ON I-FINE MODULES, I-COFINE MODULES AND LOCAL COHOMOLOGY MODULES

TRAN TUAN NAM[1], NGUYEN MINH TRI[2,3], AND NGUYEN THANH NAM[4]

Abstract. In this paper, we introduce I-fine modules and I-cofine modules. Some results of local cohomology modules concerning to these modules will be shown.

Key words: Local cohomology, coatomic module, I-fine module.
2010 Mathematics subject classification: 13D45.

1. Introduction

Throughout this paper, R is a noetherian commutative (with non-zero identity) ring and I is an ideal of R. It is well-known that the local cohomology theory of Grothendieck is an important tool in commutative algebra and algebraic geometry. For an R-module M, the I-torsion submodule of M is

$$\Gamma_I(M) = \{x \in M \mid I^n x = 0 \text{ for some positive integer } n\}.$$

The functor Γ_I from the category of R-modules to itself is covariant, left exact and R-linear. The i-th right derived functor of Γ_I is called the i-th local cohomology functor H^i_I with respect to ideal I. When M is a finitely generated R-module, many properties of $H^i_I(M)$ have been studied in [3], [5], [7] and [9]. In [1] the authors studied some properties of the local cohomology modules $H^i_I(M)$ relating to coatomic modules. An R-module M is called coatomic if every proper submodule of M is contained in a maximal submodule of M. The coatomic modules were introduced by H. Zöschinger in [10]. An important result on coatomic modules is implied from [10, Satz 2.4] that if M is a coatomic over a local ring $(R, \mathfrak{m})$, then there is a short exact sequence

$$0 \to 0 :_M \mathfrak{m}^t \to M \to M/(0 :_M \mathfrak{m}^t) \to 0,$$

where $M/(0 :_M \mathfrak{m}^t)$ is finitely generated for some integer t. It is known that finitely generated modules are coatomic. Another extension of

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 101.04-2018.304.

finitely generated modules are minimax modules. Minimax modules were first introduced by H. Zöschinger in [11]. An R-module M is minimax if there is a finitely generated submodule N of M such that M/N is artinian. Base on the coatomic modules and minimax modules, we introduce I-fine R-modules and I-cofine R-modules. An R-module M is I-fine if it has an I-torsion submodule N such that M/N is finitely generated. In a local ring $(R, \mathfrak{m})$, coatomic modules are $\mathfrak{m}$-fine. An R-module M is I-cofine if it has a finitely generated submodule N such that M/N is I-torsion. The minimax R-modules over a local ring $(R, \mathfrak{m})$ are $\mathfrak{m}$-cofine.

The organization of our paper is as follows. In Section 2, we deal with I-fine modules. An equivalent condition of I-torsion modules when M is an I-fine R-module is shown in Theorem 2.5. Next, Theorem 2.8 shows that $H^i_I(M)$ is finitely generated or coatomic for all $i \geq t$ if and only if $H^i_I(M) = 0$ for all $i \geq t$. When studying the finiteness of supports of local cohomology modules with respect to an ideal, Aghapournahr and Melkersson in [1] or Saremi in [6] proved that $\operatorname{Supp}(H_I^{\dim M-1}(M))$ is a finite set. Now, we verify in Theorem 2.14 that in a semi-local ring, the set $\operatorname{Supp}(H_I^{\dim M-1}(M))$ is finite when M is an I-fine R-module. Section 2 is closed by Theorem 2.16 which affirms that if M is an I-fine R-module with $d = \dim M > 0$ or an I-cofine R-module with $d = \dim M > 1$ over a local ring $(R, \mathfrak{m})$, then the module $H^d_I(M)$ is artinian and

$$\operatorname{Att}(H^d_I(M)) = \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{cd}(I, R/\mathfrak{p}) = d\}.$$

The last section is devoted to the study of I-cofine modules. Theorem 3.5 asserts that in a local ring $(R, \mathfrak{m})$ and M a non-zero $\mathfrak{m}$-cofine R-module, the module $H^i_{\mathfrak{m}}(M)$ is artinian for all $i > 0$ and $H_{\mathfrak{m}}^{\dim M}(M) \neq 0$. Finally, the set of attached primes of $H_I^{\dim M}(M)$ is established in Theorem 3.8 when M is an I-cofine module.

2. I-fine modules

An R-module M is called I-torsion if $M = \Gamma_I(M)$. When M is a finitely generated R-module then M is I-torsion if and only if $I^tM = 0$ for some integer t. Firstly, we extend this result in the case M is belong to a class of R-modules which contains class of finitely generated R-modules.

