BOUNDS FOR HILBERT COEFFICIENTS

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Abstract. Let $(A, \mathfrak{m})$ be a noetherian local ring with $\dim(A) = d \geq 1$ and $\operatorname{depth}(A) \geq d-1$. Let J be an $\mathfrak{m}$-primary ideal and write $\sigma_J(k) = \operatorname{depth}(G(J^k))$. Elias [4] proved that $\sigma_J(k)$ is constant for $k \gg 0$ and denoted this number by $\sigma(J)$. In this paper, we investigate the non-negativity and non-positivity for the Hilbert coefficients $e_i(J)$ under some conditions for $\sigma_J(r)$, where $r = \operatorname{reg}(G(J)) + 1$. In case of $J = Q$ is a parameter ideal, we establish bounds for the Hilbert coefficients of Q in terms of dimension and the first Hilbert coefficient $e_1(Q)$.

Introduction

Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension d and J an $\mathfrak{m}$-primary ideal of A. Let $\ell(.)$ denote the length of an A-module. The Hilbert-Samuel function of A with respect to J is the function $H_J : \mathbb{Z} \longrightarrow \mathbb{N}_0$ given by

$$H_J(n) = \begin{cases} \ell(A/J^n) & \text{if } n \geq 0; \\ 0 & \text{if } n < 0. \end{cases}$$

There exists a unique polynomial $P_J(x) \in \mathbb{Q}[x]$ (called the *Hilbert- Samuel polynomial*) of degree d such that $H_J(n) = P_J(n)$ for $n \gg 0$ and it is written by

$$P_J(n) = \sum_{i=0}^{d} (-1)^i \binom{n+d-i-1}{d-i} e_i(J).$$

Then, the integers $e_i(J)$ is called *Hilbert coefficients* of J. Let denote by $G(J) = \oplus_{n\geq 0} J^n/J^{n+1}$ the associated graded ring of A with respect to J. In [4], Elias denoted $\sigma_J(k) = \operatorname{depth}(G(J^k))$ and proved that $\sigma_J(k)$ is constant for $k \gg 0$. We call this number $\sigma(J)$.

The aim of this paper is to investigate the sign of $e_i(J)$ for $i = 3, ..., d$ under assumption $\sigma_J(r) \geq d-2$, here $r = \operatorname{reg}(G(J)) + 1$. In case $J = Q$ is a parameter ideal and $\operatorname{depth}(A) \geq d-1$, we establish bounds for the Hilbert coefficients $e_i(Q)$, for $i = 2, ..., d$, in terms of the dimension and the first Hilbert coefficient $e_1(Q)$.

First, we study the non-negativity of the Hilbert coefficients. It is well known that $e_0(J)$ is always positive. There were several results on the non-negativity of the Hilbert coefficients $e_i(J)$. If A is Cohen-Macaulay, Northcott [19] proved the non-negativity of the first Hilbert coefficient $e_1(J)$. Narita [18] proved the non-negativity of the second Hilbert coefficient $e_2(J)$ and he also showed that it is possible for $e_3(J)$

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to be negative. However, Itoh [9] showed that if J is a normal parameter ideal, then $e_3(J) \geq 0$. Later, Corso-Polini-Rossi [3] improved the proof the of Itoh on the non-negativity of Hilbert coefficients of $\mathfrak{m}$-primary asymptotically normal ideal in the case $\dim(A) = 3$.

The first main result of this paper is to prove the non-negativity $e_d(J)$ under condition $\sigma_J(r) \geq d-1$.

Theorem 2.2 *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring of dimension $d \geq 2$. Let J be an $\mathfrak{m}$-primary ideal such that $\sigma_J(r) \geq d-1$. Then $e_d(J) \geq 0$.*

It is well known that $\sigma_J(r) \geq \operatorname{depth}(G(J))$. Thus, Theorem 2.2 implies an early result of Marley [16] on the non-negativity of all Hilbert coefficients $e_i(J)$ with assumption $\operatorname{depth}(G(J)) \geq d-1$.

Next we study the non-positivity of the Hilbert coefficients. If $J = Q$ is a parameter ideal of a Cohen-Macaulay ring A, then $e_i(Q) = 0$ for $i = 1, ..., d$. If A is an arbitrary ring, Mandal-Singh-Verma [15] showed that $e_1(Q) \leq 0$ for all parameter ideals of A. If $\operatorname{depth}(A) \geq d-1$, McCune [17] showed that $e_2(Q) \leq 0$ and Saikia-Saloni [24] proved that $e_3(Q) \leq 0$ for every parameter ideal Q. In [17], McCune also proved that if Q is a parameter ideal such that $\operatorname{depth}(G(Q)) \geq d-1$, then $e_i(Q) \leq 0$ for $i = 1, ..., d$. Later, Saikia-Saloni [24] and Linh-Trung [12] proved that if $\operatorname{depth}(A) \geq d-1$ and Q is a parameter ideal such that $\operatorname{depth}(G(Q)) \geq d-2$, then $e_i(Q) \leq 0$ for $i = 1, ..., d$. In [21], Puthenpurakal obtained remarkable results about non-positivity of $e_3(J)$.

