

INFINITESIMAL CR AUTOMORPHISMS AND STABILITY GROUPS OF NONMINIMAL INFINITE TYPE MODELS IN \mathbb{C}^2

VAN THU NINH, THI NGOC OANH DUONG, VAN HOANG PHAM, AND HYESEON KIM

ABSTRACT. We determine infinitesimal CR automorphisms and stability groups of real hypersurfaces in \mathbb{C}^2 in the case when the hypersurface is nonminimal and of infinite type at the reference point.

1. INTRODUCTION AND THE STATEMENT OF MAIN RESULTS

The purpose of this article is to describe the spaces of infinitesimal CR automorphisms and stability groups of real hypersurfaces in \mathbb{C}^2 such that they are nonminimal in the sense of Tumanov [12] and of infinite type at the origin in the sense of D'Angelo [2].

We now introduce some notations which are needed to state our main results. Let (M, p) be the germ at p of a C^∞ -smooth real hypersurface M in \mathbb{C}^n , $n \geq 2$. We denote by $\text{Aut}(M)$ the CR automorphism group of M . For each $p \in M$, we denote by $\text{Aut}(M, p)$ the set of germs at p of biholomorphisms mapping M into itself and fixing the point p . In addition, we denote by $\mathbf{aut}(M, p)$ the set of germs of holomorphic vector fields in \mathbb{C}^n at p whose real part is tangent to M . With this notation, a smooth vector field germ (X, P) on M is called an *infinitesimal CR automorphism germ at p of M* if there exists an element in $\mathbf{aut}(M, p)$ such that its real part is equal to X on M . We also denote by $\mathbf{aut}_0(M, p)$ the set of all elements $H \in \mathbf{aut}(M, p)$ for which H vanishes at p .

The study of CR geometry on real hypersurfaces in \mathbb{C}^n is relatively well-developed in the case of *rigid hypersurfaces* (see [7], [8], [11] and the references therein). Here, we say that a C^∞ -smooth real hypersurface M through the origin in \mathbb{C}^n is *rigid* if there exist coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ and a C^∞ -smooth function F near the origin such that M is given by an equation of the form

$$(1) \quad \text{Re } w = F(z, \bar{z})$$

(cf. [1] and [11]). For a certain class of rigid hypersurfaces of finite type in the sense of D'Angelo in \mathbb{C}^2 , we refer the reader to [10] which addresses the existence of infinitesimal CR automorphisms. However, if we move our attention to the case of *rigid hypersurfaces of infinite type*, then we necessarily encounter more complicated procedure to get such geometric object due to the computational difficulty and the lack of literatures in the setting of infinite type (see [3] and the references therein). As a significant result which has inspired the present paper, Hayashimoto and Ninh [3] investigated an *infinite type model* $(M'_P, 0)$ in \mathbb{C}^2 which is defined by

$$(2) \quad M'_P := \{(z, w) \in \mathbb{C}^2 : \text{Re } w + P(z) = 0\},$$

Date: August 29, 2019.

2000 Mathematics Subject Classification. Primary 32M05; Secondary 32H02, 32H50, 32T25.

Key words and phrases. automorphism group, holomorphic vector field, infinite type point, real hypersurface.

where P is a non-zero germ of a real-valued C^∞ -smooth function at the origin vanishing to infinite order at $z = 0$. More precisely, the associated $\text{Aut}(M'_P, 0)$, $\mathbf{aut}(M'_P, 0)$, $\mathbf{aut}_0(M'_P, 0)$ were explicitly described under the variance of the zero set of the function P defined in (2). Furthermore, it follows from the definition that M'_P given in (2) is a rigid real hypersurface of infinite type.

We now employ the concept of a nonminimal hypersurface (this term is coined in [12]) which has also inspired the present paper. By following the definition in [12], a CR manifold N is *minimal at a point* $p \in N$ if there are no submanifolds passing through p of smaller dimension but with the same CR dimension. In this sense, one can say that a real hypersurface $N \in \mathbb{C}^2$ is *nonminimal at a point* $p \in N$ if there exists a germ of a complex hypersurface E through p which is contained in N (cf. [4] and [6]). In addition, a germ at the origin of a real hypersurface $(N, 0)$ in \mathbb{C}^2 is a *ruled hypersurface* if there exist coordinates $(z, w) \in \mathbb{C}^2$ such that N is given by an equation of the form

$$\text{Im } w = (\text{Re } w)A(z, \bar{z}),$$

where $A(z, \bar{z})$ does not vanish identically (for more details on A in the case when N is a ruled real analytic hypersurface of infinite type, see Eq. (6) and the consecutive arguments in [7, Section 3]). Moreover, a ruled hypersurface is known as a crucial prototype in considering local equivalence problem of nonminimal real analytic hypersurfaces in \mathbb{C}^2 . We further say that a germ at p of a real hypersurface (N, p) in \mathbb{C}^2 is *m -nonminimal* ($m \geq 1$) at p if there exist local coordinates $(z, w) \in \mathbb{C}^2$, p corresponds to 0, close by 0, such that N is given by an equation of the form

$$(3) \quad \text{Im } w = (\text{Re } w)^m \psi(z, \bar{z}, \text{Re } w),$$

where $\psi(z, 0, \text{Re } w) = \psi(0, \bar{z}, \text{Re } w) = 0$ and $\psi(z, \bar{z}, 0)$ does not vanish identically (cf. [4] and [8]). In particular, if (N, p) is a germ at p of a real analytic hypersurface which is 1-nonminimal at p , then $\text{Aut}(N, p)$ constitutes a finite dimensional Lie group (see [4, Theorem 1]). Moreover, a class of real analytic 1-nonminimal hypersurface in \mathbb{C}^2 is also meaningful in the sense that such nonminimal condition is related to the degeneration of the Levi form as the natural second-order invariant of a real analytic hypersurface (cf. [4, Introduction]).

In this paper, we first investigate the spaces of infinitesimal CR automorphisms and stability groups of a 1-nonminimal infinite type model $(M_P, 0)$ in \mathbb{C}^2 which is defined by

$$(4) \quad M_P := \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_1 + (\text{Im } z_1)P(z_2) = 0\},$$

where P is a non-zero germ of a real-valued C^∞ -smooth function at 0 vanishing to infinite order at $z_2 = 0$.

Before stating our main results, we now prepare further notations. For each $r > 0$, let us denote by Δ_{ϵ_0} the complex disk of radius r centred at the origin in \mathbb{C} . We also denote by Δ_r^* the punctured disk $\Delta_r \setminus \{0\}$. For a sufficiently small $\epsilon_0 > 0$ and a C^∞ -smooth function $P : \Delta_{\epsilon_0} \rightarrow \mathbb{R}$, we denote by $S_\infty(P)$ the set of all points $z \in \Delta_{\epsilon_0}$ for which $\nu_z(P) = +\infty$, where $\nu_z(P)$ is the vanishing order of $P(z + \zeta) - P(z)$ at $\zeta = 0$. In addition, we denote by $P_\infty(M_P)$ the set of all points of infinite type in M_P . We note that it is not hard to see that

$$(5) \quad P_\infty(M_P) = \{(it - tP(z_2), z_2) : t \in \mathbb{R}, z_2 \in S_\infty(P)\}$$

(see [3, Remark 1] for the difference with the rigid infinite type model $(M'_P, 0)$ defined in Eq. (2)).

We now ready to state our main results. For the case of a 1-nonminimal infinite type model, we have three main theorems in this paper. Theorem 1.1 comes under the case that special conditions on holomorphic vector fields determine the precise form of local defining functions. The other two main theorems explain the converse situation. Such division on the main results is originated in the work of Hayashimoto and Ninh [3]. In what follows, as commented in [3, Introduction], all functions, mappings, hypersurfaces, and so on, will be understood as germs at the reference points unless stated otherwise.

Theorem 1.1. *Let $(M_P, 0)$ be a C^∞ -smooth hypersurface in \mathbb{C}^2 defined by the equation $\rho(z) = \rho(z_1, z_2) := \operatorname{Re} z_1 + (\operatorname{Im} z_1)P(z_2) = 0$, where P is a C^∞ -smooth function on a neighborhood of the origin in \mathbb{C} satisfying:*

- (i) *The connected component of $z_2 = 0$ in the zero set of P is $\{0\}$;*
- (ii) *P vanishes to infinite order at $z_2 = 0$.*

Then any holomorphic vector field vanishing at the origin tangent to $(M_P, 0)$ is either of the form $\alpha z_1 \partial z_1$ for some $\alpha \in \mathbb{R}$, or after a change of variable in z_2 , of the form $\alpha z_1 \partial z_1 + i\beta z_2 \partial z_2$ for some $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^$, in which case M_P is rotationally symmetric, that is, $P(z_2) \equiv P(|z_2|)$.*

Remark 1.2. The condition (i) in Theorem 1.1 simply shows that the set $\{z_2 \in \mathbb{C} : P(z_2) = 0\}$ does not contain any curve in \mathbb{C} . In contrast to this theorem, Theorem 1.5 below allows the possibility that the curve $\operatorname{Re} z_2 = 0$ is contained in the zero set of P . Moreover, the condition (ii) and consideration of the points given in (5) provide a first step for the proof of Theorem 1.1.

Theorem 1.3. *Let $(M_P, 0)$ be a C^∞ -smooth hypersurface in \mathbb{C}^2 defined by the equation $\rho(z) = \rho(z_1, z_2) := \operatorname{Re} z_1 + (\operatorname{Im} z_1)P(z_2) = 0$, where P is a C^∞ -smooth function on a neighborhood of 0, vanishing to infinite order at $z_2 = 0$, and satisfying:*

- (i) *$P(z_2) \not\equiv 0$ on a neighborhood of $z_2 = 0$;*
- (ii) *the connected component of 0 in $S_\infty(P)$ is $\{0\}$.*

Then the following assertions hold:

- (a) $\mathbf{aut}(M_P, 0) = \mathbf{aut}_0(M_P, 0)$.
- (b) *If $\mathbf{aut}_0(M_P, 0) = \{\alpha z_1 \partial z_1 : \alpha \in \mathbb{R}\}$, then*

$$\mathbf{Aut}(M_P, 0) = G_2(M_P, 0),$$

where $G_2(M_P, 0)$ is the set of all CR automorphisms of M_P defined by

$$(z_1, z_2) \mapsto (Cz_1, g_2(z_2))$$

for some constant $C \in \mathbb{R}^$ and some holomorphic function g_2 with $g_2(0) = 0$ and $|g_2'(0)| = 1$ defined on a neighborhood of the origin in \mathbb{C} satisfying that $P(g_2(z_2)) \equiv P(z_2)$.*

Remark 1.4. We note that the connected component of 0 in $P_\infty(M_P)$ is $\{(0, is) : s \in \mathbb{R}\}$. This fact is a crucial ingredient in the proof of Theorem 1.3.

In the case when $S_\infty(P)$ contains a non-trivial connected component of 0 which contrasts with the condition (ii) of Theorem 1.3, for instance M_P is *tubular*, we obtain the following theorem.

Theorem 1.5. *Let \tilde{P} be a C^∞ -smooth function defined on a neighborhood of 0 in \mathbb{C} satisfying:*

- (i) $\tilde{P}(x) \neq 0$ on a neighborhood of $x = 0$ in \mathbb{R} ;
- (ii) the connected component of 0 in $S_\infty(\tilde{P})$ is $\{0\}$.

