(x,y))+\mathscr{C}_{\mathbb{G}}$ for any $x\neq y$, which implies that

$$s_0\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}|\mathbb{G}(x,y)|\leqslant\frac{s_0}{8\pi^2}\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}\log\frac{1}{d(x,y)}+8\pi^2\mathscr{C}_{\mathbb{G}}s_0$$

for any $x\neq y$. From this we can estimate

$$\begin{aligned}&\int_M\exp\Big[s_0\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}|\mathbb{G}(x,y)|\Big]\,\mathrm{d}\mu_{g_0}(x)\\&\qquad\leqslant e^{8\pi^2\mathscr{C}_{\mathbb{G}}s_0}\int_M\big(d(x,y)\big)^{-(s_0/8\pi^2)\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}}\,\mathrm{d}\mu_{g_0}(x).\end{aligned}$$

The last integral in the preceding inequality is uniformly bounded because

$$(s_0/8\pi^2)\|Q^+_{g_k}\chi_{\widetilde B_i}\|_{L^1(M,g_k)}<4,$$

thanks to (4.21). So we have shown that

$$\int_{B_i} e^{s_0 v_k^{(+)}}\,\mathrm{d}\mu_{g_k}<+\infty$$

for $1\leqslant i\leqslant N$. Since M is covered by finitely many B_i's, we conclude that

$$\int_M e^{s_0 v_k^{(+)}}\,\mathrm{d}\mu_{g_k}<+\infty. \tag{4.22}$$

This completes the first step.

Step 2. In this step, we claim from the key estimate (4.22) that $v_k^{(+)}$, defined in (4.20), is uniformly bounded. To see this, we let $p=2s_0/(s_0+4)$. Then it follows from $s_0>4$ that

$$1<p<2,\quad 4p<s_0. \tag{4.23}$$

With this real number p, Minkowski's inequality, and (4.14), we can estimate

$$\begin{aligned}\big\|\mathbf{P}_{g_0}v_k^{(+)}\big\|_{L^p(M,g_0)}&=\big\|\mathbf{P}_{g_0}w_k^{(+)}\big\|_{L^p(M,g_0)}\\&=\big\|Q^+_{g_k}e^{4w_k}\mp\mu_k\big\|_{L^p(M,g_0)}\\&\leqslant\alpha_k\lambda_k\big\|e^{4w_k}\big\|_{L^p(M,g_0)}+\big\|h_ke^{4w_k}\big\|_{L^p(M,g_0)}+\mu_k\\&=I+II+128\pi^2+o(1)_{k\nearrow+\infty}.\end{aligned}$$

<u>Estimate of I</u>: By Jensen's inequality and the fact $w_k\in X^*_{f_{\lambda_k}}$, we know that

$$\overline{w}_k=\int_M w_k\,\mathrm{d}\mu_{g_0}\leqslant\frac14\log\Big(\int_M e^{4w_k}\,\mathrm{d}\mu_{g_0}\Big)=0.$$

Hence, $w_k = v_k + \overline{w}_k \leqslant v_k \leqslant v_k^{(+)}$. This together with the fact that $\alpha_k \lambda_k \leqslant 64\pi^2 + o(1)_{k \nearrow +\infty}$, (4.22) , (4.23), and Hölder's inequality implies that

$$I \leqslant 65\pi^2 \|e^{4v_k^{(+)}}\|_{L^p(M,g_0)} \leqslant 65\pi^2 \|e^{s_0 v_k^{(+)}}\|_{L^1(M,g_0)}^{4/s_0}.$$

Thus $I = O(1)_{k \nearrow +\infty}$.

Estimate of II: To estimate this term, we make use of Hölder's inequality and the facts that $1 < p < 2$ and $s_0 = 4p/(2-p)$ to get

$$\begin{aligned} II = \|h_k e^{4w_k}\|_{L^p(M,g_0)} &= \Big(\int_M |h_k|^p e^{2pw_k} e^{2pw_k} \, \mathrm{d}\mu_{g_0} \Big)^{1/p} \\ &\leqslant \|h_k\|_{L^2(M,g_k)} \Big(\int_M e^{s_0 w_k} \, \mathrm{d}\mu_{g_0} \Big)^{1/p - 1/2}. \end{aligned}$$

By (4.22) and (4.7), we deduce that

$$II = o(1)_{k \nearrow +\infty}.$$

Combining the estimates of I and II gives

$$\int_M |\mathbf{P}_{g_0} v_k^{(+)}|^p \, \mathrm{d}\mu_{g_0} < +\infty.$$

In addition, it follows from (4.22) that

$$v_k^{(+)} \in L^q(M, g_0)$$

for any $q \geqslant 1$. Thus, by standard elliptic theory, we have shown that $v_k^{(+)}$ is bounded in $W^{4,p}(M, g_0)$ for some $p > 1$. Again by Sobolev's embedding, we conclude that $v_k^{(+)}$ is bounded in $C^{0,\alpha}(M, g_0)$ for some $\alpha \in [0, 4 - 4/p]$. The claim is proved.

Step 3. In this step, we show that the sequence (α_k) is unbounded. Indeed, suppose that (α_k) is bounded, namely, $\alpha_k = O(1)_{k \to +\infty}$. Mimicking the argument used in (4.10) to get

$$0 \leqslant \int_M Q_{g_k}^+ e^{4w_k} \, \mathrm{d}\mu_{g_0} \leqslant \alpha_k \lambda_k + \|h_k\|_{L^2(M,g_k)} = o(1)_{k \nearrow +\infty},$$

which tells us that (4.13) holds at any point in M. From this, we repeat the arguments in Steps 1 and 2 to realize that $(v_k^{(+)})$ is uniformly bounded. It is now possible to bound w_k uniformly from above as follows

$$w_k = v_k + \overline{w}_k \leqslant v_k \leqslant v_k^{(+)} \leqslant C.$$

In view of the estimate $se^s \geqslant -1$ for $s \leqslant 0$ we find that

$$\alpha_k f_{\lambda_k} e^{4w_k} w_k \leqslant C \quad \text{and} \quad |e^{2w_k} w_k| \leqslant C.$$

uniformly in M. But then by multiplying (4.5) with w_k we obtain the bound

$$\begin{aligned} \beta_{\lambda_k} &\leqslant 2 \langle \mathbf{P}_{g_0} w_k, w_k \rangle \\ &\leqslant 2 \int_M \alpha_k f_{\lambda_k} e^{4w_k} w_k \, \mathrm{d}\mu_{g_0} + 2 \int_M h_k e^{4w_k} w_k \, \mathrm{d}\mu_{g_0} \\ &\leqslant C + 2 \|h_k\|_{L^2(M,g_k)} \|e^{2w_k} w_k\|_{L^2(M,g_0)} \leqslant C, \end{aligned}$$

provided k is large enough, which contradicts Lemma 4.1. Thus, (α_k) is unbounded as claimed.

Step 4. Now, we are in position to obtain a contradiction and show that there are finitely many points $x_\infty^{(i)} \in M$ with $1 \leqslant i \leqslant I$ such that (4.12) holds. Keep in mind that $\alpha_k \nearrow +\infty$ as $k \to +\infty$. Depending on the size of $\overline{w}_k$, there are two possible cases as follows.

<u>Case 1</u>. Suppose that $\overline{w}_k \to -\infty$ as $k \to +\infty$. By the uniform boundedness of $v_k^{(+)}$ established before, we have

$$w_k = v_k + \overline{w}_k \leqslant v_k^{(+)} + \overline{w}_k \leqslant C + \overline{w}_k.$$

Consequently, $w_k \to -\infty$ uniformly on M as $k \to +\infty$. This contradicts the fact that $\int_M e^{4w_k}\,\mathrm{d}\mu_{g_0} = 1$.

<u>Case 2</u>. Suppose that $\overline{w}_k \geqslant -C$ for some positive constant C. In view of (4.10), we choose $\gamma = 1/17$ so that

$$\gamma \int_M |Q_{g_k}|\,\mathrm{d}\mu_{g_k} < 8\pi^2$$

holds for large k. This estimate plays a similar role as that of (4.13). Therefore, we can repeat the previous argument to get that

$$\big\| e^{-s_1\gamma v_k^{(-)}} \big\|_{L^1(M,g_0)} < +\infty \tag{4.24}$$

for some $s_1 > 4$. This together with (4.5) and $\mathrm{vol}(M, g_0) = 1$ implies that

$$\begin{aligned}
&\int_M \big(\alpha_k |f_{\lambda_k}|\big)^{2^{-1}} e^{2v_k - 2\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0}\\
&\qquad\leqslant \Big(\int_M \alpha_k |f_{\lambda_k}| e^{4v_k}\,\mathrm{d}\mu_{g_0}\Big)^{1/2}\Big(\int_M e^{-4\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0}\Big)^{1/2}\\
&\qquad= e^{-2\overline{w}_k}\Big(\int_M \alpha_k |f_{\lambda_k}| e^{4w_k}\,\mathrm{d}\mu_{g_0}\Big)^{1/2}\Big(\int_M e^{-4\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0}\Big)^{1/2}\\
&\qquad\leqslant e^{-2\overline{w}_k}\Big(\int_M (|Q_{g_k}| + |h_k|) e^{4w_k}\,\mathrm{d}\mu_{g_0}\Big)^{1/2}\Big(\int_M e^{-4\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0}\Big)^{1/2}\\
&\qquad\leqslant e^{-2\overline{w}_k}\Big(\int_M |Q_{g_k}|\,\mathrm{d}\mu_{g_k} + \big\|h_k\big\|_{L^2(M,g_k)}\Big)^{1/2}\Big(\int_M e^{-4\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0}\Big)^{1/2}.
\end{aligned}$$

This, the lower bound of $\overline{w}_k$, (4.7), (4.10), and (4.24) imply that

$$\int_M \big(\alpha_k |f_{\lambda_k}|\big)^{2^{-1}} e^{2v_k - 2\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0} \leqslant C \tag{4.25}$$

for some constant $C > 0$. For any integer $m \geqslant 1$, thanks to (4.24) and (4.25), we do iteration to get that

$$\begin{aligned}
&\int_M \big(\alpha_k |f_{\lambda_k}|\big)^{2^{-m}} e^{2^{2-m}v_k - 2\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0}\\
&\quad\leqslant \Big(\int_M \big(\alpha_k |f_{\lambda_k}|\big)^{2^{1-m}} e^{2^{3-m}v_k - 2\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0}\Big)^{2^{-1}}\Big(\int_M e^{-4\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0}\Big)^{2^{-1}}\\
&\quad\leqslant \cdots \leqslant C\Big(\int_M \big(\alpha_k |f_{\lambda_k}|\big)^{2^{-1}} e^{2v_k - 2\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0}\Big)^{2^{1-m}} \leqslant C
\end{aligned} \tag{4.26}$$

for some new constant $C > 0$. By choosing $m \geqslant 1$ large enough such that $2^{1-m} < \gamma$ and fixing it, we have

$$2^{2-m}v_k - 2\gamma v_k^{(-)} = 2\big[2^{1-m}v_k^{(+)} + (2^{1-m} - \gamma)v_k^{(-)}\big] \geqslant 2^{2-m}v_k^{(+)} \geqslant 0.$$

This implies that

$$\liminf_{k\to+\infty} \int_M |f_{\lambda_k}|^{2^{-m}} e^{2^{2-m}v_k - 2\gamma v_k^{(-)}}\,\mathrm{d}\mu_{g_0} \geqslant \int_M |f_0|^{2^{-m}}\,\mathrm{d}\mu_{g_0} > 0.$$

Substituting this estimate into (4.26) gives

$$\liminf_{k\to+\infty} \alpha_k^{2^{-m}} \leqslant C\Big(\int_M |f_0|^{2^{-m}}\,\mathrm{d}\mu_{g_0}\Big)^{-1} < +\infty,$$

which contradicts the fact that $\alpha_k \to +\infty$ as $k \to +\infty$ established in Step 3. This contradiction implies that there exists at least one point $x \in M$ such that (4.12) holds. Moreover, the bound of total Q-curvature (4.10) shows that there can only have finitely many points in M such that (4.12) holds. Let us denote by I the number of such points and for clarity these points will be denoted by $x_\infty^{(i)}$ with $1 \leqslant i \leqslant I$. This completes PART 1.

PART 2. Proof of $f_0(x_\infty^i) = 0$ for $1 \leqslant i \leqslant I$ and $I \leqslant 8$.

Suppose that, for some $i \in \{1, 2, ..., I\}$, we have $f_0(x_\infty^{(i)}) < 0$. Then, on one hand, for sufficiently small $\varepsilon > 0$, we may find some $r > 0$ such that

$$f_{\lambda_k} \leqslant -\varepsilon/2$$

on $B_r(x_\infty^i)$ for k sufficiently large. On the other hand, again we make use of the estimate $Q_{g_k}^+ \leqslant (\alpha_k f_{\lambda_k})^+ + |h_k| \leqslant |h_k|$ to get

$$\int_{B_r(x_\infty^i)} Q_{g_k}^+ \, \mathrm{d}\mu_{g_k} \leqslant \int_{B_r(x_\infty^i)} \big(\alpha_k f_{\lambda_k}\big)^+ \mathrm{d}\mu_{g_k} + \int_M |h_k| \, \mathrm{d}\mu_{g_k} \leqslant \|h_k\|_{L^2(M,g_k)}^2,$$

thanks to Hölder's inequality and $\mathrm{vol}(M, g_k) = 1$. Thus $\int_{B_r(x_\infty^i)} Q_{g_k}^+ \, \mathrm{d}\mu_{g_k} \to 0$ as $k \to +\infty$, which contradicts (4.12). Thus, (4.11) holds. Finally, the estimate $I \leqslant 8$ follows from the inequality

$$\limsup_{k \to +\infty} \int_M Q_{g_k}^+ e^{4w_k} \, \mathrm{d}\mu_{g_0} \leqslant 64\pi^2$$

in (4.10) and the inequality (4.12). □

An immediate consequence of Lemma 4.6 is the following

$$8\pi^2 - o(1)_{k \nearrow +\infty} \leqslant \alpha_k \lambda_k \leqslant 64\pi^2 + o(1)_{k \nearrow +\infty}. \tag{4.27}$$

4.3. **Blow-up analysis.** In this subsection, we derive the blow-up behavior for the functions w_k in (4.5), namely

$$\mathbf{P}_{g_0} w_k = Q_{g_k} e^{4w_k} = \big(\alpha_k f_{\lambda_k} + h_k\big) e^{4w_k}$$

under the following two hypotheses

$$\limsup_{k \to +\infty} (\lambda_k \alpha_k) \leqslant 64\pi^2$$

and

$$\|h_k\|_{L^2(M,g_k)} = o(1)_{k \nearrow +\infty}.$$

We also characterize the shape of the associated conformal metrics $g_k = e^{2w_k} g_0$ as $k \to +\infty$.

First we consider the non-degenerate case.