Definition 2.1. An R-module M is I-fine if it has an I-torsion submodule N such that M/N is finitely generated.

The following examples may be implied immediately from 2.1.

Example 2.2. (i) Finitely generated modules are I-fine modules.

(ii) Coatomic modules over local ring $(R, \mathfrak{m})$ are $\mathfrak{m}$-fine.

(iii) Let M be a finitely generated R-module. It follows from [3, 2.2.6] that there is a short exact sequence

$$0 \to M/\Gamma_I(M) \to D_I(M) \to H_I^1(M) \to 0.$$

Therefore, the ideal transform $D_I(M)$ is I-fine.

Here are some elementary properties of this concept.

Proposition 2.3. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence. Then M is I-fine if and only if L, N are both I-fine.*

Proof. We can assume that L is a submodule of M and $N = M/L$. Firstly, if M is an I-fine R-module, then there is an I-torsion submodule K of M such that M/K is finitely generated. It is clear that $K \cap L$ is an I-torsion submodule of L and

$$L/K \cap L \cong K + L/K \subseteq M/K.$$

Hence $L/K \cap L$ is finitely generated and then L is I-fine. Next we show that M/L is I-fine. Since $K + L/L \cong K/K \cap L$, it follows that $K + L/L$ is I-torsion. On the other hand,

$$\frac{M/L}{K + L/L} \cong \frac{M}{K + L}$$

and $M/K + L$ is a homomorphic image of finitely generated R-module M/K. Therefore, M/L is an I-fine R-module.

Now assume that $L, M/L$ are both I-fine R-modules. By 2.1, L has an I-torsion submodule K such that L/K is finitely generated and there is a submodule P of M containing L such that P/L is I-torsion and M/P is finitely generated. Note that K is an I-torsion submodule of M and M/K is finitely generated since

$$\frac{M}{K} \cong \frac{M/P}{P/K}.$$

Consequently, M is an I-fine R-module. □

From 2.3, the I-fine modules is a Serre subcategory of the category of R-modules.

Corollary 2.4. *The following statements hold:*

(i) *Direct sum of finite I-fine R-modules is I-fine.*

(ii) *Let M be a finitely generated R-module and N an I-fine R-module. Then* $\operatorname{Ext}^i_R(M,N)$ *and* $\operatorname{Tor}^R_i(M,N)$ *are I-fine for all* $i \geq 0$.

Now we give some equivalent conditions on the I-torsionness relating to I-fine R-modules.

Theorem 2.5. *Let M be an I-fine R-module. The following statements are equivalent:*

(i) *M is I-torsion;*
(ii) $H^i_I(M) = 0$ *for all* $i > 0$.

Proof. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (i). By 2.1, there is a short exact sequence

$$0 \to N \to M \to P \to 0,$$

where N is I-torsion and P is finitely generated. By applying the functor Γ_I to the above exact squence, we have

$$H^i_I(M) \cong H^i_I(P)$$

for all $i > 0$ since $H^i_I(N) = 0$ for all $i > 0$ by [3, 2.1.7(i)].

It follows from the assumption that $H^i_I(P) = 0$ for all $i > 0$. If $P \neq \Gamma_I(P)$, then $P/\Gamma_I(P)$ is an I-torsion-free R-module. There is an element $x \in I$ which is $P/\Gamma_I(P)$-regular. Hence $H^i_I(P/\Gamma_I(P)) \neq 0$ for some $i \geq 1$. On the other hand, $H^i_I(P/\Gamma_I(P)) \cong H^i_I(P)$ for all $i > 0$ and this is a contradition. Therefore, P is I-torsion and then so is M. □

Let M be a finitely generated module over local ring $(R, \mathfrak{m})$, then $H^i_{\mathfrak{m}}(M)$ is artinian for all $i \geq 0$. Now, we extend this property in the case M is a $\mathfrak{m}$-fine R-module.

Proposition 2.6. *Let $(R, \mathfrak{m})$ be a local ring and M a non-zero $\mathfrak{m}$-fine R-module. Then*

(i) $H^i_{\mathfrak{m}}(M)$ *is artinian for all* $i > 0$.
(ii) $H^{\dim M}_{\mathfrak{m}}(M) \neq 0$.