Denote by P a function defined by setting $P(z_2) := \tilde{P}(\operatorname{Re} z_2)$ with a further condition that $P(z_2)$ vanishes to infinite order at $z_2 = 0$. Then the following assertions hold:

- (a) $\mathbf{aut}_0(M_P, 0) = \{\alpha z_1 \partial_{z_1} : \alpha \in \mathbb{R}\}$ and the Lie algebra $\mathfrak{g} = \mathbf{aut}(M_P, 0)$ admits the decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0,$$

where $\mathfrak{g}_1 = \{\alpha z_1 \partial_{z_1} : \alpha \in \mathbb{R}\}$ and $\mathfrak{g}_0 = \{i\beta \partial_{z_2} : \beta \in \mathbb{R}\}$.

- (b) $\operatorname{Aut}(M_P, 0)$ is either $\{(z_1, z_2) \mapsto (tz_1, z_2) : t \in \mathbb{R}^*\}$ or $\{(z_1, z_2) \mapsto (tz_1, \pm z_2) : t \in \mathbb{R}^*\}$, where the latter case happens only if $P(z_2) = P(-z_2)$.
- (c) If $S_\infty(\tilde{P}) = \{0\}$, then $\operatorname{Aut}(M_P)$ can be decomposed into either

$$T^1(M_P) \oplus T^2(M_P)$$

or

$$T^2(M_P) \oplus T^3(M_P),$$

where $T^1(M_P) = \{(z_1, z_2) \mapsto (sz_1, z_2) : s \in \mathbb{R}^*\}$, $T^2(M_P) = \{(z_1, z_2) \mapsto (z_1, z_2 + it) : t \in \mathbb{R}\}$ and $T^3(M_P) = \{(z_1, z_2) \mapsto (sz_1, \pm z_2) : s \in \mathbb{R}^*\}$. The latter case happens only if $P(z_2) = P(-z_2)$.

In addition, we also investigate an analogue of Theorem 1.1 for an m -nonminimal infinite type model $(M_{P,m}, 0)$ with $m > 1$ in \mathbb{C}^2 which is defined by

$$(6) \quad M_{P,m} := \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 - (\operatorname{Re} z_1)^m P(z_2) = 0\},$$

where P is a non-zero germ of a real-valued C^∞ -smooth function at the origin, which vanishes to infinite order at $z_2 = 0$. Due to the variance of the choice of the constant m in (6), the procedure to analyze the associated holomorphic vector fields becomes more complicated than that of a 1-nonminimal infinite type model $(M_P, 0)$ defined above. For the convenience of exposition, we shall proceed the assertion for the case of $(M_{P,m}, 0)$ with $m > 1$ separately in Appendix.

The organization of the paper is described as follows: In Section 2, we provide the proof of Theorem 1.1 for which certain conditions on holomorphic vector fields determine the precise form of local defining functions. As the converse of this situation, we next provide the proofs of Theorem 1.3 and Theorem 1.5 in Section 3. We further present several examples as analogues of those in [3] in Section 4. In addition, an analogue of Theorem 1.1 for an m -nonminimal infinite type model $(M_{P,m}, 0)$ with $m > 1$ will be investigated in Appendix.

2. ANALYSIS OF HOLOMORPHIC TANGENT VECTOR FIELDS

This section is devoted to the proof of our first main result Theorem 1.1. Let us first prepare two crucial technical ingredients for the proof of Theorem 1.1. The following proposition will be treated also in the assertion for an m -nonminimal infinite type model $(M_{P,m}, 0)$ with $m > 1$ in Appendix.

Proposition 2.1 ([3, Lemma 7]). *Let $P : \Delta_{\epsilon_0} \rightarrow \mathbb{R}$ be a C^∞ -smooth function satisfying that the connected component of $z = 0$ in the zero set of P is $\{0\}$ and that P vanishes to infinite order at $z = 0$. If a, b are complex numbers and if g_0, g_1, g_2 are C^∞ -smooth functions defined on Δ_{ϵ_0} satisfying:*

- (A1) $g_0(z) = O(|z|), g_1(z) = O(|z|^\ell)$, and $g_2(z) = o(|z|^m)$;
(A2) $\operatorname{Re} [(az^m + g_2(z))P^{n+1}(z) + bz^\ell(1 + g_0(z))P_z(z) + g_1(z)P(z)] = 0$ on Δ_{ϵ_0}

for any non-negative integers l, m and n except for the following two cases

- (E1) $\ell = 1$ and $\operatorname{Re} b = 0$;
(E2) $m = 0$ and $\operatorname{Re} a = 0$,

then $ab = 0$.

The proof of this proposition proceeds along the similar lines as that of Lemma 3 in [5]. (Notice that P was assumed to be positive on $\Delta_{\epsilon_0}^*$ in [5].) For the sake of brevity we shall omit routine arguments, except (7) below. The following lemma assures the existence of a modification of Eq. (7) in [5], which is a main ingredient for the proof. For the convenience of the reader, we provide the proof of the following lemma.

Lemma 2.2 ([3, Lemma 8]). *Let P, a, b, g_0, g_1, g_2 be as in Proposition 2.1. Suppose that for each $t_0 \in \mathbb{R}$, $\gamma : [t_0, t_\infty) \rightarrow \Delta_{\epsilon_0}^*$, where t_∞ satisfies either $t_\infty \in \mathbb{R}$ or $t_\infty = +\infty$, is a solution of the initial-value problem*

$$(7) \quad \frac{d\gamma}{dt}(t) = b\gamma^\ell(t)(1 + g_0(\gamma(t))), \quad \gamma(t_0) = z_0,$$

where $z_0 \in \Delta_{\epsilon_0}^*$ with $P(z_0) \neq 0$, such that $\lim_{t \uparrow t_\infty} \gamma(t) = 0$. Then $P(\gamma(t)) \neq 0$ for all $t \in (t_0, t_\infty)$.

Proof. Aiming for a contradiction, we suppose that P has a zero on the curve γ . Then since the connected component of $z = 0$ in the zero set of P is $\{0\}$, without loss of generality, we may further assume that there exists a $t_1 \in (t_0, t_\infty)$ such that $P(\gamma(t)) \neq 0$ for all $t \in (t_0, t_1)$ and $P(\gamma(t_1)) = 0$.

Let $u(t) := \frac{1}{2} \log |P(\gamma(t))|$ for $t_0 < t < t_1$. Then it follows from (7) and (A2) that

$$u'(t) = -P^n(t) (\operatorname{Re}(a\gamma^m(t) + o(|\gamma(t)|^m))) + O(|\gamma(t)|^\ell)$$

for all $t_0 < t < t_1$. Combining this with the assumption for the vanishing order of P at $z = 0$, one can deduce that $u'(t)$ is bounded on (t_0, t_1) . This after applying the fundamental theorem of ordinary differential equations in turn yields the boundedness of $u(t)$ on (t_0, t_1) , which is absurd since $u(t) \rightarrow -\infty$ as $t \uparrow t_1$. Hence our proof is complete. \square

Before going further, we shall fix the notations. In what follows, we denote by \mathbb{N}^0 and \mathbb{N}^* the set of all non-negative integers and the set of all positive integers, respectively.

2.1. Proof of Theorem 1.1. The CR hypersurface germ $(M_P, 0)$ at the origin in \mathbb{C}^2 under consideration is defined by the equation

$$\rho(z) = \rho(z_1, z_2) := \operatorname{Re} z_1 + (\operatorname{Im} z_1)P(z_2) = 0,$$

where P is a C^∞ -smooth function satisfying the two above conditions (i) and (ii). Then we consider a holomorphic vector field $H = h_1(z_1, z_2)\partial z_1 + h_2(z_1, z_2)\partial z_2$ defined near the

origin in \mathbb{C}^2 . We focus only on H which is tangent to M_P . This means that H satisfies the identity

$$(8) \quad (\operatorname{Re} H)\rho(z) = 0, \quad \forall z \in M_P.$$

Expanding h_1 and h_2 into the Taylor series at the origin, we get

$$\begin{aligned} h_1(z_1, z_2) &= \sum_{j,k=0}^{\infty} a_{j,k} z_1^j z_2^k = \sum_{j=0}^{\infty} a_j(z_2) z_1^j; \\ h_2(z_1, z_2) &= \sum_{j,k=0}^{\infty} b_{j,k} z_1^j z_2^k = \sum_{j=0}^{\infty} b_j(z_2) z_1^j, \end{aligned}$$

where $a_{j,k}, b_{j,k} \in \mathbb{C}$ and a_j, b_j are holomorphic functions for all $j \in \mathbb{N}^0$. We note that

$$a_{0,0} = b_{0,0} = 0$$

since $h_1(0,0) = h_2(0,0) = 0$. A direct computation shows that

$$\rho_{z_1}(z_1, z_2) = \frac{1}{2} + \frac{1}{2i}P(z_2); \quad \rho_{z_2}(z_1, z_2) = (\operatorname{Im} z_1)P_{z_2}(z_2),$$

and hence (8) can be re-written as

$$\operatorname{Re} \left[\left(\frac{1}{2} + \frac{1}{2i}P(z_2) \right) h_1(z_1, z_2) + (\operatorname{Im} z_1)P_{z_2}(z_2)h_2(z_1, z_2) \right] = 0$$

for all $(z_1, z_2) \in M_P$.

Since $(it - tP(z_2), z_2) \in M_P$ with t small enough, the previous equation again admits a new form

$$(9) \quad \operatorname{Re} \left[\left(\frac{1}{2} + \frac{1}{2i}P(z_2) \right) \sum_{j,k=0}^{\infty} a_{j,k}(it - tP(z_2))^j z_2^k + tP_{z_2}(z_2) \sum_{m,n=0}^{\infty} b_{m,n}(it - tP(z_2))^m z_2^n \right] = 0$$

for all $z_2 \in \mathbb{C}$ and $t \in \mathbb{R}$ with $z_2 \in \Delta_{\epsilon_0}$ and $|t| < \delta_0$, where $\epsilon_0, \delta_0 > 0$ are small enough.