4.3.1. *Non-degenerate case.*

Theorem 4.7. *Assume that the Paneitz operator $\mathbf{P}_{g_0}$ is positive with kernel consisting of constant functions. Let $f_0 \leqslant 0$ be a smooth, non-constant function with $\max_M f_0 = 0$ having only non-degenerate maximum points. Then for w_k as in (4.5) above and suitable $I \in \mathbb{N}$, $r_k^{(i)} \searrow 0$, $x_k^{(i)} \to x_\infty^{(i)} \in M$ with $f_0(x_\infty^{(i)}) = 0$, $1 \leqslant i \leqslant I$, as $k \to +\infty$ the following hold:*

(i) *$w_k \to -\infty$ locally uniformly on $M_\infty = M \backslash \{x_\infty^{(i)}; 1 \leqslant i \leqslant I\}$.*
(ii) *In normal coordinates around $x_\infty^{(i)}$, set $z_k^{(i)} = \exp_{x_\infty^{(i)}}^{-1}(x_k^{(i)})$ and $\widetilde{w}_k = w_k \circ \exp_{x_\infty^{(i)}}$. Then for each $1 \leqslant i \leqslant I$, either*

(a) $\limsup_{k\to+\infty} r_k^{(i)}/\sqrt{\lambda_k}=0$ *and*

$$\widehat{w}_k(z):=\widetilde{w}\big(z_k^{(i)}+r_k^{(i)}z\big)+\log r_k^{(i)}\to\widehat{w}_\infty(z)$$

strongly in $H^4_{\rm loc}(\mathbf{R}^4)$, *where* $\widehat{w}_\infty$, *up to a translation and a scaling, is given by*

$$\widehat{w}_\infty(z)=\log\Big(\frac{4\sqrt{6}}{4\sqrt{6}+|z|^2}\Big)$$

and it induces a spherical metric

$$\widehat{g}_\infty=e^{4\widehat{w}_\infty}g_{\mathbf{R}^4}$$

of Q*-curvature*

$$Q_{\widehat{g}_\infty}\equiv 1$$

on $\mathbf{R}^4$ *and* $1\leqslant I\leqslant 4$, *or*

(b) $\limsup_{k\to+\infty} r_k^{(i)}/\sqrt{\lambda_k}>0$ *and*

$$\widehat{w}_k(z):=\widetilde{w}\big(z_k^{(i)}+r_k^{(i)}z\big)+\log(r_k^{(i)})\to\widehat{w}_\infty(z)$$

strongly in $H^4_{\rm loc}(\mathbf{R}^4)$, *where* $\widehat{w}_\infty$, *up to a translation and a scaling, solves*

$$\Delta_z^2\widehat{w}_\infty(z)=\Big(1+\frac{1}{2}{\rm Hess}_{f_0}(x_\infty^{(i)})\big[z,z\big]\Big)e^{4\widehat{w}_\infty}.$$

In addition, the metric

$$\widehat{g}_\infty=e^{4\widehat{w}_\infty}g_{\mathbf{R}^4}$$

on $\mathbf{R}^4$ *has finite volume and finite total* Q*-curvature*

$$Q_{\widehat{g}_\infty}(z)=1+\frac{1}{2}{\rm Hess}_{f_0}(x_\infty^{(i)})\big[z,z\big]$$

and $1\leqslant I\leqslant 8$.

Proof. Our proof consists of two parts.

PART 1. We establish Part (i) of the theorem. Recall that

$$M_\infty=M\backslash\{x_\infty^{(i)}:1\leqslant i\leqslant I\}$$

and let $x\in M_\infty$ be arbitrary. Then it follows from Lemma 4.6 that there exists a radius $r_x>0$ perhaps depending on x such that for large k we have

$$\int_{B_{r_x}(x)}Q_{g_k}^+\,{\rm d}\mu_{g_k}<8\pi^2.$$

Following the same notations defined in (4.20), we split

$$v_k=v_k^{(+)}+v_k^{(-)}.$$

Also, we let $\gamma=1/17$. Since the preceding estimate serves the same role as that of (4.13) in the proof of Lemma 4.6, by repeating a similar argument used in the proof of Lemma 4.6 to get (4.22), we find that

$$\big\|e^{v_k^{(+)}}\big\|_{L^{s_x}(B_{r_x}(x))}+\big\|e^{-\gamma v_k^{(-)}}\big\|_{L^{s_x}(B_{r_x}(x))}\leqslant C$$

for some $s_x>4$, which could also depend on x.

Now, given any open subset $\Omega\subset\overline{\Omega}\subset M_\infty$, our aim in this part is to show that $w_k\to-\infty$ uniformly in Ω. To this purpose, we first cover $\overline{\Omega}$ by finitely many balls $B_{r_j/2}(x_j)$, $1\leqslant j\leqslant N_0$, in such a way that for each ball $B_j:=B_{r_j}(x_j)$ with $1\leqslant j\leqslant N_0$ we still have

$$\int_{B_j}Q_{g_k}^+\,{\rm d}\mu_{g_k}<8\pi^2.$$

Set $s = \min_{1 \leqslant j \leqslant N_0} s_{x_j}$. Then it follows from the uniform boundedness of $(v_k^{(+)})$ in M that

$$\|e^{v_k^{(+)}}\|_{L^s(\Omega)} \leqslant \sum_{1 \leqslant j \leqslant N_0} \|e^{v_k^{(+)}}\|_{L^s(B_j)} \leqslant C.$$

We may assume that Ω is connected and large enough so that $\int_\Omega f_0 \, d\mu_{g_0} < 0$. If there holds $\overline{w}_k \geqslant -C > -\infty$, then we may argue as in Case 2 of Step 4 in PART 1 of the proof of Lemma 4.6 to obtain

$$\liminf_{k \to +\infty} \alpha_k^{2^{-m}} \leqslant C\Big(\int_\Omega |f_0|^{2^{-m}} \, d\mu_{g_0}\Big)^{-1} < +\infty,$$

which contradicts the fact that $\alpha_k \to +\infty$ as $k \to +\infty$. Hence, we must have

$$\overline{w}_k \to -\infty$$

as $k \to +\infty$. Then it follows from the uniform boundedness of $(v_k^{(+)})$ that

$$w_k = v_k + \overline{w}_k \leqslant v_k^{(+)} + \overline{w}_k \to -\infty$$

as $k \to +\infty$. This finishes the proof of Part (i).

PART 2. Starting from now to the rest of the proof, we establish Part (ii) of the theorem, namely, the blow-up behavior near each point $x_\infty^{(i)}$ with $1 \leqslant i \leqslant I$. Since the proof of this part is rather long, we also divide it into several claims. Before doing so, we devote ourselves to preliminaries necessary for the blow-up analysis below. For simplicity, we denote $x_0 = x_\infty^{(i)}$. Let i_g be the injectivity radius of M. Clearly $i_g > 0$ since M is compact and the restriction of $\exp_{x_0}$ to $\{X \in T_{x_0}M : \|X\|_{g_0} < i_g\}$ induces a diffeomorphism onto $B_{i_g}(x_0)$. Assimilating $(T_{x_0}M, g_0(x_0))$ to $(\mathbf{R}^n, dz^2)$ isometrically, one can then consider $\exp_{x_0}$ as a local chart around the point x_0. This allows us to select $\delta_0 \in (0, i_g/2)$ sufficiently small such that for all $x \in M$ and all $y, z \in \mathbf{R}^4$, if $|y| \leqslant \delta_0$ and $|z| \leqslant \delta_0$, then

$$\frac{|y - z|}{2} \leqslant d_g(\exp_x(y), \exp_x(z)) \leqslant 2|y - z|; \tag{4.28}$$

see [DER04, page 43]. Let $\delta < \min\{1, \delta_0\}$ and denote by $\widehat{B}_\delta(0)$ the open ball $\{x \in \mathbf{R}^4 : |x| < \delta\}$ in $\mathbf{R}^4$. As always, we often use either a hat or a tilde to denote quantities in $\mathbf{R}^4$. We now consider the exponential map

$$\exp_{x_0} : \widehat{B}_\delta(0) \to M$$

with $\exp_{x_0}(0) = x_0$. We can also assume that $\delta > 0$ is chosen sufficiently small in order to guarantee that the only maxima of f_0 in $\exp_{x_0}(\widehat{B}_\delta(0))$ is x_0. Since $\exp_{x_0}$ is an isometric diffeomorphism onto $B_{i_g}(x_0)$, we deduce that

$$\exp_{x_0}(\widehat{B}_r(0)) = B_r(x_0)) \tag{4.29}$$

whenever $r < \delta_0$, while by (4.28) it is not hard to see that

$$\exp_{x_0}(\widehat{B}_r(z)) \subset B_{2r}(\exp_{x_0}(z)) \tag{4.30}$$

whenever $|z| + r < \delta_0$ and that

$$B_r(\exp_{x_0}(z)) \subset \exp_{x_0}(\widehat{B}_{2r}(z)) \tag{4.31}$$

whenever $|z| + r < \delta_0$. Combining (4.30) and (4.31) gives

$$\widehat{B}_{r/2}(\exp_{x_0}^{-1}(y)) \subset \exp_{x_0}^{-1}(B_r(y)) \subset \widehat{B}_{2r}(\exp_{x_0}^{-1}(y)), \tag{4.32}$$

which is often used throughout the paper. Given a function h on M we denote

$$\widetilde{h} = h \circ \exp_{x_0};$$

we also consider the pull-back metric

$$\widetilde{g}_0 = \exp_{x_0}^* g_0.$$

Since x_0 is a non-degenerate maxima of f_0, up to an action of the orthogonal group, we may assume that $\widetilde{f}_0$ has the following expansion

$$\widetilde{f}_0(z) = -\sum_{i=1}^{4} a_i \big(z^i\big)^2 + O(|z|^3), \tag{4.33}$$

for any $z = (z^1, ..., z^4) \in \widehat{B}_\delta(0)$ with $0 < a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. If we choose δ even smaller, then we can further assume that

$$-\frac{3a_4}{2}|z|^2 \leqslant \widetilde{f}_0(z) \leqslant -\frac{a_1}{2}|z|^2$$

for all $z \in \widehat{B}_\delta(0)$. From now on let us consider large k in such a way that

$$\lambda_k/a_1 < \delta^4/24.$$

We also set

$$A_k = \big\{x \in M : \alpha_k f_{\lambda_k}(x) + \overline{\overline{h}}_k \geqslant 0\big\} \cap \overline{B_\delta(x_0)},$$

where $\overline{\overline{h}}_k$ is defined by (4.8). Clearly,

$$\big\{x \in M : \alpha_k f_{\lambda_k}(x) + \overline{\overline{h}}_k \geqslant 0\big\} = \Big\{x \in M : \lambda_k + \frac{\overline{\overline{h}}_k}{\alpha_k} \geqslant -f_0(x)\Big\}.$$

Combining (4.9) and (4.27) gives $\lambda_k + |\overline{\overline{h}}_k|/\alpha_k \leqslant (3/2)\lambda_k$. From this, the estimate $-\widetilde{f}_0(z) \geqslant (a_1/2)|z|^2$ in $\widehat{B}_\delta(0)$, and (4.29) we conclude that

$$\exp_{x_0}^{-1}(A_k) \subset \widehat{B}_{\sqrt{3\lambda_k/a_1}}(0). \tag{4.34}$$

Claim 1. There exists a constant $\rho_0 > 0$ such that for each $\rho \in (0, \rho_0)$, there exists a sequence of positive numbers $(r_k)_k$ and a sequence of points $(x_k)_k \subset \overline{B_\delta(x_0)}$ satisfying

$$\begin{cases} 0 < r_k \leqslant \sqrt{3\lambda_k/a_1}, \\ \displaystyle\int_{B_{r_k}(x_k)} e^{4w_k}\, \mathrm{d}\mu_{g_0} = \rho, \\ x_k \to x_0 \quad \text{and} \quad w_k(x_k) \to +\infty, \text{ as } k \to \infty, \\ \displaystyle\int_{B_{r_k}(y)} e^{4w_k}\, \mathrm{d}\mu_{g_0} \leqslant \rho, \qquad \text{for all } \ y \in B_{\sqrt{r_k}}(x_k). \end{cases}$$

Proof of Claim 1. It follows from Lemma 4.6 and $Q_{g_k}^+ \leqslant (\alpha_k f_{\lambda_k} + \overline{\overline{h}}_k)^+ + |h_k - \overline{\overline{h}}_k|$ that

$$8\pi^2 - o(1)_{k\nearrow+\infty} \leqslant \int_{B_\delta(x_0)} (\alpha_k f_{\lambda_k} + \overline{\overline{h}}_k)^+ e^{4w_k}\, \mathrm{d}\mu_{g_0} + \int_{B_\delta(x_0)} |h_k - \overline{\overline{h}}_k| e^{4w_k}\, \mathrm{d}\mu_{g_0}\,.$$

Making use of (4.9) we further obtain

$$8\pi^2 - o(1)_{k\nearrow+\infty} \leqslant \int_{B_\delta(x_0)} (\alpha_k f_{\lambda_k} + \overline{\overline{h}}_k)^+ e^{4w_k}\, \mathrm{d}\mu_{g_0} + o(1)_{k\nearrow+\infty}.$$

On the other hand, by (4.34), (4.9), and (4.27) we can estimate

$$\begin{aligned} \int_{B_\delta(x_0)} (\alpha_k f_{\lambda_k} + \overline{\overline{h}}_k)^+ e^{4w_k}\, \mathrm{d}\mu_{g_0} &= \int_{A_k} (\alpha_k f_{\lambda_k} + \overline{\overline{h}}_k) e^{4w_k}\, \mathrm{d}\mu_{g_0} \\ &= \int_{\exp_{x_0}^{-1}(A_k)} (\alpha_k \widetilde{f}_{\lambda_k} + \overline{\overline{h}}_k) e^{4\widetilde{w}_k}\, \mathrm{d}\mu_{\widetilde{g}_0} \\ &\leqslant \int_{\exp_{x_0}^{-1}(A_k)} \big(\alpha_k \lambda_k + o(1)_{k\nearrow+\infty}\big) e^{4\widetilde{w}_k}\, \mathrm{d}\mu_{\widetilde{g}_0} \end{aligned}$$

$$\leqslant \big(64\pi^2 + o(1)_{k\nearrow+\infty}\big) \int_{B_{\sqrt{3\lambda_k/a_1}}(x_0)} e^{4w_k}\, \mathrm{d}\mu_{g_0}.$$

Putting all these estimates together, we eventually get

$$8\pi^2 - o(1)_{k\nearrow+\infty} \leqslant \big(64\pi^2 + o(1)_{k\nearrow+\infty}\big) \int_{B_{\sqrt{3\lambda_k/a_1}}(x_0)} e^{4w_k}\, \mathrm{d}\mu_{g_0}.$$

Hence, we have just shown that

$$\int_{B_{\sqrt{3\lambda_k/a_1}}(x_0)} e^{4w_k}\, \mathrm{d}\mu_{g_0} \geqslant \frac{7}{65}. \tag{4.36}$$

Now, if we let

$$\tau(s) = \sup_{x\in\overline{B_\delta(x_0)}} \int_{B_s(x)} e^{4w_k}\, \mathrm{d}\mu_{g_0},$$

then $\tau(0) = 0$ and by (4.36) we know that

$$\tau(\sqrt{3\lambda_k/a_1}) \geqslant \frac{7}{65}.$$

By setting $\rho_0 = 7/65$ and by the continuity of τ, we thus have for each $\rho \in (0, \rho_0)$, there exists some $r_k \in (0, \sqrt{3\lambda_k/a_1})$ such that $\tau(r_k) = \rho$. Furthermore, the compactness of $\overline{B_\delta(x_0)}$ allows us to choose $x_k \in \overline{B_\delta(x_0)}$ such that

$$\int_{B_{r_k}(x_k)} e^{4w_k}\, \mathrm{d}\mu_{g_0} = \sup_{x\in\overline{B_\delta(x_0)}} \int_{B_{r_k}(x)} e^{4w_k}\, \mathrm{d}\mu_{g_0}$$

for each $k \in \mathbb{N}$. This finishes the proof of (4.35a) and (4.35b).