Proof. (i). We have a short exact sequence

$$0 \to N \to M \to K \to 0,$$

where N is $\mathfrak{m}$-torsion and K is finitely generated. Apply the functor $\Gamma_{\mathfrak{m}}$ to above exact sequence, we get an exact sequence

$$0 \to \Gamma_{\mathfrak{m}}(N) \to \Gamma_{\mathfrak{m}}(M) \to \Gamma_{\mathfrak{m}}(K) \to 0$$

and $H^i_{\mathfrak{m}}(K) \cong H^i_{\mathfrak{m}}(M)$ for all $i > 0$. Since K is finitely generated, it follows that $H^i_{\mathfrak{m}}(K)$ is artinian for all $i \geq 0$. Therefore $H^i_{\mathfrak{m}}(M)$ is artinian for all $i > 0$.

(ii) If $\dim M = 0$, then M is $\mathfrak{m}$-torsion. Thus $H^0_{\mathfrak{m}}(M) = M$ and the assertion follows from the hypothesis on M.

Let $\dim M > 0$. Note that $\dim M = \dim K$ and $H^{\dim M}_{\mathfrak{m}}(K) \cong H^{\dim M}_{\mathfrak{m}}(M)$. By [3, 6.1.4], $H^{\dim M}_{\mathfrak{m}}(K) \neq 0$ and the proof is complete. □

Corollary 2.7. [1, 3.2, 3.4] *Let $(R, \mathfrak{m})$ be a local ring and M a non-zero coatomic R-module. Then*

(i) $H^i_{\mathfrak{m}}(M)$ *is artinian for all* $i > 0$.
(ii) $H^{\dim M}_{\mathfrak{m}}(M) \neq 0$.

The following theorem is a generalization of [1, 3.9] which shows a relationship on the vanishing, the finiteness of $H^i_I(M)$.

Theorem 2.8. *Let $(R, \mathfrak{m})$ be a local ring, M an I-fine R-module and t a positive integer. The following statements are equivalent:*

(i) $H^i_I(M) = 0$ *for all* $i \geq t$;
(ii) $H^i_I(M)$ *is finitely generated for all* $i \geq t$;
(iii) $H^i_I(M)$ *is coatomic for all* $i \geq t$.

Proof. (i) ⇒ (ii) ⇒ (iii). Trivial.

(iii) ⇒ (i). There is a short exact sequence

$$0 \to L \to M \to N \to 0,$$

where L is I-torsion and N is finitely generated. Apply the functor Γ_I to the above exact sequence, we get an exact sequence

$$0 \to \Gamma_I(L) \to \Gamma_I(M) \to \Gamma_I(N) \to 0$$

and $H^i_I(N) \cong H^i_I(M)$ for all $i > 0$. By the assumption, $H^i_I(N)$ is coatomic for all $i \geq t$. It follows from [1, 3.9] that $H^i_I(N) = 0$ for all $i \geq t$, which completes the proof. □

Corollary 2.9. *Let $(R, \mathfrak{m})$ be a local ring and M an I-fine R-module with* $\operatorname{cd}(I, M) > 0$. *Then* $H^{cd(I,M)}_I(M)$ *is not finitely generated.*

Proposition 2.10. *Let M be an I-fine R-module and t a positive integer such that $H^i_I(M) = 0$ for all $i > t$. Then $H^t_I(M)/IH^t_I(M) = 0$.*

Proof. Since M is an I-fine R-module, there is an I-torsion submodule N such that M/N is finitely generated. The proof above gives

$$H^t_I(M) \cong H^t_I(M/N).$$

Combining the hypothesis with [2, 3.1] we have $H_I^t(M/N)/IH_I^t(M/N) = 0$ and the assertion follows. □

Corollary 2.11. *Let* $(R, \mathfrak{m})$ *be a local ring and* M *an* I*-fine* R*-module. Assume that* $\mathrm{cd}(I, M) > 0$ *and* K *is a proper submodule of* $H_I^{\mathrm{cd}(I,M)}(M)$*. Then* $H_I^{\mathrm{cd}(I,M)}(M)/K$ *is not a coatomic* R*-module.*

Proof. Suppose that the conclusion is false. By the definition of coatomic modules, there exists a submodule L of $H_I^{\mathrm{cd}(I,M)}(M)$ such that we have a short exact sequence

$$0 \to L/K \to H_I^{\mathrm{cd}(I,M)}(M)/K \to R/\mathfrak{m} \to 0.$$

By applying the functor $R/I \otimes_R -$ to the above exact sequence, there is a following exact sequence

$$\cdots \to L/IL + K \to H_I^{\mathrm{cd}(I,M)}(M)/IH_I^{\mathrm{cd}(I,M)}(M) + K \to R/\mathfrak{m} \to 0.$$

