ASYMPTOTIC EXPANSION IN GEVREY SPACES FOR SOLUTIONS OF THE NAVIER-STOKES-α EQUATIONS

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ABSTRACT. In this paper, we study the asymptotic behavior of solutions to the three-dimensional incompressible Navier-Stokes-α equations with periodic boundary conditions and non-potential body forces. We prove that if the body force possesses a large-time asymptotic expansion or, resp., finite asymptotic approximation in Sobolev-Gevrey spaces in terms of polynomial and decaying exponential functions of time, then any weak solution admits an asymptotic expansion, or resp., finite asymptotic approximation of the same type. The result obtained reveals precisely how the structure of the force influences the asymptotic behavior of the solutions.

1. Introduction

In this paper, we consider the following 3D Navier-Stokes-α (NS-α) equations introduced in [16] with space periodic boundary conditions

$$\begin{cases} \partial_t v - \nu\Delta v + \nabla p = u \times (\nabla \times v) + f, & x \in \Omega, t > 0, \\ v = u - \alpha^2 \Delta u, & \\ \nabla \cdot u = \nabla \cdot v = 0, & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ represents the unknown "filtered" fluid velocity vector, $p = p(x,t)$ is the unknown "filtered" pressure and the body force $f = f(x,t)$; the positive constant ν is the kinematic viscosity and α is a length scale parameter that represents the width of the filter. At the limit $\alpha = 0$, we obtain the three-dimensional Navier-Stokes equations with periodic boundary conditions.

We assume that u, f and p are periodic in each direction with period 2π, we denote by Ω the cube of period $(0, 2\pi)^3$. From (1.1) one can easily see, after integration by parts, that

$$\frac{d}{dt}\int_\Omega (u - \alpha^2\Delta u)dx = \int_\Omega f(x,t)dx.$$

On the other hand, because of the spatial periodicity of the solution, we have $\int_\Omega \Delta u dx = 0$. As a result, we have

$$\frac{d}{dt}\int_\Omega u dx = \int_\Omega f(x,t)dx,$$

that is, the mean of the solution is invariant provided the mean of the forcing term is zero. In this paper we will consider forcing terms and initial values with spatial means that are zero, i.e., we will assume

$$\int_\Omega u_0(x)dx = \int_\Omega f(x,t)dx = 0$$

and hence

$$\int_\Omega u(x,t)dx = 0. \tag{1.2}$$

Current scientific methods and tools are unable to compute either the turbulent behavior of three-dimensional fluids or via direct numerical simulation due to the large range of scales of motion that need to be resolved when the Reynolds number is high. Over the last decades, several turbulence models have been proposed for capturing the physical phenomenon of turbulence at computably low resolution. Navier-Stokes-α equations (also known as the viscous Camassa-Holm equations or the Lagrangian-averaged Navier-Stokes-α)

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is considered as a closure model of turbulence by modifying the nonlinearity in the Navier-Stokes equations to stop the cascading of turbulence at scales smaller than a certain length, but without introducing any extra dissipation (see [20], [25] and references therein for a nice detailed description of the development of the NS-α model).

The inviscid ($\nu = 0$) case, known as the Lagrangian averaged Euler (LAE-α) or Euler-α equations, was introduced in [19] as a natural mathematical generalization of the integrable inviscid 1D Camassa-Holm equation discovered in [5] through a variational formulation. In [16], the authors first added viscous dissipation to the equations, they argued on physical grounds that the momentum $u + \alpha^2 Au$ rather than the velocity u, need be diffused. In the past years, the existence and long-time behavior of solutions to the NS-α equations have attracted the attention of many mathematicians. In bounded domains with Dirichlet or periodic boundary conditions, there are many results on the existence of solutions and existence of global attractors for NS-α equations, see e.g. [8, 16, 22, 26, 30] and references therein. The time decay rates of solutions on the whole space were investigated in [4] and more recently in [3]. The numerical simulations of both forced and decaying isotropic turbulence using the NS-α model were investigated in [7, 28]. We also refer the interested reader to [1, 23] for recent results on the data assimilation to the NS-α equations with periodic boundary conditions and to [2] for the optimal control problem of the 3D viscous Camassa-Holm equations.

In studying the dynamics of Navier-Stokes equations, the function $u(x, t)$ of several variables can be viewed as a function of t valued in some functional space. For time-dependent functions of such type, their asymptotic properties, as time goes to infinity, can be understood most precisely if some form of asymptotic expansion is established. In recent decades, there has been a great deal of interest dedicated to the large time behavior of solutions of the Navier-Stokes equations. In an early work, Dyer and Edmunds [9] prove that any non-trivial, regular solution u has $|u(t)|^2$ bounded below by an exponential function of t. However, this answer is far from being definitive in describing the exact asymptotic behavior of a non-trivial, regular solution. Later, Foias and Saut established a description of the large time behavior of solutions of the Navier-Stokes equations with potential forces. They proved that in bounded or periodic domains the regular, non-trivial solutions of the Navier-Stokes equations decay exponentially at an exact rate which is an eigenvalue of the Stokes operator (see [13]). They go on to show the corresponding normal form which provides a complete description of the large time behavior of solutions in [14, 15]. After these works, a number of subsequent studies on this expansion, as well as the associated normal form of the Navier-Stokes equations, its normalization map, and invariant nonlinear manifolds have been studied extensively in [10, 11, 12, 24, 27] and references therein. To the best of our knowledge, however, there are no results on the asymptotic expansion for solutions of NS-α equations. This is our motivation.

The type of asymptotic expansion that we study here is defined as follows.

Definition 1.1. *Let X be a real vector space.*
(i) An X-valued polynomial is a function $t \in \mathbb{R} \mapsto \sum_{n=1}^{d} a_n t^n$, for some $d \geq 0$, and a_n's belonging to X.
(ii) When $(X, \|\cdot\|_X)$ is a normed space, a function $g : (0, \infty) \to X$ is said to have the asymptotic expansion

$$g(t) \sim \sum_{n=1}^{\infty} g_n(t) e^{-nt} \text{ in } X, \tag{1.3}$$

where $g_n(t)$'s are X-valued polynomials, if for all $N \geq 1$, there exists $\varepsilon_N > 0$ such that

$$\|g(t) - \sum_{n=1}^{N} g_n(t) e^{-nt}\|_X = \mathcal{O}(e^{-(N+\varepsilon_N)t}) \text{ as } t \to \infty.$$

The notation, $\mathcal{O}(f(t))$, above is defined as

$$\Psi(t) = \mathcal{O}\Big(f(t)\Big) \text{ as } t \to \infty \text{ if and only if } \exists T, C > 0, \Psi(t) \leq Cf(t), \forall t > T,$$

where Ψ and f are non-negative scalar quantities. We note that the times T, and constants C may depend on the parameters appearing in f.

In this paper, following the general lines of the strategy introduced in [18], we study the asymptotic behavior of the solution $u(x, t)$ as $t \to \infty$ for a certain class of forces $f(x, t)$ belonging to a Sobolev-Gevrey space $G_{\rho,\sigma}$ for any $\rho \ge 1/2$ and $\sigma \ge 0$. More precisely, if

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\nu n t} \quad \text{in } G_{\rho,\sigma},$$

then any weak solution $u(t)$ of (1.1) will admit the following expansion

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\nu n t} \quad \text{in } G_{\rho,\sigma}. \tag{1.4}$$

This expansion indicate how each term in the expansion of the force integrates into the expansion of the solution. On the other hand, when $f(t)$ have a finite asymptotic approximation, then we show that the corresponding solution, $u(t)$, admits a finite sum approximation of the same type. The main contribution of the result is Theorem 4.1, that is for each $\sigma > 0$ and $N \ge 1$, the estimate (4.11) is conjectured and then proved, by induction in N, to hold true for all $\rho \ge 1/2$. This is crucial due to the estimate of the nonlinear term $\mathcal{B}(u, v)$ which always requires the regularity of one more derivative u or v. However, since the Gevrey norm in $|u|_{0,\sigma}$ for any $\sigma > 0$ is stronger than all Sobolev norms $|A^\rho u|$ (see (3.3)), this obstacle becomes a non issue. When $\alpha = 0$, we formally obtain the results on asymptotic expansion for solutions of the Navier-Stokes equations with non-potential body forces [18]. Moreover, our results recover the case when the force $f(x, t)$ is potential, i.e., $f(x, t) = -\nabla\phi(x, t)$, for some scale function $\phi(x, t)$ (because of its vanishing when we apply the Leray projection $\mathbb{P}(-\nabla\phi) = 0$). It is also noticed that our arguments can be applied to some other NS-α models, such as Leray-α model [6], the modified Leray-α model [21].

The paper is organized as follows. In Section 2, for convenience of the reader, we recall some auxiliary results on function spaces and inequalities for the nonlinear terms related to the NS-α equations. Section 3 contains some basic inequalities for Sobolev and Gevrey norms, estimates for nonlinear term $\mathcal{B}(u, v)$ and the exponential decay for the weak solutions in Sobolev and Gevrey spaces (Propositions 3.1 and 3.2). In Section 4, the asymptotic expansion (1.4) is obtained, either as an infinite sum in Theorem 4.1, or a finite sum in Theorem 4.2.

2. Preliminaries

We will denote by $(\cdot, \cdot)$ and $|\cdot|$, respectively, the scalar product and the associated norm in $L^2(\Omega)^3$, and by $(\nabla u, \nabla v)$ the scalar product in $L^2(\Omega)^3$ of the gradients of u and v. We consider the scalar product in $H_0^1(\Omega)^3$ defined by $((u, v)) = (\nabla u, \nabla v)$, for $u, v \in H_0^1(\Omega)^3$, and its associated norm, which is in fact equivalent to the usual gradient norm, will be denoted by $\|\cdot\|$.

Let us define the spaces

$$\mathcal{V} = \left\{ u = 2\pi\text{-periodic trigonometric polynomial vector fields}, \nabla \cdot u = 0, \int_\Omega u dx = 0 \right\},$$

$$\mathbb{L}^p(\Omega) = L^p(\Omega)^3, \quad \mathbb{H}^s(\Omega) = H^s(\Omega)^3, \quad \mathbb{H}_0^1(\Omega) = H_0^1(\Omega)^3.$$

We denote by H the closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega)$, and by V the closure of $\mathcal{V}$ in $\mathbb{H}_0^1(\Omega)$. Then, H is a Hilbert space equipped with the inner product of $\mathbb{L}^2(\Omega)$, and V is a Hilbert space equipped with the inner product of $\mathbb{H}_0^1(\Omega)$.

We use the following embeddings and identification

$$V \subset H \equiv H' \subset V'$$

where each space is dense in the next one, and the embeddings are compact.