Inserting $t = 0$ into (9), we have

$$(10) \quad \operatorname{Re} \left[\left(\frac{1}{2} + \frac{1}{2i}P(z_2) \right) \sum_{k=0}^{\infty} a_{0,k} z_2^k \right] \equiv 0$$

on Δ_{ϵ_0} . Combining this with the assumption that P vanishes to infinite order at $z_2 = 0$, one can assert that

$$(11) \quad a_{0,k} = 0, \quad \forall k \in \mathbb{N}^*.$$

Moreover, setting the coefficient of t^{m+1} in (9) equals zero for each $m \in \mathbb{N}^0$, we obtain

$$(12) \quad \operatorname{Re} \left[\left(\frac{1 + P^2(z_2)}{2} \right) \sum_{k=0}^{\infty} i a_{m+1,k} (i - P(z_2))^m z_2^k + P_{z_2}(z_2) \sum_{n=0}^{\infty} b_{m,n} (i - P(z_2))^m z_2^n \right] \equiv 0$$

for each $m \in \mathbb{N}^0$ on Δ_{ϵ_0} . Since both $P(z_2)$ and $P_{z_2}(z_2)$ vanish to infinite order at $z_2 = 0$, (12) yields

$$(13) \quad \operatorname{Re} \left[\sum_{k=0}^{\infty} i^{m+1} a_{m+1,k} z_2^k \right] \equiv 0$$

for each $m \in \mathbb{N}^0$ on Δ_{ϵ_0} . Then it follows from (11) and (13) that

$$(14) \quad a_{j,k+1} = 0, \quad \forall j, k \in \mathbb{N}^0; \quad \operatorname{Re}(i^\ell a_{\ell,0}) = 0, \quad \forall \ell \in \mathbb{N}^*.$$

Considering again the assumption for the vanishing order of $P(z_2)$ at $z_2 = 0$, we indeed have

$$h_1(z_1, z_2) = \alpha z_1$$

for some $\alpha \in \mathbb{R}$, if $h_2(z_1, z_2) \equiv 0$. Therefore, in the remaining of the proof, we always assume that $h_2 \not\equiv 0$ without loss of generality.

Let m_0 be the smallest integer such that $b_{m_0,n} \neq 0$ for some $n \in \mathbb{N}^0$. Then we let n_0 be the smallest integer such that $b_{m_0,n_0} \neq 0$. Since $b_{0,0} = 0$, it is clear that $m_0 \geq 1$ if $n_0 = 0$. With this setting, (12) and (14) yield

$$(15) \quad \operatorname{Re} \left[\left(\frac{1 + P^2(z_2)}{2} \right) i a_{m_0+1,0} (i - P(z_2))^{m_0} + P_{z_2}(z_2) \sum_{n=n_0}^{\infty} b_{m_0,n} (i - P(z_2))^{m_0} z_2^n \right] \equiv 0$$

on Δ_{ϵ_0} . Since $P(z_2) = o(|z_2|^j)$ for any $j \in \mathbb{N}^*$, it follows from (15) that

$$\operatorname{Re} \left[\left(\frac{1 + P^2(z_2)}{2} \right) i a_{m_0+1,0} (i - P(z_2))^{m_0} + i^{m_0} b_{m_0,n_0} (z_2^{n_0} + o(z_2^{n_0})) P_{z_2}(z_2) \right] \equiv 0$$

on Δ_{ϵ_0} .

Now we shall consider the following two cases.

Case 1. $m_0 = 0$. In this case, by [3, Corollary 4], we first obtain $n_0 = 1$ and $b_{0,1} = i\beta$ for some $\beta \in \mathbb{R}^*$. Then, by a change of variables (cf. [9, Lemma 1]), we may assume that

$$b_0(z_2) = \sum_{n=0}^{\infty} b_{0,n} z_2^n = i\beta z_2.$$

Therefore, we get from (15) that

$$(16) \quad \operatorname{Re} [i\beta z_2 P_{z_2}(z_2)] \equiv 0$$

on Δ_{ϵ_0} . This implies that $P(z_2) \equiv P(|z_2|)$ on Δ_{ϵ_0} .

We now prove that $b_m = 0$ for every $m \in \mathbb{N}^*$. Suppose otherwise. Then there exists the smallest number $m_1 \in \mathbb{N}^*$ such that $b_{m_1} \neq 0$. By the same argument as above, we may assume that $b_{m_1}(z_2) = i^{1-m_1} \beta_1 z_2 + o(|z_2|)$ for some $\beta_1 \in \mathbb{R}^*$ on Δ_{ϵ_0} . Moreover, we indeed have $b_{m_1}(z_2) = i^{1-m_1} \beta_1 z_2$ for some $\beta_1 \in \mathbb{R}^*$: suppose otherwise. Then there exist $k_0 \geq 2$ and $c_{k_0} \in \mathbb{C}^*$ such that

$$b_{m_1}(z_2) = i^{1-m_1} \beta_1 z_2 + c_{k_0} z_2^{k_0} + o(|z_2|^{k_0}).$$

Putting $m = m_1$ in (12) and then subtracting the associated modification of (12) from the equation

$$(17) \quad \operatorname{Re} [i\beta_1 z_2 P_{z_2}(z_2)] \equiv 0$$

on Δ_{ϵ_0} induced by (16), we obtain on Δ_{ϵ_0}

$$\operatorname{Re} \left[\left(\frac{1 + P^2(z_2)}{2} \right) i a_{m_1+1,0} (i - P(z_2))^{m_1} + i^{m_1} c_{k_0} (z_2^{k_0} + o(|z_2|^{k_0})) P_{z_2}(z_2) \right] \equiv 0$$

on Δ_{ϵ_0} , which contradicts to Proposition 2.1. Hence we have $b_{m_1}(z_2) = i^{1-m_1}\beta_1 z_2$ for some $\beta_1 \in \mathbb{R}^*$. Substituting this into (12), one gets

$$(18) \quad \operatorname{Re} \left[\left(\frac{1 + P^2(z_2)}{2} \right) ia_{m_1+1,0}(i - P(z_2))^{m_1} + i^{1-m_1}\beta_1 z_2 P_{z_2}(z_2)(i - P(z_2))^{m_1} \right] \equiv 0$$

on Δ_{ϵ_0} . Subtracting (18) from (17), we have

$$\operatorname{Re} \left[\left(\frac{1 + P^2(z_2)}{2} \right) ia_{m_1+1,0}(i - P(z_2))^{m_1} - m_1\beta_1(z_2 + o(|z_2|))P_{z_2}(z_2)P(z_2) \right] \equiv 0$$

on Δ_{ϵ_0} , which again contradicts to Proposition 2.1.

Altogether, in this case, we obtain $h_2(z_1, z_2) = i\beta z_2$ and $P(z_2) \equiv P(|z_2|)$ for some $\beta \in \mathbb{R}^*$ on Δ_{ϵ_0} .

Case 2. $m_0 \geq 1$. In this case, by Proposition 2.1, we first obtain $n_0 = 1$ and $b_{m_0,1} = i^{1-m_0}\beta z_2$ for some $\beta \in \mathbb{R}^*$. Then, by a change of variables, we may assume that

$$b_{m_0}(z_2) = \sum_{n=0}^{\infty} b_{m_0,n} z_2^n = i^{1-m_0}\beta z_2.$$

Therefore, in this case, (15) can be re-written as

$$(19) \quad \operatorname{Re} \left[\left(\frac{1 + P^2(z_2)}{2} \right) ia_{m_0+1,0}(i - P(z_2))^{m_0} + i^{1-m_0}\beta z_2 P_{z_2}(z_2)(i - P(z_2))^{m_0} \right] \equiv 0$$

on Δ_{ϵ_0} .

We now divide the argument into two subcases as follows.

Subcase 2.1. $a_{m_0+1,0} = 0$. In this subcase, it follows from (19) that

$$(20) \quad \operatorname{Re} [i^{1-m_0}\beta z_2 P_{z_2}(z_2)(i - P(z_2))^{m_0}] \equiv 0$$

on Δ_{ϵ_0} .

Let $r \in (0, \epsilon_0)$ such that $P(r) \neq 0$. Then we let $\gamma : [t_0, +\infty) \rightarrow \mathbb{C}$ be a curve such that $\gamma'(t) = i^{1-m_0}\beta\gamma(t)(i - P(\gamma(t)))^{m_0}$ and $\gamma(t_0) = r$. Then setting $u(t) = P(\gamma(t))$, (20) shows that $u'(t) \equiv 0$, and hence $u(t) \equiv P(r)$. Therefore, we have

$$\gamma'(t) = a\gamma(t); \quad \gamma(t_0) = r,$$

where $a := i^{1-m_0}\beta(1 - P(r))^{m_0}$. This yields $\gamma(t) = r \exp(a(t - t_0))$. Since $|\gamma(t)| = r \exp((\operatorname{Re} a)(t - t_0))$ and $\gamma(t_0) = r \neq 0$, we momentarily assume that $\operatorname{Re} a < 0$. Since $0 \leq |\gamma(t)| = r \exp((\operatorname{Re} a)(t - t_0))$, we get $\gamma(t) \rightarrow 0$ as $t \rightarrow \infty$, and hence $P(r) \equiv P(\gamma(t)) \rightarrow P(0) = 0$, which contradicts to our choice of $r \in (0, \epsilon_0)$. In the case when $\operatorname{Re} a > 0$, one can proceed the same argument as above (by considering a curve $\tilde{\gamma} : (-\infty, t_0] \rightarrow \mathbb{C}$ instead of the above curve γ).

Subcase 2.2. $a_{m_0+1,0} \neq 0$. In this subcase, it follows from (19) that on Δ_{ϵ_0}

$$(21) \quad \operatorname{Re} [i^{1-m_0}\beta z_2 P_{z_2}(z_2)(i - P(z_2))^{m_0}] \equiv (\delta + \epsilon(z_2))P(z_2),$$

where $\delta := \operatorname{Re}(m_0 i^{m_0} a_{m_0+1,0}/2) \in \mathbb{R}^*$ and $\epsilon : \Delta_{\epsilon_0} \rightarrow \mathbb{R}$ is a smooth function with the condition that $\epsilon(z_2) \rightarrow 0$ as $z_2 \rightarrow 0$. Without loss of generality, we may assume that $\delta < 0$ and $|\epsilon(z_2)| < |\delta|/2$ on Δ_{ϵ_0} .

Let $r \in (0, \epsilon_0)$ such that $P(r) \neq 0$. Then we let $\gamma : [t_0, +\infty) \rightarrow \mathbb{C}$ such that $\gamma'(t) = i^{1-m_0} \beta \gamma(t) (i - P(\gamma(t)))^{m_0}$ and $\gamma(t_0) = r$. Then setting $u(t) = \log |P(\gamma(t))|$, (21) shows that $u'(t) = \delta + \epsilon(\gamma(t))$. Hence, we get

$$(22) \quad u(t) - u(t_0) = \delta(t - t_0) + \int_{t_0}^t \epsilon(\gamma(\tau)) d\tau, \quad \forall t \geq t_0.$$

This implies that $u(t) \rightarrow -\infty$ as $t \rightarrow +\infty$, and hence $\gamma(t) \rightarrow 0$ as $t \rightarrow +\infty$. Moreover, since $|\epsilon(z_2)| < |\delta|/2$ on Δ_{ϵ_0} , it follows from (22) that

$$u(t) < u(t_0) + \frac{\delta}{2}(t - t_0), \quad \forall t > t_0.$$

This inequality yields

$$|P(\gamma(t))| \lesssim \exp(\delta t/2), \quad \forall t > t_0.$$

Therefore, $\gamma(t)$ satisfies the following:

$$\gamma'(t) = \gamma(t)(i\beta + g(t)),$$

where $g : (t_0, +\infty) \rightarrow \mathbb{C}$ is a smooth function satisfying that $|g(t)| \lesssim \exp(\delta t/2)$. Then this yields

$$\gamma(t) = \exp(i\beta t + O(\exp(\delta t/2)));$$

hence $\gamma(t) \rightarrow 0$ as $t \rightarrow +\infty$, which contradicts to the discussion right after (22).

Hence, all the possible cases for the choice of h_2 are considered.