Note that $H_I^{\mathrm{cd}(I,M)}(M)/IH_I^{\mathrm{cd}(I,M)}(M) + K$ is a homomorphic image of $H_I^{\mathrm{cd}(I,M)}(M)/IH_I^{\mathrm{cd}(I,M)}(M)$. Consequently, we can conclude that $H_I^{\mathrm{cd}(I,M)}(M)/IH_I^{\mathrm{cd}(I,M)}(M) + K = 0$ by 2.10. This implies that $R/\mathfrak{m} = 0$ which is a contradiction. □

We see in [9, 2.2] that $H_I^i(M)$ is artinian for all $i < t$ if M is a finitely generated R-module such that $\mathrm{Supp}(H_I^i(M)) \subseteq \mathrm{Max}$ for all $i < t$. Now, we consider the case M is an I-fine module.

Proposition 2.12. *Let* M *be an* I*-fine* R*-module and* t *a non-negative integer such that* $\mathrm{Supp}(H_I^i(M)) \subseteq \mathrm{Max}(R)$ *for all* $i < t$*. Then* $H_I^0(M)$ *is minimax and* $H_I^i(M)$ *is artinian for all* $0 < i < t$*.*

Proof. It follows from 2.1 that there is an I-torsion submodule N of M such that M/N is finitely generated. From the exactness of the sequence

$$0 \to N \to M \to M/N \to 0$$

we deduce a short exact sequence

$$0 \to N \to \Gamma_I(M) \to \Gamma_I(M/N) \to 0$$

and $H_I^i(M) \cong H_I^i(M/N)$ for all $i > 0$. Since $\mathrm{Supp}(H_I^i(M)) \subseteq \mathrm{Max}(R)$ for all $i < t$, we infer that $\mathrm{Supp}(H_I^i(M/N)) \subseteq \mathrm{Max}(R)$ for all $i < t$. Since M/N is a finitely generated R-module, we can conclude by [9, 2.2] that $H_I^i(M/N)$ is artinian for all $i < t$ and which completes the proof. □

Corollary 2.13. *Let* $(R, \mathfrak{m})$ *be a local ring,* M *a coatomic* R*-module. Assume that* t *is a non-negative integer such that* $\mathrm{Supp}(H_I^i(M)) \subseteq \{\mathfrak{m}\}$ *for all* $i < t$*. Then* $H_I^0(M)$ *is minimax and* $H_I^i(M)$ *is artinian for all* $0 < i < t$*.*

Next, we will consider the dimension of $H_I^i(M)$ and the support of $H_I^{d-1}(M)$ where $d = \dim M$. In [1, 3.3] or [6, 2.3], when studying the local cohomology modules with respect to an ideal, the authors showed that $\dim H_I^i(M) \leq d - i$ and $\mathrm{Supp}(H_I^{d-1}(M))$ is a finite set. The proof of next theorem is based on these results. .

Theorem 2.14. *Let* M *be an* I*-fine* R*-module with* $d = \dim M < \infty$*. Then*

(i) $\dim H_I^i(M) \leq d - i$*.*
(ii) *If* R *is a semi-local ring, then* $\mathrm{Supp}(H_I^{d-1}(M))$ *is finite.*

Proof. (i) The short exact sequence

$$0 \to L \to M \to N \to 0,$$

where L is I-torsion and N is finitely generated, induces an exact sequence

$$0 \to \Gamma_I(L) \to \Gamma_I(M) \to \Gamma_I(N) \to 0$$

and $H_I^i(N) \cong H_I^i(M)$ for all $i > 0$. It follows from [1, 3.3(a)] that $\dim H_I^i(M) \leq d - i$ for all $0 < i \leq d$. Note that $\dim \Gamma_I(N) \leq d$ and $\dim \Gamma_I(L) = \dim L \leq d$. Hence $\dim \Gamma_I(M) \leq d$, as required.