The second main result of this paper is a generalization a recent result of Linh [13, Proposition 3.5] on the non-positivity of the d-th Hilbert coefficients $e_d(J)$.

Theorem 2.4 *Let $(A, \mathfrak{m})$ be a noetherian local ring with $\dim(A) = d \geq 3$ and $\operatorname{depth}(A) \geq d-1$. Let J be an $\mathfrak{m}$-primary ideal such that $r(J) \leq d-1$ and $\sigma_J(r) \geq d-2$. Then $e_d(J) \leq 0$.*

Theorem 2.4 implies an early result of Mafy and Nadery [14] that if A is a Cohen-Macaulay ring of dimension 4 and J an $\mathfrak{m}$-primary asymptotically normal ideal such that $r(J) \leq 3$, then $e_4(J) \leq 0$. From Theorem 2.4, we get some interesting results and one of them is a generalization of an early results of Saikia-Saloni [24, Corollary 3.2] and Linh-Trung [12, Theorem 2.9].

Corollary 2.9 *Let $(A, \mathfrak{m})$ a noetherian local ring with $\dim(A) = d \geq 3$ and $\operatorname{depth}(A) \geq d-1$. Let J be an $\mathfrak{m}$-primary ideal of A such that $r(J) \leq 2$ and $\operatorname{depth}(G(J)) \geq d-2$. Then*

$$e_i(J) \leq 0 \quad \text{for} \quad i = 3, ..., d.$$

Finally, we want to bound the Hilbert coefficients in terms of several common invariants. If A is Cohen-Macaulay and generalized Cohen-Macaulay, Srinivas and Trivedi [25]-[27] gave bounds for the Hilbert coefficients of $\mathfrak{m}$-primary ideals in terms of the dimension and multiplicity. If A is an arbitrary ring, Rossi, Trung and Valla [22] established bounds for the Hilbert coefficients of the maximal ideal in terms of the dimension and an extended degree. In [11], the author gave bounds for the Hilbert coefficients of $\mathfrak{m}$-primary ideals in terms of the degree of nilpotency. Recently, Goto and Ozeki [6] established uniform bounds for the Hilbert coefficients of parameter ideals in a generalized Cohen-Macaulay ring.

The next main result of the paper is to establish bounds for the Hilbert coefficients of parameter ideals in terms of the dimension and the first coefficient $e_1(Q)$.

Theorem 3.4 *Let A be a noetherian local ring of dimension $d \geq 2$ and* $\operatorname{depth}(A) \geq d-1$. *Let Q be a parameter ideal of A. Then*

$$|e_i(Q)| \leq 3.2^{i-2} r^{i-1} |e_1(Q)| \quad \textit{for} \quad i = 2, ..., d,$$

where $r = \max\{[-4e_1(Q)]^{(d-1)!} + e_1(Q) - 1, 0\} + 1.$

The paper is divided into three sections. In Section 1, we prepare some facts relate to the Hilbert coefficients and regularity. In Section 2, we prove the non-negativity and non-positivity for the Hilbert coefficients of $\mathfrak{m}$-primary ideals. In Section 3, we establish bounds for the Hilbert coefficients of parameter ideals in terms of the dimension and the first Hilbert coefficient.

1. Preliminary

Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension d and J be an $\mathfrak{m}$-primary ideal of A. A numerical function

$$H_J : \mathbb{Z} \longrightarrow \mathbb{N}_0$$

$$n \longmapsto H_J(n) = \begin{cases} \ell(A/J^n) & \text{if } n \geq 0; \\ 0 & \text{if } n < 0 \end{cases}$$

is said to be a *Hilbert-Samuel function* of A with respect to the ideal J. It is well known that there exists a polynomial $P_J \in \mathbb{Q}[x]$ of degree d such that $H_J(n) = P_J(n)$ for $n \gg 0$. The polynomial P_J is called the *Hilbert-Samuel polynomial* of A with respect to the ideal J and it is written of the form

$$P_J(n) = \sum_{i=0}^{d} (-1)^i \binom{n+d-i-1}{d-i} e_i(J),$$

where $e_i(J)$ for $i = 0, ..., d$ are integers, called *Hilbert coefficients* of J. In particular, $e(J) = e_0(J)$ and $e_1(J)$ are called the *multiplicity* and *Chern coefficient* of J, respectively. Denote

$$n(J) = \max\{n \mid H_J(n) \neq P_J(n)\}.$$

An element $x \in J \backslash \mathfrak{m}J$ is said to be *superficial* for J if there exists a number $c \in \mathbb{N}$ such that $(J^n : x) \cap J^c = J^{n-1}$ for $n > c$. If $A/\mathfrak{m}$ is infinite, then a superficial element for J always exists. A sequence of elements $x_1, ..., x_r \in J \backslash \mathfrak{m}J$ is said to be a *superficial sequence* for J if x_i is superficial for $J/(x_1, ...x_{i-1})$ for $i = 1, ...r$.