Now we shall show that h_1 has the form of

$$h_1(z_1, z_2) = \alpha z_1, \quad \alpha \in \mathbb{R},$$

if $P(z_2) \equiv P(|z_2|)$ and $h_2(z_1, z_2) = i\beta z_2$ for some $\beta \in \mathbb{R}^*$. Suppose otherwise. Then there exists $j_0 \geq 2$ such that $h_1(z_1, z_2) = \alpha z_1 + a_{j_0,0} z_1^{j_0} + o(|z_1|^{j_0})$ with $a_{j_0,0} \neq 0$. Combining (9) with (14), $a_{0,0} = 0$, and $P(z_2) \equiv P(|z_2|)$, we have

$$(23) \quad \begin{aligned} & \operatorname{Re} \left[\left(\frac{1}{2} + \frac{1}{2i} P(z_2) \right) \sum_{j=1}^{\infty} a_{j,0} (it - tP(z_2))^j + it\beta z_2 P_{z_2}(z_2) \right] \\ & \operatorname{Re} \left[\left(\frac{1}{2} + \frac{1}{2i} P(z_2) \right) \sum_{j=1}^{\infty} a_{j,0} (it - tP(z_2))^j \right] \\ & = 0 \end{aligned}$$

for all $z_2 \in \mathbb{C}$ and $t \in \mathbb{R}$ with $z_2 \in \Delta_{\epsilon_0}$ and $|t| < \delta_0$, where $\epsilon_0, \delta_0 > 0$ are small enough. Considering the coefficient of t^j in (23), for each $j \in \mathbb{N}^*$, we get

$$(24) \quad \operatorname{Re} \left[\left(\frac{1}{2} + \frac{1}{2i} P(z_2) \right) a_{j,0} (i - P(z_2))^j \right] \equiv 0$$

on Δ_{ϵ_0} . Then one may regard (24) as an equation with a variable $P(z_2)$. For this reason, considering the coefficient of degree 1 with respect to the variable $P(z_2)$ in (24), we have

$$\operatorname{Re} \left[-\frac{1}{2}(j-1)i^{j-1}a_{j,0} \right] = 0$$

for each $j \in \mathbb{N}^*$. This conjunction with (14) yields

$$\operatorname{Re}[ia_{1,0}] = 0; \quad a_{s,0} = 0, \quad \forall s \geq 2.$$

Moreover, we note that if P is rotationally symmetric, then

$$H := \alpha z_1 \frac{\partial}{\partial z_1} + i\beta z_2 \frac{\partial}{\partial z_2}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^*$, always satisfies the condition (8). Hence we complete the proof.

3. PROOFS OF THEOREMS 1.3 AND 1.5

In this section, we continue the study of a 1-nonminimal infinite type model $(M_P, 0)$. As mentioned above, Theorems 1.3 and 1.5 present the investigation of the associated holomorphic vector fields under certain conditions of local defining functions. For the proofs of these main theorems, we now prepare the following two technical lemmas. For the sake of brevity, we omit the proofs (see [3] for the details of the proofs).

Lemma 3.1 (Lemma 1 in [3]). *Let $P : \Delta_{\epsilon_0} \rightarrow \mathbb{R}$ be a C^∞ -smooth function satisfying $\nu_0(P) = +\infty$ and $P(z) \neq 0$. Suppose that there exists a conformal map g on Δ_{ϵ_0} with $g(0) = 0$ such that*

$$P(g(z)) = (\beta + o(1))P(z), \quad z \in \Delta_{\epsilon_0},$$

for some $\beta \in \mathbb{R}^*$. Then $|g'(0)| = 1$.

Lemma 3.2 (Lemma 3 in [3]). *Let P be a non-zero C^∞ -smooth function with $P(0) = 0$ and let g be a conformal map satisfying $g(0) = 0$, $|g'(0)| = 1$, and $g \neq \text{id}$. If there exists a real number $\delta \in \mathbb{R}^*$ such that $P(g(z)) \equiv \delta P(z)$, then $\delta = 1$. Moreover, we have either $g'(0) = \exp(2\pi ip/q)$ ($p, q \in \mathbb{Z}$) and $g^q = \text{id}$ or $g'(0) = \exp(2\pi i\theta)$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

3.1. Proof of Theorem 1.3. (a) Let $H = h_1(z_1, z_2)\partial_{z_1} + h_2(z_1, z_2)\partial_{z_2} \in \mathfrak{aut}(M_P, 0)$ be arbitrary. That is, H is a holomorphic vector field near the origin in \mathbb{C}^2 such that

$$(\text{Re } H)\rho(z) = 0$$

for all $z \in M_P$. We assume that $\{\phi_t\}_{t \in \mathbb{R}} \subset \text{Aut}(M_P, 0)$ is the associated subgroup generated by H . Since ϕ_t is biholomorphic for every $t \in \mathbb{R}$, the set $\{\phi_t(0, 0) : t \in \mathbb{R}\}$ is contained in $P_\infty(M_P)$. Moreover, since the connected component of 0 in $P_\infty(M_P)$ is $\{(is, 0) : s \in \mathbb{R}\}$, one gets $\phi_t(0, 0) \in \{(is, 0) : s \in \mathbb{R}\}$ for every $t \in \mathbb{R}$. This relation yields

$$(25) \quad \text{Re } h_1(0, 0) = h_2(0, 0) = 0.$$

Then we immediately prove the assertion (a), if $\text{Im } h_1(0, 0) = 0$.

For this reason, we shall now consider the case when $\text{Im } h_1(0, 0) \neq 0$. Expanding the functions h_1 and h_2 into the Taylor series at the origin,

$$h_1(z_1, z_2) = \sum_{j,k=0}^{\infty} a_{j,k} z_1^j z_2^k;$$

$$h_2(z_1, z_2) = \sum_{j,k=0}^{\infty} b_{j,k} z_1^j z_2^k,$$

where $a_{j,k}, b_{j,k} \in \mathbb{C}$ as in the proof of Theorem 1.1. Since $P(z_2)$ vanishes to infinite order at $z_2 = 0$, Eq. (10) in the proof of Theorem 1.1 yields

$$(26) \quad \text{Re } a_{0,0} = 0; \quad a_{0,\ell} = 0, \quad \forall \ell \in \mathbb{N}^*.$$

If $\text{Im } a_{0,0} \neq 0$, then (10) and (26) imply that

$$\begin{aligned} \text{Re} \left[\left(\frac{1}{2} + \frac{P(z_2)}{2i} \right) \sum_{k=0}^{\infty} a_{0,k} z_2^k \right] &= \text{Re} \left[\left(\frac{1}{2} + \frac{P(z_2)}{2i} \right) i(\text{Im } a_{0,0}) \right] \\ &= \frac{(\text{Im } a_{0,0})}{2} P(z_2) \\ &\equiv 0 \end{aligned}$$

on Δ_{ϵ_0} ; hence, $P(z_2) \equiv 0$ on Δ_{ϵ_0} which contradicts to our assumption (i). Combining this fact with (25), we obtain the vanishing property of H at the origin in \mathbb{C}^2 .

In addition, from the definition of $\mathbf{aut}_0(M_P, 0)$, it is clear that

$$\mathbf{aut}_0(M_P, 0) \subset \mathbf{aut}(M_P, 0).$$

This completes the proof of (a).

(b) We first assume that

$$\mathbf{aut}_0(M_P, 0) = \{\alpha z_1 \partial_{z_1} : \alpha \in \mathbb{R}\}.$$

Then it follows from the assertion (a) that

$$\mathbf{aut}(M_P, 0) = \{\alpha z_1 \partial_{z_1} : \alpha \in \mathbb{R}\}$$

also holds.

Now let us denote by $\{T_t\}_{t \in \mathbb{R}}$ the 1-parameter subgroup generated by $z_1 \partial_{z_1}$, that is,

$$T_t(z_1, z_2) = (\exp(t)z_1, z_2), \quad t \in \mathbb{R}.$$

For any $f = (f_1, f_2) \in \text{Aut}(M_P, 0)$, we define a family $\{F_t\}_{t \in \mathbb{R}}$ of automorphisms by setting

$$F_t := f \circ T_{-t} \circ f^{-1}.$$

Then it follows that $\{F_t\}_{t \in \mathbb{R}}$ is a 1-parameter subgroup of $\text{Aut}(M_P)$. Moreover, since $\mathbf{aut}(M_P, 0) = \{\alpha z_1 \partial_{z_1} : \alpha \in \mathbb{R}\}$, the holomorphic vector field H generated by $\{F_t\}_{t \in \mathbb{R}}$ belongs to $\{\alpha z_1 \partial_{z_1} : \alpha \in \mathbb{R}\}$. This means that there exists a real number δ such that

$$H = \delta z_1 \partial_{z_1},$$

which yields

$$F_t(z_1, z_2) = (\exp(\delta t)z_1, z_2), \quad t \in \mathbb{R}.$$

This implies that for $t \in \mathbb{R}$

$$f = T_{\delta t} \circ f \circ T_t$$

which is equivalent to

$$(27) \quad f_1(z_1, z_2) = \exp(\delta t) f_1(\exp(t)z_1, z_2);$$

$$(28) \quad f_2(z_1, z_2) = f_2(\exp(t)z_1, z_2).$$

Taking the derivative of both sides of (27) with respect to t , we have

$$0 = \delta \exp(\delta t) f_1(\exp(t)z_1, z_2) + \exp(\delta t) \exp(t) z_1 \frac{\partial f_1(\exp(t)z_1, z_2)}{\partial(\exp(t)z_1)}.$$

This relation yields

$$0 = \delta f_1(z_1, z_2) + z_1 \frac{\partial f_1}{\partial z_1}(z_1, z_2);$$

hence one can deduce that

$$f_1(z_1, z_2) = z_1^{-\delta} g_1(z_2),$$

where g_1 is a holomorphic function on a neighborhood of $z_2 = 0$. Moreover, since f_1 is a biholomorphism, the constant δ should be -1 .

Applying the same procedure as above to (28), one can also deduce that

$$f_2(z_1, z_2) = g_2(z_2),$$

where g_2 is a holomorphic function on a neighborhood of $z_2 = 0$ with $g_2(0) = 0$.

Now we shall determine f more precisely. Since $(it - tP(z_2), z_2) \in M_P$, $t \in \mathbb{R}$, and M_P is invariant under f , we get

$$(29) \quad \begin{aligned} 0 &= \operatorname{Re}(f_1(it - tP(z_2), z_2)) + \operatorname{Im}(f_1(it - tP(z_2), z_2)) P(f_2(it - tP(z_2), z_2)) \\ &= \operatorname{Re}((it - tP(z_2))g_1(z_2)) + \operatorname{Im}((it - tP(z_2))g_1(z_2)) P(g_2(z_2)). \end{aligned}$$

Since the case $g_1 \equiv 0$ contradicts to the fact that f is biholomorphic near the origin, we may assume that $g_1 \not\equiv 0$. Then (29) implies that

$$(30) \quad P(g_2(z_2)) = -\frac{\operatorname{Re}(g_1(z_2)(i - P(z_2)))}{\operatorname{Im}(g_1(z_2)(i - P(z_2)))}.$$

Since $P(g_2(z_2))$ vanishes to infinite order at $z_2 = 0$, $\operatorname{Re}(g_1(z_2)(i - P(z_2)))$ also has the same property at $z_2 = 0$. Moreover, by the same reason, we can further say that $\operatorname{Re}(ig_1(z_2))$ vanishes to infinite order at $z_2 = 0$. Combining this with the fact that g_1 is holomorphic near $z_2 = 0$, we obtain

$$g_1(z_2) \equiv \text{a constant } C \in \mathbb{R}^*.$$

Therefore, (30) can be re-written as

$$P(g_2(z_2)) = P(z_2).$$

near the origin. Applying Lemma 3.1 to this relation, we also obtain $|g_2'(0)| = 1$ which finishes the proof of (b).