We denote $\mathbb{P}$ the orthogonal (Leray) projection in $\mathbb{L}^2(\Omega)$ onto H and by A the Stokes operator, with domain $D(A) = \mathbb{H}^2(\Omega) \cap V$, defined by $Au = -\mathbb{P}(\Delta u), \forall u \in D(A)$. The operator A can be extended continuously to be defined on V with values in V' such that

$$\langle Au, w \rangle_{V'} = ((u, w)), \quad \text{for } u, w \in V.$$

Similarly, the operator A^2 can be defined on $D(A)$ with values in $D(A)'$, the dual space of the Hilbert space $D(A)$, such that

$$\langle A^2 u, w \rangle_{D(A)'} = (Au, Aw), \quad \text{for every } u, w \in D(A).$$

Notice that in the case of periodic boundary condition $A = -\Delta$ is a selfadjoint positive operator with compact inverse. Hence the space H has an orthonormal basis $\{w_j\}_{j=1}^{\infty}$ of eigenfunctions of A, i.e.,

$$Aw_j = \lambda_j w_j, \quad j = 1, 2, \cdots .$$

It is known that the spectrum of the Stokes operator A is $\sigma(A) = \{\lambda_j : j \in \mathbb{N}\}$, where λ_j is strictly increasing in j and is an eigenvalue of A. In fact, these eigenvalues have the form $|k|^2$ with $k \in \mathbb{Z}^3 \setminus \{0\}$. Note that $\sigma(A) \subset \mathbb{N}$ and $1 \in \sigma(A)$, hence, the additive semigroup generated by $\sigma(A)$ is $\mathbb{N}$.

By virtue of Poincaré inequality one can show that there is a constant $c > 0$, such that

$$c|Au| \leq \|u\|_{\mathbb{H}^2(\Omega)} \leq c^{-1}|Au|, \quad \forall u \in D(A)$$

and so $D(A)$ is a Hilbert space with the scalar product $(u, v)_{D(A)} = (Au, Av)$.

The orthogonal projection in H on the linear span of $w_1, \cdots, w_m$ will be denoted by

$$P_m = R_1 + R_2 + \cdots + R_m,$$

where R_n will stand for the orthogonal projection in H on the eigenspace of A corresponding to n if $n \in \sigma(A)$, or otherwise $R_n = 0$. Then, we have of course

$$R_i R_j = 0 \text{ if } i \neq j \quad \text{and} \quad R_1 + R_2 + \cdots = I.$$

For $\rho, \sigma \in \mathbb{R}$ and $u = \sum_{k\neq 0} \hat{u}(k) e^{ik\cdot x}$, define

$$A^\rho u = \sum_{k\neq 0} |k|^{2\rho} \hat{u}(k) e^{ik\cdot x},$$

$$A^\rho e^{\sigma A^{1/2}} u = \sum_{k\neq 0} |k|^{2\rho} e^{\sigma|k|} \hat{u}(k) e^{ik\cdot x},$$

where $\hat{u}(k)$ denotes the Fourier coefficient of u at wavenumber k.

We then define the Gevrey spaces by

$$G_{\rho,\sigma} = D(A^\rho e^{\sigma A^{1/2}}) = \left\{u \in H : |u|_{\rho,\sigma} := |A^\rho e^{\sigma A^{1/2}} u| < \infty\right\}$$

and the domain of the fractional operator A^ρ by

$$D(A^\rho) = G_{\rho,0} = \{u \in H : |A^\rho u| = |u|_{\rho,0} < \infty\}.$$

Thanks to the zero-average condition (1.2), the norm $|A^{m/2}u|$ is equivalent to $\|u\|_{\mathbb{H}^m(\Omega)}$ on the space $D(A^{m/2})$ for $m = 0, 1, 2, ...$ Note that $D(A^0) = H$, $D(A^{1/2}) = V$ and the spaces $G_{\rho,\sigma}$ are decreasing in ρ and σ.

Denote for $\sigma \in \mathbb{R}$ the space

$$E^{\infty,\sigma} = \bigcap_{\rho\geq 0} G_{\rho,\sigma} = \bigcap_{m\in\mathbb{N}} G_{m,\sigma}.$$

We will say that an asymptotic expansion (1.3) holds in $E^{\infty,\sigma}$ if it holds in $G_{\rho,\sigma}$ for all $\rho \geq 0$.

Let us also denote by $\mathcal{P}^{\rho,\sigma}$ the space of $G_{\rho,\sigma}$-valued polynomials in case $\rho \in \mathbb{R}$, and the space of $E^{\infty,\sigma}$-valued polynomials in case $\rho = \infty$.

We define the trilinear form b by

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

whenever the integrals make sense. It is easy to check that if $u, v, w \in V$, then

$$b(u, v, w) = -b(u, w, v).$$

Hence

$$b(u, v, v) = 0.$$

We now consider the trilinear form defined by

$$\overline{b}(u, v, w) = b(u, v, w) - b(w, v, u), \quad \forall (u, v, w) \in D(A) \times \mathbb{L}^2(\Omega) \times \mathbb{H}_0^1(\Omega),$$

and we define a continuous bilinear operator $\mathcal{B}$ from $V \times V$ into V' by

$$\langle \mathcal{B}(u, v), w\rangle = \overline{b}(u, v, w).$$

Next, using the identity

$$u\cdot\nabla v+\sum_{j=1}^{3}v_j\nabla u_j=-u\times(\nabla\times v)+\nabla(u\cdot v)$$

and using the fact that $\nabla\cdot u=0$, it is immediate to check that

$$\begin{aligned}\big(-u\times(\nabla\times v),w\big)&=\big(u\cdot\nabla v,w\big)+\big(v\cdot\nabla u^T,w\big)\\&=b(u,v,w)+b(w,u,v)=\bar b(u,v,w).\end{aligned}$$

We also have the trilinear form $\bar b$ satisfies

$$\bar b(u,v,w)=-\bar b(w,v,u),\quad (u,v,w)\in D(A)\times\mathbb{L}^2(\Omega)\times D(A),$$

and consequently,

$$\bar b(u,v,u)=0,\quad \text{for all } (u,v)\in D(A)\times\mathbb{L}^2(\Omega). \tag{2.1}$$

By applying the Leray projection $\mathbb{P}$ to (1.1) and use the above notation to obtain the equivalent system of equations

$$\begin{cases}\partial_t v+\nu Av+\mathcal{B}(u,v)=\mathbb{P}f, & x\in\Omega, t>0,\\ v=u-\alpha^2\Delta u, & \\ \nabla\cdot u=\nabla\cdot v=0, & x\in\Omega, t>0,\\ u(x,0)=u_0(x), & x\in\Omega,\end{cases} \tag{2.2}$$

We will assume that $\mathbb{P}f=f$, otherwise we add the gradient part of f to the modified pressure and rename $\mathbb{P}f$ by f.

Definition 2.1. *Let $f\in L^2(0,T;V')$. For given $T>0$, $u_0\in V$ (or $v_0\in V'$), a function*

$$u\in C([0,T];V)\cap L^2(0,T;D(A)) \text{ with } \frac{du}{dt}\in L^2(0,T;H)$$

or equivalently

$$v\in C([0,T];V')\cap L^2(0,T;H) \text{ with } \frac{dv}{dt}\in L^2(0,T;D(A)')$$

is said to be a weak solution to problem (2.2) in the interval $(0,T)$ if it satisfies

$$\left\langle\frac{d}{dt}v,w\right\rangle_{D(A)'}+\nu\left\langle Av,w\right\rangle_{D(A)'}+\left\langle\mathcal{B}(u,v),w\right\rangle_{D(A)'}=(f,w) \tag{2.3}$$

for every $w\in D(A)$ and for almost every $t\in[0,T]$.

3. Exponential decay in Gevrey and Sobolev spaces

In this section, we derive the exponential decay for weak solutions in both Gevrey and Sobolev spaces, particular. First, we state some basic inequalities concerning the Gevrey and Sobolev norms.

For all $\rho,\sigma\geq 0$,

$$|u|\leq|A^\rho u|\qquad\text{(Poincaré's inequality)}, \tag{3.1}$$

and

$$|u|\leq e^{-\sigma}|e^{\sigma A^{1/2}}u|. \tag{3.2}$$

When $\rho,\sigma>0$, one has

$$\max_{x\geq 0}(x^{2\rho}e^{-\sigma x})=\left(\frac{2\rho}{e\sigma}\right)^{2\rho},$$

hence

$$|A^\rho u|\leq|(A^\rho e^{-\sigma A^{1/2}})e^{\sigma A^{1/2}}u|\leq\left(\frac{2\rho}{e\sigma}\right)^{2\rho}|e^{\sigma A^{1/2}}u|. \tag{3.3}$$

Lemma 3.1. *Let $\sigma\geq 0$ and $\rho\geq 1/2$. There exists an absolute constant $K>1$, independent of ρ,σ, such that*

$$|\mathcal{B}(u,v)|_{\rho,\sigma}\leq K^\rho|u|_{\rho+1/2,\sigma}|v|_{\rho+1/2,\sigma},\quad \forall u,v\in G_{\rho+1/2,\sigma}. \tag{3.4}$$

Proof. The proof is mainly based on the work of Hoang and Martinez [17].

Let u, v, w be in H with

$$u=\sum_{k\in\mathbb{Z}^3}u_ke^{ik\cdot x},\quad v=\sum_{l\in\mathbb{Z}^3}v_le^{il\cdot x},\quad w=\sum_{m\in\mathbb{Z}^3}w_me^{im\cdot x}.$$

Define the scalar functions

$$u_*=\sum_{k\in\mathbb{Z}^3}|u_k|e^{ik\cdot x},\quad v_*=\sum_{l\in\mathbb{Z}^3}|v_l|e^{il\cdot x},\quad w_*=\sum_{m\in\mathbb{Z}^3}|w_m|e^{im\cdot x},$$

where

$$|u_k|=e^{\rho|k|}u_k,\quad |v_l|=e^{\rho|l|}v_l,\quad |w_m|=e^{\rho|m|}w_m.$$

Then

$$|A^\rho u|=|(-\Delta)^\rho u_*| \text{ for all } \rho\geq 0.$$

We have

$$\langle A^\rho e^{\sigma A^{1/2}}\mathcal{B}(u,v),w\rangle=(2\pi)^3i\sum_{k+l+m=0}|m|^{2\rho}e^{\sigma|m|}\Big((u_k\cdot l)(v_l\cdot w_m)+(w_m\cdot l)(v_l\cdot u_k)\Big).$$

Therefore

$$\Big|\langle A^\rho e^{\sigma A^{1/2}}\mathcal{B}(u,v),w\rangle\Big|\leq 16\pi^3\sum_{k+l+m=0}|m|^{2\rho}e^{\sigma|m|}|l||u_k||v_l||w_m|.$$

By using the same arguments in the proof of [17, Lemma 2.1] we deduce

$$\Big|\langle A^\rho e^{\sigma A^{1/2}}\mathcal{B}(u,v),w\rangle\Big|\leq K^\rho|u|_{\rho+1/2,\sigma}|v|_{\rho+1/2,\sigma}|w|,\quad \text{for } \rho\geq 1/2,$$

and, hence, we obtain (3.4). □

Remark 3.1. As a consequence of Lemma 3.1, we have

$$\mathcal{B}(G_{\rho+1/2,\sigma},G_{\rho+1/2,\sigma})\subset G_{\rho,\sigma}\quad \text{for } \rho\geq 1/2,\sigma>0, \tag{3.5}$$

$$\mathcal{B}(E^{\infty,\sigma},E^{\infty,\sigma})\subset E^{\infty,\sigma}\quad \text{for } \sigma\geq 0. \tag{3.6}$$

Next, we will establish the global existence of the solution in Gevrey spaces and its exponential decay as time goes to infinity.