(ii) If $\dim M > 1$, then the claim follows from [1, 3.3(b)] and the isomorphism $H_I^i(N) \cong H_I^i(M)$ for all $i > 0$. Now assume that $\dim M = 1$. Note that $\mathrm{Supp}(H_I^0(N))$ is finite by [1, 3.3(b)]. Since N is I-torsion and $\dim N \leq 1$, then $\mathrm{Supp}(N)$ is finite. Now from the equality

$$\mathrm{Supp}(H_I^0(M)) = \mathrm{Supp}(H_I^0(L)) \cup \mathrm{Supp}(H_I^0(N))$$

we can conclude that $\mathrm{Supp}(H_I^0(M))$ is finite, which completes the proof. □

Corollary 2.15. *Let* M *be an* I*-fine* R*-module with finite dimension. Then*

$$\mathrm{Supp}(H_I^{\dim M - 1}(M)) \subseteq \mathrm{Ass}(H_I^{\dim M - 1}(M)) \cup \mathrm{Max}(R).$$

Proof. Since $\dim(H_I^{\dim M - 1}(M)) \leq 1$, we see that $\mathrm{Supp}(H_I^{\dim M - 1}(M))$ contains minimal prime ideals of $\mathrm{Ass}(H_I^{\dim M - 1}(M))$ and maximal ideals, which completes the proof. □

It is well-known that if M is a finitely generated R-module with $\dim M = d$, then $H^d_I(M)$ is artinian. In [5], Dibaei and Yassemi proved that

$$\mathrm{Att}(H^d_I(M)) = \{\mathfrak{p} \in \mathrm{Ass}(M) \mid \mathrm{cd}(I, R/\mathfrak{p}) = n\},$$

where $\mathrm{cd}(I, N) = \sup\{n \mid H^n_I(N) \neq 0\}$.

Theorem 2.16. *Let M be an I-fine R-module with $d = \dim M > 0$. Then $H^d_I(M)$ is artinian and*

$$\mathrm{Att}(H^d_I(M)) = \{\mathfrak{p} \in \mathrm{Supp}(M) \mid \mathrm{cd}(I, R/\mathfrak{p}) = d\}.$$

Proof. There is an I-torsion R-submodule N of M such that M/N is finitely generated by 2.1. From the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

we have

$$H^i_I(M) \cong H^i_I(M/N)$$

for all $i > 0$. If $\dim M/N = \dim M > 0$, then $H^d_I(M/N)$ is artinian. Hence $H^d_I(M)$ is also artinian. Now by [5, Theorem A],

$$\begin{aligned}\mathrm{Att}(H^d_I(M)) &= \mathrm{Att}(H^d_I(M/N)) \\ &= \{\mathfrak{p} \in \mathrm{Supp}(M/N) \mid \mathrm{cd}(I, R/\mathfrak{p}) = d\}.\end{aligned}$$

Since N is I-torsion, it follows that $\mathrm{Supp}(N) \subseteq V(I)$ and then

$$\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(M/N) \subseteq \mathrm{Supp}(M/N) \cup V(I).$$

Let $\mathfrak{p} \in \mathrm{Supp}(M)$ such that $\mathrm{cd}(I, R/\mathfrak{p}) = d > 0$. We see that $\mathfrak{p} \notin V(I)$ since $H^i_I(R/\mathfrak{p}) = 0$ for all $i > 0$. This implies that $\mathfrak{p} \in \mathrm{Supp}(M/N)$. Therefore

$$\mathrm{Att}(H^d_I(M)) = \{\mathfrak{p} \in \mathrm{Supp}(M) \mid \mathrm{cd}(I, R/\mathfrak{p}) = d\}.$$

If $\dim M/N < \dim M$, then $\dim M = \dim N$. We see that $H^d_I(M) = 0$ and $\mathrm{Att}(H^d_I(M)) = \emptyset$. Let $\mathfrak{p} \in \mathrm{Supp}(M)$ such that $\mathrm{cd}(I, R/\mathfrak{p}) = d > 0$. We see that $\mathfrak{p} \in \mathrm{Supp}(N) \cup \mathrm{Supp}(M/N)$. Since $\mathrm{Supp}(N) \subseteq V(I)$, it follows that $\mathfrak{p} \notin \mathrm{Supp}(N)$. Hence $\mathfrak{p} \in \mathrm{Supp}(M/N)$ and $\dim(R/\mathfrak{p}) < d$ since $\dim M/N < d$. This yields $\mathrm{cd}(I, R/\mathfrak{p}) < d$. So we can conclude that

$$\{\mathfrak{p} \in \mathrm{Supp}(M) \mid \mathrm{cd}(I, R/\mathfrak{p}) = d\} = \emptyset,$$

and the proof is complete. □

It should be mentioned that the above result is not true when $\dim M = 0$. The example is similar to [1, 3.5]. On the other hand, if R is not a semi-local ring and $\dim M = 0$, then $H^0_I(M)$ is not artinian.

Let $R = \mathbb{Z}, M = (\mathbb{Z}_2)^{\mathbb{N}}$ and $I = 2\mathbb{Z}$. We see that $\dim M = 0$ and $H^0_I(M) = M$ is not artinian.