Suppose that $\dim(A) = d \geq 1$ and $x \in J \setminus \mathfrak{m}J$ is a superficial element for J, then $\ell(0 :_A x) < \infty$ and $\dim(A/(x)) = \dim(A) - 1 = d - 1$. The following lemma give us a relationship between $e_i(J)$ and $e_i(J_1)$, where $J_1 = J(A/(x))$.

Lemma 1.1. [23, Proposition 1.3.2] *Let A be a noetherian local ring of dimension $d \geq 2$ and J be an $\mathfrak{m}$-primary ideal of A. Let $x \in J \backslash \mathfrak{m}J$ be a superficial element for J and $J_1 = J(A/(x))$. Then*

(i) $e_i(J) = e_i(J_1)$ *for* $i = 0, ..., d-2$;

(ii) $e_{d-1}(J) = e_{d-1}(J_1) + (-1)^d \ell(0 : x)$.

If denote by $G(J) = \oplus_{n\geq 0} J^n/J^{n+1}$ the associated graded ring of A with respect to J and

$$a_i(G(J)) = \sup\{n \mid H^i_{G(J)_+}(G(J))_n \neq 0\},$$

then the Castelnuovo-Mumford regularity of $G(J)$ is defined by

$$\mathrm{reg}(G(J)) = \max\{a_i(G(J)) + i \mid i \geq 0\}.$$

Lemma 1.2. *Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension d and J be an $\mathfrak{m}$-primary ideal of A. Let $x \in J \setminus \mathfrak{m}J$ be a superficial element for J. Set $\bar{A} = A/(x)$ and $\bar{J} = J\bar{A}$. Then*

(i) $n(J) \leq \mathrm{reg}(G(J))$;

(ii) $\mathrm{reg}(G(\bar{J})) \leq \mathrm{reg}(G(J))$;

(iii) $J^{n+1} : x/J^n \cong (0 : x)$ *for* $n > \mathrm{reg}(G(J))$.

Proof.

(i) It is implied from [13, Lemma 2.1 and Lemma 2.2].

(ii) Let x^* be an initial form of x in $G(J)$. Then

$$\mathrm{reg}(G(J)/(x^*)) \leq \mathrm{reg}(G(J)).$$

On the other hand, there is a natural graded epimorphism from $G(J)/(x^*)$ to $G(\bar{J})$ whose kernel is

$$K = \bigoplus_{n\geq 0}(J^{n+1} + x \cap J^n)/(J^{n+1} + xJ^{n-1}).$$

Since x is superficial for J, $x \cap J^{n+1} = xJ^n$ for $n \gg 0$. Hence $K_n = 0$ for $n \gg 0$. Thus K is a module with finite length. Hence

$$\mathrm{reg}(G(\bar{J})) \leq \mathrm{reg}(G(J)/(x^*)).$$

This implies

$$\mathrm{reg}(G(\bar{J})) \leq \mathrm{reg}(G(J)).$$

(iii) From the exact sequence

$$0 \longrightarrow J^{n+1} : x/J^n \longrightarrow A/J^n \xrightarrow{x} A/J^{n+1} \longrightarrow A/(J^{n+1}, x) \longrightarrow 0,$$

we get

$$\begin{aligned} \ell(J^{n+1} : x/J^n) &= \ell(A/J^n) - \ell(A/J^{n+1}) + \ell(\bar{A}/\bar{J}^{n+1}) \\ &= \ell(\bar{A}/\bar{J}^{n+1}) - \ell(J^n/J^{n+1}). \end{aligned}$$

It is well known that $J^{n+1} : x/J^n \cong (0 : x)$ for $n \gg 0$. From (i) and (ii), we have

$$n(J) \leq \mathrm{reg}(G(J)) \ \text{ and } \ n(\bar{J}) \leq \mathrm{reg}(G(J)).$$

It follows that

$$J^{n+1} : x/J^n \cong (0 : x) \ \text{ for } \ n > \mathrm{reg}(G(J)).$$

□

Recall that an ideal $K \subseteq J$ is called a reduction of J if $J^{n+1} = KJ^n$ for $n \gg 0$. If K is a reduction of J and no other reduction of J is contained in K, then K is said to be a minimal reduction of J. If K is a minimal reduction of J, then the reduction number of J with respect to K, $r_K(J)$, is given by

$$r_K(J) := \min\{\ n \mid J^{n+1} = KJ^n\}.$$

The reduction number of J, denoted $r(J)$, is given by

$$r(J) := \min\{r_K(J) |\ J \text{ is a minimal reduction of } J\}.$$

The following lemma give a relationship between reduction number of J and the regularity of $G(J)$.