Proposition 3.1. *Let $\delta\in(0,1)$, $\lambda\in(1-\delta,1]$ and $\sigma\geq 0,\rho\geq 1/2$. Define the positive numbers*

$$C_0=\frac{\nu\alpha^4\delta}{2K^\rho(1+\alpha^2)(2+\alpha^2)}\quad \textit{and}\quad C_1=C_0\nu\alpha\sqrt{\delta(\lambda+\delta-1)}.$$

Suppose that

$$|A^{\rho+1/2}u_0|\leq C_0 \tag{3.7}$$

and

$$|f(t)|_{\rho-1,\sigma}\leq C_1e^{-\nu\lambda t},\quad \forall t\geq 0. \tag{3.8}$$

Then there exists a unique solution $u(t)$ of (2.2) that satisfies $u\in C([0,\infty),D(A^{\rho+1/2}))$ and

$$|u(t)|^2_{\rho,\sigma}+\alpha^2|u(t)|^2_{\rho+1/2,\sigma}\leq C_0^2(2+\alpha^2)e^{-2\nu(1-\delta)t},\quad \forall t\geq t^*, \tag{3.9}$$

where $t^*=\dfrac{8\sigma}{\nu\delta}$. Moreover, one has for all $t\geq t^*$ that

$$\int_t^{t+1}\Big(|u(s)|^2_{\rho+1/2,\sigma}+\alpha^2|u(s)|^2_{\rho+1,\sigma}\Big)ds\leq\frac{3+\alpha^2}{2\nu(1-\delta)}C_0^2e^{-2\nu(1-\delta)t}. \tag{3.10}$$

Proof. We use the Galerkin approximations and the standard passage to the limit (see e.g. [29]) to prove global existence and to establish the necessary a priori estimates.

Part I: case $\sigma>0$. Let $\varphi(t)$ be a function in $C^\infty(\mathbb{R})$ such that $\varphi(t)=0$ for $t\leq 0$, $\varphi(t)>0$ for $t>0$, $\varphi(t)=\sigma$ for $t\geq t^*$, and

$$0<\varphi'(t)<2\frac{\sigma}{t^*}=\frac{1}{4}\nu\delta\quad \text{for all } t\in(0,t^*).$$

From the first two equations of (2.2), we have

$$\begin{aligned} &\frac{d}{dt}\Big(A^\rho e^{\varphi(t)A^{1/2}}(u(t)+\alpha^2Au(t))\Big)-\varphi'(t)A^{1/2}A^\rho e^{\varphi(t)A^{1/2}}(u+\alpha^2Au)\\ &\qquad+\nu A^\rho e^{\varphi(t)A^{1/2}}A(u+\alpha^2Au)+A^\rho e^{\varphi(t)A^{1/2}}\mathcal{B}(u,u+\alpha^2Au)=A^\rho e^{\varphi(t)A^{1/2}}f. \end{aligned} \tag{3.11}$$

Taking inner product of the equation (3.11) with $A^\rho e^{\varphi(t)A^{1/2}}u(t)$ gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|u|^2_{\rho,\varphi(t)}+\alpha^2|A^{1/2}u|^2_{\rho,\varphi(t)})+\nu(|A^{1/2}u|^2_{\rho,\varphi(t)}+\alpha^2|Au|^2_{\rho,\varphi(t)})=\langle A^\rho e^{\varphi(t)A^{1/2}}f,A^\rho e^{\varphi(t)A^{1/2}}u\rangle\\ &+\varphi'(t)\langle A^{\rho+1/2}e^{\varphi(t)A^{1/2}}(u+\alpha^2Au),A^\rho e^{\varphi(t)A^{1/2}}u\rangle-\langle A^\rho e^{\varphi(t)A^{1/2}}\mathcal{B}(u,u+\alpha^2Au),A^\rho e^{\varphi(t)A^{1/2}}u\rangle \end{aligned} \tag{3.12}$$

Applying the Cauchy and Poincaré inequalities and Lemma 3.1 to the terms on the right-hand side to obtain

$$\big|\varphi'(t)\langle A^{\rho+1/2}e^{\varphi(t)A^{1/2}}u,A^\rho e^{\varphi(t)A^{1/2}}u\rangle\big|\le\frac{1}{4}\nu\delta|A^{1/2}u|_{\rho,\varphi(t)}|u|_{\rho,\varphi(t)}\le\nu\delta|A^{1/2}u|^2_{\rho,\varphi(t)},$$

$$\alpha^2\big|\varphi'(t)\langle A^{\rho+1/2}e^{\varphi(t)A^{1/2}}Au,A^\rho e^{\varphi(t)A^{1/2}}u\rangle\big|\le\frac{\alpha^2}{4}\nu\delta|Au|_{\rho,\varphi(t)}|A^{1/2}u|_{\rho,\varphi(t)}\le\frac{\alpha^2}{4}\nu\delta|Au|^2_{\rho,\varphi(t)},$$

$$\begin{aligned} \big|-\langle A^\rho e^{\varphi(t)A^{1/2}}\mathcal{B}(u,u+\alpha^2Au),A^\rho e^{\varphi(t)A^{1/2}}u\rangle\big|&=\big|\langle A^{\rho-1/2}e^{\varphi(t)A^{1/2}}\mathcal{B}(u,u+\alpha^2Au),A^{\rho+1/2}e^{\varphi(t)A^{1/2}}u\rangle\big|\\ &\le K^\rho|u|_{\rho,\varphi(t)}|(u+\alpha^2Au)|_{\rho,\varphi(t)}|Au|_{\rho,\varphi(t)}\\ &\le K^\rho(1+\alpha^2)|A^{1/2}u|_{\rho,\varphi(t)}|Au|^2_{\rho,\varphi(t)}, \end{aligned}$$

and

$$\begin{aligned} \big|\langle A^\rho e^{\varphi(t)A^{1/2}}f,A^\rho e^{\varphi(t)A^{1/2}}u\rangle\big|&\le|f|_{\rho-1,\varphi(t)}|u|_{\rho+1,\varphi(t)}\\ &\le\frac{1}{\nu\alpha^2\delta}|f|^2_{\rho-1,\varphi(t)}+\frac{\nu\alpha^2\delta}{4}|u|^2_{\rho+1,\varphi(t)}. \end{aligned}$$

From all estimates above, inserting all of them on the right-hand side of (3.12), we obtain

$$\begin{aligned} &\frac{d}{dt}(|u|^2_{\rho,\varphi(t)}+\alpha^2|A^{1/2}u|^2_{\rho,\varphi(t)})+2\nu(1-\delta)|A^{1/2}u|^2_{\rho,\varphi(t)}\\ &\qquad+2\nu\alpha^2\left(1-\frac{\delta}{2}-K^\rho\frac{1+\alpha^2}{\nu\alpha^2}|A^{1/2}u|_{\rho,\varphi(t)}\right)|Au|^2_{\rho,\varphi(t)}\le\frac{2}{\nu\alpha^2\delta}|f|^2_{\rho-1,\varphi(t)}. \end{aligned} \tag{3.13}$$

We claim that

$$K^\rho\frac{1+\alpha^2}{\nu\alpha^2}|A^{1/2}u(t)|_{\rho,\varphi(t)}\le\delta/2,\quad\forall t\ge0. \tag{3.14}$$

Suppose (3.14) is not true, then by (3.7), there exists $T\in(0,\infty)$ such that

$$K^\rho\frac{1+\alpha^2}{\nu\alpha^2}|A^{1/2}u(t)|_{\rho,\varphi(t)}<\delta/2,\quad\forall t\in[0,T) \tag{3.15}$$

and

$$K^\rho\frac{1+\alpha^2}{\nu\alpha^2}|A^{1/2}u(T)|_{\rho,\varphi(T)}=\delta/2. \tag{3.16}$$

By (3.15) and (3.8), we have for $t\in(0,T)$ that

$$\frac{d}{dt}(|u|^2_{\rho,\varphi(t)}+\alpha^2|A^{1/2}u|^2_{\rho,\varphi(t)})+2\nu(1-\delta)(|A^{1/2}u|^2_{\rho,\varphi(t)}+\alpha^2|Au|^2_{\rho,\varphi(t)})\le\frac{2}{\nu\alpha^2\delta}|f|^2_{\rho-1,\varphi(t)}\le\frac{2C_1^2}{\nu\alpha^2\delta}e^{-2\nu\lambda t}. \tag{3.17}$$

Applying the Gronwall inequality in (3.17), we deduce for all $t\in(0,T)$ that

$$\begin{aligned} |u(t)|^2_{\rho,\varphi(t)}+\alpha^2|A^{1/2}u(t)|^2_{\rho,\varphi(t)}&\le e^{-2\nu(1-\delta)t}(|u_0|^2_{\rho,0}+\alpha^2|A^{1/2}u_0|^2_{\rho,0})+\frac{2C_1^2}{\nu\alpha^2\delta}e^{-2\nu(1-\delta)t}\int_0^t e^{-2\nu(\lambda+\delta-1)s}ds\\ &\le e^{-2\nu(1-\delta)t}(|u_0|^2_{\rho,0}+\alpha^2|A^{1/2}u_0|^2_{\rho,0})+\frac{C_1^2}{\nu^2\alpha^2\delta(\lambda+\delta-1)}e^{-2\nu(1-\delta)t}\\ &\le(|u_0|^2_{\rho,0}+\alpha^2|A^{1/2}u_0|^2_{\rho,0}+C_0^2)e^{-2\nu(1-\delta)t}. \end{aligned}$$

Combining this with condition (3.7) for the initial data, we obtain

$$|u(t)|^2_{\rho,\varphi(t)}+\alpha^2|A^{1/2}u(t)|^2_{\rho,\varphi(t)}\le(2+\alpha^2)C_0^2e^{-2\nu(1-\delta)t} \tag{3.18}$$

or

$$|A^{1/2}u(t)|_{\rho,\varphi(t)} \leq C_0 \frac{\sqrt{2+\alpha^2}}{\alpha} e^{-\nu(1-\delta)t}, \quad \text{for all } t \in (0,T). \tag{3.19}$$

In particular, letting $t \to T^-$ in (3.19) gives

$$\lim_{t\to T^-} |A^{1/2}u(t)|_{\rho,\varphi(t)} \leq \frac{\sqrt{2+\alpha^2}}{\alpha} C_0 < \frac{2+\alpha^2}{\alpha^2} C_0 = \frac{\nu\alpha^2\delta}{2K^\rho(1+\alpha^2)},$$

which contradicts (3.16). Therefore, (3.15) holds true. Consequently, we obtain (3.19). Since $\varphi(t) = \sigma$ for $t \geq t^*$, from (3.18) we deduce (3.9).