Next, let us show (4.35c). To see this, we assume by contradiction that $w_k(x_k) \leqslant C_w$ for some constant $C_w > 0$. On one hand, by the estimate $(\alpha_k f_{\lambda_k} + \overline{\overline{h}}_k)^+ \geqslant (\alpha_k f_{\lambda_k} + h_k)^+ - |h_k - \overline{\overline{h}}_k|$, Lemma 4.6, and (4.9), we get

$$\begin{aligned}
&\liminf_{k\to+\infty} \int_{B_\delta(x_0)} (\alpha_k f_{\lambda_k} + \overline{\overline{h}}_k)^+ e^{4w_k}\, \mathrm{d}\mu_{g_0}\\
&\qquad\geqslant \liminf_{k\to+\infty} \Big(\int_{B_\delta(x_0)} (\alpha_k f_{\lambda_k} + h_k)^+ e^{4w_k}\, \mathrm{d}\mu_{g_0} - \|h_k - \overline{\overline{h}}_k\|_{L^1(M,g_k)} \Big)\\
&\qquad\geqslant 8\pi^2.
\end{aligned}$$

On the other hand, we have, by (4.27) and (4.9), that

$$\begin{aligned}
\liminf_{k\to+\infty} \int_{B_\delta(x_0)} (\alpha_k f_{\lambda_k} + \overline{\overline{h}}_k)^+ e^{4w_k}\, \mathrm{d}\mu_{g_0} &= \liminf_{k\to+\infty} \int_{A_k} (\alpha_k f_{\lambda_k} + \overline{\overline{h}}_k) e^{4w_k}\, \mathrm{d}\mu_{g_0}\\
&\leqslant \liminf_{k\to+\infty} \Big(e^{4C_w} \alpha_k \lambda_k \int_{A_k} \mathrm{d}\mu_{g_0} \Big)\\
&\leqslant O(\delta^4).
\end{aligned}$$

We thus obtain a contradiction if we choose δ small at the beginning. Thus, we have already established the unboundedness of $w_k(x_k)$. To see why $x_k \to x_0$ as $k \to +\infty$, we assume by contradiction that $x_k \to x_* \neq x_0$. Clearly, $x_* \in M_\infty$ because $(x_k) \subset \overline{B_\delta(x_0)}$. By the result in Part (i) we know that $w_k(x_*) \to -\infty$ which contradicts the fact that $w_k(x_k) \to +\infty$.

Finally, keep in mind that $r_k < \sqrt{3\lambda_k/a_1} < (\delta/2)^2$. This together with the proved fact (4.35c) above immediately implies that $B_{\sqrt{r_k}}(x_k) \subset B_\delta(x_0)$. Hence from our choice of ρ we have

$$\int_{B_{r_k}(y)} e^{4w_k}\, \mathrm{d}\mu_{g_0} \leqslant \rho$$

for all $y \in B_{\sqrt{r_k}}(x_k)$. Thus (4.35d) is proved and we complete the proof of Claim 1. $\square$

Set $z_k = \exp_{x_0}^{-1}(x_k)$. With the choice of r_k and z_k above, we consider in $\mathbf{R}^4$ the translation–dilation

$$\Gamma_k : z \mapsto z_k + r_k z.$$

Clearly, $\Gamma_k(\widehat{D}_{k,\delta}) = \widehat{B}_\delta(0)$, where, given $r > 0$, the set $\widehat{D}_{k,r}$ is defined as follows

$$\widehat{D}_{k,r} := \{z \in \mathbf{R}^4 : |z_k + r_k z| < r\}.$$

Clearly we may rewrite $\widehat{D}_{k,r}$ as $\widehat{D}_{k,r} = \widehat{B}_{r/r_k}(-z_k/r_k)$. Recall that $r_k \to 0$ and $z_k \to 0$ as $k \to +\infty$ by Claim 1. This implies that $z_k/r_k = o(r/r_k)_{k\nearrow+\infty}$. From this we deduce that, for each $r > 0$ fixed, the set $\widehat{D}_{k,r}$ exhausts $\mathbf{R}^4$ as $k \to +\infty$. Next, we consider the scaled metrics

$$\widehat{g}_k = r_k^{-2}\Gamma_k^* \widetilde{g}_0$$

on $\widehat{D}_{k,r}$. Also, we define

$$\widehat{w}_k = \widetilde{w}_k \circ \Gamma_k + \log r_k. \tag{4.37}$$

Making use of (4.32) gives

$$\widehat{B}_{R/2}(0) \subset (\exp_{x_0} \circ \Gamma_k)^{-1}(B_{Rr_k}(x_k)) \subset \widehat{B}_{2R}(0). \tag{4.38}$$

In view of the conformally covariant property of $\mathbf{P}$, there holds $\mathbf{P}_{\widehat{g}_k} = r_k^4 \mathbf{P}_{\Gamma_k^*\widetilde{g}_0}$. Then by a direct computation, the function $\widehat{w}_k$ satisfies

$$\begin{aligned} \mathbf{P}_{\widehat{g}_k} \widehat{w}_k(z) &= r_k^4 \mathbf{P}_{\Gamma_k^*\widetilde{g}_0}\big(\widetilde{w}_k(\Gamma_k(z))\big) \\ &= \big(\alpha_k \widetilde{f}_{\lambda_k}(\Gamma_k(z)) + \widetilde{h}_k \circ \Gamma_k\big) e^{4[\widetilde{w}_k(\Gamma_k(z)) + \log r_k]} \\ &= \widehat{f}_k(z) e^{4\widehat{w}_k(z)}, \end{aligned} \tag{4.39}$$

where

$$\widehat{f}_k = \alpha_k \widetilde{f}_{\lambda_k} \circ \Gamma_k + \widetilde{h}_k \circ \Gamma_k.$$

Using the exponential map, we can rewrite the identity in (4.35b) as follows

$$\rho = \int_{(\exp_{x_0} \circ \Gamma_k)^{-1}(B_{r_k}(x_k))} e^{4\widehat{w}_k} \, \mathrm{d}\mu_{\widehat{g}_k} . \tag{4.40}$$

To rewrite the inequality in (4.35d), first we make use of (4.32) to get

$$\widehat{B}_{r_k/2}(\exp_{x_0}^{-1}(y)) \subset \exp_{x_0}^{-1}(B_{r_k}(y)).$$

Since $\Gamma_k^{-1} : z \mapsto (z - z_k)/r_k$, we deduce that

$$\widehat{B}_{1/2}\Big(\frac{\exp_{x_0}^{-1}(y) - z_k}{r_k}\Big) \subset (\exp_{x_0} \circ \Gamma_k)^{-1}(B_{r_k}(y)).$$

Therefore, the last inequality in (4.35) gives

$$\int_{\widehat{B}_{1/2}\big(\frac{\exp_{x_0}^{-1}(y) - z_k}{r_k}\big)} e^{4\widehat{w}_k} \, \mathrm{d}\mu_{\widehat{g}_k} \leqslant \rho$$

for all $y \in B_{\sqrt{r_k}}(x_k)$. By (4.38), notice that

$$(\exp_{x_0} \circ \Gamma_k)\big(\widehat{B}_{1/(2\sqrt{r_k})}(0)\big) = \exp_{x_0}\big(\widehat{B}_{\sqrt{r_k}/2}(z_k)\big) \subset B_{\sqrt{r_k}}(x_k).$$

Hence, substituting $y = \exp_{x_0}(\Gamma_k(z))$ into the inequality above yields

$$\int_{\widehat{B}_{1/2}(z)} e^{4\widehat{w}_k} \, \mathrm{d}\mu_{\widehat{g}_k} \leqslant \rho \tag{4.41}$$

for any $z \in \widehat{B}_{1/(2\sqrt{r_k})}(0)$. Since the set $\widehat{B}_{1/(2\sqrt{r_k})}(0)$ exhausts $\mathbf{R}^4$ as $k \to +\infty$, we can freely use (4.41) for arbitrary z in any fixed ball provided k is suitably large. In the next step, we provide a more precise estimate on $d(x_k, x_0)$ in terms of λ_k.

Claim 2. There exists some constant $C > 0$ such that

$$d(x_k, x_0) \leqslant C\sqrt{\lambda_k} \tag{4.42}$$

for all k large.

Proof of Claim 2. If this were not true, then we would have $d(x_k, x_0)/\sqrt{\lambda_k} \to +\infty$ as $k \to +\infty$. From the expansion of $\widetilde{f}_0$ in (4.33), the bound for $\alpha_k\lambda_k$ in (4.27), and the inequality $r_k^2/\lambda_k \leqslant 3/a_1$ we obtain for any fixed $R > 2$ with $|z| \leqslant R$

$$\begin{aligned}
\alpha_k|\widetilde{f}_0(\Gamma_k(z))| &\geqslant (\alpha_k\lambda_k)\lambda_k^{-1}\Big(\frac{a_1}{2}|z_k + r_k z|^2\Big)\\
&\geqslant (\alpha_k\lambda_k)\lambda_k^{-1}\frac{a_1}{2}\Big(\frac{1}{2}|z_k|^2 - r_k^2|z|^2\Big)\\
&\geqslant 7\pi^2\Big(\frac{a_1}{4}\frac{|z_k|^2}{\lambda_k} - \frac{3}{2}R^2\Big).
\end{aligned}$$

This together with the fact that $|z_k|/\sqrt{\lambda_k} \to +\infty$ as $k \to +\infty$ implies that

$$\alpha_k|\widetilde{f}_0(\Gamma_k(z))| \to +\infty$$

uniformly in the ball $\widehat{B}_R(0)$. Thus for $K = 65\pi^2/\rho$ there holds $\alpha_k|\widetilde{f}_0(\Gamma_k(z))| \geqslant K$ for all $z \in \widehat{B}_R(0)$ provided k is large enough. Note that $|f_0| = \lambda_k - f_{\lambda_k}$. From this we may write

$$\alpha_k|\widetilde{f}_0(\Gamma_k(z))| = \alpha_k\lambda_k - \alpha_k\widetilde{f}_{\lambda_k}(\Gamma_k(z)).$$

Then by (4.35b), $R > 2$, (4.38), and (4.9), for k large enough, we have the estimate

$$\begin{aligned}
65\pi^2 &\leqslant K\int_{B_{Rr_k/2}(x_k)} e^{4w_k}\,\mathrm{d}\mu_{g_0}\\
&= K\int_{(\exp_{x_0}\circ\Gamma_k)^{-1}(B_{Rr_k/2}(x_k))} e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k}\\
&\leqslant \int_{\widehat{B}_R(0)} \alpha_k|\widetilde{f}_0(\Gamma_k(z))|e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k}\\
&= \int_{\widehat{B}_R(0)} \big(\alpha_k\lambda_k - \alpha_k\widetilde{f}_{\lambda_k}(\Gamma_k(z))\big)e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k}\\
&\leqslant \int_{(\exp_{x_0}\circ\Gamma_k)^{-1}(B_{2Rr_k}(x_k))} \big(\alpha_k\lambda_k - \alpha_k\widetilde{f}_{\lambda_k}(\Gamma_k(z))\big)e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k}\\
&= \int_{B_{2Rr_k}(x_k)} \big(\alpha_k\lambda_k - \alpha_k f_{\lambda_k}\big)e^{4w_k}\,\mathrm{d}\mu_{g_0}\\
&\leqslant \int_M \big(\alpha_k\lambda_k - (\alpha_k f_{\lambda_k} + h_k)\big)e^{4w_k}\,\mathrm{d}\mu_{g_0} + o(1)\\
&\leqslant \alpha_k\lambda_k + o(1)\\
&\leqslant 64\pi^2 + o(1),
\end{aligned}$$

which is impossible for k sufficiently large. This proves (4.42). ☐

Using the expansion of $\widetilde{f}_0$ in (4.33) we may write $\alpha_k\widetilde{f}_{\lambda_k}\circ\Gamma_k$ as

$$\begin{aligned}
\alpha_k\widetilde{f}_{\lambda_k}\big(\Gamma_k(z)\big) &= \alpha_k\lambda_k + \alpha_k\widetilde{f}_0(\Gamma_k(z))\\
&= \alpha_k\lambda_k\big[\lambda_k^{-1}\widetilde{f}_0(\Gamma_k(z)) + 1\big]\\
&= \alpha_k\lambda_k\left[\begin{array}{l} -\displaystyle\sum_{i=1}^4 a_i\Big(\frac{z_k^i}{\sqrt{\lambda_k}} + \frac{r_k}{\sqrt{\lambda_k}}z^i\Big)^2\\ + O\Big(\sqrt{\lambda_k}\Big|\frac{z_k}{\sqrt{\lambda_k}} + \frac{r_k}{\sqrt{\lambda_k}}z\Big|^3\Big) + 1\end{array}\right].
\end{aligned} \tag{4.43}$$

It follows from (4.35) and (4.42) that

$$\frac{r_k}{\sqrt{\lambda_k}} \leqslant \sqrt{3/a_1} \quad \text{and} \quad \frac{|z_k|}{\sqrt{\lambda_k}} \leqslant C.$$

Plugging these into (4.43) and using (4.27) we can find a positive constant C_R such that

$$\alpha_k \big| \widetilde{f}_{\lambda_k}(\Gamma_k(z)) \big| \leqslant C_R \tag{4.44}$$

for any $z \in \widehat{B}_R(0)$.

Claim 3. Let $\widehat{w}_k$ be given in (4.37). Then $\widehat{w}_k$ is bounded in $W^{4,s_0}_{\rm loc}(\mathbf{R}^4)$ for some $s_0 > 1$. Thus, there exists a function $\widehat{w}_\infty$ such that $\widehat{w}_k \to \widehat{w}_\infty$ strongly in $C^{0,\alpha}_{\rm loc}(\mathbf{R}^4) \cap H^2_{\rm loc}(\mathbf{R}^4)$ for any $0 < \alpha < 1 - 1/s_0$. Moreover, there holds

$$\int_{\mathbf{R}^4} e^{4\widehat{w}_\infty} \, {\rm d}z \leqslant 1. \tag{4.45}$$

Proof of Claim 3: We borrow the method used in the proof of [Mal06, Proposition 3.4]. Let $R > 8$ be arbitrary but fixed. Then we define a smooth cut-off function η_R with

$$\eta_R(z) = \begin{cases} 1 & \text{if } z \in \widehat{B}_{R/2}(0), \\ 0 & \text{if } z \in \mathbf{R}^4 \setminus \widehat{B}_{2R}(0). \end{cases}$$

Set

$$\begin{cases} a_k = \dfrac{1}{|\widehat{B}_R(0)|} \displaystyle\int_{\widehat{B}_R(0)} \widehat{w}_k \, {\rm d}\mu_{\widehat{g}_k}, \\ \Phi_k = \eta_R \widehat{w}_k + (1-\eta_R) a_k, & \text{and} \\ \widehat{\Phi}_k = \Phi_k - a_k. \end{cases}$$

Then

$$\Phi_k = \begin{cases} \widehat{w}_k & \text{on } \widehat{B}_{R/2}(0), \\ a_k & \text{on } \mathbf{R}^4 \setminus \widehat{B}_{2R}(0). \end{cases}$$

In particular, $\widehat{\Phi}_k = 0$ in $\mathbf{R}^4 \setminus \widehat{B}_{2R}(0)$. Hence, $\widehat{\Phi}_k$ has a uniform compact support. Observe that $\widehat{\Phi}_k = \eta_R\big(\widehat{w}_k - a_k\big)$. From this and the equation satisfied by $\widehat{w}_k$ in (4.39), it is not hard to see that $\widehat{\Phi}_k$ satisfies the following equation

$$\mathbf{P}_{\widehat{g}_k} \widehat{\Phi}_k = \eta_R \mathbf{P}_{\widehat{g}_k} \widehat{w}_k + L_k\big(\widehat{w}_k - a_k\big) = \varphi_k, \tag{4.46}$$

where

$$\varphi_k = \eta_R \widehat{f}_k e^{4\widehat{w}_k} + L_k\big(\widehat{w}_k - a_k\big).$$