Lemma 1.3. [28, Proposition 3.2]

$$a_d(G(J)) + d \leq r(J) \leq \operatorname{reg}(G(J)).$$

2. The sign of Hilbert coefficients

Through this section, let $(A, \mathfrak{m})$ be a noetherian local ring of dimension d, J be an $\mathfrak{m}$-primary ideal of A and $r = \operatorname{reg}(G(J)) + 1$. In this section, we investigate the sign of Hilbert coefficients $e_i(J)$.

In [4, Proposition 2.2], Elias denoted $\sigma_J(k) = \operatorname{depth}(G(J^k))$ and proved that $\sigma_J(k)$ is constant for $k \gg 0$. We call this number $\sigma(J)$. By [7, Lemma 2.4],

$$a_i(G(J^k)) \leq [a_i(G(J))/k] \quad \text{for all} \quad i \leq d \quad \text{and} \quad k \geq 1,$$

where $[a] = \max\{m \in \mathbb{Z} \mid m \leq a\}$. Therefore

$$a_i(G(J^k)) \leq 0 \quad \text{for all} \quad i \leq d \quad \text{and} \quad k \geq r = \operatorname{reg}(G(J)) + 1 \tag{1}$$

and

$$\sigma_J(k) \geq \operatorname{depth}(G(J)) \quad \text{for} \quad k \geq 1. \tag{2}$$

The following lemma gives whenever the number $\sigma_J(k)$ is positive.

Lemma 2.1. *Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension $d \geq 1$ and J an $\mathfrak{m}$-primary ideal of A. If* $\operatorname{depth}(A) \geq 1$*, then $\sigma_J(k) \geq 1$ for all $k \geq r = \operatorname{reg}(G(J)) + 1$.*

Proof. From (1), we have $a_i(G(J^k)) \leq 0$ for all $i = 0, ..., d$ and for $k \geq r$. But by [8, Theorem 5.2], $a_0(G(J^k)) < a_1(G(J^k)) \leq 0$. Hence $H^0_{G(J^k)_+}(G(J^k)) = 0$ for $k \geq r$. This implies that $\sigma_J(k) = \operatorname{depth}(G(J^k)) \geq 1$ for all $k \geq r$. □

Theorem 2.2. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring of dimension $d \geq 1$. Let J be an $\mathfrak{m}$-primary ideal such that $\sigma_J(r) \geq d - 1$. Then $e_d(J) \geq 0$.*

Proof. Let $I = J^r$, $R = A[It] = \oplus_{n\geq 0} I^n$ denote the Rees algebra of A with respect to I, $R_+ = \oplus_{n>0} R_n$. By [1, Theorem 4.1] and [1, Theorem 3.8], we have

$$\begin{aligned}(-1)^d e_d(J) = (-1)^d e_d(I) &= P_I(0) - H_I(0) \\ &= \sum_{i=0}^{d} (-1)^i \ell(H^i_{R_+}(R)_0) \\ &= \sum_{i=0}^{d} (-1)^i \ell(H^i_{G(I)_+} G(I)_0).\end{aligned}$$

Since $\sigma_J(r) \geq d - 1$, $\operatorname{depth}(G(I)) \geq d - 1$. Thus $H^i_{G(I)_+}(G(I)) = 0$ for all $i = 0, ..., d-2$. From (1), $a_i(G(I)) \leq 0$ for all $i \geq 0$. On the other hand, by [8, Theorem 5.2], $a_{d-1}(G(I)) < a_d(G(I)) \leq 0$; that is, $a_{d-1}(G(I)) < 0$. Hence

$$(-1)^d e_d(J) = (-1)^d \ell(H^d_{G(I)_+}(G(I))_0).$$

This implies that

$$e_d(J) = \ell(H^d_{G(I)_+}(G(I))_0) \geq 0.$$

$\square$

Theorem 2.2 implies an early result of Marley [16, Corollary 2].

Corollary 2.3. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring of dimension $d \geq 1$. Let J be an $\mathfrak{m}$-primary ideal such that* $\operatorname{depth}(G(J)) \geq d-1$. *Then*

$$e_i(J) \geq 0 \quad \text{for} \quad i = 1, ..., d.$$

Proof. From (2), $\sigma_J(r) \geq \operatorname{depth}(G(J)) \geq d-1$. Applying Theorem 2.2, we get $e_d(J) \geq 0$.