Altogether, we complete the proof of Theorem 1.3.

3.2. Proof of Theorem 1.5. (a) Let $H = h_1(z_1, z_2)\partial_{z_1} + h_2(z_1, z_2)\partial_{z_2} \in \mathbf{aut}_0(M_P, 0)$ be arbitrary. That is, H is a holomorphic vector field near the origin such that

$$H(0) = 0; \quad (\operatorname{Re} H)\rho(z) = 0$$

for all $z \in M_P$. Then we define a holomorphic vector field \tilde{H} by setting

$$\tilde{H} := H - \alpha z_1 \partial_{z_1}, \quad \alpha \in \mathbb{R}.$$

Now we expand the functions $h_1 - \alpha z_1$ and h_2 into the Taylor series at the origin:

$$\begin{aligned} h_1(z_1, z_2) - \alpha z_1 &= \sum_{j,k=0}^{\infty} a_{j,k} z_1^j z_2^k; \\ h_2(z_1, z_2) &= \sum_{j,k=0}^{\infty} b_{j,k} z_1^j z_2^k, \end{aligned}$$

where $a_{j,k}, b_{j,k} \in \mathbb{C}$. Then it follows from $\tilde{H}(0) = 0$ that

$$a_{0,0} = b_{0,0} = 0.$$

Moreover, since $(it - tP(z_2), z_2) \in M_P$ with a small enough $t \in \mathbb{R}$, the tangency condition for \tilde{H} can be written as

$$\begin{aligned} & (\operatorname{Re} \tilde{H})\rho(z) \\ &= \operatorname{Re} \left[\left(\frac{1}{2} + \frac{P(z_2)}{2i} \right) \sum_{j,k=0}^{\infty} a_{j,k} (it - tP(z_2))^j z_2^k + tP_{z_2}(z_2) \sum_{m,n=0}^{\infty} b_{m,n} (it - tP(z_2))^m z_2^n \right] \\ &= 0 \end{aligned}$$

for all $z_2 \in \mathbb{C}$ and $t \in \mathbb{R}$ with $z_2 \in \Delta_\epsilon$ and $|t| < \delta_0$, where $\epsilon_0, \delta_0 > 0$ are small enough.

Applying the same argument as in the proof of Theorem 1.1, one can obtain the following: for all $m \in \mathbb{N}^0$ and $\ell, \ell' \in \mathbb{N}^*$,

$$\begin{aligned} & a_{0,m} = 0; \\ (31) \quad & \operatorname{Re} \left[\left(\frac{1 + P^2(z_2)}{2} \right) \sum_{k=0}^{\infty} ia_{m+1,k} z_2^k + P_{z_2}(z_2) \sum_{n=0}^{\infty} b_{m,n} z_2^n \right] (i - P(z_2))^m = 0; \\ & \operatorname{Re} [i^\ell a_{\ell,0}] = 0; \\ & a_{\ell,\ell'} = 0. \end{aligned}$$

With these observations, we note that the coefficients $a_{\ell,0}, \ell \in \mathbb{N}^*$, only can be candidates to be non-zero among all the coefficients $a_{j,k}$.

Now we shall show that $\tilde{H} \equiv 0$. Aiming for a contradiction, we suppose that $\tilde{H} \not\equiv 0$. Since $P(z_2)$ and $P_{z_2}(z_2)$ vanish to infinite order at $z_2 = 0$, one can see that if $h_2 \equiv 0$, then the above tangency condition yields $h_1(z_1, z_2) = \alpha z_1, \alpha \in \mathbb{R}$. Therefore, in the remaining of the proof, we focus our attention only on the case when $h_2 \not\equiv 0$.

We shall divide our argument into the following two cases.

Case 1. $h_1(z_1, z_2) - \alpha z_1 \not\equiv 0$. In this case, let m_0 be the smallest integer such that $b_{m_0,n} \neq 0$ for some integer n , and then let n_0 be the smallest integer such that $b_{m_0,n_0} \neq 0$. Since $h_2(0,0) = b_{0,0} = 0$, we first observe that $m_0 \geq 1$ if $n_0 = 0$. Since $P(z_2) = o(|z_2|^\ell)$ for any $\ell \in \mathbb{N}^*$, (31) yields

$$(32) \quad \operatorname{Re} \left[\left(\frac{1 + P^2(z_2)}{2} \right) ia_{m_0+1,0} + P_{z_2}(z_2) b_{m_0,n_0} (z_2^{n_0} + o(|z_2|^{n_0})) \right] (i - P(z_2))^{m_0} \equiv 0$$

on Δ_{ϵ_0} . Moreover, we remark that $P_{z_2}(z_2) = \frac{1}{2}P'(x)$, where $x := \operatorname{Re}(z_2)$. In addition, if $b_{m_0,n_0} \neq 0$, then we get

$$\operatorname{Re} [P_{z_2}(z_2) b_{m_0,n_0} (i - P(z_2))^{m_0} (z_2^{n_0} + o(|z_2|^{n_0}))] \neq 0$$

on Δ_{ϵ_0} . Indeed, if $\operatorname{Re} [P_{z_2}(z_2) b_{m_0,n_0} (i - P(z_2))^{m_0} (z_2^{n_0} + o(|z_2|^{n_0}))] = 0$, then the binomial theorem shows that $b_{m_0,n_0} = 0$ since $\tilde{P}'(x) \neq 0$ near $x = 0$, $P(z_2)$ and $P_{z_2}(z_2)$ vanish to infinite order at $z_2 = 0$. This contradicts to the choice of the pair (m_0, n_0) such that $b_{m_0,n_0} \neq 0$. Then it follows from (32) that

$$(33) \quad \tilde{P}'(x) = - \frac{\operatorname{Re} \left[\left(\frac{1 + P^2(z_2)}{2} \right) i (i - P(z_2))^{m_0} a_{m_0+1,0} \right]}{\operatorname{Re} [b_{m_0,n_0} (i - P(z_2))^{m_0} (z_2^{n_0} + o(|z_2|^{n_0}))]},$$

for all $z_2 := x + iy \in \Delta_{\epsilon_0}$ satisfying

$$\tilde{P}'(x) \neq 0; \operatorname{Re} [b_{m_0,n_0} (i - P(z_2))^{m_0} (z_2^{n_0} + o(|z_2|^{n_0}))] \neq 0.$$

If $n_0 \geq 1$, then the right-hand side of (33) depends on x and y ; however, the left-hand side of (33) is independent of y which leads to a contradiction.

In addition, if $n_0 = 0$, then (33) yields

$$\tilde{P}'(x) = -\frac{\operatorname{Re} \left[\left(\frac{1+P^2(z_2)}{2} \right) i(i-P(z_2))^{m_0} a_{m_0+1,0} \right]}{\operatorname{Re} [b_{m_0, n_0} (i-P(z_2))^{m_0} (1+o(1))]} \sim \tilde{P}(x)$$

near the origin. This implies that $\frac{\tilde{P}'(x)}{\tilde{P}(x)}$ becomes a bounded function near the origin.

Integrating $\frac{\tilde{P}'(x)}{\tilde{P}(x)}$, we get

$$\tilde{P}(x) = \tilde{P}(x_0) \exp \left(\int_{x_0}^x \frac{\tilde{P}'(t)}{\tilde{P}(t)} dt \right),$$

where x and x_0 are in a neighborhood of the origin. In this case, we obtain $\tilde{P}(x) \rightarrow 0$ as $x \rightarrow 0$, which contradicts to our assumption (ii). Hence, in this case, one must have $h_1(z_1, z_2) - \alpha z_1 \equiv 0$.

Case 2. $h_1(z_1, z_2) - \alpha z_1 \equiv 0$. Let m_0 and n_0 be as in Case 1. Since $P(z_2) = o(|z_2|^\ell)$ for any $\ell \in \mathbb{N}^*$, (31) implies that

$$\frac{1}{2} \tilde{P}'(x) \operatorname{Re} [b_{m_0, n_0} (i-P(z_2))^{m_0} (z_2^{n_0} + o(|z_2|^{n_0}))] = 0$$

for all $z_2 := x + iy \in \Delta_{\epsilon_0}$. Moreover, since $\tilde{P}'(x) \neq 0$ near the origin, we get

$$\operatorname{Re} [b_{m_0, n_0} (i-P(z_2))^{m_0} (z_2^{n_0} + o(|z_2|^{n_0}))] = 0$$

for all $z_2 \in \Delta_{\epsilon_0}$, which is absurd as we observed in the previous case.

Altogether, one can say that $\mathbf{aut}_0(M_P, 0) = \{\alpha z_1 \partial_{z_1} : \alpha \in \mathbb{R}\}$.

Now it remains to show that

$$\mathbf{aut}(M_P, 0) = \mathfrak{g}_1 \oplus \mathfrak{g}_0,$$

where $\mathfrak{g}_1 = \{\alpha z_1 \partial_{z_1} : \alpha \in \mathbb{R}\}$ and $\mathfrak{g}_0 = \{i\beta \partial_{z_2} : \beta \in \mathbb{R}\}$.

In what follows, by abuse of notation, let $H = h_1(z_1, z_2) \partial_{z_1} + h_2(z_1, z_2) \partial_{z_2}$ stand for an arbitrary element of $\mathbf{aut}(M_P, 0)$ and then let $\{\phi_t\}_{t \in \mathbb{R}} \subset \operatorname{Aut}(M_P)$ be the 1-parameter subgroup generated by the vector field H . Since ϕ_t is biholomorphic for every $t \in \mathbb{R}$, the set $\{\phi_t(0, 0) : t \in \mathbb{R}\}$ is contained in $P_\infty(M_P)$.

Furthermore, since the connected component of 0 in $S_\infty(\tilde{P})$ is $\{0\}$, one can deduce that the connected component of 0 in $P_\infty(M_P)$ is $\{(0, is) : s \in \mathbb{R}\}$. Therefore, we have

$$\phi_t(0, 0) \subset \{(0, is) : s \in \mathbb{R}\}.$$

This yields

$$\operatorname{Re} h_2(0, 0) = h_1(0, 0) = 0.$$

Hence, the holomorphic vector field

$$H(z_1, z_2) - i\beta \partial_{z_2},$$

where $\beta := \operatorname{Im} h_2(0, 0)$, belongs to $\mathbf{aut}_0(M_P, 0)$. This ends the proof of the assertion (a).

(b) By (a), we see that $\mathbf{aut}(M_P, 0) = \mathfrak{g}_1 \oplus \mathfrak{g}_0$, that is, $z_1 \partial_{z_1}$ and $i \partial_{z_2}$ generate $\mathbf{aut}(M_P, 0)$.