Note that in (4.46), $(L_k)_k$ are linear operators containing derivatives of order $0, 1, 2$ and 3 with uniformly bounded and smooth coefficients. Therefore, by Lemma 3.1 and some scaling argument one can easily find that

$$\int_{\widehat{B}_{2R}(0)} \big(|\nabla^3 \widehat{w}_k|^s + |\nabla^2 \widehat{w}_k|^s + |\nabla \widehat{w}_k|^s \big) \, {\rm d}\mu_{\widehat{g}_k} \leqslant C_R \tag{4.47}$$

for any $k \in \mathbb{N}$ and any $s \in [1, 4/3)$. Since $\widehat{\Phi}_k$ has compact support and $\widehat{\Phi}_k = \widehat{w}_k - a_k$ in $\widehat{B}_{R/2}(0)$, we can apply L^s-Poincaré's inequality to get

$$\int_{\widehat{B}_{R/2}(0)} |\widehat{\Phi}_k|^s \, {\rm d}\mu_{\widehat{g}_k} \leqslant C_R \tag{4.48}$$

for any $k \in \mathbb{N}$ and any $s \in [1, 4/3)$. It follows from (4.47) and (4.48) that

$$\int_{\widehat{B}_{R/2}(0)} \big| L_k\big(\widehat{w}_k - a_k\big) \big|^s \, {\rm d}\mu_{\widehat{g}_k} \leqslant C_R \tag{4.49}$$

for any $k \in \mathbb{N}$ and any $s \in [1, 4/3)$. We now use (4.49) together with Hölder's inequality to conclude, for any $z \in \widehat{B}_{R/4}(0)$ and $r > 0$ sufficiently small, that

$$\int_{\widehat{B}_r(z)} \big|L_k\big(\widehat{w}_k - a_k\big)\big| \,\mathrm{d}\mu_{\widehat{g}_k} \leqslant O(r). \tag{4.50}$$

On the other hand, it follows from the boundedness of $\|h_k\|_{L^2(M,g_k)}$ in (4.7), the boundedness of $\alpha_k \widetilde{f}_{\lambda_k} \circ \Gamma_k$ in (4.44), and the estimate of $\int e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k}$ in (4.41) that

$$\begin{aligned}\int_{\widehat{B}_r(z)} \big|\widehat{f}_k e^{4\widehat{w}_k}\big|\,\mathrm{d}\mu_{\widehat{g}_k} &\leqslant \int_{\widehat{B}_r(z)} \alpha_k \big|\widetilde{f}_{\lambda_k} \circ \Gamma_k\big| e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k} + \int_{\widehat{B}_r(z)} \big|\widetilde{h}_k \circ \Gamma_k\big| e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k}\\ &\leqslant C_R \int_{\widehat{B}_r(z)} e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k} + \big\|h_k\big\|_{L^2(M,g_k)}\\ &\leqslant C_R\rho + \varepsilon_k \leqslant 2C_R\rho\end{aligned} \tag{4.51}$$

for any $z \in \widehat{B}_{R/4}(0)$, $r > 0$ small, and k large. Hence, by choosing $r > 0$ and $\rho > 0$ suitably small, we obtain from (4.50) and (4.51) the following estimate

$$\int_{\widehat{B}_r(z)} |\varphi_k|\,\mathrm{d}\mu_{\widehat{g}_k} < 8\pi^2$$

for all $z \in \widehat{B}_{R/4}(0)$. Then, it follows from the equation solved by $\widehat{\Phi}_k$ in (4.46), Remark 3.3 and a finite covering argument that there exists some $s_1 > 1$ such that

$$\int_{\widehat{B}_{R/4}(0)} e^{4s_1\widehat{\Phi}_k}\,\mathrm{d}\mu_{\widehat{g}_k} \leqslant C, \tag{4.52}$$

where $C > 0$ is a fixed constant.

Next, we show that a_k is bounded. To see this, it follows from Jensen's inequality, (4.30) and the fact that $\int_M e^{4w_k}\,\mathrm{d}\mu_{g_0} = 1$ that

$$\begin{aligned} a_k &= \frac{1}{|\widehat{B}_R(0)|}\int_{\widehat{B}_R(0)} \widehat{w}_k\,\mathrm{d}\mu_{\widehat{g}_k}\\ &\leqslant \frac{1}{4}\log\Big(\frac{1}{|\widehat{B}_R(0)|}\int_{\widehat{B}_R(0)} e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k}\Big)\\ &\leqslant \frac{1}{4}\log\Big(\frac{1}{|\widehat{B}_R(0)|}\int_{B_{2Rr_k}(x_k)} e^{4w_k}\,\mathrm{d}\mu_{g_0}\Big) \leqslant C_R.\end{aligned}$$

To bound a_k from below, we recall from (4.40) the following

$$\int_{(\exp_{x_0}\circ\Gamma_k)^{-1}(B_{r_k}(x_k))} e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k} = \rho.$$

Making use of (4.38) gives

$$(\exp_{x_0}\circ\Gamma_k)^{-1}(B_{r_k}(x_k)) \subset \widehat{B}_2(0).$$

Consequently, for k large and because $R/4 > 2$, we arrive at

$$\int_{\widehat{B}_{R/4}(0)} e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k} \geqslant \rho.$$

This together with the fact that $\Phi_k = \widehat{w}_k$ in $\widehat{B}_{R/4}(0)$, we obtain

$$\rho \leqslant \int_{\widehat{B}_{R/4}(0)} e^{4\Phi_k}\,\mathrm{d}\mu_{\widehat{g}_k} = e^{4a_k}\int_{\widehat{B}_{R/4}(0)} e^{4\widehat{\Phi}_k}\,\mathrm{d}\mu_{\widehat{g}_k},$$

which implies by (4.52) that $a_k \geqslant -C_R$ and hence we find

$$|a_k| \leqslant C_R.$$

Using this fact, we have by (4.48) and Minkowski's inequality that

$$\int_{\widehat{B}_{R/2}(0)} |\widehat{w}_k|^s \, d\mu_{\widehat{g}_k} \leqslant C_R \tag{4.53}$$

for all $s \in [1, 4/3)$ and by (4.52) that

$$\int_{\widehat{B}_{R/4}(0)} e^{4s_1 \widehat{w}_k} \, d\mu_{\widehat{g}_k} \leqslant C_R. \tag{4.54}$$

Now, we take $1 < s_2 < \min\{s_1, 2\}$ and let $s_0 = 2s_2/(1+s_2)$. Then $1 < s_0 < \min\{4/3, s_1\}$ and $s_2 = s_0/(2-s_0)$. Using the boundedness of $\alpha_k \widetilde{f}_{\lambda_k} \circ \Gamma_k$ in (4.44), Hölder's inequality, the estimate of $\|h_k\|_{L^2(M,g_k)}$ in (4.7), and (4.54) we can bound

$$\begin{aligned}
\int_{\widehat{B}_{R/4}(0)} \big|\widehat{f}_k e^{4\widehat{w}_k}\big|^{s_0} d\mu_{\widehat{g}_k} \leqslant & C \int_{\widehat{B}_{R/4}(0)} \big|\alpha_k \widetilde{f}_{\lambda_k} \circ \Gamma_k\big|^{s_0} e^{4s_0 \widehat{w}_k} \, d\mu_{\widehat{g}_k} \\
& + C \int_{\widehat{B}_{R/4}(0)} |\widetilde{h}_k \circ \Gamma_k|^{s_0} e^{2s_0 \widehat{w}_k} e^{2s_0 \widehat{w}_k} \, d\mu_{\widehat{g}_k} \\
\leqslant & C_R \Big(\int_{\widehat{B}_{R/4}(0)} e^{4s_1 \widehat{w}_k} \, d\mu_{\widehat{g}_k} \Big)^{s_0/s_1} \\
& + C \Big(\int_{\widehat{B}_{R/4}(0)} |\widetilde{h}_k \circ \Gamma_k|^2 e^{4\widehat{w}_k} \, d\mu_{\widehat{g}_k} \Big)^{s_0/2} \\
& \times \Big(\int_{\widehat{B}_{R/4}(0)} e^{4s_2 \widehat{w}_k} \, d\mu_{\widehat{g}_k} \Big)^{s_0/(2s_2)} \\
\leqslant & C_R + C_R \|h_k\|_{L^2(M,g_k)} \Big(\int_{\widehat{B}_{R/4}(0)} e^{4s_1 \widehat{w}_k} \, d\mu_{\widehat{g}_k} \Big)^{s_0 s_1/(2s_2^2)} \\
\leqslant & C_R.
\end{aligned} \tag{4.55}$$

Plugging (4.55) into (4.39) gives

$$\int_{\widehat{B}_{R/4}(0)} |\mathbf{P}_{\widehat{g}_k} \widehat{w}_k|^{s_0} \, d\mu_{\widehat{g}_k} \leqslant C_R,$$

which together with (4.53) implies that $\widehat{w}_k$ is bounded in $W^{4,s_0}_{\rm loc}(\mathbf{R}^4)$. In particular, Sobolev embedding theorem implies that $\widehat{w}_k \to \widehat{w}_\infty$ strongly in $C^{0,\alpha}_{\rm loc}(\mathbf{R}^4) \cap H^2_{\rm loc}(\mathbf{R}^4)$ with $0 < \alpha < 1 - 1/s_0$. It remains to establish (4.45). Indeed, by Fatou's lemma, (4.38) and the fact that $\int_M e^{4w} \, d\mu_{g_0} = 1$ we obtain

$$\begin{aligned}
\int_{\widehat{B}_R(0)} e^{4\widehat{w}_\infty} \, dz & \leqslant \liminf_{k\to+\infty} \int_{\widehat{B}_R(0)} e^{4\widehat{w}_k} \, d\mu_{\widehat{g}_k} \\
& \leqslant \liminf_{k\to+\infty} \int_{B_{2Rr_k}(x_k)} e^{4w_k} \, d\mu_{g_0} \leqslant 1.
\end{aligned}$$

Passing to the limit $R \to +\infty$ we find

$$\int_{\mathbf{R}^4} e^{4\widehat{w}_\infty} \, dz \leqslant 1.$$

We thus finish the proof of Claim 3. □

Claim 4. The assertions in Theorem 4.7(ii) hold true.

Proof of Claim 4. Since $0 < r_k/\sqrt{\lambda_k} \leqslant \sqrt{3/a_1}$ by (4.35), we have two possibilities.

Case 1. There holds $\limsup_{k\to+\infty} r_k/\sqrt{\lambda_k} = 0$. In this scenario, recall that the estimate $d(x_k, x_0) = O(\sqrt{\lambda_k})_{k\nearrow+\infty}$ in (4.42) implies that $|z_k| = O(\sqrt{\lambda_k})_{k\nearrow+\infty}$ if k is large enough. This together with (4.27) and (4.43) implies that there exists some $r_0 \leqslant 64\pi^2$ such that, up to a subsequence,

$$\lim_{k\to+\infty} \alpha_k \widetilde{f}_{\lambda_k}\big(\Gamma_k(z)\big) = r_0 \tag{4.56}$$

uniformly in any fixed ball $\widehat{B}_R(0)$. Next we derive the equation for the limit function $\widehat{w}_\infty$ in Claim 3. We multiply by a smooth function φ with compact support on the both sides of equation (4.39) and then do integrating by parts to obtain

$$\langle \mathbf{P}_{\widehat{g}_k}\widehat{w}_k, \varphi\rangle = \int_{\mathbf{R}^4} \alpha_k \widetilde{f}_{\lambda_k}\circ\Gamma_k e^{4\widehat{w}_k}\varphi \,\mathrm{d}\mu_{\widehat{g}_k} + \int_{\mathbf{R}^4} \widetilde{h}_k\circ\Gamma_k e^{4\widehat{w}_k}\varphi \,\mathrm{d}\mu_{\widehat{g}_k}.$$

By the fact that $\widehat{g}_k \to (dz)^2$ in $C^\infty_{\rm loc}(\mathbf{R}^4)$ and the estimate of $\|h_k\|_{L^2(M,g_k)}$ in (4.7) we send k to infinity in the equality above to conclude that the function $\widehat{w}_\infty$ solves the equation

$$\Delta_z^2 \widehat{w}_\infty = r_0 e^{4\widehat{w}_\infty} \tag{4.57}$$

in $\mathbf{R}^4$.

Since in this case we can obtain a very precise form for w_∞ from (4.57), we need more work by showing that $r_0 > 0$. Indeed, suppose that this is not true, then we are led to two cases: $r_0 = 0$ or $r_0 < 0$. When $r_0 = 0$, it follows from (4.57) and (4.45) that the function $\widehat{w}_\infty$ solves

$$\Delta_z^2 \widehat{w}_\infty = 0$$

with the finite energy condition

$$\int_{\mathbf{R}^4} e^{4\widehat{w}_\infty}\,\mathrm{d}z < +\infty.$$

Now it follows from [Mar09, Theorem 3] that $\widehat{w}_\infty$ is a polynomial of order exactly two, which is also bounded in $\mathbf{R}^4$. Consequently, $\widehat{w}_\infty$ is at most linear. Therefore, we can make use of [ARS06, Theorem 2.4] to conclude that

$$\Delta_z \widehat{w}_\infty \equiv c_0 > 0$$

everywhere in $\mathbf{R}^4$. Using this fact, on one hand, the strong convergence $\Delta_{\widehat{g}_k}\widehat{w}_k \to \Delta_z\widehat{w}_\infty$ in $L^2_{\rm loc}(\mathbf{R}^4)$ implies, for arbitrary but fixed $R > 0$, that

$$\lim_{k\to+\infty}\int_{\widehat{B}_{R/2}(0)} |\Delta_{\widehat{g}_k}\widehat{w}_k|\,\mathrm{d}\mu_{\widehat{g}_k} = \int_{\widehat{B}_{R/2}(0)} |\Delta_z\widehat{w}_\infty|\,\mathrm{d}z = \frac{c_0}{4}\pi^2\Big(\frac{R}{2}\Big)^4.$$

However, on the other hand, we can estimate

$$\begin{aligned}
&\int_{\widehat{B}_{R/2}(0)} |\Delta_{\widehat{g}_k}\widehat{w}_k|\,\mathrm{d}\mu_{\widehat{g}_k}\\
&\quad= r_k^{-2}\int_{\widehat{B}_{Rr_k}(z_k)} |\Delta_{\widetilde{g}_0}\widetilde{w}_k|\,\mathrm{d}\mu_{\widetilde{g}_0}\\
&\quad\leqslant r_k^{-2}\int_{B_{2Rr_k}(x_k)} |\Delta_{g_0} w_k|\,\mathrm{d}\mu_{g_0}\\
&\quad= r_k^{-2}\int_{B_{2Rr_k}(x_k)}\int_M |\Delta_{g_0}\mathbb{G}(x,y)|\big|\alpha_k f_{\lambda_k}(y) + h_k\big|e^{4w_k(y)}\,\mathrm{d}\mu_{g_0}(y)\,\mathrm{d}\mu_{g_0}(x)\\
&\quad\leqslant Cr_k^{-2}\int_M \big|\alpha_k f_{\lambda_k}(y) + h_k\big|e^{4w_k(y)}\int_{B_{2Rr_k}(x_k)} d(x,y)^{-2}\,\mathrm{d}\mu_{g_0}(x)\,\mathrm{d}\mu_{g_0}(y)\\
&\quad\leqslant CR^2\int_M \big|\alpha_k f_{\lambda_k} + h_k\big|e^{4w_k}\,\mathrm{d}\mu_{g_0}\\
&\quad= O(R^2).
\end{aligned}$$

Putting these facts together, we eventually obtain

$$\frac{c_0}{4}\pi^2\Big(\frac{R}{2}\Big)^4 = O(R^2), \tag{4.58}$$

which is impossible if we let R sufficiently large. We now rule out the case $r_0 < 0$. Indeed, in this scenario, we apply [Mar08, Theorem 2] to (4.57) to get

$$\lim_{t\to+\infty} \Delta\widehat{w}_\infty(t\xi) = c_1 \neq 0$$

uniformly in $\xi \in K$ where $K \subset \mathbb{S}^3$ is any compact set with positive Hausdorff measure. Then, for k large enough, we have the following estimates similar to the ones leading to (4.58)

$$\begin{aligned} C\Big(\frac{R}{2}\Big)^4 &\leqslant \int_{\widehat{B}_{R/2}(0)\cap(\mathbf{R}+K)} |\Delta_z \widehat{w}_\infty| \,\mathrm{d}z \\ &\leqslant \lim_{k\to+\infty} \int_{\widehat{B}_{R/2}(0)} |\Delta_{\widehat{g}_k}\widehat{w}_k| \,\mathrm{d}\mu_{\widehat{g}_k} \\ &= O(R^2), \end{aligned}$$

which, again, is a contradiction if R is sufficiently large. Hence, we have proved that $r_0 > 0$. Since $e^{4\widehat{w}_\infty} \in L^1(\mathbf{R}^4)$ by (4.45), the well-known classification theorem in [Lin98] then implies that either there exists a constant $c_0 > 0$ such that

$$-\Delta_z\widehat{w}_\infty \geqslant c_0$$

everywhere in $\mathbf{R}^4$ or there exist some $\mu_0 > 0$ and $z_0 \in \mathbf{R}^4$ such that

$$\widehat{w}_\infty(z) = \log\Big(\frac{2\mu_0}{1+\mu_0^2|z-z_0|^2}\Big) - \frac{1}{4}\log\frac{r_0}{6}. \tag{4.59}$$

We can rule out the first alternative in the same way as (4.58). Hence, the second alternative must occur. Now, recall by Claim 3 that we have the strong convergence $\widehat{w}_k \rightharpoonup \widehat{w}_\infty$ in $C^{0,\alpha}_{\rm loc}(\mathbf{R}^4) \cap H^2_{\rm loc}(\mathbf{R}^4)$ for some $0 < \alpha < 1$. This together with the decomposition

$$\begin{aligned} \mathbf{P}_{\widehat{g}_k}(\widehat{w}_k - \widehat{w}_\infty) + (\mathbf{P}_{\widehat{g}_k} - \Delta_z^2)\widehat{w}_\infty =& \widetilde{h}_k \circ \Gamma_k e^{4\widehat{w}_k} + \big(\alpha_k \widetilde{f}_{\lambda_k} \circ \Gamma_k - r_0\big)e^{4\widehat{w}_k} \\ &+ r_0(e^{4\widehat{w}_k} - e^{4\widehat{w}_\infty}), \end{aligned}$$

(4.7) and (4.56) implies that $\widehat{w}_k \to \widehat{w}_\infty$ strongly in $H^4_{\rm loc}(\mathbf{R}^4)$.