Without loss of generality, assume that $A/\mathfrak{m}$ is infinite and $x_1, ..., x_{d-1}$ is a superficial sequence for J. For $i = 1, ..., d-1$, set $A_i = A/(x_1, ..., x_i)$ and $J_i = JA_i$. Then $e_i(J) = e_i(J_{d-i})$ from Lemma 1.1. By assumption, we have

$$\dim(A_{d-i}) = i \quad \text{and} \quad \operatorname{depth}(G(J_{d-i})) \geq i-1.$$

By [4, Proposition 2.2], $\sigma_{J_{d-i}}(r') \geq \operatorname{depth}(G(J_{d-i})) \geq i-1$, where $r' = \operatorname{reg}(G(J_{d-i}))$. Applying Theorem 2.2, we get $e_i(J) \geq 0$ for $1 = 1, ..., d-1$. $\square$

The following theorem is a generalization of [13, Proposition 3.5].

Theorem 2.4. *Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension $d \geq 3$ and* $\operatorname{depth}(A) \geq d-1$. *Let J be an $\mathfrak{m}$-primary ideal such that $r(J) \leq d-1$ and $\sigma_J(r) \geq d-2$. Then $e_d(J) \leq 0$.*

Proof. Let $I = J^r$. By arguing as the proof in Proposition 2.2, we have

$$\begin{aligned}(-1)^d e_d(J) = (-1)^d e_d(I) &= P_I(0) - H_I(0) \\ &= \sum_{i=0}^{d} (-1)^i \ell(H^i_{G(I)_+} G(I)_0).\end{aligned}$$

Since $\sigma_J(r) = \operatorname{depth}(G(I)) \geq d-2$, $H^i_{G(I)_+}(G(I)) = 0$ for $i = 0, ..., d-3$. By Lemma 1.3, we have $a_d(G(I)) + d \leq r(I)$. From [7, Lemma 2.7],

$$r(I) \leq \frac{]r(J)+1-s(J)[}{r} + s(I) - 1 = \frac{]r(J)+1-d[}{r} + d - 1 \leq d-1.$$

Hence $a_d(G(I)) < 0$. On the other hand, $a_i(G(I)) \leq 0$ for all $i \geq 0$ from (1). By applying [8, Theorem 5.2], we get $a_{d-2}(G(I)) < a_{d-1}(G(I)) \leq 0$. It follows that

$$(-1)^d e_d(J) = (-1)^{d-1} \ell(H^{d-1}_{G(I)_+} G(I)_0).$$

This implies that $e_d(J) = -\ell(H^{d-1}_{G(I)_+}(G(I))_0) \leq 0$. $\square$

From Theorem 2.2 and Theorem 2.4, we obtain the following corollary.

Corollary 2.5. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring of dimension $d \geq 2$. Let J be an $\mathfrak{m}$-primary ideal such that $r(J) \leq d-1$ and $\sigma_J(r) \geq d-1$. Then $e_d(J) = 0$.*

An ideal J is said to be asymptotically normal if there exists an integer $k \geq 1$ such that J^n is integrally closed for all $n \geq k$. If J is an asymptotically normal ideal of A, $\sigma(J) \geq 2$ by [20, Theorem 7.3]. In [14, Theorem 1.5], Mafi and Naderi proved that if A is a Cohen-Macaulay ring of dimension 4 and J be an $\mathfrak{m}$-primary asymptotically normal ideal such that $r(J) \leq 3$, then $e_4(J) \leq 0$. For $k \gg 0$, set $I = J^k$. By similarly argument as the proof of Theorem 2.4, we get the following corollary

Corollary 2.6. *Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension $d = 4$ and* $\operatorname{depth}(A) \geq 3$. *Let J be an $\mathfrak{m}$-primary asymptotically normal ideal of A such that $r(J) \leq 3$. Then $e_4(J) \leq 0$.*

Notice that the hypothesis of the ring A in Corollary 2.6 is not necessarily Cohen-Macaulay.

Corollary 2.7. *Let $(A, \mathfrak{m})$ be a noetherian ring of dimension $d = 4$ and* $\operatorname{depth}(A) \geq 3$. *Let J be an $\mathfrak{m}$-primary ideal of A. If $r(J) \leq 2$ and $\sigma_J(r) \geq 2$, then*

$$e_i(J) \leq 0 \quad \textit{for} \quad i = 3, 4.$$

Proof. Applying Theorem 3.2, we get $e_4(J) \leq 0$.