Now let us denote by $\{T_t^1\}_{t \in \mathbb{R}}$ and $\{T_t^2\}_{t \in \mathbb{R}}$ the 1-parameter subgroups generated by $z_1 \partial_{z_1}$ and $i \partial_{z_2}$ respectively, that is,

$$T_t^1 = (\exp(t)z_1, z_2); \quad T_t^2(z_1, z_2) = (z_1, z_2 + it)$$

for $t \in \mathbb{R}$. For any $f = (f_1, f_2) \in \text{Aut}(M_P, 0)$, we define families $\{F_t^j\}_{t \in \mathbb{R}}$ of automorphisms by setting

$$F_t^j := f \circ T_{-t}^j \circ f^{-1} \quad (j = 1, 2).$$

Then it follows that $\{F_t^j\}_{t \in \mathbb{R}}$, $j = 1, 2$, are 1-parameter subgroups of $\text{Aut}(M_P)$.

Moreover, since $\mathfrak{aut}(M_P, 0) = \mathfrak{g}_1 \oplus \mathfrak{g}_0$, each holomorphic vector field H^j generated by $\{F_t^j\}_{t \in \mathbb{R}}$ ($j = 1, 2$), surely belongs to $\mathfrak{g}_1 \oplus \mathfrak{g}_0$. This means that there exist real numbers δ_1^j, δ_2^j , $j = 1, 2$, such that

$$H^j = \delta_1^j z_1 \partial_{z_1} + i \delta_2^j \partial_{z_2} \quad (j = 1, 2),$$

which yields

$$F_t^j(z_1, z_2) = (\exp(\delta_1^j t)z_1, z_2 + i \delta_2^j t) = T_{\delta_1^j t}^1 \circ T_{\delta_2^j t}^2(z_1, z_2)$$

for $j = 1, 2$ and $t \in \mathbb{R}$. This implies that

$$f = T_{\delta_1^j t}^1 \circ T_{\delta_2^j t}^2 \circ f \circ T_t^j \quad (j = 1, 2),$$

which is equivalent to

$$(34) \quad f_1(z_1, z_2) = \exp(\delta_1^1 t) f_1(\exp(t)z_1, z_2);$$

$$(35) \quad f_2(z_1, z_2) = f_2(\exp(t)z_1, z_2) + i \delta_2^1 t;$$

$$(36) \quad f_1(z_1, z_2) = \exp(\delta_1^2 t) f_1(z_1, z_2 + it);$$

$$(37) \quad f_2(z_1, z_2) = f_2(z_1, z_2 + it) + i \delta_2^2 t.$$

Taking the derivative of both sides of (34) with respect to t , we have

$$0 = \delta_1^1 \exp(\delta_1^1 t) f_1(\exp(t)z_1, z_2) + \exp(\delta_1^1 t) \exp(t) z_1 \frac{\partial f_1(\exp(t)z_1, z_2)}{\partial(\exp(t)z_1)}.$$

This implies that $0 = \delta_1^1 f_1(z_1, z_2) + z_1 \frac{\partial f_1}{\partial z_1}(z_1, z_2)$; hence, one gets

$$(38) \quad f_1(z_1, z_2) = z_1^{-\delta_1^1} g_1(z_2),$$

where g_1 is a holomorphic function on a neighborhood of $z_2 = 0$. Moreover, since f_1 is a biholomorphism, δ_1^1 should be -1 .

Now we apply the same procedure as above to (36). Then we first get

$$0 = \delta_1^2 \exp(\delta_1^2 t) f_1(z_1, z_2 + it) + \exp(\delta_1^2 t) i \frac{\partial f_1(z_1, z_2 + it)}{\partial(z_2 + it)},$$

which yields

$$(39) \quad 0 = \delta_1^2 f_1(z_1, z_2) + i \frac{\partial f_1}{\partial z_2}(z_1, z_2).$$

Substituting (38) into (39), we obtain

$$(40) \quad 0 = \delta_1^2 z_1 g_1(z_2) + i z_1 \frac{dg_1}{dz_2}(z_2).$$

Then (40) tells us that g_1 has a form

$$g_1(z_2) = C_1 \exp(i \delta_1^2 z_2),$$

where C_1 is a constant which will be determined more precisely later on.

Next, applying the same argument as above to (35) and (37) again, one can deduce that

$$(41) \quad 0 = z_1 \frac{\partial f_2}{\partial z_1}(z_1, z_2) + i\delta_2^1;$$

$$(42) \quad 0 = i \frac{\partial f_2}{\partial z_2}(z_1, z_2) + i\delta_2^2.$$

It follows from (42) that

$$(43) \quad f_2(z_1, z_2) = -\delta_2^2 z_2 + h_1(z_1),$$

where h_1 is a holomorphic function on a neighborhood of $z_1 = 0$, fixing the origin. Substituting (43) into (41), we get

$$0 = z_1 \frac{dh_1}{dz_1}(z_1) + i\delta_2^1.$$

This clearly forces that h_1 should be identically zero since h_1 is a biholomorphism fixing the origin in \mathbb{C} ; hence $0 = z_1 \frac{dh_1}{dz_1}(z_1) + i\delta_2^1 = i\delta_2^1$. Therefore, we obtain

$$f_2(z_1, z_2) = -\delta_2^2 z_2.$$

Altogether, we have

$$f(z_1, z_2) = (C_1 z_1 \exp(i\delta_1^2 z_2), -\delta_2^2 z_2).$$

Now we shall determine f more precisely. Since M_P is invariant under f , one can deduce that

$$(44) \quad \begin{aligned} 0 &= \operatorname{Re}(f_1(it - tP(z_2), z_2)) + \operatorname{Im}((it - tP(z_2))g_1(z_2))P(f_2(it - tP(z_2), z_2))) \\ &= \operatorname{Re}((it - tP(z_2))g_1(z_2)) + \operatorname{Im}((it - tP(z_2))g_1(z_2))P(-\delta_2^2 z_2) \end{aligned}$$

for sufficiently small $|z_2|, t \in \mathbb{R}$. Since the case $g_1 \equiv 0$ contradicts to the fact that f is biholomorphic near the origin, we may assume that $g_1 \neq 0$; hence (44) implies that

$$(45) \quad P(-\delta_2^2 z_2) = -\frac{\operatorname{Re}((i - P(z_2))g_1(z_2))}{\operatorname{Im}((i - P(z_2))g_1(z_2))}.$$

Since $P(-\delta_2^2 z_2)$ vanishes to infinite order at $z_2 = 0$, $\operatorname{Re}((i - P(z_2))g_1(z_2))$ also has the same property at $z_2 = 0$. In addition, since $P(z_2)$ vanishes to infinite order at $z_2 = 0$, one can further say that $\operatorname{Re}(ig_1(z_2))$ vanishes to infinite order at $z_2 = 0$. Combining this with the fact that g_1 is holomorphic near $z_2 = 0$, we obtain

$$g_1(z_2) = \text{a constant } C \in \mathbb{R}^*.$$

Note that this yields the constants

$$C = C_1; \delta_1^2 = 0.$$

Therefore, (45) can be re-written as

$$P(-\delta_2^2 z_2) = P(z_2).$$

Then it follows from Lemma 3.1 that $|\delta_2^2| = 1$. Thus, since δ_2^2 was chosen in \mathbb{R} , the only two cases appeared in the statement of this theorem can occur as desired.

(c) Now let $f \in \text{Aut}(M_P)$ be arbitrary. Then $f(0,0)$ is of infinite type. It follows from the assumption $S_\infty(\tilde{P}) = \{0\}$ that $P_\infty(M_P) = \{(0, is) : s \in \mathbb{R}\}$. Therefore, we get

$$f(0,0) = (0, is_0) \text{ for some } s_0 \in \mathbb{R}.$$

Combining this with the assertion (b), one can deduce that

$$T_{-s_0}^2 \circ f \in \text{Aut}(M_P, 0),$$

where $T_{-s_0}^2$ and $\text{Aut}(M_P, 0)$ are explicitly described in the proof of (b). This completes the proof of (c).

Altogether, we finish the proof of Theorem 1.5.

4. EXAMPLES

In this section, we shall investigate several examples as analogues of those in [3, Section 6].

Example 4.1. Consider the model M_{P_1} , where P_1 is defined by setting

$$P_1(z) := \begin{cases} \exp(-1/|z|^a) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

where $a > 0$. Then it is easily seen that $S_\infty(P_1) = \{0\}$. Since $P_1 \not\equiv 0$ near the origin in \mathbb{C} and P_1 is rotationally symmetric, Theorem 1.1 and Theorem 1.3 (a) show that

$$\mathbf{aut}(M_{P_1}, 0) = \mathbf{aut}_0(M_{P_1}, 0) = \{\alpha z_1 \partial_{z_1} + i\beta z_2 \partial_{z_2} : \alpha, \beta \in \mathbb{R}\}.$$

In addition, we obtain

$$\text{Aut}(M_{P_1}, 0) = \{(z_1, z_2) \mapsto (sz_1, \exp(it)z_2) : s \in \mathbb{R}^*, t \in \mathbb{R}\},$$

which is clear from Theorem 1.3 (b).

Example 4.2. Consider the model M_{P_2} , where P_2 is defined by setting

$$P_2(z) := \begin{cases} \exp(-1/|z|^a + \text{Re } z) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

where $a > 0$.

In this case we first observe that, by definition, $S_\infty(P_2) = \{0\}$. In contrast with the previous example, P_2 is not rotationally symmetric, but P_2 is also not identically zero near the origin in \mathbb{C} . Then Theorem 1.1 and Theorem 1.3 (a) imply that

$$\mathbf{aut}(M_{P_2}, 0) = \mathbf{aut}_0(M_{P_2}, 0) = \{\alpha z_1 \partial_{z_1} : \alpha \in \mathbb{R}\}.$$

In addition, it follows from Theorem 1.3 (b) that

$$\text{Aut}(M_{P_2}, 0) = G_2(M_{P_2}, 0).$$

Example 4.3. Consider the model M_{P_3} , where P_3 is defined by setting

$$P_3(z) := \begin{cases} \exp(-1/|\text{Re } z|^a) & \text{if } \text{Re } z \neq 0, \\ 0 & \text{if } \text{Re } z = 0, \end{cases}$$

where $a > 0$. Let us define a function $\tilde{P}(z)$ by setting

$$\tilde{P}(z) := P_1(z),$$

where P_1 is given in the above Example 4.1. Then it is easy to check that \tilde{P} satisfies the conditions (i) and (ii) in Theorem 1.5, and $P_3(z) = \tilde{P}(\operatorname{Re} z)$. Combining the discussion in Example 4.1 with Theorem 1.5, one can see that

$$\begin{aligned}\mathbf{aut}_0(M_{P_3}, 0) &= \{\alpha z_1 \partial_{z_1} : \alpha \in \mathbb{R}\}; \\ \mathbf{aut}(M_{P_3}, 0) &= \{\alpha z_1 \partial_{z_1} + i\beta \partial_{z_2} : \alpha, \beta \in \mathbb{R}\}; \\ \operatorname{Aut}(M_{P_3}, 0) &= \{(z_1, z_2) \mapsto (sz_1, \pm z_2) : s \in \mathbb{R}^*\}; \\ \operatorname{Aut}(M_{P_3}) &= \{(z_1, z_2) \mapsto (sz_1, \pm z_2 + it) : s \in \mathbb{R}^*, t \in \mathbb{R}\}.\end{aligned}$$

APPENDIX

In this Appendix, we shall describe an analogue of Theorem 1.1 for *m-nonminimal infinite type models* with $m > 1$ in \mathbb{C}^2 . Let us consider a C^∞ -smooth hypersurface $(M_{P,m}, 0)$ with $m > 1$ in \mathbb{C}^2 defined by

$$M_{P,m} := \{(w, z) \in \mathbb{C}^2 : \rho(w, z) := \operatorname{Im} w - (\operatorname{Re} w)^m P(z) = 0\},$$

where $P(z)$ is a non-zero germ of a real-valued C^∞ -smooth function at the origin, which vanishes to infinite order at $z = 0$.