3. On I-cofine modules

Next, we give another definition that is an extension of finitely generated R-modules.

Definition 3.1. An R-module M is I-cofine if it has a finitely generated submodule N such that M/N is I-torsion.

We see that finitely generated modules are I-cofine modules. In a local ring $(R, \mathfrak{m})$, minimax modules are $\mathfrak{m}$-cofine modules. The reversion will be shown in the following proposition.

Proposition 3.2. *Let $(R, \mathfrak{m})$ be a local ring and M a $\mathfrak{m}$-cofine R-module. If $0 :_M \mathfrak{m}$ is artinian, then M is minimax.*

Proof. There is a short exact sequence

$$0 \to L \to M \to N \to 0,$$

where L is finitely generated and N is $\mathfrak{m}$-torsion. From this, we get an exact sequence

$$0 \to 0 :_L \mathfrak{m} \to 0 :_M \mathfrak{m} \to 0 :_N \mathfrak{m} \to \mathrm{Ext}^1_R(R/\mathfrak{m}, L).$$

Since L is finitely generated, it follows that $\mathrm{Ext}^1_R(R/\mathfrak{m}, L)$ is artinian. By the assumption, $0 :_N \mathfrak{m}$ is an artinian R-module. It follows from [4, 1.3] that N is artinian, which infers that M is a minimax R-module. $\square$

Let us mention an important property of this concept.

Proposition 3.3. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence. Then M is I-cofine if and only if L, N are both I-cofine.*

Proof. We can assume that L is a submodule of M and $N = M/L$. Let M be an I-cofine R-module. By 3.1 there is a finitely generated submodule K of M such that M/K is I-torsion. This implies that $K \cap L$ a finitely generated submodule of L. Moreover,

$$\frac{L}{K \cap L} \cong \frac{K + L}{K} \subseteq \frac{M}{K}$$

and by [3, 2.1.3] $L/K \cap L$ is I-torsion. Hence L is an I-cofine R-module. Now we prove that M/L is I-cofine. From the isomorphism

$$\frac{K + L}{L} \cong \frac{K}{K \cap L}$$

we get that $K + L/L$ is a finitely generated submodule of M/L since K is finitely generated. Next, combining the isomorphisms

$$\frac{M/L}{K + L/L} \cong \frac{M}{K + L} \cong \frac{M/K}{K + L/K}$$

with [3, 2.1.3], $(M/L)/(K + L/L)$ is an I-torsion R-module. Therefore M/L is an I-cofine R-module.

Now, assume that L and M/L are both I-cofine. There is a finitely generated submodule K of L such that L/K is I-torsion and a submodule P of M containing L such that P/L is finitely generated and $(M/L)/(P/L)$ is I-torsion. This induces that K is a finitely generated submodule of M. The isomorphism

$$\frac{M}{K} \cong \frac{M/P}{P/K}$$

shows that M/K is I-torsion. Hence M is an I-cofine R-module. □

From 3.3, the I-cofine modules is a Serre subcategory of the category of R-modules.

Corollary 3.4. *The following statements hold:*

(i) *Direct sum of finite I-cofine R-modules is I-cofine.*
(ii) *Let M be a finitely generated R-module and N an I-cofine R-module. Then $\mathrm{Ext}^i_R(M, N)$ and $\mathrm{Tor}^R_i(M, N)$ are I-cofine for all $i \geq 0$.*

Let M be a finitely generated module over local ring $(R, \mathfrak{m})$, then $H^i_{\mathfrak{m}}(M)$ is artinian for all $i \geq 0$. Now, we extend this property in the case M is an $\mathfrak{m}$-cofine R-module.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a local ring and M a non-zero $\mathfrak{m}$-cofine R-module. Then*

(i) *$H^i_{\mathfrak{m}}(M)$ is artinian for all $i > 0$.*
(ii) *If $\dim M \neq 1$, then $H^{\dim M}_{\mathfrak{m}}(M) \neq 0$.*

Proof. (i). We have a short exact sequence

$$0 \to N \to M \to K \to 0,$$

where K is $\mathfrak{m}$-torsion and N is finitely generated. Apply the functor $\Gamma_{\mathfrak{m}}$ to above exact sequence, we get an exact sequence

$$0 \to \Gamma_{\mathfrak{m}}(N) \to \Gamma_{\mathfrak{m}}(M) \to \Gamma_{\mathfrak{m}}(K) \to H^1_{\mathfrak{m}}(N) \to H^1_{\mathfrak{m}}(M) \to 0$$

and $H^i_{\mathfrak{m}}(N) \cong H^i_{\mathfrak{m}}(M)$ for all $i > 1$. Since N is finitely generated, it follows that $H^i_{\mathfrak{m}}(N)$ is artinian for all $i \geq 0$. Therefore $H^i_{\mathfrak{m}}(M)$ is artinian for all $i > 0$.