For $t \geq t^*$, integrating (3.17) from t to $t+1$ gives

$$\begin{aligned} 2\nu(1-\delta)\int_t^{t+1} (|A^{1/2}u(s)|^2_{\rho,\sigma} + \alpha^2|Au(s)|^2_{\rho,\sigma})ds &\leq |u(t)|^2_{\rho,\sigma} + \alpha^2|A^{1/2}u(t)|^2_{\rho,\sigma} + \frac{2C_1^2}{\nu\alpha^2\delta}\int_t^{t+1} e^{-2\nu\lambda s}ds \\ &\leq (2+\alpha^2)C_0^2 e^{-2\nu(1-\delta)t} + \frac{C_1^2}{\nu^2\alpha^2\delta\lambda} e^{-2\nu\lambda t} \\ &\leq (2+\alpha^2 + \frac{\lambda+\delta-1}{\lambda})C_0^2 e^{-2\nu(1-\delta)t} \leq (3+\alpha^2)C_0^2 e^{-2\nu(1-\delta)t}. \end{aligned}$$

Thus, we obtain (3.10).

Part II: case $\sigma = 0$. We use the same arguments in **Part I** without using the function $\varphi(t)$. Here, we provide some necessary calculations. First, using Sobolev norms, we have

$$\begin{aligned} &\frac{d}{dt}(|A^\rho u|^2 + \alpha^2|A^{\rho+1/2}u|^2) + 2\nu(1-\delta)|A^{\rho+1/2}u|^2 \\ &\qquad + 2\nu\alpha^2\left(1 - \frac{\delta}{2} - K^\rho\frac{1+\alpha^2}{\nu\alpha^2}|A^{\rho+1/2}u|\right)|A^{\rho+1}u|^2 \leq \frac{1}{\nu\alpha^2\delta}|A^{\rho-1}f|^2. \end{aligned} \tag{3.20}$$

From (3.14), (3.8) and (3.7), we have for $t \in (0,T)$ that

$$\begin{aligned} |A^\rho u|^2 + \alpha^2|A^{\rho+1/2}u|^2 &\leq e^{-2\nu(1-\delta)t}(|A^\rho u_0|^2 + \alpha^2|A^{\rho+1/2}u_0|^2) + \frac{C_1^2}{\nu\alpha^2\delta}e^{-2\nu(1-\delta)t}\int_0^t e^{-2\nu(\lambda+\delta-1)s}ds \\ &\leq e^{-2\nu(1-\delta)t}(|u_0|^2_{\rho,0} + \alpha^2|A^{1/2}u_0|^2_{\rho,0}) + \frac{C_1^2}{2\nu^2\alpha^2\delta(\lambda+\delta-1)}e^{-2\nu(1-\delta)t} \\ &\leq (|u_0|^2_{\rho,0} + \alpha^2|A^{1/2}u_0|^2_{\rho,0} + C_0^2)e^{-2\nu(1-\delta)t}. \end{aligned}$$

This implies $T = \infty$ and then also proves (3.9). And for $t \geq t^*$, from the inequality above, we also have

$$\begin{aligned} 2\nu(1-\delta)\int_t^{t+1} (|A^{\rho+1/2}u(s)|^2 + \alpha^2|A^{\rho+1}u(s)|^2)ds &\leq |A^\rho u(t)|^2 + \alpha^2|A^{\rho+1/2}u(t)|^2 + \frac{C_1^2}{\nu\alpha^2\delta}\int_t^{t+1} e^{-2\nu\lambda s}ds \\ &\leq (2+\alpha^2)C_0^2 e^{-2\nu(1-\delta)t} + \frac{C_1^2}{2\nu^2\alpha^2\delta\lambda}e^{-2\nu\lambda t} \\ &\leq (2+\alpha^2 + \frac{\lambda+\delta-1}{\lambda})C_0^2 e^{-2\nu(1-\delta)t} \leq (3+\alpha^2)C_0^2 e^{-2\nu(1-\delta)t}, \end{aligned}$$

hence we get (3.10) and the proof is complete. □

Remark 3.2. Proposition 3.1 provides information of decay rates of the solution and the particular effect on it from the body force.

Next, we will give estimate for the Gevrey norms of the weak solution of problem (2.2) with the optimal exponential decay for large time.

Proposition 3.2. *Assume that there are numbers $M, \kappa_0 > 0$ suth that*

$$|f(t)| \leq Me^{-\nu(1+\kappa_0)t/2}, \quad \text{for all } t \geq 0, \tag{3.21}$$

and, additionally, that there are $\sigma \geq 0, \rho \geq 1/2$ and $\lambda_0 \in (0,1)$ such that

$$|f(t)|_{\rho,\sigma} = \mathcal{O}(e^{-\nu\lambda_0 t}) \quad \text{as } t \to \infty. \tag{3.22}$$

Let $u(t)$ be a weak solution of (2.2)*. Then, for any $\delta \in (1-\lambda_0, 1)$ there exist $T^* > 0$ and a positive constant $K_0 = \max(K_2, K_3)$ such that*

$$|u(t)|_{\rho+3/2,\sigma} \le K_0 e^{-\nu(1-\delta)t}, \quad \textit{for all } t \ge T^*, \tag{3.23}$$

where K_2, K_3 are positive numbers in the proof below.

Proof. Taking inner product of (2.2) with u and using the orthogonality property (2.1), the Cauchy and Poincaré inequalities, we have

$$\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2) + \nu(\|u\|^2 + \alpha^2|Au|^2) \le \frac{1}{\nu}\|f\|^2_{V'}. \tag{3.24}$$

Applying Gronwall's inequality, we obtain

$$|u(t)|^2 + \alpha^2\|u(t)\|^2 \le e^{-\nu t}(|u_0|^2 + \alpha^2\|u_0\|^2) + \frac{e^{-\nu t}}{\nu}\int_0^t e^{\nu s}\|f(s)\|^2_{V'}ds, \quad \forall t > 0.$$

From (3.21), we deduce

$$|u(t)|^2 + \alpha^2\|u(t)\|^2 \le e^{-\nu t}(|u_0|^2 + \alpha^2\|u_0\|^2 + \frac{M^2}{\nu^2\kappa_0}), \quad \forall t > 0. \tag{3.25}$$

Moreover, by integrating (3.24) over (t_0, t) with $t \ge t_0 \ge 0$ we obtain

$$|u(t)|^2 + \alpha^2\|u(t)\|^2 + \nu\int_{t_0}^t(\|u(s)\|^2 + \alpha^2|Au(s)|^2)ds \le |u(t_0)|^2 + \alpha^2\|u(t_0)\|^2 + \frac{1}{\nu}\int_{t_0}^t \|f(s)\|^2_{V'}ds.$$

Using (3.25) and (3.21) we get

$$\begin{aligned}\nu\int_{t_0}^{t_0+1}(\|u(s)\|^2 + \alpha^2|Au(s)|^2)ds &\le e^{-\nu t_0}\left(|u_0|^2 + \alpha^2\|u_0\|^2 + \frac{M^2}{\nu^2\kappa_0}\right) + \frac{M^2}{\nu^2(1+\kappa_0)}e^{-\nu(1+\kappa_0)t_0} \\ &\le e^{-\nu t_0}\Big(|u_0|^2 + \alpha^2\|u_0\|^2 + \frac{2M^2}{\nu^2\kappa_0}\Big).\end{aligned} \tag{3.26}$$

For any $t \ge 0$, let $\{t_n\}_{n=1}^\infty$ be a sequence in $(0,\infty)$ converging to t such that (3.26) holds for $t_0 = t_n$. Then letting $n \to \infty$ yields

$$\nu\int_t^{t+1}(\|u(s)\|^2 + \alpha^2|Au(s)|^2)ds \le e^{-\nu t}\left(|u_0|^2 + \alpha^2\|u_0\|^2 + \frac{2M^2}{\nu^2\kappa_0}\right). \tag{3.27}$$

Define $\lambda = \dfrac{1-\delta+\lambda_0}{2} \in (1-\delta, \lambda_0)$. We consider each case $\sigma > 0$ and $\sigma = 0$ in turn.

(i) Case $\sigma > 0$. By (3.27) and (3.22), there exists $t_0 > 0$ such that

$$|Au(t_0)| < C_0(\nu, \alpha, M, \kappa_0),$$

and

$$|f(t)|_{-1/2,\sigma} \le C_1(\delta, \lambda)e^{-\nu\lambda t}, \quad \forall t \ge t_0.$$

Applying Proposition 3.1 to $\rho := 1/2$ we obtain

$$|u(t)|^2_{1/2,\sigma} + \alpha^2|u(t)|^2_{1,\sigma} \le K_1 e^{-2\nu(1-\delta)t}, \quad \forall t \ge t_0 + t^* = t_0 + \frac{8\sigma}{\nu\delta},$$

where $K_1 := C_0(\nu, \alpha, M, \kappa_0)(2+\alpha^2)$. Then, by using (3.3), for all $t \ge t_0 + t^*$, we have that

$$|A^{\rho+3/2}u(t)| \le \left(\frac{2\rho+1}{e\sigma}\right)^{2\rho+1}|u|_{1,\sigma} \le \left(\frac{2\rho+1}{e\sigma}\right)^{2\rho+1}K_1 e^{-\nu(1-\delta)t}. \tag{3.28}$$

Next, from (3.28) and (3.22), we deduce that there exists a sufficiently large $T_0 > t_0 + t^*$ such that

$$|A^{\rho+3/2}u(T_0)| < C_0(\rho, \sigma, e, K_1),$$

and

$$|f(t)|_{\rho,\sigma} \le C_1(\delta, \lambda)e^{-\nu\lambda t}, \quad \forall t \ge T_0.$$

Applying Proposition 3.1 again to $\rho := \rho + 1$ we obtain that there exists $T^* > T_0 + t^*$ so that

$$|u(t)|^2_{\rho+1,\sigma} + \alpha^2|u(t)|^2_{\rho+3/2,\sigma} \le K_2 e^{-2\nu(1-\delta)t}, \quad \forall t \ge T^*,$$

where $K_2 := C_0^2(\rho,\sigma,e,K_1)(2+\alpha^2)$. This implies (3.23) and completes the proof of **Case (i)**.

(ii) Case $\sigma = 0$. We will apply Proposition 3.1 recursively to obtain the exponential decay for $u(t)$ in higher Sobolev norms.