Without loss of generality, assume that $A/\mathfrak{m}$ is infinite and x_1 is a superficial sequence for J. Let $A_1 = A/(x_1)$ and $J_1 = JA_1$. Then $\dim(A_1) = 3$, J_1 is a $\mathfrak{m}$-primary ideal of A_1 and $e_3(J) = e_3(J_1)$. Since $\operatorname{depth}(A) \geq 3$, $\operatorname{depth}(A_1) \geq 2$. By Lemma 2.1, $\sigma_{J_1}(r_1) \geq 1$, where $r_1 = \operatorname{reg}(G(J_1)) + 1$. Moreover, $r(J_1) \leq r(J) \leq 2$. By applying Theorem 2.4, we obtain $e_3(J) = e_3(J_1) \leq 0$. □

In case of A is a Cohen-Macaulay ring of dimension $d = 3$ and $r(I) = 2$, Puthenpurakal [21, Theorem 9.1] proved that $e_3(J) \leq 0$. The following corollary is a extension the result of Puthenpurakal.

Corollary 2.8. *Let $(A, \mathfrak{m})$ be a noetherian ring with* $\dim(A) = d \geq 3$ *and* $\operatorname{depth}(A) \geq d - 1$. *If J be an $\mathfrak{m}$-primary ideal of A such that $r(J) \leq 2$, then $e_3(J) \leq 0$.*

Proof. By Lemma 2.1, one has $\sigma_J(r) \geq 1$. If $d = 3$, by applying Theorem 2.4 we get $e_3(J) \leq 0$.

If $d > 3$, without loss of generality, assume that $A/\mathfrak{m}$ is infinite and $x_1, ..., x_{d-3}$ is a superficial sequence for J. Let $\bar{A} = A/(x_1, ..., x_{d-3})$ and $\bar{J} = J\bar{A}$. Then $\dim(\bar{A}) = 3$, $\operatorname{depth}(\bar{A}) \geq 2$ and $r(\bar{J}) \leq r(J) \leq 2$. Sine $\operatorname{depth}(\bar{A}) \geq 2$ and by Lemma 2.1, $\sigma_{\bar{J}}(r') \geq 1$, where $r' = \operatorname{reg}(G(\bar{J})) + 1$. It follows $e_3(J) = e_3(\bar{J})$ from Lemma 1.1. Applying Theorem 2.4, we obtain $e_3(J) = e_3(\bar{J}) \leq 0$. □

By (2), $\sigma_J(r) \geq \operatorname{depth} G(J)$. From Theorem 2.4, we get the following corollary.

Corollary 2.9. *Let $(A, \mathfrak{m})$ a noetherian local ring with* $\dim(A) = d \geq 3$ *and* $\operatorname{depth}(A) \geq d - 1$. *Let J be an $\mathfrak{m}$-primary ideal of A such that $r(J) \leq 2$ and* $\operatorname{depth}(G(J)) \geq d - 2$. *Then*

$$e_i(J) \leq 0 \quad \textit{for} \quad i = 3, ..., d.$$

Proof. It is well known that $e_d(J) \leq 0$. If $d \leq 4$, the corollary is proved by Corollary 2.7. If $d > 4$, we need to prove that $e_{d-i}(J) \leq 0$ for $i = 1, ..., d - 2$. Indeed, without loss of generality, assume that $A/\mathfrak{m}$ is infinite and $x_1, ..., x_d$ is a superficial sequence for J. For each $i = 1, ..., d-2$, let $A_i = A/(x_1, ..., x_i)$, $J_i = JA_i$ and $r_i = \operatorname{reg}(G(J_i))+1$. From hypothesis, we have $\dim(A_i) = d-i$, $\operatorname{depth}(A_i) \geq d-i-1$. and $r(J_i) \leq r(J) \leq 2$. Since $\operatorname{depth}(G(J)) \geq d-2$, $\operatorname{depth}(G(J_i)) \geq d-i-2$. From (2), $\sigma_{J_i}(r_i) \geq \operatorname{depth}(G(J_i)) \geq d-i-2$. By applying Theorem 2.4, we get

$$e_{d-i}(J) = e_{d-i}(J_i) \leq 0 \quad \text{for} \quad i = 1,,,, d-2.$$

Hence $e_i(J) \leq 0$ for $i = 2, ..., d-1$. □

Corollary 2.9 is a generalization of an early results of Saikia-Saloni [24, Corollary 3.2] and Linh-Trung [12, Theorem 2.9].

Example 2.10. Let $A = \mathbb{Q}[x, y, z]_{(x,y,z)}$ and $J = (x^3, y^3, z^3, x^2y + z^3, xz^2, y^2z + x^2z, xyz)$. Then $K = (x^3, y^3, z^3)$ is a minimal reduction of J and $r_K(J) = 2$. Using Macaulay 2, we compute $\operatorname{depth}(G(J)) = 0$. Hence $\sigma_J(k) \geq 1$ for all $k \geq r$. On the other hand, the Hilbert series $P_{G(J)}(t)$ of $G(J)$ is