(ii) If $\dim M = 0$, then M is $\mathfrak{m}$-torsion. Consequently $H^0_{\mathfrak{m}}(M) = M$. The assertion follows from the hypothesis on M.

Let $\dim M > 1$. Note that $\dim N = \dim M$ and $H^{\dim M}_{\mathfrak{m}}(N) \cong H^{\dim M}_{\mathfrak{m}}(M)$. By [3, 6.1.4], $H^{\dim N}_{\mathfrak{m}}(N) \neq 0$ and the proof is complete. □

If $\dim M = 1$, then $H^1_I(M)$ can be vanished.

Example 3.6. Let $R = \mathbb{Z}, M = (\mathbb{Z}_2)^{\mathbb{N}}$ and $I = 2\mathbb{Z}$. Then $\dim M = 1$ and M is I-torsion. Hence $H^1_I(M) = 0$.

Corollary 3.7. *Let $(R, \mathfrak{m})$ be a local ring and M a minimax R-module. Then the following statements hold:*

(i) $H^i_{\mathfrak{m}}(M)$ *is artinian for all* $i > 0$;
(ii) $H^{\dim M}_{\mathfrak{m}}(M) \neq 0$ *where* $\dim M \neq 1$.

Theorem 3.8. *Let M be an I-cofine R-module with $d = \dim M$. The following statements hold:*

(i) $\dim H^i_I(M) \leq d - i$.
(ii) *If R is a semi-local ring, then* $\operatorname{Supp}(H^{d-1}_I(M))$ *is finite.*
(iii) *If* $\dim M > 1$, *then* $H^d_I(M)$ *is artinian and*

$$\operatorname{Att}(H^d_I(M)) = \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{cd}(I, R/\mathfrak{p}) = d\}.$$

Proof. (i) By 3.1, there exists a short exact sequence

$$0 \to N \to M \to A \to 0$$

where N is finitely generated and A is I-torsion. By applying the functor Γ_I to the above exact sequence, there is an exact sequence

$$0 \to H^0_I(N) \to H^0_I(M) \to H^0_I(A) \to H^1_I(N) \to H^1_I(M) \to 0$$

and

$$H^i_I(N) \cong H^i_I(M)$$

for all $i \geq 2$. It follows from [1, 3.3] that $\dim H^i_I(N) \leq d - i$. This implies that $\dim H^i_I(M) \leq d - i$.

(ii). Using again [1, 3.3], $\operatorname{Supp}(H^{d-1}_I(N))$ is finite. The assertion holds in the case $\dim M \geq 2$. Now, assume that $\dim M = 1$. We see that $\dim N \leq 1$ and $\dim H^0_I(A) = \dim A \leq 1$. Then $\operatorname{Supp}(H^0_I(N))$ and $\operatorname{Supp}(H^0_I(A))$ are finite. Consequently, $\operatorname{Supp}(H^0_I(M))$ is finite and the claim follows.

(iii). If $\dim M = \dim N$, then $H^d_I(N)$ is artinian and so is $H^d_I(M)$. By using [5, Theorem A] again, we have

$$\begin{aligned}\operatorname{Att}(H^d_I(M)) &= \operatorname{Att}(H^d_I(N)) \\ &= \{\mathfrak{p} \in \operatorname{Supp}(N) \mid \operatorname{cd}(I, R/\mathfrak{p}) = d\}.\end{aligned}$$

Note that

$$\operatorname{Supp}(M) = \operatorname{Supp}(N) \cup \operatorname{Supp}(A) \subseteq \operatorname{Supp}(N) \cup V(I)$$

since A is an I-torsion R-module. Let $\mathfrak{p} \in \operatorname{Supp}(A)$, we have $H^i_I(R/\mathfrak{p}) = 0$ for all $i > 0$. Hence $\operatorname{cd}(I, R/\mathfrak{p}) \leq 0$. This implies that

$$\operatorname{Att}(H^d_I(M)) = \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{cd}(I, R/\mathfrak{p}) = d\}.$$