For $j \in \mathbb{N}$, suppose

$$\lim_{t\to\infty} \nu\alpha^2 \int_t^{t+1} |A^{(j+1)/2}u(s)|^2 ds = 0, \tag{3.29}$$

and

$$|A^{j/2-1}f(t)| = \mathcal{O}(e^{-\nu\lambda_0 t}) \quad \text{as } t\to\infty. \tag{3.30}$$

Then there exists $T>0$ so that

$$|A^{(j+1)/2}u(T)| \le C_0(j,\nu,\alpha),$$

and

$$|A^{j/2-1}f(t)| \le C_1(j,\delta,\lambda)e^{-\nu\lambda t}, \quad \forall t\ge T.$$

Applying Proposition 3.1 to $\rho := j/2, \sigma := 0$, we get

$$|A^{j/2}u(t)|^2 + \alpha^2|A^{(j+1)/2}u(t)|^2 \le C_0(j,\nu,\alpha)(2+\alpha^2)e^{-2\nu(1-\delta)t} \le K_1 e^{-2\nu(1-\delta)t}, \quad \forall t\ge T,$$

where $K_1 := C_0(j,\nu,\alpha)(2+\alpha^2)$ and

$$\int_t^{t+1} \Big(|A^{(j+1)/2}u(s)|^2 + \alpha^2|A^{(j+2)/2}u(s)|^2\Big)ds = \mathcal{O}(e^{-2\nu(1-\delta)t}) \quad \text{as } t\to\infty. \tag{3.31}$$

Notice that, (3.27) implies that (3.29) holds true for $j=1$. Let $2\le m\in\mathbb{N}$ be given such that

$$\rho \le \frac{m}{2} < \rho + \frac{1}{2}. \tag{3.32}$$

Then we have $(m-1)/2 < \rho$. Hence, from (3.22), condition (3.30) is satisfied for $j = 1,2,\cdots,m+1$. We now repeat the arguments from (3.29) to (3.31) for $j=1,2,\cdots,m+1$. Particularly, as $j=m+1$ we obtain from (3.31) that

$$\int_t^{t+1} \Big(|A^{(m+2)/2}u(s)|^2 + \alpha^2|A^{(m+3)/2}u(s)|^2\Big)ds = \mathcal{O}(e^{-2\nu(1-\delta)t}) \quad \text{as } t\to\infty.$$

Since $\rho\le m/2$, this implies

$$\int_t^{t+1} \Big(|A^{\rho+1}u(s)|^2 + \alpha^2|A^{\rho+3/2}u(s)|^2\Big)ds = \mathcal{O}(e^{-2\nu(1-\delta)t}) \quad \text{as } t\to\infty. \tag{3.33}$$

Thus, together with (3.22), there is $T_1\ge T$ such that

$$|A^{\rho+3/2}u(T_1)| < C_0(\rho,\alpha,\nu),$$

and

$$|f(t)|_{\rho,\sigma} \le C_1(\rho,\delta,\lambda)e^{-\nu\lambda t}, \quad \forall t\ge T_1.$$

Applying Proposition 3.1 to $\rho := \rho+1$ we obtain that there exists $T^* > T_1$ so that

$$|u(t)|^2_{\rho+1,\sigma} + \alpha^2|u(t)|^2_{\rho+3/2,\sigma} \le K_3 e^{-2\nu(1-\delta)t}, \quad \forall t\ge T^*,$$

where $K_3 := C_0(\rho,\alpha,\nu)(2+\alpha^2)$. This implies (3.23) and completes the proof of **Case (ii)**. □

Corollary 3.1. *Let the assumptions of Proposition 3.2 be fulfilled. Then*

$$|\mathcal{B}(u(t),v(t))|_{\rho,\sigma} \le K^\rho(1+\alpha^2)K_0^2 e^{-2\nu(1-\delta)t}, \quad \textit{for all } t\ge T^*. \tag{3.34}$$

Proof. We deduce from the Lemma 3.1 that

$$|\mathcal{B}(u(t),v(t))|_{\rho,\sigma} \le K^\rho(1+\alpha^2)|u|_{\rho+1/2,\sigma}|u|_{\rho+3/2,\sigma}, \quad \forall t\ge 0,$$

where $v = u+\alpha^2 Au$. Apply the Poincaré inequality, we obtain (3.34) with the use of (3.23). □

The next lemma is a building block of the construction of the polynomials $q_n(t)$'s. It summarizes and reformulates the facts used in [14] and [18, Lemma 4.2].

Lemma 3.2. *Let $(X, \|\cdot\|_X)$ be a Banach space. Suppose $y(t)$ is a function in $C([0,\infty);X)$ that solves the following ODE*

$$y'(t) + \beta y(t) = p(t) + g(t)$$

in the X-valued distribution sense on $(0,\infty)$. Here, $\beta \in \mathbb{R}$ is a fixed constant, $p(t)$ is X-valued polynomial in t, and $g \in L^1(0,T;X)$ satisfies

$$\|g(t)\|_X \le Me^{-\delta t}, \quad \forall t \ge 0, \text{ for some } M, \delta > 0.$$

Define $q(t)$ for $t \in \mathbb{R}$ by

$$q(t) = \begin{cases} e^{-\beta t}\int_{-\infty}^t e^{\beta s}p(s)ds & \text{if } \beta > 0,\\ y(0) + \int_0^\infty g(s)ds + \int_0^t p(s)ds & \text{if } \beta = 0,\\ -e^{-\beta t}\int_t^\infty e^{\beta s}p(s)ds & \text{if } \beta < 0.\end{cases}$$

Then $q(t)$ is an X-valued polynomial of degree at most $\deg(p)+1$ that satisfies

$$q'(t) + \beta q(t) = p(t), \quad t \in \mathbb{R},$$

and the following estimates hold:
(i) If $\beta > 0$ then

$$\|y(t) - q(t)\|_X \le \left(\|y(0) - q(0)\|_X + \frac{M}{|\beta - \delta|}\right) e^{-\min(\delta,\beta)t}, \quad t \ge 0, \text{ for } \beta \ne \delta,$$

and

$$\|y(t) - q(t)\|_X \le (\|y(0) - q(0)\|_X + Mt)\, e^{-\delta t}, \quad t \ge 0, \text{ for } \beta = \delta.$$

(ii) If $\beta = 0$ then

$$\|y(t) - q(t)\|_X \le \frac{M}{\delta} e^{-\delta t}, \quad t \ge 0.$$

(iii) If $\beta < 0$ and $\lim_{t\to\infty}\left(e^{\beta t}y(t)\right) = 0$, then

$$\|y(t) - q(t)\|_X \le \frac{M}{|\beta| + \delta} e^{-\delta t}, \quad t \ge 0.$$

4. Main results

We start by providing the following elementary identities: for $\beta > 0$, integer $d \ge 0$ and $t \in \mathbb{R}$.

$$\int_{-\infty}^t s^d e^{\beta s} ds = \frac{e^{\beta t}}{\beta}\sum_{n=0}^d \frac{(-1)^{d-n} d!}{n!\beta^{d-n}} t^n, \tag{4.1}$$

$$\int_t^\infty s^d e^{-\beta s} ds = \frac{e^{-\beta t}}{\beta}\sum_{n=0}^d \frac{d!}{n!\beta^{d-n}} t^n. \tag{4.2}$$

Theorem 4.1. *Assume that there exist a number $\sigma_0 \ge 0$ and polynomials $f_n \in \mathcal{P}^{\infty,\sigma_0}$ for all $n \ge 1$, such that $f(t)$ has the asymptotic expansion*

$$f(t) \sim \sum_{n=1}^\infty f_n(t) e^{-\nu n t} \quad \text{in } E^{\infty,\sigma_0}. \tag{4.3}$$

Let $u(t)$ be a weak solution of (2.2). Then
(i) There exist polynomials $q_n \in \mathcal{P}^{\infty,\sigma_0}$ for all $n \ge 1$, such that $u(t)$ has the asymptotic expansion

$$u(t) \sim \sum_{n=1}^\infty q_n(t) e^{-\nu n t} \quad \text{in } E^{\infty,\sigma_0}, \tag{4.4}$$

Moreover, the mappings

$$u_n(t) := q_n(t)e^{-\nu n t} \quad \text{and} \quad F_n(t) := f_n(t)e^{-\nu n t} \tag{4.5}$$

satisfy the following ordinary differential equations in the space E^{∞,σ_0}

$$\frac{d}{dt}v_n(t) + \nu A v_n(t) + \sum_{\substack{i,j\ge 1\\ i+j=n}} \mathcal{B}(u_i(t), v_j(t)) = F_n(t), \quad t \in \mathbb{R}, \forall n \ge 1 \tag{4.6}$$

where $v_n = u_n + \alpha^2 A u_n$.
(ii) If all $f_n(t)$*'s belong to* $\mathcal{V}$*, then so do all* $q_n(t)$*'s and the ODEs* (4.6) *hold in* $\mathcal{V}$.

Remark 4.1. Observe that since the expansion (4.3) is an infinite sum, it immediately implies the following remainder estimate:

$$\begin{aligned}\left|f(t) - \sum_{n=1}^{N} f_n(t)e^{-\nu nt}\right|_{\rho,\sigma_0} &\le \left|f_{N+1}(t)e^{-\nu(N+1)t}\right|_{\rho,\sigma_0} + \left|f(t) - \sum_{n=1}^{N+1} f_n(t)e^{-\nu nt}\right|_{\rho,\sigma_0} \\ &= \mathcal{O}\left(e^{-\nu(N+\varepsilon)t}\right) + \mathcal{O}\left(e^{-\nu(N+1+\delta_{N+1,\rho})t}\right),\end{aligned}$$

which holds for each $N \ge 1$, $\rho \ge 0$, $\varepsilon \in (0,1)$ and some $\delta_{N+1,\rho} \in (0,1)$. Therefore, we have for each $N \ge 1$ and $\rho \ge 0$, there exists a number $\delta_{N,\rho} \in (0,1)$ such that

$$\left|f(t) - \sum_{n=1}^{N} f_n(t)e^{-\nu nt}\right|_{\rho,\sigma_0} = \mathcal{O}\left(e^{-\nu(N+\delta_{N,\rho})t}\right) \quad \text{as } t \to \infty. \tag{4.7}$$

Thus, we have the following consequences:
(i) The relation (4.7) implies for each $\rho \ge 0$ that $f(t)$ belongs to G_{ρ,σ_0} for t large.
(ii) Note that, when $N = 1$, the function $f(t)$ itself satisfies

$$\left|f(t) - f_1(t)e^{-\nu t}\right|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu(1+\delta_{1,\rho})t}).$$

Since $f_1(t)$ is a polynomial, it follows that

$$|f(t)|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu\lambda t}), \quad \forall\lambda \in (0,1) \text{ and } \forall\rho \ge 0. \tag{4.8}$$

Consequently, for any $\varepsilon > 0$, $\rho \ge 0$ and $\lambda \in (0,1)$, applying (4.8) with $(\lambda+1)/2$ replacing λ, it follows that there is $T > 0$ such that

$$|f(t)|_{\rho,\sigma_0} \le \varepsilon e^{-\nu\lambda t} \quad \forall t \ge T. \tag{4.9}$$

(iii) Combining (4.8) for $\rho = 0$, with $f \in L^2(0,T;H)$, we assume, without loss of generality, for each $\lambda \in (0,1)$ that

$$|f(t)| \le M_\lambda e^{-\nu\lambda t}, \quad \forall t \ge 0, \text{ for some } M_\lambda > 0. \tag{4.10}$$

Similarly, the expansion (4.4) implies for any $N \ge 1$ and $\rho \ge 0$ that

$$\left|u(t) - \sum_{n=1}^{N} q_n(t)e^{-\nu nt}\right|_{\rho,\sigma_0} = \mathcal{O}\left(e^{-\nu(N+\varepsilon)t}\right) \quad \text{as } t \to \infty, \forall\varepsilon \in (0,1).$$

Proof. **(i)** It suffices to prove that there exist polynomials q_n's for all $n \ge 1$ such that for each $N \ge 1$, the following properties $(\mathcal{H}_1), (\mathcal{H}_2)$ and $(\mathcal{H}_3)$ hold true:
$(\mathcal{H}_1)$ $q_N \in \mathcal{P}^{\infty,\sigma_0}$,
$(\mathcal{H}_2)$ For $\rho \ge 1/2$,

$$\left|u(t) - \sum_{n=1}^{N} q_n(t)e^{-\nu nt}\right|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu(N+\varepsilon)t}) \quad \text{as } t \to \infty, \forall\varepsilon \in (0, \delta^*_{N,\rho}), \tag{4.11}$$

where the number $\delta^*_{n,\rho}$'s, for $\rho \ge 1/2$, are defined recursively by

$$\delta^*_{n,\rho} = \begin{cases} \delta_{1,\rho} & \text{for } n = 1, \\ \min(\delta_{n,\rho}, \delta^*_{n-1,\rho+3/2}), & \text{for } n \ge 2. \end{cases}$$

$(\mathcal{H}_3)$ The ODE (4.6) holds in E^{∞,σ_0} for $n = N$.