Let $H = h_1(w, z)\partial_w + h_2(w, z)\partial_z \in \mathbf{aut}_0(M_{P,m}, 0)$ be arbitrary. That is, H is a holomorphic vector field near the origin in \mathbb{C}^2 such that

$$(\operatorname{Re} H)\rho(w, z) = 0; \quad H(0) = 0$$

for all $(w, z) \in M_{P,m}$. Expanding the functions h_1 and h_2 into the Taylor series at the origin,

$$\begin{aligned}h_1(w, z) &= \sum_{j,k=0}^{\infty} a_{j,k} w^j z^k; \\ h_2(w, z) &= \sum_{\ell,n=0}^{\infty} b_{\ell,n} w^\ell z^n,\end{aligned}$$

where $a_{j,k}, b_{\ell,n} \in \mathbb{C}$.

Since $(t + it^m P(z), z) \in M_{P,m}$ with t small enough, the above tangency condition admits the following form:

$$(46) \quad \operatorname{Re} \left[\left(\frac{1}{2i} - \frac{mt^{m-1}P(z)}{2} \right) \sum_{j,k=0}^{\infty} a_{j,k} (t + it^m P(z))^j z^k - t^m P_z(z) \sum_{\ell,n=0}^{\infty} b_{\ell,n} (t + it^m P(z))^\ell z^n \right] \equiv 0$$

for all $z \in \mathbb{C}$ and $t \in \mathbb{R}$ with $z_2 \in \Delta_{\epsilon_0}$ and $|t| < \delta_0$, where $\epsilon_0, \delta_0 > 0$ are sufficiently small. Since $P(z)$ and $P_z(z)$ vanish to infinite order at $z = 0$, it follows from (46) that

$$(47) \quad \begin{cases} a_{0,0} = 0; \\ \operatorname{Im} a_{s,0} = 0, \quad \forall s \in \mathbb{N}^*; \\ a_{s',\ell'+1} = 0, \quad \forall s', \ell' \in \mathbb{N}^0. \end{cases}$$

Then (46) can be re-written as

$$(48) \quad \begin{aligned} & \operatorname{Re} \left[\frac{1}{2} P(z) \sum_{j=0}^{\infty} (j-m) a_{j,0} t^{m+j-1} - P_z(z) \sum_{\ell,n=0}^{\infty} b_{\ell,n} z^n t^{m+\ell} \right. \\ & \quad \left. - iP(z)P_z(z) \sum_{\ell'=1}^{\infty} \sum_{n=0}^{\infty} \ell' b_{\ell',n} z^n t^{2m+\ell'-1} + o(|P(z)|) \right] \\ & \equiv 0 \end{aligned}$$

for all $z \in \Delta_{\epsilon_0}$ and $t \in \mathbb{R}$ sufficiently small.

If $h_2 \equiv 0$, then after considering the coefficient of t^{m+j-1} for each $0 \leq j \leq \infty$ in (48), one can deduce that $a_{j,0} = 0$ for all $j \neq m$. Then we obtain

$$H = a_{m,0} w^m \partial_w.$$

Let us denote by $\{\varphi(t)\}_{t \in \mathbb{R}} := \{(\varphi^1(t), \varphi^2(t))\}_{t \in \mathbb{R}} \subset \operatorname{Aut}(M_{P,m}, 0)$ the 1-parameter subgroup generated by $w^m \partial_w$, that is, for $t \in \mathbb{R}$

$$\frac{d\varphi^1}{dt}(t) = (\varphi^1(t))^m; \quad \frac{d\varphi^2}{dt}(t) = 0$$

with $(\varphi^1(0), \varphi^2(0)) = (w, z) \in M_{P,m}$. On the other hand, since $m > 1$, the solution $\varphi(t)$ of this initial value problem is not invertible, hence $\{\varphi(t)\}_{t \in \mathbb{R}} \not\subset \operatorname{Aut}(M_{P,m}, 0)$ which leads to a contradiction. Hence, if $h_2 \equiv 0$, then we must have $h_1 \equiv 0$. For this reason, in the remaining of the proof, we always assume that $h_2 \neq 0$ without loss of generality.

Let m_0 be the smallest integer such that $b_{m_0,n} \neq 0$ for some $n \in \mathbb{N}^0$. Then we let n_0 be the smallest integer such that $b_{m_0,n_0} \neq 0$. Since $b_{0,0} = 0$, it is clear that $m_0 \geq 1$ if $n_0 = 0$. We shall divide the argument into the following three cases.

Case 1. $0 \leq m_0 < m - 1$. Considering the coefficient of t^{m+m_0} in (48), we get

$$(49) \quad \operatorname{Re} \left[\frac{1}{2} P(z) (m_0 + 1 - m) a_{m_0+1,0} - P_z(z) \sum_{n=0}^{\infty} b_{m_0,n} z^n \right] \equiv 0$$

on Δ_{ϵ_0} . In this case, by Proposition 2.1, we obtain $n_0 = 1$ and $b_{m_0,1} = i\beta$ for some $\beta \in \mathbb{R}^*$. Then, by a change of variables (cf. [9, Lemma 1]), we may assume that

$$b_{m_0}(z) := \sum_{n=0}^{\infty} b_{m_0,n} z^n = i\beta z.$$

Let $r \in (0, \epsilon_0)$ be an arbitrary number such that $P(r) \neq 0$ and then let $v(t) := P(r \exp(it))$ for all $t \in \mathbb{R}$. Combining these relations with the above condition (49), one gets

$$\frac{v'(t)}{v(t)} = (m_0 + 1 - m) \frac{a_{m_0+1,0}}{\beta}.$$

Integrating this, we obtain

$$v(t) = v(0) \exp \left((m_0 + 1 - m) \frac{a_{m_0+1,0}}{\beta} t \right)$$

for all $t \in \mathbb{R}$. Here, without loss of generality, we may take $\frac{a_{m_0+1,0}}{\beta}$ as a positive number. Then as $t \rightarrow +\infty$, we get $v(t) \rightarrow 0$ and hence $P(r \exp(it)) \rightarrow P(0) = 0$, which contradicts to our choice of $r \in (0, \epsilon_0)$ (consider the associated limit of $v(t)$ as $t \rightarrow -\infty$ if $\frac{a_{m_0+1,0}}{\beta} < 0$).

Case 2. $m_0 = m - 1$. In this case, we first note that $m + m_0 = 2m - 1$. Considering the coefficient of t^{2m-1} in (48), we have

$$(50) \quad \operatorname{Re} \left[P_z(z) \sum_{n=0}^{\infty} b_{m_0,n} z^n \right] \equiv 0$$

on Δ_{ϵ_0} . Applying [3, Corollary 4] to this relation, one can obtain

$$n_0 = 1; \operatorname{Re} b_{m_0,n_0} = \operatorname{Re} b_{m_0,1} = 0.$$

Therefore, by a change of variables, we may assume that

$$b_{m_0}(z) := \sum_{n=0}^{\infty} b_{m_0,n} z^n = i\tilde{\beta}z$$

for some $\tilde{\beta} \in \mathbb{R}^*$. Combining this with the above condition (50), we get

$$\operatorname{Re} \left[i\tilde{\beta}z P_z(z) \right] \equiv 0$$

on Δ_{ϵ_0} . This implies that $P(z) \equiv P(|z|)$ on Δ_{ϵ_0} .

We now prove that $b_{\ell}(z) := \sum_{n=0}^{\infty} b_{\ell,n} z^n = 0$ for every $\ell \geq m$: suppose otherwise. Then there exists the smallest number $m_1 \in \mathbb{N}^*$ such that $b_{m_1} \neq 0$ and $m_1 \geq m$. By the same argument as above, we may assume that $b_{m_1}(z) = i\tilde{\beta}_1 z + o(|z|)$ for some $\tilde{\beta}_1 \in \mathbb{R}^*$ on Δ_{ϵ_0} . Moreover, we indeed have $b_{m_1}(z) = i\tilde{\beta}_1 z$ for some $\tilde{\beta}_1 \in \mathbb{R}^*$: suppose otherwise. Then there exist $k_0 \geq 2$ and $\tilde{c}_{k_0} \in \mathbb{C}^*$ such that

$$(51) \quad b_{m_1}(z) = i\tilde{\beta}_1 z + \tilde{c}_{k_0} z^{k_0} + o(|z|^{k_0}).$$

Putting $\ell = m_1$ in (46) and then using (48), we get

$$(52) \quad \operatorname{Re} \left[\left(\frac{1}{2i} - \frac{mt^{m-1}P(z)}{2} \right) \sum_{j=0}^{\infty} a_{j,0} (t + it^m P(z))^j - t^m P_z(z) \sum_{n=0}^{\infty} b_{m_1,n} (t + it^m P(z))^{m_1} z^n \right] \equiv 0$$

for all $z \in \Delta_{\epsilon_0}$ and $t \in \mathbb{R}$ sufficiently small. Considering the coefficient of t^{m+m_1} in (52) and then using (47) and (51), one can get

$$(53) \quad \operatorname{Re} \left[\frac{1}{2} P(z) (m_1 + 1 - m) a_{m_1+1,0} - P_z(z) (i\tilde{\beta}_1 z + \tilde{c}_{k_0} z^{k_0} + o(|z|^{k_0})) + o(|P(z)|) \right] \equiv 0$$

on Δ_{ϵ_0} . Considering again the coefficient of t^{m+m_1} in (52) and then using (47), for some $\tilde{n}_0 \in \mathbb{N}^0$ we have

$$(54) \quad \operatorname{Re} \left[\frac{1}{2} P(z) (m_1 + 1 - m) a_{m_1+1,0} - P_z(z) b_{m_1,\tilde{n}_0} (z^{\tilde{n}_0} + o(|z|^{\tilde{n}_0})) + o(|P(z)|) \right] \equiv 0$$

on Δ_{ϵ_0} . If $a_{m_1+1,0} \neq 0$, then Proposition 2.1 yields $\tilde{n}_0 = 1$ and $b_{m_1,\tilde{n}_0} = i\tilde{\beta}_1$. In addition, if $a_{m_1+1,0} = 0$, then Proposition 2.1 and [3, Corollary 4] (apply this corollary to the case when $\operatorname{Re} \left[\frac{1}{2} P(z) (m_1 + 1 - m) a_{m_1+1,0} + o(|P(z)|) \right] \equiv 0$ on Δ_{ϵ_0}) also yield the same conclusion. Combining this with the subtraction of (54) from (53), we get

$$\operatorname{Re} \left[P_z(z) \tilde{c}_{k_0} (z^{k_0} + o(|z|^{k_0})) \right] \equiv 0$$

on Δ_{ϵ_0} , which contradicts to [3, Corollary 4].