$$P_{G(J)}(t) = \sum_{n \geq 0} \ell(J^n/J^{n+1})t^n = \frac{h(t)}{(1-t)^3},$$

where $h(t) = a_0 + a_1t + \cdots + a_s \in \mathbb{Z}[t]$. It follows that

$$h(t) = a_0 + a_1t + \cdots + a_s = (1 - 3t + 3t^2 - t^3)P_{G(J)}(t).$$

Hence

$$\begin{aligned} a_0 &= \ell(A/J); \\ a_1 &= \ell(J/J^2) - 3\ell(A/J); \\ a_2 &= \ell(J^2/J^3) - 3\ell(J/J^2) + 3\ell(A/I); \\ a_i &= \ell(I^i/I^{i+1}) - 3\ell(I^{i-1}/J^i) + 3\ell(I^{i-2}/J^{i-1}) - \ell(I^{i-3}/J^{i-2}) \quad \text{for} \quad i \geq 3. \end{aligned}$$

By using Macaulay 2, we get

$$a_0 = 13, a_1 = 6, a_2 = 13, a_3 = -6, a_4 = 1, a_5 = a_6, = \cdots = 0.$$

That means

$$h(t) = 13 + 6t + 13t^2 - 6t^3 + t^4.$$

So,

$$\begin{aligned} &e_0(J) = h(1) = 27; \quad e_1(J) = h'(1) = 18; \\ &e_2(J) = h"(1)/2! = 1; \quad e_3(J) = h^{(3)}(1)/3! = -2. \end{aligned}$$

This implies that $\sigma_J(k) = \sigma(J) = 1$ for all $k \geq r = \operatorname{reg}(G(J)) + 1$.

3. Bound for Hilbert coefficients of parameter ideals

Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension d and $\operatorname{depth}(A) \geq d-1$. In this section, we will establish bounds for the Hilbert coefficients of parameter ideals.

Lemma 3.1. *Let A be a noetherian local ring of dimension $d \geq 2$ and* $\operatorname{depth}(A) \geq d-1$. *Let Q be a parameter ideal of A and x a superficial element for Q. For all $n \geq 1$, we have*

$$\ell(Q^{n+1} : x/Q^n) \leq -\binom{n+d-3}{d-2} e_1(Q).$$

Proof. Suppose that $Q = (x_1, ..., x_d)$ and $x = x_1$ is superficial for Q. Set $J = (x_1, ..., x_{d-1})$, we have

$$\begin{aligned} Q^{n+1} : x/Q^n &= ((xQ^n + J^nQ) : x)/Q^n \\ &= (Q^n + (J^nQ : x))/Q^n \\ &\cong (J^nQ : x)/(Q^n \cap (J^nQ : x)). \end{aligned}$$

Since

$$J^n \subseteq Q^n \cap (J^nQ : x),$$

we obtain

$$\ell(Q^{n+1} : x/Q^n) \leq \ell(J^n : x/J^n).$$

By [13, Corollary 4.4],

$$\ell(J^n : x/J^n) \leq -\binom{n+d-3}{d-2} e_1(Q).$$

This implies that

$$\ell(Q^{n+1} : x/Q^n) \leq -\binom{n+d-3}{d-2} e_1(Q).$$

□

Lemma 3.2. *Let A be a noetherian local ring of dimension $d \geq 2$ and* $\operatorname{depth}(A) \geq 1$. *Let I be an $\mathfrak{m}$-primary ideal of A and x a superficial element for I. Then*

$$(-1)^d e_d(I) = \sum_{k=0}^{r} (H_{\bar{I}}(k) - P_{\bar{I}}(k)) - \sum_{k=0}^{r} \ell(I^{k+1} : x/I^k),$$

where some $r \geq \operatorname{reg}(G(I)) + 1$, $\bar{A} = A/(x)$ and $\bar{I} = I\bar{A}$.

Proof. From [17, Lemma 3.2], we have

$$(-1)^d e_d(I) = \sum_{k=0}^{\infty} (H_{\bar{I}}(k) - P_{\bar{I}}(k) - \sum_{k=0}^{\infty} \ell(I^{k+1} : x/I^k).$$

By Lemma 1.2, $n(\bar{I}) \leq \operatorname{reg}(G(\bar{I})) \leq \operatorname{reg}(G(I)) < r$ and $\ell(I^{k+1} : x/I^k) = \ell(0 :_A x) = 0$ for $k \geq r$. Thus

$$(-1)^d e_d(I) = \sum_{k=0}^{r} (H_{\bar{I}}(k) - P_{\bar{I}}(k) - \sum_{k=0}^{r} \ell(I^{k+1} : x/I^k),$$

□

In [13], the author gave a bound for the regularity of associated graded ring with respect to parameter ideals in terms of the first coefficient $e_1(Q)$.