We prove these statements by constructing the polynomials $q_N(t)$'s recursively.
Base case: $N = 1$. Let $k \ge 1$. By taking $w \in R_k H$ in the weak formulation (2.3), we obtain

$$\frac{d}{dt}R_k u + \nu k R_k u = \frac{1}{1+\alpha^2 k} R_k\Big(f(t) - \mathcal{B}(u(t), v(t))\Big) \quad \text{in } R_k H, \text{ for a.e. } t \in (0,\infty). \tag{4.12}$$

Let $w_0(t) = e^{\nu t}u(t)$ and $w_{0,k}(t) = R_k w_0(t)$. Since $u \in C([0,T];H)$, we also have $w_{0,k} \in C([0,T];R_kH)$. It follows from (4.12) that

$$\frac{d}{dt}w_{0,k} + \nu(k-1)w_{0,k} = \frac{1}{1+\alpha^2 k}\Big(R_k f_1 + R_k H_0(t)\Big), \text{ in } R_kH, \text{ for a.e. } t \in (0,\infty), \tag{4.13}$$

where

$$H_0(t) = e^{\nu t}\Big(f(t) - F_1(t) - \mathcal{B}(u(t), v(t))\Big), \tag{4.14}$$

with F_1 is defined in (4.5) and $R_k f_1(t)$ is an R_kH-valued polynomial in t.

Let $\rho \geq 1/2$ be fixed. By using (4.7) for $N = 1$ and applying Corollary 3.1 with the use of (4.9), in fact that (3.22) holds for $\sigma = \sigma_0$, $\rho \geq 1/2$ and $\lambda_0 \in (0,1)$, we have for $\delta = (1-\delta_{1,\rho})/4$ that there are $T_0 > 0$ and $D_0 \geq 1$ such that for $t \geq T_0$,

$$e^{\nu t}|f(t) - F_1(t)|_{\rho,\sigma_0} \leq D_0 e^{-\nu\delta_{1,\rho}t}, \tag{4.15}$$

$$e^{\nu t}|\mathcal{B}(u(t), v(t))|_{\rho,\sigma_0} \leq K^\rho K_0^2(1+\alpha^2)e^{\nu(2\delta-1)t} \leq Je^{-\nu(1+\delta_{1,\rho})t/2} \leq Je^{-\nu\delta_{1,\rho}t}, \tag{4.16}$$

where $J := K^\rho K_0^2(1+\alpha^2)$. Then, by putting $D_1 = \max(D_0, J)$, we have

$$|H_0(t)|_{\rho,\sigma_0} \leq D_1 e^{-\nu\delta_{1,\rho}t}, \quad \forall t \geq T_0. \tag{4.17}$$

We now identify the components of the desired polynomial, $q_1(t)$, belonging to each eigenspace R_kH.

Case $k = 1$: Applying Lemma 3.2 (ii) to equation (4.13) with $X = R_1H, \|\cdot\|_X = |\cdot|_{\rho,\sigma_0}, \beta = 0$,

$$y(t) = w_{0,1}(T_0 + t), \quad p(t) = \frac{1}{1+\alpha^2}R_1 f_1(T_0 + t), \quad g(t) = \frac{1}{1+\alpha^2}R_1 H_0(T_0 + t),$$

we infer that there is an R_1H-valued polynomial $q_{1,1}(t)$ such that for any $t \geq 0$

$$|w_{0,1}(T_0 + t) - q_{1,1}(t)|_{\rho,\sigma_0} \leq \frac{D_1}{\nu\delta_{1,\rho}(1+\alpha^2)}e^{-\nu\delta_{1,\rho}t},$$

hence

$$|R_1 w_0(t) - q_{1,1}(t - T_0)|_{\rho,\sigma_0} \leq \frac{D_1 e^{\nu\delta_{1,\rho}T_0}}{\nu\delta_{1,\rho}(1+\alpha^2)}e^{-\nu\delta_{1,\rho}t}, \quad \forall t \geq T_0, \tag{4.18}$$

and

$$q_{1,1}(t) = R_1 w_0(T_0) + \frac{1}{1+\alpha^2}\left(\int_0^\infty R_1 H_0(T_0 + s)ds + \int_0^t R_1 f_1(T_0 + s)ds\right).$$

Thanks to (4.17), the improper integral $\frac{1}{1+\alpha^2}\int_0^\infty R_1 H_0(T_0 + s)ds$ exists and belongs to R_1H. Thus

$$q_{1,1}(t) = \xi_1 + \frac{1}{1+\alpha^2}\int_0^t R_1 f_1(T_0 + s)ds \quad \text{for some } \xi_1 \in R_1H. \tag{4.19}$$

Case $k \geq 2$: We apply Lemma 3.2 (i) to equation (4.13) with $X = R_kH, \|\cdot\|_X = |\cdot|_{\rho,\sigma_0}$, $\beta = \nu(k-1) > \nu\delta_{1,\rho}$ and

$$y(t) = w_{0,k}(T_0 + t), \quad p(t) = \frac{1}{1+\alpha^2 k}R_k f_1(T_0 + t), \quad g(t) = \frac{1}{1+\alpha^2 k}R_k H_0(T_0 + t).$$

In particular, there is an R_kH-valued polynomial, $q_{1,k}(t)$, such that for all $t \geq T_0$

$$|R_k w_0(t) - q_{1,k}(t - T_0)|_{\rho,\sigma_0} \leq e^{-\nu\delta_{1,\rho}(t-T_0)}\left(|R_k w_0(T_0)|_{\rho,\sigma_0} + |q_{1,k}(0)|_{\rho,\sigma_0} + \frac{D_1}{\nu(k-1-\delta_{1,\rho})(1+\alpha^2)}\right), \tag{4.20}$$

and

$$q_{1,k}(t) = \frac{e^{-\nu(k-1)t}}{1+\alpha^2 k}\int_{-\infty}^t e^{\nu(k-1)s}R_k f_1(T_0 + s)ds, \quad \text{for } k \geq 2. \tag{4.21}$$

We now define the polynomial $q_1(t)$

$$q_1(t) = \sum_{k=1}^\infty q_{1,k}(t - T_0), \quad t \in \mathbb{R}. \tag{4.22}$$

Next, we prove that $q_1 \in \mathcal{P}^{\infty,\sigma_0}$. Write

$$f_1(T_0+t) = \sum_{d=0}^{m} a_d t^d, \quad \text{for some } a_d \in E^{\infty,\sigma_0}.$$

By (4.19), we have that $R_1 q_1(T_0+t) = q_{1,1}(t)$ is a $\mathcal{V}$-valued polynomial, and hence,

$$\text{the mapping } t \mapsto R_1 q_1(T_0+t) \text{ belongs to } \mathcal{P}^{\infty,\sigma_0}. \tag{4.23}$$

We consider the remaining part $(I-R_1)q_1(T_0+t)$. By using the formula (4.1)

$$\begin{aligned}(I-R_1)q_1(T_0+t) &= \sum_{k=2}^{\infty} q_{1,k}(t) = \sum_{k=2}^{\infty} \frac{e^{-\nu(k-1)t}}{1+\alpha^2 k} \int_{-\infty}^{t} e^{\nu(k-1)s} \left(\sum_{d=0}^{m} R_k a_d s^d\right) ds \\ &= \sum_{k=2}^{\infty} \frac{1}{\nu(k-1)(1+\alpha^2 k)} \sum_{d=0}^{m}\sum_{n=0}^{d} \frac{(-1)^{d-n} d!}{n![\nu(k-1)]^{d-n}} t^n R_k a_d \\ &= \sum_{k=2}^{\infty} \frac{1}{\nu(k-1)(1+\alpha^2 k)} \sum_{n=0}^{d} \left(\sum_{d=n}^{m} \frac{(-1)^{d-n} d!}{n![\nu(k-1)]^{d-n}} R_k a_d\right) t^n = \sum_{n=0}^{d} b_n t^n,\end{aligned}$$

where the coefficient $b_n, 0 \le n \le d$ is

$$b_n = \sum_{k=2}^{\infty} \frac{1}{\nu(k-1)(1+\alpha^2 k)} \left(\sum_{d=n}^{m} \frac{(-1)^{d-n} d!}{n![\nu(k-1)]^{d-n}} R_k a_d\right).$$

For any $\theta \ge 0$, we have

$$\begin{aligned}|b_n|^2_{\theta+2,\sigma_0} = |A^2 b_n|^2_{\theta,\sigma_0} &= \frac{1}{\nu^2} \sum_{k=2}^{\infty} \left| \frac{1}{(k-1)(1+\alpha^2 k)} \sum_{d=n}^{m} \frac{(-1)^{d-n} d!}{n![\nu(k-1)]^{d-n}} k^2 . R_k a_d \right|^2_{\theta,\sigma_0} \\ &\le \frac{1}{\nu^2} \sum_{k=2}^{\infty} \left(\frac{k^2}{(k-1)(1+\alpha^2 k)}\right)^2 \left(\sum_{d=n}^{m} \frac{m!}{n!\nu^{d-n}} |R_k a_d|_{\theta,\sigma_0}\right)^2 \\ &\le \frac{16(m!)^2}{\nu^2 (n!)^2 (1+2\alpha^2)^2} \sum_{k=2}^{\infty}\sum_{d=n}^{m} \frac{1}{\nu^{2(d-n)}} \sum_{d=n}^{m} |R_k a_d|^2_{\theta,\sigma_0}.\end{aligned}$$

Thus

$$|b_n|^2_{\theta+2,\sigma_0} \le \frac{16(m!)^2}{\nu^2 (n!)^2 (1+2\alpha^2)^2} \sum_{d=n}^{m} \frac{1}{\nu^{2(d-n)}} \sum_{d=n}^{m} |(I-R_1)a_d|^2_{\theta,\sigma_0} < \infty. \tag{4.24}$$

Hence, $b_n \in E^{\infty,\sigma_0}$ for all $0 \le n \le d$, and the mapping $t \mapsto (I-R_1)q_1(T_0+t)$ belongs to $\mathcal{P}^{\infty,\sigma_0}$. This, together with (4.23) implies that the mapping $t \mapsto q_1(T_0+t)$ belongs to $\mathcal{P}^{\infty,\sigma_0}$ and we obtain that $t \mapsto q_1(t)$ belongs to $\mathcal{P}^{\infty,\sigma_0}$ as well.