Up to this point, we are ready to estimate the number of blow-up points. Recall that we have already had $I \leqslant 8$, however, in the present case, we aim to show that indeed $I \leqslant 4$. Clearly at each blow-up point, say x_0 as before with the same notations used up to this position for simplicity, from the explicit formula (4.59) we can compute

$$\int_{\mathbf{R}^4} e^{4\widehat{w}_\infty}\,\mathrm{d}z = \frac{6}{r_0}\int_{\mathbf{R}^4}\Big(\frac{2\mu_0}{1+\mu_0^2|z-z_0|^2}\Big)^4 \mathrm{d}z = \frac{16\pi^2}{r_0}.$$

Since $e^{\widehat{w}_k} \to e^{4\widehat{w}_\infty}$ strongly in $L^1_{\rm loc}(\mathbf{R}^4)$ as $k\to+\infty$ and $r_0 \leqslant 64\pi^2$, we have for R and k sufficiently large

$$\frac{15}{64} \leqslant \int_{\widehat{B}_R(0)} e^{4\widehat{w}_k}\,\mathrm{d}\mu_{\widehat{g}_k} \leqslant \int_{B_{2Rr_k}(x_k)} e^{4w_k}\,\mathrm{d}\mu_{g_0}.$$

Since the number of blow-up points is finite, if we choose k even larger, we deduce that the sets $B_{2Rr_k}(x_k) \subset B_\delta(x_0)$ and they are non-overlap at different blow-up points. Keep in my that $\int_M e^{4w_k}\,\mathrm{d}\mu_{g_0} = 1$. From this we deduce that the number of blow-up points must less than or equal to 4, namely $I \leqslant 4$.

Finally, we notice that, up to a translation and a scaling, $\widehat{w}_\infty$ has the form

$$\widehat{w}_\infty(z) = \log\Big(\frac{4\sqrt{6}}{4\sqrt{6}+|z|^2}\Big).$$

Indeed, it suffices to substituting (4.59) into the expression

$$\widehat{w}^*_\infty(z) := \widehat{w}_\infty\Big(e^{-\widehat{w}_\infty(z_0)} r_0^{-1/4} z + z_0\Big) - \widehat{w}_\infty(z_0) - \frac{1}{4}\log\frac{r_0}{6}.$$

We thus obtain the alternative (ii)(a) in Theorem 4.7.

Case 2. We now suppose that $\limsup_{k\to+\infty} r_k/\sqrt{\lambda_k} > 0$. Since $r_k/\sqrt{\lambda_k}$ is bounded from above and $|z_k| = O(\sqrt{\lambda_k})_{k\nearrow+\infty}$, we may assume that

$$\limsup_{k\to+\infty} \frac{r_k}{\sqrt{\lambda_k}} = d_0 > 0$$

and that

$$\limsup_{k\to+\infty} \frac{z_k}{\sqrt{\lambda_k}} = \vec{c}_0$$

for some constant vector $\vec{c}_0$. This together with (4.27) and (4.43) implies that there exists some constant r_0 with $8\pi^2 \leqslant r_0 \leqslant 64\pi^2$ such that

$$\limsup_{k\to+\infty} \widehat{f}_k(z) = r_0\Big(1 + \frac{1}{2}\mathrm{Hess}_{f_0}(x_0)\big[\vec{c}_0 + d_0 z, \vec{c}_0 + d_0 z\big]\Big)$$

uniformly in $\widehat{B}_R(0)$. Arguing the same way as in the proof of Case 1 to obtain (4.57), the limiting function $\widehat{w}_\infty$ solves the equation

$$\Delta_z^2 \widehat{w}_\infty = r_0\Big(1 + \frac{1}{2}\mathrm{Hess}_{f_0}(x_0)\big[\vec{c}_0 + d_0 z, \vec{c}_0 + d_0 z\big]\Big) e^{4\widehat{w}_\infty} \tag{4.60}$$

in $\mathbf{R}^4$. Furthermore, in view of (4.45) and the L^1-bound (4.10), we have

$$\int_{\mathbf{R}^4} e^{4\widehat{w}_\infty}\, \mathrm{d}z < +\infty$$

and

$$\int_{\mathbf{R}^4} \Big|1 + \frac{1}{2}\mathrm{Hess}_{f_0}(x_0)[\vec{c}_0 + d_0 z, \vec{c}_0 + d_0 z]\Big| e^{4\widehat{w}^*_\infty(z)}\, \mathrm{d}z < +\infty.$$

By denoting

$$F_\infty = r_0\Big(1 + \frac{1}{2}\mathrm{Hess}_{f_0}(x_0)[\vec{c}_0 + d_0 z, \vec{c}_0 + d_0 z]\Big),$$

it follows from the decomposition

$$\begin{aligned}\mathbf{P}_{\widehat{g}_k}(\widehat{w}_k - \widehat{w}_\infty) + (\mathbf{P}_{\widehat{g}_k} - \Delta_z^2)\widehat{w}_\infty =& \big(\alpha_k \widetilde{f}_{\lambda_k} \circ \Gamma_k + \widetilde{h}_k \circ \Gamma_k - F_\infty\big) e^{4\widehat{w}_k(z)}\\ &+ F_\infty\big(e^{4\widehat{w}_k} - e^{4\widehat{w}_\infty}\big)\end{aligned}$$

and (4.7) that $\widehat{w}_k \to \widehat{w}_\infty$ strongly in $H^4_{\mathrm{loc}}(\mathbf{R}^4)$ as $k \to +\infty$.

Finally, by performing a translation and a scaling, equation (4.60) can be reduced as

$$\Delta_z^2 \widehat{w}_\infty = \Big(1 + \frac{1}{2}\mathrm{Hess}_{f_0}(x_0)\big[z, z\big]\Big) e^{4\widehat{w}_\infty}.$$

We thus obtain the alternative (ii)(b) in Theorem 4.7. □

The proof of Theorem 4.7 is complete. □

4.3.2. *Degenerate case.* Now we consider the degenerate case. An analogue of Theorem 4.7 is the following result.

Theorem 4.8. *Assume all the conditions, expcept for the assumption of the non-degeneracy of the function f_0 at some maxima, in Theorem 4.7 above. If, in addition, (M, g_0) is locally conformally flat and f_0 satisfies the* **Condition A** *with $d_0, A_0 > 0$, then for w_k defined as in the Theorem 4.7 there exist suitable $I \in \mathbb{N}$ with $I \leqslant 8$, $r_k^{(i)} \searrow 0$ and $x_k^{(i)} \to x_\infty^{(i)} \in M$ with $f_0(x_\infty^{(i)}) = 0$, $1 \leqslant i \leqslant I$ such that the following hold*

(i) $w_k \to -\infty$ *locally uniformly on*

$$M_\infty = M\backslash\{x_\infty^{(i)} : 1 \leqslant i \leqslant I\}.$$

(ii) *For each* $1 \leqslant i \leqslant I$*, we have*

$$\widehat{w}_k(z) := \widetilde{w}_k\big(z_k^{(i)} + r_k^{(i)} z\big) + \log r_k^{(i)} \to \widehat{w}_\infty(z)$$

strongly in $H^4_{\rm loc}(\mathbf{R}^4)$*, where* $z_k^{(i)} = \exp^{-1}_{x_\infty^{(i)}}(x_k^{(i)})$ *and* $\widehat{w}_\infty$ *induces a metric*

$$g_\infty = e^{4\widehat{w}_\infty} g_{\mathbf{R}^4}$$

on $\mathbf{R}^4$ *of locally bounded curvature and of volume less than or equal* 1.

Proof. For simplicity and clarity, we still use the notations in the proof of Theorem 4.7. We first notice that Lemma 4.6 and the upper bound for $\int_M |Q_{g_k}|\, {\rm d}\mu_{g_k}$ as in (4.10) continue to hold even if $f_0(x)$ has a degenerate maxima. Consequently, the bounds for $\alpha_k\lambda_k$ as in (4.27) also holds as well.

PART 1. The proof of statement (i) in Theorem 4.8 is then identical with that of the corresponding statement in Theorem 4.7.

PART 2. We now examine the blow-up behavior of w_k near the blow-up point x_0. Since (M, g_0) is locally comformally flat, we may assume that M is flat around x_0, namely

$$(g_0)_{ij} = \delta_{ij}$$

in $B_\delta(x_0)$ for some fixed but small $\delta > 0$.

Claim 1. There is a constant $\rho_0 > 0$ such that for each $\rho \in (0, \rho_0)$ to be determined later, there exists a sequence of positive numbers $(r_k)_k$ and a sequence of points $(x_k)_k \subset \overline{B_\delta(x_0)}$ satisfying

$$\begin{cases} \lim\limits_{k\to+\infty} r_k = 0,\\ \displaystyle\int_{B_{r_k}(x_k)} e^{4w_k}\, {\rm d}\mu_{g_0} = \rho,\\ x_k \to x_0 \quad \text{and} \quad w_k(x_k) \to +\infty \text{ as } k\to\infty,\\ \displaystyle\int_{B_{r_k}(y)} e^{4w_k}\, {\rm d}\mu_{g_0} \leqslant \rho \quad \text{for all} \quad y \in B_{\sqrt{r_k}}(x_k). \end{cases}$$

Proof of Claim 1. The proof of (4.61) is essentially similar to the proof of (4.35). Notice that in the degenerate case we cannot assert an upper bound for $r_k/\sqrt{\lambda_k}$ as shown in (4.35a). However, we still have the estimate $r_k = o(1)_{k\nearrow+\infty}$ shown in (4.61a). To realize this, we first notice by Lemma 4.6, the estimate $Q^+_{g_k} \leqslant \alpha_k\lambda_k + |h_k|$, (4.27) and (4.7) that

$$8\pi^2 - o(1)_{k\nearrow+\infty} \leqslant \int_{B_r(x_0)} Q^+_{g_k} e^{4w_k}\ {\rm d}\mu_{g_0} \leqslant (64\pi^2 + o(1)_{k\nearrow+\infty}) \int_{B_r(x_0)} e^{4w_k}\ {\rm d}\mu_{g_0} \tag{4.62}$$

for all $r > 0$. In particular, we have

$$\int_{B_\delta(x_0)} e^{4w_k}\ {\rm d}\mu_{g_0} \geqslant \frac{7}{65} =: \rho_0$$

for k large. With the help of the above estimate, we can follow the proof of Claim 1 in Theorem 4.7 to obtain that for each $\rho \in (0, \rho_0)$ there exists $r_k \in (0, \delta)$ such that

$$\sup_{x\in\overline{B_\delta(x_0)}} \int_{B_{r_k}(x)} e^{4w_k}\ {\rm d}\mu_{g_0} = \rho.$$

This implies that

$$\int_{B_{r_k}(x_0)} e^{4w_k} \, \mathrm{d}\mu_{g_0} \leqslant \rho. \tag{4.63}$$

We are now ready to conclude (4.61a). Indeed, by the way of contradiction and up to a subsequence, we may assume that $\lim_{k\to+\infty} r_k = r_0 > 0$. Then, there holds $B_{r_0/2}(x_0) \subset B_{r_k}(x_0)$ provided k is sufficiently large. This together with (4.63) and the choice of ρ yields

$$\int_{B_{r_0/2}(x_0)} e^{4w_k} \, \mathrm{d}\mu_{g_0} \leqslant \rho < \frac{7}{65}.$$

But this contradicts with (4.62) and the proof of (4.61a) is complete. As for the other assertions in (4.61), their proofs are identical with those of Claim 1 in Theorem 4.7. ☐

Lacking of a bound for $r_k/\sqrt{\lambda_k}$ brings us difficulty to obtain a local bound for $\alpha_k \widetilde{f}_{\lambda_k} \circ \Gamma_k$ as in (4.44). However, under an additional hypothesis on the flatness of (M, g_0) we can proceed with some tools developed in [Mar09] together with **Condition A** to regain its local boundedness; see Claim 3 below.

Now, since (M, g_0) is locally comformally flat, we may assume that M is flat around x_0, namely

$$(g_0)_{ij} = \delta_{ij}$$

in $B_\delta(x_0)$ for some fixed but small $\delta > 0$. On one hand, this helps us to conclude that

$$\mathrm{d}\mu_{\widetilde{g}_0} = \exp^*_{x_0}(\mathrm{d}\mu_{g_0}) = \mathrm{d}z\,.$$

This and the relation $\widehat{g}_k = r_k^{-2}\Gamma_k^*\widetilde{g}_0$ imply that

$$\mathrm{d}\mu_{\widehat{g}_k} = r_k^{-4}\,\mathrm{d}\mu_{\Gamma_k^*\widetilde{g}_0} = \mathrm{d}z\,.$$

On the other hand, the Paneitz operator becomes the bi-Laplace operator in $B_\delta(x_0)$. The equation (4.5) becomes

$$\Delta^2 w_k = \alpha_k f_{\lambda_k} e^{4w_k} + h_k e^{4w_k}$$

in $B_\delta(x_0)$. Let $\widehat{w}_k$ be defined as in (4.37), then $\widehat{w}_k$ solves

$$\Delta^2 \widehat{w}_k = \widehat{f}_k e^{4\widehat{w}_k} \tag{4.64}$$

in $\widehat{D}_{k,\delta}$, where, as before,

$$\widehat{f}_k = \alpha_k \widetilde{f}_{\lambda_k} \circ \Gamma_k + \widetilde{h}_k \circ \Gamma_k$$

and

$$\widehat{D}_{k,\delta} := \{z \in \mathbf{R}^4 : |z_k + r_k z| < \delta\}.$$

Also because $\widehat{B}_{1/2}(0) \subset (\exp_{x_0} \circ \Gamma_k)^{-1}(B_{r_k}(x_k)) \subset \widehat{B}_2(0)$ it follows from (4.61b) that

$$\int_{\widehat{B}_{1/2}(0)} e^{4\widehat{w}_k}\,\mathrm{d}z \leqslant \rho \leqslant \int_{\widehat{B}_2(0)} e^{4\widehat{w}_k}\,\mathrm{d}z \tag{4.65}$$

and, similar to (4.41), we rewrite (4.61d) to get

$$\int_{\widehat{B}_{1/2}(z)} e^{4\widehat{w}_k}\,\mathrm{d}z \leqslant \rho, \tag{4.66}$$

for all $z \in \widehat{B}_{1/(2\sqrt{r_k})}(0)$.