Theorem 3.3. [13, Theorem 4.5] *Let A be a noetherian local ring of dimension $d \geq 1$ and* $\operatorname{depth}(A) \geq d-1$. *Let Q be a parameter ideal of A. Then*

$$\operatorname{reg}(G(Q)) \leq \max\{-e_1(Q) - 1, 0\} \quad \text{if } \ d = 1;$$
$$\operatorname{reg}(G(Q)) \leq \max\{[-4e_1(Q)]^{(d-1)!} + e_1(Q) - 1, 0\} \quad \text{if } \ d \geq 2.$$

Using the bound for the regularity of $G(Q)$ in Theorem 3.3, we will establish bounds for Hilbert coefficients $e_i(Q)$.

Theorem 3.4. *Let A be a noetherian local ring of dimension $d \geq 2$ and* $\operatorname{depth}(A) \geq d-1$. *Let Q be a parameter ideal of A. Then*

$$|e_i(Q)| \leq 3.2^{i-2} r^{i-1} |e_1(Q)| \quad \text{for } \ i = 2, ..., d,$$

where $r = \max\{[-4e_1(Q)]^{(d-1)!} + e_1(Q) - 1, 0\} + 1$.

Proof. By Lemma 3.2, we have

$$\begin{aligned}(-1)^d e_d(Q) &= \sum_{k=0}^{r} [H_{\bar{A}}(k) - P_{\bar{A}}(k)] - \sum_{k=0}^{r} \ell(Q^{k+1} : x/Q^k) \\ &= \sum_{k=0}^{r} [\ell(\bar{A}/\overline{Q}^k) - \sum_{i=0}^{d-1} (-1)^i \binom{k+d-i-2}{d-i-1} e_i(\overline{Q})] - \sum_{k=0}^{r} \ell(Q^{k+1} : x/Q^k).\end{aligned}$$

By [13, Lemma 4.1],

$$0 \leq \ell(\bar{A}/\overline{Q}^k) - \binom{k+d-2}{d-1} e_0(\bar{Q}) \leq -\binom{k+d-3}{d-2} e_1(\bar{Q}).$$

From [13, Corollary 4.3],

$$\ell(Q^{k+1} : x/Q^k) \leq \binom{k+d-3}{d-2} |e_1(Q)|.$$

Thus

$$\begin{aligned}|e_d(Q)| &\leq 3 \sum_{k=0}^{r} \binom{k+d-3}{d-2} |e_1(Q)| + \sum_{k=0}^{r} \sum_{i=2}^{d-1} \binom{k+d-i-2}{d-i-1} |e_i(\overline{Q})| \\ &\leq 3 \binom{r+d-2}{d-1} |e_1(Q)| + \sum_{i=2}^{d-1} \sum_{k=0}^{r} \binom{k+d-i-2}{d-i-1} |e_i(\overline{Q})| \\ &= 3 \binom{r+d-2}{d-1} |e_1(Q)| + \sum_{i=2}^{d-1} \binom{r+d-i-1}{d-i} |e_i(\overline{Q})|.\end{aligned}$$

Notice that

$$\binom{r+d-2}{d-1} \leq r^{d-1} \quad \text{and} \quad \binom{r+d-i-1}{d-i} \leq r^{d-i}.$$

Hence

$$|e_d(Q)| \leq 3.r^{d-1} |e_1(Q)| + \sum_{i=2}^{d-1} r^{d-i} e_i(\overline{Q}).$$

By induction on d, we may assume that

$$|e_i(\overline{Q})| \leq 3.2^{i-2}.r^{i-1} |e_1(\overline{Q})| \quad \text{for} \quad i = 2, ..., d-1.$$

But $e_i(Q) = e_i(\overline{Q})$ for $i = 1, ..., d-1$, from Lemma 1.1. This implies that

$$|e_i(Q)| \leq 3.2^{i-2}.r^{i-1}|e_1(\overline{Q})| = 3.2^{i-2}.r^{i-1}|e_1(Q)| \quad \text{for} \quad i = 2, ..., d-1.$$

It remains to prove the bound for $e_d(Q)$. Indeed, from inductive hypothesis we have

$$\begin{aligned}|e_d(Q)| &\leq 3.r^{d-1}|e_1(Q)| + \sum_{i=2}^{d-1} r^{d-i}.3.2^{i-2}.r^{i-1}|e_1(Q)| \\ &= 3.r^{d-1}|e_1(Q)| + \sum_{i=2}^{d-1} 3.r^{d-1}.2^{i-2}|e_1(Q)| \\ &= 3.r^{d-1}|e_1(Q)| + 3.r^{d-1}|e_1(Q)|(\sum_{i=2}^{d-1} 2^{i-2}) \\ &= 3.r^{d-1}|e_1(Q)| + 3.r^{d-1}|e_1(Q)|.(2^{d-2} - 1) \\ &= 3.2^{d-2}.r^{d-1}|e_1(Q)|.\end{aligned}$$

This finishes the proof. □

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