Next, we estimate $|u(t) - q_1(t)e^{-\nu t}|_{\rho,\sigma_0}$. From (4.18) we have

$$|R_1(w_0(t) - q_1(t))|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu\delta_{1,\rho} t}). \tag{4.25}$$

Moreover, by (4.20) we deduce that

$$\begin{aligned}\sum_{k=2}^{\infty} |R_k(w_0(t) - q_1(t))|^2_{\rho,\sigma_0} &\le 3e^{2\nu\delta_{1,\rho}T_0} e^{-2\nu\delta_{1,\rho}t} \sum_{k=2}^{\infty} \left(|R_k w_0(T_0)|^2_{\rho,\sigma_0} + |R_k q_1(T_0)|^2_{\rho,\sigma_0} + \frac{D_1^2(1+\alpha^2)^{-2}}{\nu^2(k-1-\delta_{1,\rho})^2}\right) \\ &\le D_2^2 e^{-2\nu\delta_{1,\rho}t},\end{aligned}$$

for all $t \ge T_0$ and

$$D_2^2 = 3e^{2\nu\delta_{1,\rho}T_0} \left(|(I-R_1)w_0(T_0)|^2_{\rho,\sigma_0} + |(I-R_1)q_1(T_0)|^2_{\rho,\sigma_0} + \frac{D_1^2}{\nu^2(1+\alpha^2)^2} \sum_{k=2}^{\infty} \frac{1}{(k-1-\delta_{1,\rho})^2}\right) < \infty.$$

This implies

$$|(I-R_1)(w_0(t) - q_1(t))|_{\rho,\sigma_0} \le D_2 e^{-\nu\delta_{1,\rho}t}, \quad \forall t \ge T_0. \tag{4.26}$$

Combining (4.25) with (4.26) yields

$$|w_0(t) - q_1(t)|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu\delta_{1,\rho}t}),$$

and consequently,

$$|u(t) - q_1(t)e^{-\nu t}|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu(1+\delta_{1,\rho})t}), \tag{4.27}$$

where the polynomial $q_1(t)$ is independent of ρ, thus the same $q_1(t)$ satisfies (4.27) for all $\rho \geq 1/2$, which proves $(\mathcal{H}_2)$ for $N = 1$.

We next establish the ODE (4.6). By Lemma 3.2, we have the polynomial $q_1(t)$ satisfies

$$\frac{d}{dt}(1+\alpha^2 k)R_k q_1(T_0+t) + \nu(k-1)(1+\alpha^2 k)R_k q_1(T_0+t) = R_k f_1(T_0+t), \quad \forall k \geq 1 \text{ and } \forall t \in \mathbb{R}$$

or

$$\frac{d}{dt}(I+\alpha^2 A)R_k q_1(T_0+t) + \nu(k-1)(I+\alpha^2 A)R_k q_1(T_0+t) = R_k f_1(T_0+t), \forall k \geq 1 \text{ and } \forall t \in \mathbb{R}. \tag{4.28}$$

For each $\theta \geq 0$, we have $A^2 q_1(T_0+t)$ and $f_1(T_0+t)$ belong to G_{θ,σ_0}. Hence, we can sum over k in (4.28) and obtain

$$\frac{d}{dt}(q_1(t) + \alpha^2 A q_1(t)) + \nu(A-1)(q_1(t) + \alpha^2 A q_1(t)) = f_1(t) \quad \text{in } G_{\theta,\sigma_0}, \forall t \in \mathbb{R},$$

which implies that the differential equation (4.6) holds in E^{∞,σ_0}. And q_1 satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ for $N = 1$.

Recursive step. Let $N \geq 1$. Suppose that there already exist $q_1, q_2, \cdots, q_N \in \mathcal{P}^{\infty,\sigma_0}$ that satisfies $(\mathcal{H}_2)$ and the ODE (4.6) holds in E^{∞,σ_0} for each $n = 1, 2, \cdots, N$. We will construct a polynomial $q_{N+1}(t)$ that satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ for $n = N+1$.

Let $\rho \geq 1/2$ be given and $\varepsilon^* \in (0, \delta^*_{N+1,\rho})$ arbitrary. Define

$$\overline{u}_N = \sum_{n=1}^{N} u_n \quad \text{and} \quad e_N = u - \overline{u}_N,$$

$$\overline{v}_N = \sum_{n=1}^{N} v_n \quad \text{and} \quad r_N = v - \overline{v}_N,$$

where $v_n = u_n + \alpha^2 A u_n$ and $v = u + \alpha^2 A u$. By assumption $(\mathcal{H}_2)$ we deduce

$$\begin{aligned} |e_N(t)|_{\rho+3/2,\sigma_0} &= \mathcal{O}(e^{-\nu(N+\varepsilon)t}), \quad \forall \varepsilon \in (0, \delta^*_{N,\rho+3/2}), \\ |r_N(t)|_{\rho+1/2,\sigma_0} &= \mathcal{O}(e^{-\nu(N+\varepsilon)t}), \quad \forall \varepsilon \in (0, \delta^*_{N,\rho+3/2}). \end{aligned} \tag{4.29}$$

Subtracting (4.6) for $n = 1, 2, \cdots, N$ from (2.2), we have

$$\frac{d}{dt}(e_N + \alpha^2 A e_N) + \nu A(e_N + \alpha^2 A e_N) + \mathcal{B}(u, v) - \sum_{\substack{1 \leq i,j \\ i+j \leq N}} \mathcal{B}(u_i, v_j) = f - \sum_{n=1}^{N} F_n, \tag{4.30}$$

where the functions F_n's are defined in (4.5). We reformulate (4.30) as

$$\frac{d}{dt}(e_N + \alpha^2 A e_N) + \nu A(e_N + \alpha^2 A e_N) + \sum_{i+j=N+1} \mathcal{B}(u_i, v_j) = F_{N+1} + h_N, \tag{4.31}$$

where

$$\begin{aligned} h_N &= -\mathcal{B}(u, v) + \sum_{\substack{1 \leq i,j \leq N \\ i+j \leq N+1}} \mathcal{B}(u_i, v_j) + f - \sum_{n=1}^{N+1} F_n \\ &= -\Big(\mathcal{B}(u, v) - \mathcal{B}(\overline{u}_N, \overline{v}_N)\Big) - \left(\mathcal{B}(\overline{u}_N, \overline{v}_N) - \sum_{\substack{1 \leq i,j \leq N \\ i+j \leq N+1}} \mathcal{B}(u_i, v_j)\right) + \left(f - \sum_{n=1}^{N+1} F_n\right). \end{aligned}$$

Then we can rewrite h_N as

$$h_N = -\mathcal{B}(e_N, v) - \mathcal{B}(\overline{u}_N, r_N) - \sum_{\substack{1\le i,j\le N\\ i+j\ge N+2}} \mathcal{B}(u_i, v_j) + \overline{F}_{N+1}, \tag{4.32}$$

where

$$\overline{F}_{N+1}(t) = f(t) - \sum_{n=1}^{N+1} F_n(t).$$

By assumption (4.3) we have

$$|\overline{F}_{N+1}(t)|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu(N+1+\delta_{N+1,\rho})t}) = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}). \tag{4.33}$$

And it easy to deduce from (3.34) that for $\delta \in (0,1)$,

$$\begin{aligned}\sum_{\substack{1\le i,j\le N\\ i+j\ge N+2}} |\mathcal{B}(u_i, v_j)|_{\rho,\sigma_0} &= \sum_{\substack{1\le i,j\le N\\ i+j\ge N+2}} e^{-\nu(i+j)t} |\mathcal{B}(q_i, (I+\alpha^2 A)q_j)|_{\rho,\sigma_0} \\ &= \mathcal{O}(e^{-\nu(N+2+2(1-\delta))t}) = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}).\end{aligned} \tag{4.34}$$

Take $\varepsilon \in (\varepsilon^*, \delta^*_{N+1,\rho}) \subset (0, \delta^*_{N,\rho+3/2})$ in (4.29), and set $\delta = \varepsilon - \varepsilon^* \in (0,1)$. Then we have from (3.23) that

$$|\overline{u}_N(t)|_{\rho+3/2,\sigma_0} = \mathcal{O}(e^{-\nu(1-\delta)t}) \quad \text{and} \quad |u(t)|_{\rho+3/2,\sigma_0} = \mathcal{O}(e^{-\nu(1-\delta)t}).$$

Since $v = (I+\alpha^2 A)u$ and $(I+\alpha^2 A)$ is a bounded linear operator from $D(A^{\rho+3/2})$ to $D(A^{\rho+1/2})$, thus, we deduce

$$|\overline{v}_N(t)|_{\rho+1/2,\sigma_0} = \mathcal{O}(e^{-\nu(1-\delta)t}) \quad \text{and} \quad |v(t)|_{\rho+1/2,\sigma_0} = \mathcal{O}(e^{-\nu(1-\delta)t}). \tag{4.35}$$

By Corollary 3.1 and estimates (4.29), (4.35), it follows that

$$\begin{aligned}|\mathcal{B}(e_N, v)|_{\rho,\sigma_0} &= \mathcal{O}(e^{-\nu(N+\varepsilon+1-\delta)t}) = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}),\\ |\mathcal{B}(\overline{u}_N, r_N)|_{\rho,\sigma_0} &= \mathcal{O}(e^{-\nu(N+\varepsilon+1-\delta)t}) = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}).\end{aligned} \tag{4.36}$$

Therefore, by (4.32)-(4.34) and (4.36), we obtain

$$|h_N(t)|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}). \tag{4.37}$$

Next, by using the weak formulation (4.31), and then taking the test function $w \in R_k H$, we obtain

$$\frac{d}{dt}(1+\alpha^2 k)R_k e_N + \nu k(1+\alpha^2 k)R_k e_N + \sum_{i+j=N+1} R_k\mathcal{B}(u_i, v_j) = R_k F_{N+1} + R_k h_N \text{ in } R_k H, \text{ for a.e. } t \in (0,\infty). \tag{4.38}$$

Let $w_N(t) = e^{\nu(N+1)t} e_N(t)$ and $w_{N,k} = R_k w_N(t)$. Then we have from (4.38) that

$$\frac{d}{dt} w_{N,k} + \nu(k-(N+1))w_{N,k} = \frac{1}{1+\alpha^2 k}\Big(R_k f_{N+1} - e^{\nu(N+1)t} \sum_{i+j=N+1} R_k\mathcal{B}(u_i, v_j) + e^{\nu(N+1)t} R_k h_N(t)\Big).$$

From (4.37) we obtain

$$e^{\nu(N+1)t} |R_k h_N(t)|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu\varepsilon^* t}).$$

Then there exists $T_N > 0$ and $D_3 > 0$ such that

$$e^{\nu(N+1)t} |R_k h_N(t)|_{\rho,\sigma_0} \le D_3 e^{-\nu\varepsilon^* t}, \quad \forall t \ge T_N. \tag{4.39}$$

Moreover, since $u \in C([0,T]; H)$, each $w_{N,k}(t)$ is continuous from $[0,\infty)$ to $R_k H$. We apply Lemma 3.2 with $X = R_k H$, $\|\cdot\|_X = |\cdot|_{\rho,\sigma_0}$, constant $\beta = \nu(k-(N+1))$, solution $y(t) = w_{N,k}(T_N+t)$, polynomial

$$p(t) = \frac{1}{1+\alpha^2 k}\left[R_k f_{N+1}(T_N+t) - e^{\nu(N+1)t} \sum_{i+j=N+1} R_k\mathcal{B}\Big(u_i(T_N+t), v_j(T_N+t)\Big)\right],$$

and function $g(t) = \dfrac{1}{1+\alpha^2 k} e^{\nu(N+1)t} R_k h_N(t)$.