Claim 2. The sequence $\widehat{w}_k$ is bounded in $W^{3,s}_{\mathrm{loc}}(\mathbf{R}^4)$ for any $1 < s < 4/3$.

Proof of Claim 2. Fix any $R > 8$, we let $\widehat{w}_k^{(\pm)}$ solve

$$\begin{cases} \Delta^2 \widehat{w}_k^{(\pm)} = (\Delta^2 \widehat{w}_k)^{\pm} & \text{in } \widehat{B}_R(0),\\ \widehat{w}_k^{(\pm)} = 0 & \text{on } \partial\widehat{B}_R(0),\\ \Delta\widehat{w}_k^{(\pm)} = 0 & \text{on } \partial\widehat{B}_R(0). \end{cases} \tag{4.67}$$

Using the maximum principle twice, we obtain $\widehat{w}_k^{(+)} \geqslant 0 \geqslant \widehat{w}_k^{(-)}$. In addition, $\widehat{w}_k$ can be decomposed as

$$\widehat{w}_k = \widehat{w}_k^{(+)} + \widehat{w}_k^{(-)} + \widehat{w}_k^{(0)}, \tag{4.68}$$

where $\widehat{w}_k^{(0)}$ solves

$$\begin{cases} \Delta^2 \widehat{w}_k^{(0)} = 0 & \text{in } \widehat{B}_R(0), \\ \widehat{w}_k^{(0)} = \widehat{w}_k & \text{on } \partial \widehat{B}_R(0), \\ \Delta \widehat{w}_k^{(0)} = \Delta \widehat{w}_k & \text{on } \partial \widehat{B}_R(0). \end{cases}$$

The next goal is to show the boundedness of $\widehat{w}_k^{(+)}$ in $W^{4,s_0}_{\rm loc}(\mathbf{R}^4)$ for some $s_0 > 1$. To see this, we observe, by (4.27), (4.66) and (4.7), the bound

$$\begin{aligned} \int_{\widehat{B}_r(z)} (\Delta^2 \widehat{w}_k)^+ \,{\rm d}z &= \int_{\widehat{B}_r(z)} \widehat{f}_k^+ e^{4\widehat{w}_k} \,{\rm d}z \\ &\leqslant \int_{\widehat{B}_r(z)} \alpha_k \big(\widetilde{f}_k \circ \Gamma_k\big)^+ e^{4\widehat{w}_k} \,{\rm d}z + \int_{\widehat{B}_r(z)} \big|\widetilde{h}_k \circ \Gamma_k\big| e^{4\widehat{w}_k} \,{\rm d}z \\ &\leqslant 65\pi^2 \rho + o(1)_{k\nearrow+\infty} \end{aligned}$$

for all $z \in \widehat{B}_{R/2}(0)$ and for $r > 0$ small. Hence, by choosing ρ sufficiently small we obtain the bound

$$\int_{\widehat{B}_r(z)} \big(\Delta^2 \widehat{w}_k\big)^+ \,{\rm d}z < 8\pi^2$$

for all $z \in \widehat{B}_{R/2}(0)$ and for $r > 0$ small. In view of the equation (4.67) satisfied by $\widehat{w}_k^{(\pm)}$, we can apply [Lin98, Lemma 2.3] and a finite covering argument to find a positive constant $s_1 > 1$ such that

$$\int_{\widehat{B}_{R/2}(0)} e^{4s_1 \widehat{w}_k^{(+)}} \,{\rm d}z \leqslant C_R. \tag{4.69}$$

Keep in mind that $\widehat{f}_k^+ \leqslant \alpha_k \lambda_k + |\widetilde{h}_k \circ \Gamma_k|$. Hence, by repeating an argument used in (4.55) together with (4.69) we can find some $1 < s_0 < \min\{4/3, s_1\}$ such that

$$\begin{aligned} &\int_{\widehat{B}_{R/2}(0)} \big(\widehat{f}_k^+ e^{4\widehat{w}_k}\big)^{s_0} \,{\rm d}z \\ &\qquad \leqslant C \int_{\widehat{B}_R(0)} (\alpha_k \lambda_k)^{s_0} e^{4s_0 \widehat{w}_k} \,{\rm d}z + C \int_{\widehat{B}_R(0)} |\widetilde{h}_k \circ \Gamma_k|^{s_0} e^{4s_0 \widehat{w}_k} \,{\rm d}z \leqslant C_R. \end{aligned}$$

Plugging the estimate above into (4.67) gives

$$\int_{\widehat{B}_{R/2}(0)} |\Delta^2 \widehat{w}_k^{(+)}|^{s_0} \,{\rm d}z \leqslant C_R.$$

This together with Sobolev's inequality implies that $\widehat{w}_k^{(+)}$ is bounded in $W^{4,s_0}_{\rm loc}(\mathbf{R}^4)$. Therefore, $\widehat{w}_k^{(+)}$ is bounded in $C^{0,\alpha}_{\rm loc}(\mathbf{R}^4)$ for some $0 < \alpha \leqslant 1 - 1/s_0$. Moreover, we let $\gamma = 1/17$. It then follows from (4.10) that

$$\gamma \int_{\widehat{B}_R(0)} |\Delta^2 \widehat{w}_k| \,{\rm d}z \leqslant \gamma \int_{\widehat{B}_R(0)} |\widehat{f}_k| e^{4\widehat{w}_k} \,{\rm d}z \leqslant \gamma \int_M |Q_{g_k}| \,{\rm d}\mu_{g_0} < 8\pi^2.$$

Then repeating the previous argument we have

$$\int_{\widehat{B}_{R/2}(0)} e^{\pm 4s_1 \gamma \widehat{w}_k^{(-)}} \,{\rm d}z \leqslant C_R. \tag{4.70}$$

Also, there holds

$$\big\|\widehat{w}_k^{(\pm)}\big\|_{W^{3,s}(\widehat{B}_R(0))} \leqslant C_R \tag{4.71}$$

for all $s \in [1, 4/3)$. Now, it follows from Jensen's inequality, the decomposition of $\widehat{w}_k$ in (4.68), Hölder's inequality, and (4.70) that

$$\begin{aligned}
\exp\Big(\fint_{\widehat{B}_r(z)} \widehat{w}_k^{(0)}\, dz\Big) &\leqslant \Big(\fint_{\widehat{B}_r(z)} e^{\frac{4s_1\gamma}{1+s_1\gamma}\widehat{w}_k^{(0)}}\, dz\Big)^{\frac{1+s_1\gamma}{4s_1\gamma}} \\
&\leqslant \Big(\fint_{\widehat{B}_r(z)} e^{\frac{4s_1\gamma}{1+s_1\gamma}(\widehat{w}_k-\widehat{w}_k^{(-)})}\, dz\Big)^{\frac{1+s_1\gamma}{4s_1\gamma}} \\
&\leqslant \Big(\fint_{\widehat{B}_r(z)} e^{4\widehat{w}_k}\, dz\Big)^{\frac{(1+s_1\gamma)^2}{4(s_1\gamma)^2}} \Big(\fint_{\widehat{B}_r(z)} e^{-4s_1\gamma\widehat{w}_k^{(-)}}\, dz\Big)^{\frac{1}{4s_1\gamma}} \\
&\leqslant C_R
\end{aligned} \tag{4.72}$$

for all $z \in \widehat{B}_{R/4}(0)$ and for $r > 0$ small. In (4.72), the symbol $\fint_\Omega h$ denotes the average of h over Ω. Notice that the estimate (4.47) also holds in the current case. This together with (4.71) implies that

$$\|\Delta\widehat{w}_k^{(0)}\|_{L^1(\widehat{B}_R(0))} \leqslant \|\Delta\widehat{w}_k\|_{L^1(\widehat{B}_R(0))} + \|\Delta\widehat{w}_k^{(+)}\|_{L^1(\widehat{B}_R(0))} + \|\Delta\widehat{w}_k^{(-)}\|_{L^1(\widehat{B}_R(0))} \leqslant C_R.$$

Since $\Delta(\Delta\widehat{w}_k^{(0)}) = 0$, we can apply [Mar09, Proposition 11] to get

$$\|\Delta\widehat{w}_k^{(0)}\|_{C^l(\widehat{B}_{R/2}(0))} \leqslant C_R(l) \tag{4.73}$$

for every $l \in \mathbb{N}$. Notice that by the mean value property for biharmonic functions, see [ARS06, Lemma 2.2], we have

$$\widehat{w}_k^{(0)}(z) = \fint_{\widehat{B}_r(z)} \widehat{w}_k^{(0)}\, dz + \frac{r^2}{12}\Delta\widehat{w}_k^{(0)}(z).$$

This together with (4.72) and (4.73) implies that

$$\widehat{w}_k^{(0)}(z) \leqslant C_R$$

for all $z \in \widehat{B}_{R/4}(0)$. In view of (4.73), we may apply weak Hanack inequality, see [GT98, Theorem 8.18], to the function $C_R - \widehat{w}_k^{(0)}$ to obtain that

- either $\widehat{w}_k^{(0)}$ uniformly converges to $-\infty$ on $\widehat{B}_{R/4}(0)$
- or $\|\widehat{w}_k^{(0)}\|_{L^1(\widehat{B}_{R/2}(0))} \leqslant C_R$.

If the first case occurs, then from the decomposition of $\widehat{w}_k$ in (4.68) and the boundedness of $\widehat{w}_k^{(+)}$ in $C^{0,\alpha}_{\rm loc}(\mathbf{R}^4)$, we know that

$$\widehat{w}_k \leqslant C_R + w_k^{(0)},$$

which immediately implies that $\widehat{w}_k \to -\infty$ uniformly on $\widehat{B}_{R/4}(0)$ as $k \to \infty$. From this we deduce that

$$\int_{\widehat{B}_{R/4}(0)} e^{4\widehat{w}_k}\, dz \to 0$$

as $k \to +\infty$, which contradicts (4.65) since we have chosen $R > 8$. Hence, we must have

$$\|\widehat{w}_k^{(0)}\|_{L^1(\widehat{B}_{R/2}(0))} \leqslant C_R.$$

We then apply [Mar09, Proposition 11] again to get

$$\|\widehat{w}_k^{(0)}\|_{C^l(\widehat{B}_{R/4}(0))} \leqslant C_R(l)$$

for every $l \in \mathbb{N}$ and

$$\|\widehat{w}_k^{(0)}\|_{W^{3,s}(\widehat{B}_{R/4}(0))} \leqslant C_R(s)$$

for every $1 < s < 4/3$. Clearly, the estimate of $\widehat{w}_k^{(0)}$ in $C^l(\widehat{B}_{R/4}(0))$ above together with the decomposition of $\widehat{w}_k$ in (4.68) implies that

$$\widehat{w}_k \leqslant C_R \tag{4.74}$$

in $\widehat{B}_{R/4}(0)$. Moreover, the estimate of $\widehat{w}_k^{(0)}$ in $W^{3,s}(\widehat{B}_{R/4}(0))$ together with (4.71) tells us that $\widehat{w}_k$ is bounded in $W^{3,s}(\widehat{B}_{R/4}(0))$. Since R is arbitrary, the sequence $\widehat{w}_k$ is bounded in $W^{3,s}_{\rm loc}(\mathbf{R}^4)$ for any $1 < s < 4/3$. □

From Claim 2, up to a subsequence, there holds

$$\widehat{w}_k \rightharpoonup \widehat{w}_\infty$$

weakly in $W^{3,s}_{\rm loc}(\mathbf{R}^4)$ for some $1 < s < 4/3$ and almost everywhere on $\mathbf{R}^4$. By Fatou's lemma and (4.30), we can deduce that

$$\begin{aligned}\int_{\widehat{B}_{R/2}(0)} e^{4\widehat{w}_\infty}\, {\rm d}z &\leqslant \liminf_{k\to\infty} \int_{\widehat{B}_{R/2}(0)} e^{4\widehat{w}_k}\, {\rm d}z \\ &= \liminf_{k\to\infty} \int_{\widehat{B}_{Rr_k/2}(z_k)} e^{4\widetilde{w}_k}\, {\rm d}z \\ &\leqslant \liminf_{k\to\infty} \int_{B_{Rr_k}(x_k)} e^{4w_k}\, {\rm d}\mu_{g_0} \\ &\leqslant 1.\end{aligned}$$

Passing to the limit as $R \to +\infty$ we find that $e^{4\widehat{w}_\infty} \in L^1(\mathbf{R}^4)$ with

$$\int_{\mathbf{R}^4} e^{\widehat{w}_\infty}\, {\rm d}z = \lim_{R\to+\infty} \int_{\widehat{B}_{R/2}(0)} e^{4\widehat{w}_\infty}\, {\rm d}z \leqslant 1.$$

Now, recall $\alpha_k \widetilde{f}_{\lambda_k} \circ \Gamma_k = \alpha_k\lambda_k + \alpha_k \widetilde{f}_0 \circ \Gamma_k$ and for simplicity, we denote

$$\widehat{f}_{0k} = \alpha_k \widetilde{f}_0 \circ \Gamma_k,$$

which is non-positive. By (4.27), we may assume, up to a subsequence, that

$$\alpha_k\lambda_k \to \mu \in [8\pi^2, 64\pi^2]$$

as $k \to +\infty$.

Claim 3. The sequence $\alpha_k \widetilde{f}_0 \circ \Gamma_k$ is locally bounded (from below).