If $\dim N < \dim M$, then $\dim M = \dim A$. It follows that $H^d_I(M) = 0$. Therefore $\operatorname{Att}(H^d_I(M)) = \emptyset$. Let $\mathfrak{p} \in \operatorname{Supp}(M)$ such that $\operatorname{cd}(I, R/\mathfrak{p}) = d$. Then $\mathfrak{p} \in \operatorname{Supp}(N)$. On the other hand, $\dim(R/\mathfrak{p}) \leq \dim N$ and $\operatorname{cd}(I, R/\mathfrak{p}) \leq \dim R/\mathfrak{p} < d$. Thus

$$\{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{cd}(I, R/\mathfrak{p}) = d\} = \emptyset$$

and the proof is complete. □

Note that, if M is an I-cofine R-module with $\dim M = 1$, then we see that

$$\operatorname{Att}(H^1_I(M)) \subseteq \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{cd}(I, R/\mathfrak{p}) = 1\}.$$

Corollary 3.9. *Let $(R, \mathfrak{m})$ be a local ring and M a minimax R-module with $d = \dim M > 1$. Then $H^d_I(M)$ is artinian and*

$$\operatorname{Att}(H^d_I(M)) = \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \operatorname{cd}(I, R/\mathfrak{p}) = d\}.$$

Proposition 3.10. *Let $(R, \mathfrak{m})$ be a local ring, M an I-cofine R-module and $t > 1$ a positive integer. The following statements are equivalent:*

(i) *$H^i_I(M) = 0$ for all $i \geq t$.*
(ii) *$H^i_I(M)$ is finitely generated for all $i \geq t$.*
(iii) *$H^i_I(M)$ is coatomic for all $i \geq t$.*

Proof. (i) ⇒ (ii) ⇒ (iii). Trivial. We now prove (iii) ⇒ (i). Since M is an I-cofine R-module, there is a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where N is finitely generated and A is I-torsion. By applying the functor Γ_I to the above exact sequence, we get a long exact sequence

$$0 \to H^0_I(N) \to H^0_I(M) \to H^0_I(A) \to H^1_I(N) \to H^1_I(M) \to 0$$

and

$$H^i_I(N) \cong H^i_I(M)$$

for all $i \geq 2$. By the hypothesis, $H^i_I(N)$ is coatomic for all $i \geq t$. It follows from 2.8 that $H^i_I(N) = 0$ for all $i \geq t$ and which completes the proof. □

Acknowledgments. The final work of this paper was done when the the first two authors visited Vietnam Institute for Advanced Study in Mathematics (VIASM). We would like to thank VIASM for hospitality and financial support.

References

[1] M. Aghapournahr and L. Melkersson, Finiteness properties of minimax and coatomic local cohomology modules, *Arch. Math.* **94**(6) (2010), 519-528.
[2] M. Asgharzadeh and M. Tousi, A unified approach to local cohomology modules using Serre classes, *Canad. Math. Bull.* **53**(4) (2010) 577-586.
[3] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge University Press, 2013.
[4] L. Melkersson, On asymptotic stability for sets of prime ideals connected with the powers of an ideal, *Math. Proc. Camb. Phil. Soc.* **107**(2) (1990) 267-271.
[5] M. T. Dibaei and S. Yassemi, Attached primes of the top local cohomology modules with respect to an ideal, *Arch. Math.* **84**(4) (2005) 292-297.
[6] H. Saremi and A. Mafi, On the finiteness dimension of local cohomology modules, *Algebra Colloq.* **21**(3) (2014) 517-520.
[7] Mafi, Some results on local cohomology modules, *Arch. Math.* **87**(3) (2006) 211-216.
[8] J. Rotman, *An introduction to homological algebra*, Academic Press, 2009.
[9] Z. Tang, Local-global principle for the artinianness of local cohomology modules, *Comm. Algebra* **40**(1) (2012) 58-63.
[10] H. Zöschinger, Koatomare moduln, *Math. Z.* **170**(3) (1980), 221-232.
[11] H. Zöschinger, Minimax moduln, *J. Algebra* **102**(1) (1986), 1-32.

[1]Department of Mathematics-Informatics, Ho Chi Minh University of Pedagogy, Ho Chi Minh city, Vietnam
E-mail address: namtuantran@gmail.com, namtt@hcmue.edu.vn

[2]Algebra and Geometry Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam

[3]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
E-mail address: nguyenminhtri@tdtu.edu.vn

[4]Faculty of Mathematics & Computer Science, University of Science, VNU-HCM, Ho Chi Minh City, Vietnam
E-mail address: hoanam8779@gmail.com