Altogether, in this case, we obtain $h_2(w, z) = i\tilde{\beta}z$ and $P(z) \equiv P(|z|)$ for some $\tilde{\beta} \in \mathbb{R}^*$ on Δ_{ϵ_0} .

Case 3. $m_0 \geq m$. Considering the coefficient of t^{m+m_0} in (46) and then using (47), we get

$$(55) \quad \begin{aligned} & \operatorname{Re} \left[\frac{1}{2} a_{m_0+1-m,0} P(z) + P_z(z) (b_{m_0,n_0} z^{n_0} + o(|z|^{n_0}) - iP(z)b_{m_0-m+1}(z) + o(|P(z)|)) \right] \\ & \equiv O(|P(z)|^2) \end{aligned}$$

on Δ_{ϵ_0} .

Since $O(|P(z)|^2) \in o(1)$ and $m_0 + 1 - m \neq 0$, if $\operatorname{Re}(a_{m_0+m-1,0}) = a_{m_0+m-1,0} = 0$, then our situation reduces to (E2) in [5, Lemma 3], which leads to a contradiction. Hence, in this case, we must have $a_{m_0+1-m,0} \neq 0$. Moreover, by [5, Lemma 3], one can get $n_0 = 1$ and $b_{m_0,1} = i\beta_2$ for some $\beta_2 \in \mathbb{R}^*$. Then (55) yields

$$(56) \quad \operatorname{Re} [i\beta_2 z P_z(z)] \equiv (\tilde{\delta} + \tilde{\epsilon}(z)) P(z),$$

where $\tilde{\delta} := -\frac{1}{2}a_{m_0+1-m,0}$ and $\tilde{\epsilon}_0 : \Delta_{\epsilon_0} \rightarrow \mathbb{R}$ is a smooth function with the condition that $\tilde{\epsilon} \rightarrow 0$ as $z \rightarrow 0$. Without loss of generality, we may assume that $\tilde{\delta} < 0$ and $|\tilde{\epsilon}(z)| < |\tilde{\delta}|/2$ on Δ_{ϵ_0} .

Let $r \in (0, \epsilon_0)$ such that $P(r) \neq 0$. Then we let $\gamma : [t_0, +\infty) \rightarrow \mathbb{C}$ such that $\gamma'(t) = i\beta_2 \gamma(t)$ and $\gamma(t_0) = r$. Then setting $u(t) = \log |P(\gamma(t))|$, (56) shows that $u'(t) = \tilde{\delta} + \tilde{\epsilon}(\gamma(t))$. Hence, we get

$$(57) \quad u(t) - u(t_0) = \tilde{\delta}(t - t_0) + \int_{t_0}^t \tilde{\epsilon}(\gamma(\tau)) d\tau, \quad \forall t \geq t_0.$$

This implies that $u(t) \rightarrow +\infty$ as $t \rightarrow \infty$, and hence $\gamma(t) \rightarrow 0$ as $t \rightarrow +\infty$.

On the other hand, $\gamma'(t) = i\beta_2 \gamma(t)$ and $\gamma(t_0) = r$ imply that $\gamma(t) = r \exp(i\beta_2 t)$. Then we get $\gamma(t) \not\rightarrow 0$ as $t \rightarrow +\infty$, which contradicts to the above discussion right after (57).

Now we shall show that

$$h_1(w, z) \equiv 0,$$

if $P(z) \equiv P(|z|)$ and $h_2(w, z) = i\tilde{\beta}z$ for some $\tilde{\beta} \in \mathbb{R}^*$ on Δ_{ϵ_0} . Suppose otherwise. Then it follows from (47) that there exists the smallest number $j_0 \in \mathbb{N}^*$ such that $a_{j_0,0} \neq 0$. Since $P(z) \equiv P(|z|)$ and $h_2(w, z) = i\tilde{\beta}z$ for some $\tilde{\beta} \in \mathbb{R}^*$ on Δ_{ϵ_0} , using (47) and (48), one can deduce that for all $t \in \mathbb{R}$ sufficiently small

$$(58) \quad \operatorname{Re} \left[\left(\frac{1}{2i} - \frac{mt^{m-1}P(z)}{2} \right) \sum_{j=j_0}^{\infty} a_{j,0} (t + it^m P(z))^j \right] \equiv 0$$

on Δ_{ϵ_0} . For a fixed $t \in \mathbb{R}$ small enough, we may regard (58) as a polynomial of a variable $P(z)$ on Δ_{ϵ_0} . Collecting the terms of degree 1 with respect to $P(z)$ in (58), for a fixed $t \in \mathbb{R}$ small enough, we get

$$P(z) \operatorname{Re} \left[\sum_{j=0}^{\infty} \frac{1}{2} (j - m) a_{j,0} t^{m+j-1} \right] \equiv 0$$

on Δ_{ϵ_0} . Since the connected component of $z = 0$ in the zero set of P is $\{0\}$, we have

$$\operatorname{Re} \left[\sum_{j=0}^{\infty} \frac{1}{2} (j-m) a_{j,0} t^{m+j-1} \right] \equiv 0$$

for a fixed $t \in \mathbb{R}$ sufficiently small. Moreover, since t can be chosen arbitrarily small, we first have

$$\operatorname{Re} [a_{j,0}] \equiv 0, \quad \forall j \in \mathbb{N}^* \setminus \{m\}.$$

Combining this with the second relation of (47), one can deduce that

$$(59) \quad a_{j,0} = 0, \quad \forall j \in \mathbb{N}^* \setminus \{m\}.$$

Then it follows from (47) and (59) that $a_{m,0}$ can be a unique candidate to be non-zero among all the possible $a_{j,k}$'s. For this reason, we now assume that

$$H(w, z) := a_{m,0} w^m \partial_w + i\tilde{\beta} z \partial_z \in \mathbf{aut}_0(M_{P,m}, 0)$$

for some $a_{m,0} \in \mathbb{R}$ and $\tilde{\beta} \in \mathbb{R}^*$.

Let us denote by $\{\varphi_t\}_{t \in \mathbb{R}} := \{(\varphi^1(t), \varphi^2(t))\}_{t \in \mathbb{R}} \subset \operatorname{Aut}(M_{P,m}, 0)$ the 1-parameter subgroup generated by the vector field H , that is, for $t \in \mathbb{R}$

$$\frac{d\varphi^1}{dt}(t) = a_{m,0} (\varphi^1(t))^m; \quad \frac{d\varphi^2}{dt}(t) = i\tilde{\beta} \varphi^2(t)$$

with $(\varphi^1(0), \varphi^2(0)) = (w, z) \in M_{P,m}$. However, we note that $\varphi(t) = (\varphi^1(t), \varphi^2(t))$ is not invertible, if $m > 1$ and $a_{m,0} \neq 0$. This tells us that if $m > 1$ and $H \in \mathbf{aut}_0(M_{P,m}, 0)$, then we should have $a_{m,0} = 0$.

Altogether, we conclude that if $P(z) = P(|z|)$ and $h_2(w, z) = i\tilde{\beta}z$ for some $\tilde{\beta} \in \mathbb{R}^*$ on Δ_{ϵ_0} , then $h_1(w, z) \equiv 0$ holds, as desired.

Acknowledgement. Parts of this work were carried out while the last named author visited VNU University of Science, Hanoi and Vietnam Institute for Advanced Study in Mathematics(VIASM). She would like to thank these institutes for their heartfelt hospitality. The last named author was supported by the National Research Foundation of Korea with grant NRF-2018R1D1A1B07044363.

REFERENCES

- [1] M. S. Baouendi, L. P. Rothschild and F. Trèves, *CR structures with group action and extendability of CR functions*, Invent. Math. 82 (1985), no. 2, 359–396.
- [2] J. P. D’Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. 115 (1982), no. 3, 615–637.
- [3] A. Hayashimoto and V. T. Ninh, *Infinitesimal CR automorphisms and stability groups of infinite-type models in \mathbb{C}^2* , Kyoto J. Math. 56 (2016), no. 2, 441–464.
- [4] R. Juhlin and B. Lamel, *On maps between nonminimal hypersurfaces*, Math. Z. 273 (2013), no. 1–2, 515–537.
- [5] K. T. Kim and V. T. Ninh, *On the tangential holomorphic vector fields vanishing at an infinite type point*, Trans. Amer. Math. Soc. 367 (2015), no. 2, 867–885.
- [6] M. Kolář, I. Kossovskiy and D. Zaitsev, *Normal forms in Cauchy-Riemann geometry*, Analysis and geometry in several complex variables, 153–177, Contemp. Math., 681, Amer. Math. Soc., Providence, RI, 2017.
- [7] M. Kolář and B. Lamel, *Holomorphic equivalence and nonlinear symmetries of ruled hypersurfaces in \mathbb{C}^2* , J. Geom. Anal. 25 (2015), no. 2, 1240–1281.

- [8] I. Kossovskiy and B. Lamel, *New extension phenomena for solutions of tangential Cauchy-Riemann equations*, Comm. Partial Differential Equations 41 (2016), no. 6, 925–951.
- [9] V. T. Ninh, *On the existence of tangential holomorphic vector fields vanishing at an infinite type point*, arXiv:1303.6156v7.
- [10] N. Stanton, *Rigid hypersurfaces in \mathbb{C}^2* , Several complex variables and complex geometry, Part 3 (Santa Cruz, CA, 1989), 347–354, Proc. Sympos. Pure Math., 52, Part 3, Amer. Math. Soc., Providence, RI, 1991.
- [11] N. Stanton, *Infinitesimal CR automorphisms of rigid hypersurfaces*, Amer. J. Math. 117 (1995), no. 1, 141–167.
- [12] A. E. Tumanov, *Extension of CR-functions into a wedge from a manifold of finite type*, (Russian) Mat. Sb. (N.S.) 136(178) (1988), no. 1, 128–139; translation in Math. USSR-Sb. 64 (1989), no. 1, 129–140.

(V. T. Ninh) DEPARTMENT OF MATHEMATICS, VIETNAM NATIONAL UNIVERSITY AT HANOI, 334 NGUYEN TRAI STR., HANOI, VIETNAM
E-mail address: `thunv@vnu.edu.vn`

(T. N. O. Duong) PHUNG HUNG SECONDARY SCHOOL, 55 NGUYEN THAI HOC, QUANG TRUNG, SON TAY, HANOI, VIETNAM
E-mail address: `duongoanh25.6@gmail.com`

(V. H. Pham) DEPARTMENT OF MATHEMATICS, VIETNAM NATIONAL UNIVERSITY AT HANOI, 334 NGUYEN TRAI STR., HANOI, VIETNAM
E-mail address: `hoangabcnpxyz95@gmail.com`

(H. Kim) RESEARCH INSTITUTE OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, 1 GWANAK-RO, GWANAK-GU, SEOUL 08826, REPUBLIC OF KOREA
E-mail address: `hop222@snu.ac.kr`