Proof of Claim 3. Suppose that for some sequence $y_k \to y_0$ in $\mathbf{R}^4$ there holds

$$\alpha_k |\widetilde{f}_0(\bar{z}_k)| \to +\infty$$

as $k \to +\infty$, where

$$\bar{z}_k = \Gamma_k(y_k) = z_k + r_k y_k.$$

Denote $p_k = \exp_{x_0}(\bar{z}_k)$. Because $\bar{z}_k \to 0$ as $k \to +\infty$, we then have $p_k \to x_0 \in M_0$ as $k \to +\infty$. From this we may assume from the beginning that $d(p_k) < d_0$. By Condition A, there exist some $A_0 > 0$ and a sequence of cones K_{p_k} with vertex at p_k such that

$$A_0 \inf_{y\in \widetilde{K}_{p_k}} \big|\alpha_k \widetilde{f}_0(\Gamma_k(y))\big| = A_0\alpha_k \inf_{z\in K_{p_k}} \big|\widetilde{f}_0(z)\big| \geqslant \alpha_k \big|\widetilde{f}_0(\bar{z}_k)\big| \to +\infty, \tag{4.75}$$

where with a suitable labeling of coordinates

$$\begin{aligned}\widetilde{K}_{p_k} &= \Gamma_k^{-1}(K_{p_k}) = \big\{z : z_k + r_k z \in K_{p_k}\big\} \\ &= \Big\{y = (y^1, ..., y^4) : \sqrt{\sum_{i=1}^3 (y^i - y^i_k)^2} < y^4 - y^4_k, \quad |y - y_k| < d_0/r_k\Big\}.\end{aligned}$$

On the other hand, by the estimate $\alpha_k|\widetilde{f}_0\circ\Gamma_k| = \alpha_k\lambda_k - \alpha_k\widetilde{f}_{\lambda_k}\circ\Gamma_k$ and the fact that $\int_M f_{\lambda_k} e^{4u_k}\,\mathrm{d}\mu_{g_0} = 0$, as routine we can apply Fatou's lemma to get that

$$\begin{aligned}\int_{\widehat{B}_{R/2}(0)} \liminf_{k\to+\infty} \big(\alpha_k|\widetilde{f}_0\circ\Gamma_k|\big)e^{4\widehat{w}_\infty}\,\mathrm{d}z &\leqslant \liminf_{k\to+\infty}\int_{\widehat{B}_{R/2}(0)} \alpha_k|\widetilde{f}_0\circ\Gamma_k|e^{4\widehat{w}_\infty}\,\mathrm{d}z\\ &\leqslant \liminf_{k\to+\infty}\int_{\widehat{B}_{Rr_k/2}(z_k)} (\alpha_k\lambda_k-\alpha_k\widetilde{f}_{\lambda_k})e^{4\widetilde{w}_k}\,\mathrm{d}z\\ &= \liminf_{k\to+\infty}\int_{B_{Rr_k}(x_k)} (\alpha_k\lambda_k-\alpha_k f_{\lambda_k})e^{4w_k}\,\mathrm{d}\mu_{g_0}\\ &\leqslant \mu.\end{aligned} \tag{4.76}$$

We thus obtain the contradiction from (4.75) and (4.76), namely, the sequence $\widehat{f}_{0k}$ is locally bounded. □

We now make use of Claim 3 together with the local upper bound of $\widehat{w}_k$ in (4.74) to ensure, up to a subsequence, that

$$\alpha_k(\widetilde{f}_0\circ\Gamma_k)e^{4\widehat{w}_k} \overset{*}{\rightharpoonup} \widehat{f}_\infty e^{4\widehat{w}_\infty}$$

weakly-* in the sense of measures, where $\widehat{f}_\infty \leqslant 0$ is locally bounded from below. By setting

$$F_\infty = \mu + \widehat{f}_\infty$$

and recall the definition of $\widehat{f}_k$ and μ we know that

$$\widehat{f}_k e^{4\widehat{w}_k} \overset{*}{\rightharpoonup} F_\infty e^{4\widehat{w}_\infty}$$

weakly-* in the sense of measures. Furthermore, we get from (4.10) the following bound

$$\int_{\mathbf{R}^4} |F_\infty|e^{4\widehat{u}_\infty}\,\mathrm{d}z \leqslant 2\mu \leqslant 128\pi^2.$$

Claim 4. The sequence $\widehat{w}_k$ is bounded in $H^4_{\rm loc}(\mathbf{R}^4)$.

Proof of Claim 4. By repeating the estimate in (4.51), it follows from the local boundedness of $\alpha_k\widetilde{f}_{\lambda_k}\circ\Gamma_k$ established in Claim 3, the local upper boundedness of $\widehat{w}_k$ in (4.74), and the smallness of $\|h_k\|_{L^2(M,g_k)}$ in (4.7) that

$$\begin{aligned}\int_{\widehat{B}_{R/4}(0)} \big|\widehat{f}_k e^{4\widehat{w}_k}\big|^2\,\mathrm{d}z &\leqslant C\int_{\widehat{B}_{R/4}(0)} \big|\alpha_k\widetilde{f}_{\lambda_k}\circ\Gamma_k\big|^2 e^{8\widehat{w}_k}\,\mathrm{d}z + C\int_{\widehat{B}_{R/2}(0)} |\widetilde{h}_k\circ\Gamma_k|^2 e^{8\widehat{w}_k}\,\mathrm{d}z\\ &\leqslant C_R + C_R\int_{\widehat{B}_{R/2}(0)} |\widetilde{h}_k\circ\Gamma_k|^2 e^{4\widehat{w}_k}\,\mathrm{d}z\\ &\leqslant C_R + C_R\|h_k\|^2_{L^2(M,g_k)} \leqslant C_R.\end{aligned}$$

This together with the equation satisfied by $\widehat{w}_k$ in (4.64) implies that $\widehat{w}_k$ is bounded in $H^4_{\rm loc}(\mathbf{R}^4)$. □

In view of Claim 4, we have that

$$\widehat{w}_k \rightharpoonup \widehat{w}_\infty$$

weakly in $H^4_{\rm loc}(\mathbf{R}^4)$ and strongly in $C^{0,\alpha}_{\rm loc}(\mathbf{R}^4)$ for some $0<\alpha<1/2$. Moreover, by passing to the limit we deduce that $\widehat{w}_\infty$ solves the equation

$$\Delta^2\widehat{w}_\infty = F_\infty e^{4\widehat{u}_\infty}$$

in $\mathbf{R}^4$. Finally, it follows from the decomposition

$$\Delta^2\widehat{w}_k - \Delta^2\widehat{w}_\infty = \big(\alpha_k\widetilde{f}_{\lambda_k}\circ\Gamma_k + \widetilde{h}_k\circ\Gamma_k - F_\infty\big)e^{4\widehat{w}_k} + F_\infty(e^{4\widehat{w}_k} - e^{4\widehat{w}_\infty})$$

that $\widehat{w}_k \to \widehat{w}_\infty$ strongly in $H^4_{\rm loc}(\mathbf{R}^4)$.

□

5. Bubbling along the flow

As in the case of Gaussian curvature flow studied by Struwe, it is unreasonable to expect that Theorem also holds for non-minimizing critical points

5.1. **Bounds for total curvature along the flow.** Bounds analogue to Lemma 4.5 can also be obtained for the solutions to the prescribed Q-curvature flow (2.6) for f_λ.

As in the static case, let $f_0 \leqslant 0$ be a smooth, non-constant function with $\max_M f_0 = 0$. Let $0 < \lambda < \lambda_0$ and let $f_\lambda = f_0 + \lambda$ as above where $\lambda_0 > 0$ is chosen in such a way that f_{λ_0} changes sign and satisfies (1.3), namely $\int_M f_{\lambda_0} \, d\mu_{g_0} < 0$. For any $0 < \lambda < \lambda_0$ and any $\sigma \in (-\sigma_0, 0)$, where the number $\sigma_0 = \sigma_0(\lambda)$ will be determined in Lemma 5.2 below, we choose $u^\sigma_{0\lambda} \in X^*_{f_\lambda}$ such that

$$\mathscr{E}(u^\sigma_{0\lambda}) \leqslant \beta_\lambda + \sigma^2, \tag{5.1}$$

where, as in (2.4), we set

$$\beta_\lambda = \min \big\{ \mathscr{E}(u) : u \in X^*_{f_\lambda} \big\}.$$

For such an initial data $u^\sigma_{0\lambda}$, it follows from Theorem 2.5 that the flow (2.6) possesses the smooth solution

$$u^\sigma_\lambda = u^\sigma_\lambda(t)$$

with $\alpha^\sigma_\lambda = \alpha^\sigma_\lambda(t)$. We also let $g^\sigma_\lambda = e^{2u^\sigma_\lambda} g_0$.

First, we establish the following simple result.

Lemma 5.1. *For any real number α, there exists a constant $\mathscr{C}_B > 0$ independent of α and time such that*

$$\int_M e^{\alpha u^\sigma_\lambda} \, d\mu_{g_0} < \mathscr{C}_B,$$

where u^σ_λ is a solution to the flow (2.6) *with the initial data $u^\sigma_{0\lambda}$ satisfying* (5.1).

Proof. Observe that $u^\sigma_\lambda \in X^*_{f_\lambda}$ and

$$\int_M e^{\alpha u^\sigma_\lambda} \, d\mu_{g_0} = e^{\alpha \overline{u}^\sigma_\lambda} \int_M e^{\alpha (u^\sigma_\lambda - \overline{u}^\sigma_\lambda)} \, d\mu_{g_0} \,.$$

Of course the case $\alpha = 0$ is trivial. If $\alpha < 0$, then as in the proof of Lemma 4.1 we apply Adam's inequality (3.2) to get

$$e^{-4\overline{u}^\sigma_\lambda} = \int_M e^{4(u^\sigma_\lambda - \overline{u}^\sigma_\lambda)} \, d\mu_{g_0} \leqslant \mathscr{C}_A \exp\Big(\frac{1}{16\pi^2} \mathscr{E}(u^\sigma_{0\lambda})\Big).$$

Using this, we can bound $\int_M \exp(\alpha u^\sigma_\lambda) \, d\mu_{g_0}$ from above as follows

$$\int_M e^{\alpha u^\sigma_\lambda} \, d\mu_{g_0} \leqslant \Big(\mathscr{C}_A \exp\Big(\frac{1}{16\pi^2} \mathscr{E}(u^\sigma_{0\lambda})\Big)\Big)^{|\alpha|/4} \mathscr{C}_A \exp\Big(\frac{\alpha^2}{256\pi^2} \mathscr{E}(u^\sigma_{0\lambda})\Big).$$

If $\alpha > 0$, then as in the proof of Lemma 4.6 we know that $\overline{u}^\sigma_\lambda \leqslant 0$, which then implies that

$$\int_M e^{\alpha u^\sigma_\lambda} \, d\mu_{g_0} \leqslant \int_M e^{\alpha (u^\sigma_\lambda - \overline{u}^\sigma_\lambda)} \, d\mu_{g_0} \leqslant \mathscr{C}_A \exp\Big(\frac{\alpha^2}{256\pi^2} \mathscr{E}(u^\sigma_{0\lambda})\Big).$$

Putting these estimates together we obtain the existence of $\mathscr{C}_B$. Clearly, $\mathscr{C}_B$ is independent of α and time, however, $\mathscr{C}_B$ depends on σ_0 and λ. □

The following lemma is the key result of this section.

Lemma 5.2. *There holds*

$$\liminf_{\lambda\searrow 0}\limsup_{\sigma\nearrow 0}\limsup_{t\to+\infty}\int_M |Q_{g_\lambda^\sigma}|\,\mathrm{d}\mu_{g_\lambda^\sigma} \leqslant 2\liminf_{\lambda\searrow 0}\limsup_{\sigma\nearrow 0}\limsup_{t\to+\infty}(\lambda\alpha_\lambda^\sigma(t)) \leqslant 2\liminf_{\lambda\searrow 0}(\lambda|\beta_\lambda'|) \leqslant 128\pi^2.$$

Proof. We split our proof into two steps as follows.

Step 1. We claim that for any $0<\lambda<\lambda_0$ we can find some $\sigma_0=\sigma_0(\lambda)>0$ sufficiently small such that for each $t\geqslant 0$ and $\sigma\in(-\sigma_0,0)$ we have

$$u_\lambda^\sigma+\sigma f_\lambda\in X_{f_\mu}$$

for some $\mu=\mu(t)>\lambda$ with

$$C^{-1}|\sigma|\leqslant|\mu-\lambda|\leqslant C|\sigma|,$$

where $C>0$ is constant independent of t and σ. To see this, we notice from (2.9) that $u_\lambda^\sigma(t)\in X_{f_\lambda}^*$ for all $t\geqslant 0$. By mean value theorem, there exists two functions σ',σ'' valued in $(\sigma,0)$ such that

$$\begin{aligned}\int_M f_\lambda e^{4(u_\lambda^\sigma+\sigma f_\lambda)}\,\mathrm{d}\mu_{g_0} &= \int_M f_\lambda\Big[e^{4(u_\lambda^\sigma+\sigma f_\lambda)}-e^{4u_\lambda^\sigma}\Big]\,\mathrm{d}\mu_{g_0}\\ &=4\sigma\int_M f_\lambda^2 e^{4(u_\lambda^\sigma+\sigma' f_\lambda)}\,\mathrm{d}\mu_{g_0}\end{aligned}$$

and

$$\begin{aligned}\int_M e^{4(u_\lambda^\sigma+\sigma f_\lambda)}\,\mathrm{d}\mu_{g_0} &= 1+\int_M \Big[e^{4(u_\lambda^\sigma+\sigma f_\lambda)}-e^{4u_\lambda^\sigma}\Big]\,\mathrm{d}\mu_{g_0}\\ &=1+4\sigma\int_M f_\lambda e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0}+8\sigma^2\int_M f_\lambda^2 e^{4(u_\lambda^\sigma+\sigma'' f_\lambda)}\,\mathrm{d}\mu_{g_0}\\ &=1+8\sigma^2\int_M f_\lambda^2 e^{4(u_\lambda^\sigma+\sigma'' f_\lambda)}\,\mathrm{d}\mu_{g_0}.\end{aligned}$$

Therefore, in view of the identity

$$\int_M f_\mu e^{4(u_\lambda^\sigma+\sigma f_\lambda)}\,\mathrm{d}\mu_{g_0}=\int_M f_\lambda e^{4(u_\lambda^\sigma+\sigma f_\lambda)}\,\mathrm{d}\mu_{g_0}+(\mu-\lambda)\int_M e^{4(u_\lambda^\sigma+\sigma f_\lambda)}\,\mathrm{d}\mu_{g_0},$$

if we let μ be

$$\mu=\lambda-\frac{4\sigma\int_M f_\lambda^2 e^{4(u_\lambda^\sigma+\sigma' f_\lambda)}\,\mathrm{d}\mu_{g_0}}{1+8\sigma^2\int_M f_\lambda^2 e^{4(u_\lambda^\sigma+\sigma'' f_\lambda)}\,\mathrm{d}\mu_{g_0}}>\lambda, \tag{5.2}$$

depending on t, then $u_\lambda^\sigma+\sigma f_\lambda\in X_{f_\mu}$. To bound $|\mu-\lambda|$, we need further estimates for numerator and denominator of μ in (5.2). First we note by Hölder's inequality that

$$\int_M f_\lambda^2 e^{4(u_\lambda^\sigma+\sigma' f_\lambda)}\,\mathrm{d}\mu_{g_0}\geqslant\Big(\int_M f_\lambda\,\mathrm{d}\mu_{g_0}\Big)^2\Big(\int_M e^{-4(u_\lambda^\sigma+\sigma' f_\lambda)}\,\mathrm{d}\mu_{g_0}\Big)^{-1}.$$

Because

$$\begin{aligned}\int_M e^{-4(u_\lambda^\sigma+\sigma' f_\lambda)}\,\mathrm{d}\mu_{g_0} &= \Big(\int_{\{f_\lambda\leqslant 0\}}+\int_{\{f_\lambda>0\}}\Big)e^{-4(u_\lambda^\sigma+\sigma' f_\lambda)}\,\mathrm{d}\mu_{g_0}\\ &\leqslant \exp\big(4|\sigma|\|f_\lambda\|_{L^\infty(M,g_0)}\big)\int_{\{f_\lambda\leqslant 0\}}e^{-4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0}+\int_{\{f_\lambda>0\}}e^{-4u_\lambda^\sigma},\end{aligned}$$

which implies that

$$\int_M e^{-4(u_\lambda^\sigma+\sigma' f_\lambda)}\,\mathrm{d}\mu_{g_0}\leqslant 2\int_M e^{-4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0}$$

if we further choose $|\sigma|$ sufficiently small. From this and Lemma 5.1 we deduce that there exists some positive constant $c(\lambda,\sigma_0)$ depending only on λ and σ_0 such that

$$\int_M f_\lambda^2 e^{4(u_\lambda^\sigma+\sigma' f_\lambda)}\,\mathrm{d}\mu_{g_0} \geqslant c(\lambda,\sigma_0) \tag{5.3}$$

for any $0<\lambda<\lambda_0$, any $\sigma\in(-\sigma_0,0)$, and any $\sigma'\in(\sigma,0)$. Moreover, it is easy to see that

$$\begin{aligned}\int_M f_\lambda^2 e^{4(u_\lambda^\sigma+\sigma' f_\lambda)}\,\mathrm{d}\mu_{g_0} &\leqslant \|f_\lambda\|^2_{L^\infty(M,g_0)} \exp\big(4|\sigma|\|f_\lambda\|_{L^\infty(M,g_0)}\big)\int_M e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0}\\ &\leqslant \|f_\lambda\|^2_{L^\infty(M,g_0)} \exp\big(4|\sigma_0|\|f_\lambda\|_{L^\infty(M,g_0)}\big)\end{aligned} \tag{5.4}$$

for any $t\geqslant 0$ and any $\sigma'\in(\sigma,0)$. From this we can bound $|\mu-\lambda|$ from above as follows

$$\begin{aligned}|\mu-\lambda| &\leqslant 4|\sigma|\int_M f_\lambda^2 e^{4(u_\lambda^\sigma+\sigma' f_\lambda)}\,\mathrm{d}\mu_{g_0}\\ &\leqslant 4\|f_\lambda\|^2_{L^\infty(M,g_0)}\exp\big(4|\sigma_0|\|f_\lambda\|_{L^\infty(M,g_0)}\big)|\sigma|.\end{aligned}$$

Moreover, we can also bound $|\mu-\lambda|$ from below, thanks to (5.3) and (5.4). Hence, for any $0<\lambda<\lambda_0$ we can find $\sigma_0=\sigma_0(\lambda)>0$ such that

$$C(\lambda)^{-1}|\sigma|\leqslant|\mu-\lambda|\leqslant C(\lambda)|\sigma|$$

for all $\sigma\in(-\sigma_0,0)$, where $C(\lambda)>0$ is independent of $t\geqslant 0$ and σ but could depend on λ. The claim is thus proved.

Step 2. It follows from [NZ17, Lemmas 4.1 and 6.3] that $\alpha_\lambda^\sigma(t)$ and $u_\lambda^\sigma(t)$ are uniformly bounded in time t and σ. Notice that by the relations

$$Q_{g_\lambda^\sigma} e^{4u_\lambda^\sigma} = \mathbf{P}_{g_0} u_\lambda^\sigma, \quad u_{\lambda,t}^\sigma = \alpha_\lambda^\sigma f_\lambda - Q_{g_\lambda^\sigma}$$

we can expand $\mathscr{E}(u_\lambda^\sigma+\sigma f_\lambda)$ to get

$$\begin{aligned}\mathscr{E}(u_\lambda^\sigma+\sigma f_\lambda) &= \mathscr{E}(u_\lambda^\sigma) + 4\sigma\int_M \mathbf{P}_{g_0} u_\lambda^\sigma f_\lambda\,\mathrm{d}\mu_{g_0} + \sigma^2\mathscr{E}(f_\lambda)\\ &= \mathscr{E}(u_\lambda^\sigma) + 4\sigma\alpha_\lambda^\sigma\int_M f_\lambda^2 e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0} - 4\sigma\int_M u_{\lambda,t}^\sigma f_\lambda e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0} + \sigma^2\mathscr{E}(f_0).\end{aligned}$$

Observing that [NZ17, Lemma 6.1] yields

$$\int_M |u_{\lambda,t}^\sigma|^2 e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0} = \int_M |\alpha_\lambda^\sigma f_\lambda - Q_\lambda^\sigma|^2 e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0} \to 0 \tag{5.5}$$

as $t\to+\infty$. Hence, by (2.8) and Hölder's inequality, we can estimate

$$\Big|\int_M u_{\lambda,t}^\sigma f_\lambda e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0}\Big| \leqslant \|f_\lambda\|_{L^\infty(M,g_0)}\Big(\int_M |u_{\lambda,t}^\sigma|^2 e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0}\Big)^{1/2}\to 0$$

as $t\to+\infty$. Since the energy $\mathscr{E}(u_\lambda^\sigma)$ is decay along the flow, we have, by (5.1) and the expansion of $\mathscr{E}(u_\lambda^\sigma+\sigma f_\lambda)$ above, that

$$\begin{aligned}\beta_\mu &\leqslant \mathscr{E}(u_\lambda^\sigma+\sigma f_\lambda)\\ &\leqslant \mathscr{E}(u_\lambda^\sigma) + 4\sigma\alpha_\lambda^\sigma\int_M f_\lambda^2 e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0} + \sigma^2\mathscr{E}(f_0) + o(1)\\ &\leqslant \beta_\lambda + 4\sigma\alpha_\lambda^\sigma\int_M f_\lambda^2 e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0} + \sigma^2(1+\mathscr{E}(f_0)) + o(1).\end{aligned}$$

However, from (5.2) we obtain

$$4\sigma\int_M f_\lambda^2 e^{4u_\lambda^\sigma}\,\mathrm{d}\mu_{g_0} = \lambda-\mu+4\sigma I,$$

with
$$I = \int_M f_\lambda^2 h e^{4u_\lambda^\sigma} \,\mathrm{d}\mu_{g_0},$$
where
$$h = 1 - \frac{e^{4\sigma' f_\lambda}}{1 + 8\sigma^2 \int_M f_\lambda^2 e^{4(u_\lambda^\sigma + \sigma'' f_\lambda)} \,\mathrm{d}\mu_{g_0}}.$$
Clearly
$$|h| \leqslant 8\sigma^2 \int_M f_\lambda^2 e^{4(u_\lambda^\sigma + \sigma'' f_\lambda)} \,\mathrm{d}\mu_{g_0} + \big|1 - e^{4\sigma' f_\lambda}\big|.$$
Because $\sigma' \in (\sigma, 0)$, there is some constant $C > 0$ independent of t and σ such that
$$\|h\|_{L^\infty(M,g_0)} \leqslant C|\sigma|.$$
Keep in mind that $u_\lambda^\sigma \in X_{f_\lambda}^*$. From this we can use (2.8) to bound I as follows
$$|I| \leqslant \|f_\lambda\|^2_{L^\infty(M,g_0)} \|h\|_{L^\infty(M,g_0)} \leqslant C(\lambda)|\mu - \lambda|,$$
where $C(\lambda) > 0$ is a uniform constant independent of t and σ. Therefore, with error $o(1) \to 0$ as $t \to +\infty$ and the uniform bound of α_λ^σ in t and in σ we arrive at the estimate
$$\begin{aligned}\beta_\mu &\leqslant \beta_\lambda + \alpha_\lambda^\sigma(\lambda - \mu + 4\sigma I) + \sigma^2(1 + \mathscr{E}(f_0)) + o(1)\\ &= \beta_\lambda - \alpha_\lambda^\sigma(\mu - \lambda) + O(1)(\mu - \lambda)^2 + o(1),\end{aligned}$$
where $O(1)$ is independent of t but could depend on λ and σ_0. This implies that
$$\limsup_{t\to+\infty} \alpha_\lambda^\sigma(t) \leqslant \limsup_{t\to+\infty} \Big(\frac{\beta_\lambda - \beta_\mu}{\mu - \lambda} + O(1)(\mu - \lambda)\Big).$$
Now, as $\sigma \nearrow 0$, we have from (5.2) that $\mu \searrow \lambda$ uniformly in time $t > 0$. So, for almost every $\lambda \in (0, \lambda_0)$ there holds
$$\limsup_{\sigma\nearrow 0} \limsup_{t\to+\infty} \alpha_\lambda^\sigma(t) \leqslant \lim_{\mu\searrow\lambda} \frac{\beta_\lambda - \beta_\mu}{\mu - \lambda} = |\beta_\lambda'|.$$
Multiplying both sides by $\lambda > 0$, as in the proof of Lemma 4.5, we find that
$$\liminf_{\lambda\searrow 0} \limsup_{\sigma\nearrow 0} \limsup_{t\to+\infty} (\lambda\alpha_\lambda^\sigma) \leqslant \liminf_{\lambda\searrow 0} (\lambda|\beta_\lambda'|) \leqslant 64\pi^2.$$
Finally, it follows from the flow equation (2.6) and (4.4) that
$$|Q_{g_\lambda^\sigma}| \leqslant \alpha_\lambda^\sigma|f_\lambda| + u_{\lambda,t}^\sigma \leqslant 2\lambda\alpha_\lambda^\sigma - \alpha_\lambda^\sigma f_\lambda + u_{\lambda,t}^\sigma,$$
which then gives
$$\int_M |Q_{g_\lambda^\sigma}| \,\mathrm{d}\mu_{g_\lambda^\sigma} \leqslant 2\lambda\alpha_\lambda^\sigma + \int_M |u_{\lambda,t}^\sigma| \,\mathrm{d}\mu_{g_\lambda^\sigma} = 2\lambda\alpha_\lambda^\sigma + o(1),$$
thanks to (2.8) and (5.5). From this the lemma follows. □

5.2. **Bubbling of the prescribed curvature flow.** In this subsection, we devote ourselves to prove the blow-up behavior along the prescribed Q-curvature flow, namely Theorem 2.6. From Lemma 5.2, it follows that there exists a sequence $\lambda_k \searrow 0$ such that
$$\sup_{k\in\mathbb{N}} \limsup_{\sigma\nearrow 0} \limsup_{t\to+\infty} (\lambda_k \alpha_{\lambda_k}^\sigma(t) - 1/k) \leqslant 64\pi^2.$$
We may then fix a sequence $\sigma_k \nearrow 0$ with
$$\sup_{k\in\mathbb{N}} \sup_{\sigma_k\leqslant\sigma<0} \limsup_{t\to+\infty} (\lambda_k \alpha_{\lambda_k}^\sigma(t) - 2/k) \leqslant 64\pi^2.$$
Choosing $\sigma = \sigma_k$ for each $k \in \mathbb{N}$, we find, for suitable $T_k \to +\infty$ satisfying
$$F_k(t) := \int_M |u_{\lambda_k,t}^{\sigma_k}(t)| \,\mathrm{d}\mu_{g_{\lambda_k}^{\sigma_k}} \leqslant 1/k$$

for $t_k \geqslant T_k$, we find the bound

$$\sup_{t \geqslant T_k} \int_M |Q_{g_{\lambda_k}^{\sigma_k}}| \, \mathrm{d}\mu_{g_{\lambda_k}^{\sigma_k}} \leqslant \sup_{t \geqslant T_k} \big(2\lambda_k \alpha_{\lambda_k}^{\sigma_k} + F_k(t)\big) \leqslant 128\pi^2 + 5/k, \tag{5.6}$$

for any $k \in \mathbb{N}$. Hence, if for each $k \in \mathbb{N}$ for any $t_k \geqslant T_k$ we let $w_k = u_{\lambda_k}^{\sigma_k}(t_k)$, then w_k satisfies (4.5) with $\alpha_k = \alpha_{\lambda_k}^{\sigma_k}(t_k)$ and $h_k = u_{\lambda_k,t}^{\sigma_k}(t_k)$. From this we can apply Theorems 4.7 and 4.8 to get the desired result. This completes the proof of Theorem 2.6.

ACKNOWLEDGMENTS

Q.A.N. would like to thank Professor Michael Struwe for sharing his preprint [Str17] at an early stage and for many fruitful discussions regarding curvature flows. Part of this work was done while Q.A.N was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM) in 2018, he gratefully acknowledges the institute for hospitality and support. He also acknowledges the support from the Vietnam National University, Hanoi (VNU) under project number QG.19.12. The research of H.Z. is supported by the Young Scientists Fund of the National Natural Science Foundation of China (Grant No. 11701544). Thanks also go to Luca Galimberti for his careful reading of the paper and a few suggestions leading to the present version. Last but not least, both authors would like to thank Professor Xingwang Xu for his interests and constant encouragement over the years.

REFERENCES

[Ada88] D. ADAMS, A sharp inequality of J. Moser for higher order derivatives, *Ann. Math.* **128** (1988), pp. 385–398. 8

[ARS06] ADIMURTHI, F. ROBERT, M. STRUWE, Concentration phenomena for Liouville's equation in dimension four, *J. Eur. Math. Soc.* **8** (2006), pp. 171–180. 30, 36

[BFR06] P. BAIRD, A. FARDOUN, R. REGBAOUI, Q-curvature flow on 4-manifolds, *Calc. Var.* **27** (2006), pp. 75–104. 3

[Bra85] T. BRANSON, Differntial operators canonically associated to a conformal structure, *Math. Scand* **57** (1985), pp. 293–345. 3

[Bre03] S. BRENDLE, Global existence and convergence for a higher order flow in conformal geometry *Ann. Math.* **158** (2003), pp. 171–180. 3

[BM91] H. BREZIS, F. MERLE, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. Partial Differential Equations* **16** (1991), pp. 1223–1253. 17

[CC01] S. Y. CHANG, W. CHEN, A note on a class of higher order conformally covariant equations, *Discrete Contin. Dynam. Systems* **7** (2001), pp. 275–281. 5

[CY95] S. Y. CHANG, P. YANG, Extremal metrics of zeta functional determinants on 4-manifolds, *Ann. Math.* **142** (1995), pp. 171–212. 3, 9

[DM08] Z. DJADLI, A. MALCHIODI, Existence of conformal metrics with constant Q-curvature, *Ann. Math.* **168** (2008), pp. 813–858. 3

[DER04] O. DRUET, E. HEBEY, F. ROBERT, *Blow-up theory for elliptic PDEs in Riemannian geometry*, Mathematical Notes, 45. Princeton University Press, Princeton, NJ, 2004 22

[Gal15] L. GALIMBERTI, Compactness issues and bubbling phenomena for the prescribed Gaussian curvature equation on the torus, *Calc. Var. Partial Differential Equations* **54** (2015), pp. 2483–2501. 2

[Gal17] L. GALIMBERTI, "Large" conformal metrics of prescribed Q-curvature in the negative case, *Nonlinear Differ. Equ. Appl.*, **24** (2017), Art. 18. 4, 13

[GX08] Y. GE, X. XU, Prescribed Q-curvature problem on closed 4-Riemannian manifolds in the null case, *Calc. Var. PDE.*, **31** (2008), pp. 549–555. 3

[GT98] D. GILBARG, N. TRUDINGER, Elliptic partial differential equations of second order, 2nd edition, Springer, 1998. 36

[KW74] J.L. KAZDAN, F.W. WARNER, Curvature function for compact 2-manifolds, *Ann. of Math.* **99** (1974), pp. 14–74. 2

[LLL12] J. LI, Y. LI, P. LIU, The Q-curvature on a 4-dimensional Riemannian manifold (M, g) with $\int_M Q_g \, dV_g = 8\pi^2$, *Adv. Math.* **231** (2012), pp. 2194–2223. 3

[Lin98] C. S. LIN, A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$, *Comm. Math. Helv.* **73** (1998), pp. 206–231. 31, 35

[Mal06] A. MALCHIODI, Compactness of solutions to some geometric fourth-order equations, *J. Reine Angew. Math.* **594** (2006), pp. 137–174. 9, 10, 27

[MS06] A. MALCHIODI, M. STRUWE, Q-curvature flow on S^4, *J. Differ. Geom.* **73** (2006), pp. 1–44. 3

[Mar08] L. MARTINAZZI, Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature, *Rend. Lincei Mat. Appl.* **19** (2008), pp. 279–292. 31

[Mar09] L. MARTINAZZI, Concentration-compactness phenomena in the higher order Liouville's equation, *J. Func. Anal.* **256** (2009), pp. 3743–3771. 30, 34, 36

[NZ17] Q. A. NGÔ, H. ZHANG, Q-curvature flow on closed manifolds of even dimension, arXiv:1701.02247v2, 2017. 4, 7, 41

[Pan82] S. PANEITZ, Essential unitarization of symplectics and applications to field quantization, *J. Funct. Anal.*, **48** (1982), pp. 310–359. 3

[Str17] M. STRUWE, Bubbling of the prescribed curvature flow on the torus, *J. Eur. Math. Soc.*, to appear. 2, 3, 4, 5, 43

[WX98] J. WEI, X. XU, On conformal deformations of metrics on S^n, *J. Funct. Anal.* **157** (1998), pp. 292–325. 3

[Yam60] H. YAMABE, On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.* **12** (1960), pp. 21–37. 1

(Q.A. Ngô) DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, VIÊT NAM NATIONAL UNIVERSITY, HA NÔI, VIÊT NAM

E-mail address: nqanh@vnu.edu.vn

E-mail address: bookworm_vn@yahoo.com

(H. Zhang) SCHOOL OF MATHEMATICS, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, NO.96 JINZHAI ROAD, HEFEI, ANHUI, CHINA, 230026.

E-mail address: matzhang@ustc.edu.cn

E-mail address: mathongzhang@gmail.com