Case $k \le N$: We have $\beta < 0$ and (4.29) implies

$$\lim_{t\to\infty}(e^{\beta t} y(t)) = \lim_{t\to\infty}(e^{\beta(t-T_N)} w_{N,k}(t)) = e^{-\beta T_N} \lim_{t\to\infty} e^{\nu k t} R_k e_N(t) = 0.$$

Then, by applying Lemma 3.2 (iii), there is a R_kH-valued polynomial $q_{N+1,k}(t)$ such that

$$|R_k w_N(t) - q_{N+1,k}(t-T_N)|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu\varepsilon^* t}), \quad \forall t \geq T_N. \tag{4.40}$$

Case $k = N+1$: We have $\beta = 0$, and Lemma 3.2 (ii) implies that there is a $R_{N+1}H$-valued polynomial $q_{N+1,N+1}(t)$ such that for any $t \geq T_N$

$$|R_{N+1} w_N(t) - q_{N+1,N+1}(t-T_N)|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu\varepsilon^* t}). \tag{4.41}$$

Case $k \geq N+2$: Then $\beta = \nu(k-(N+1)) > \nu\varepsilon^*$ and by applying Lemma 3.2 (i), there is a R_kH-valued polynomial $q_{N+1,k}(t)$ such that for all $t \geq T_N$

$$\begin{aligned} &|R_k w_N(t) - q_{N+1,k}(t-T_N)|_{\rho,\sigma_0} \\ &\qquad \leq e^{\nu\varepsilon^* T_N} e^{-\nu\varepsilon^* t} \left(|R_k w_N(T_N)|_{\rho,\sigma_0} + |q_{N+1,k}(0)|_{\rho,\sigma_0} + \frac{D_3}{\nu(1+\alpha^2)\big(k-(N+1)-\varepsilon^*\big)} \right). \end{aligned} \tag{4.42}$$

We define

$$q_{N+1}(t) = \sum_{k=1}^{\infty} q_{N+1,k}(t-T_N), \quad t \in \mathbb{R}.$$

It follows from (4.3) that $f_{N+1} \in \mathcal{P}^{\infty,\sigma_0}$ and from the recursive assumptions that $q_i, q_j \in \mathcal{P}^{\infty,\sigma_0}$ for $1 \leq i, j \leq N$, then from (3.6) we obtain

$$f_{N+1} - \sum_{i+j=N+1} \mathcal{B}(q_i, (I+\alpha^2 A)q_j) \in \mathcal{P}^{\infty,\sigma_0}.$$

By repeating the proof that shows $q_1 \in \mathcal{P}^{\infty,\sigma_0}$, we can prove that $q_{N+1} \in \mathcal{P}^{\infty,\sigma_0}$.

Next, we estimate $e_{N+1}(t)$. From (4.41) and (4.40), we deduce

$$|P_{N+1}(w_N(t) - q_{N+1}(t))|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu\varepsilon^* t}). \tag{4.43}$$

Squaring (4.42) and summing over $k \geq N+2$, we obtain for $t \geq T_N$ that

$$\begin{aligned} &|(I-P_{N+1})(w_N(t) - q_{N+1}(t))|_{\rho,\sigma_0} = \sum_{k=N+2}^{\infty} |R_k(w_N(t) - q_{N+1}(t))|^2_{\rho,\sigma_0} \\ &\leq 3e^{2\nu\varepsilon^* T_N} e^{-2\nu\varepsilon^* t} \left(\sum_{k=N+2}^{\infty} |R_k w_N(T_N)|^2_{\rho,\sigma_0} + \sum_{k=N+2}^{\infty} |R_k q_{N+1}(T_N)|^2_{\rho,\sigma_0} + \sum_{k=N+2}^{\infty} \frac{D_3^2(1+\alpha^2)^{-2}}{\nu(k-(N+1)-\varepsilon^*)^2} \right). \end{aligned}$$

Since the last three sums in the inequality above are finite, we obtain

$$|(I-P_{N+1})(w_N(t) - q_{N+1}(t))|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu\varepsilon^* t}). \tag{4.44}$$

From (4.43) and (4.44), we have

$$|w_N(t) - q_{N+1}(t)|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu\varepsilon^* t}). \tag{4.45}$$

Thus

$$|e_{N+1}(t)|_{\rho,\sigma_0} = |e_N(t) - e^{-\nu(N+1)t} q_{N+1}(t)|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}). \tag{4.46}$$

Thanks to (4.45), the polynomial $q_{N+1}(t)$ is independent of ρ and ε^*. Hence, (4.46) holds for any $\rho \geq 1/2$ and $\varepsilon^* \in (0, \delta^*_{N+1})$, which proves $(\mathcal{H}_2)$ with $n = N+1$.

Finally, we establish the ODE (4.6) for $n = N+1$. By Lemma 3.2, we have the polynomial $q_{N+1}(t)$ satisfies

$$\begin{aligned} &\frac{d}{dt}(I+\alpha^2 A)R_k q_{N+1}(T_N+t) + \nu(k-(N+1))(I+\alpha^2 A)R_k q_{N+1}(T_N+t) \\ &= R_k f_{N+1}(T_N+t) - \sum_{i+j=N+1} R_k \mathcal{B}\Big(q_i(T_N+t), (I+\alpha^2 A)q_j(T_N+t)\Big), \quad \forall k \geq 1, t \in \mathbb{R}. \end{aligned}$$

This implies for $k \geq 1$, we have

$$\begin{aligned} &\frac{d}{dt}(R_k u_{N+1}(t) + \alpha^2 A R_k u_{N+1}(t)) + \nu A(R_k u_{N+1}(t) + \alpha^2 A R_k u_{N+1}(t)) \\ &\qquad\qquad + \sum_{i+j=N+1} R_k \mathcal{B}(u_i(t), v_j(t)) = R_k F_{N+1}(t), \quad t \in \mathbb{R}. \end{aligned} \tag{4.47}$$

For any $\theta \geq 0$, since $Av_{N+1}(t)$, $\sum_{i+j=N+1} \mathcal{B}(u_i(t), v_j(t))$ and $F_{N+1}(t)$ belong to G_{θ,σ_0}. Then, we sum over k in (4.47) and obtain

$$\frac{d}{dt}v_{N+1}(t) + \nu A v_{N+1}(t) + \sum_{i+j=N+1} \mathcal{B}(u_i(t), v_j(t)) = F_{N+1}(t), \quad \text{in } G_{\theta,\sigma_0}, \forall t \in \mathbb{R}.$$

Therefore, the ODE (4.6) holds in E^{∞,σ_0} for $n = N+1$. This completes the recursive step and hence we have the completed proof of (i).

(ii) Since $f_1 \in \mathcal{V}$, there exists $N_1 \geq 1$ such that $f_1 \in R_{N_1}H$. From (4.21), we deduce

$$q_{1,k}(t) = 0, \quad \forall k > N_1.$$

Then

$$q_1(t) = \sum_{k=1}^{N_1} q_{1,k}(t - T_0) \in R_{N_1}H.$$

For the recursive step, the functions $f_{N+1} \in \mathcal{V}$ and $q_i, q_j \in \mathcal{V}$ for $1 \leq i, j, \leq N$, then we have

$$f_{N+1} - \sum_{i+j} \mathcal{B}(q_i, (I + \alpha^2 A)q_j) \in \mathcal{V}.$$

Hence $q_{N+1,k} \neq 0$ for at most finitely many k, and the fact that each $q_{N+1,k} \in \mathcal{V}$, we have $q_{N+1}(t)$ is also in $\mathcal{V}$. □

Finally, in the case $f(t)$ only possesses a finite asymptotic approximation, we will prove that the strong solution also admits a finite asymptotic approximation of the same type.

Theorem 4.2. *Suppose there exist an integer $N^* \geq 1$, a nonnegative number σ_0, a real number $\theta^* \geq \rho^* \geq 3N^*/2$ and for any $1 \leq n \leq N^*$, number $\delta_n \in (0,1)$ and polynomials $f_n \in \mathcal{P}^{\theta_n,\sigma_0}$, such that*

$$\left| f(t) - \sum_{n=1}^{N} f_n(t)e^{-\nu nt} \right|_{\rho_N,\sigma_0} = \mathcal{O}(e^{-\nu(N+\delta_N)t}) \quad \text{as } t \to \infty, \tag{4.48}$$

for $1 \leq N \leq N^*$ and

$$\theta_n = \theta^* - \frac{3n-1}{2}, \qquad \rho_n = \rho^* - \frac{3n-1}{2}.$$

Let $u(t)$ be a weak solution of (2.2). Then
(i) There exist polynomials $q_n \in \mathcal{P}^{\theta_n+2,\sigma_0}$, for $1 \leq n \leq N^*$ such that

$$\left| u(t) - \sum_{n=1}^{N} q_n(t)e^{-\nu nt} \right|_{\rho_N,\sigma_0} = \mathcal{O}(e^{-\nu(N+\varepsilon)t}) \quad \text{as } t \to \infty, \forall \varepsilon \in (0, \delta_N^*), \tag{4.49}$$

for $1 \leq N \leq N^*$ and $\delta_N^* = \min\{\delta_1, \delta_2, \ldots, \delta_N\}$. Moreover, the ODEs (4.6) hold in G_{θ_n,σ_0} for $1 \leq n \leq N^*$ where

$$u_n(t) := q_n(t)e^{-\nu nt}, \qquad F_n(t) := f_n(t)e^{-\nu nt}.$$

(ii) If all $f_n(t)$'s belong to $\mathcal{V}$, respectively E^{∞,σ_0}, then so do all $q_n(t)$'s and the ODEs (4.6) hold in $\mathcal{V}$, respectively E^{∞,σ_0}.

Proof. We adapt from the proof of Theorem 4.1 so we only show some necessary modifications. We prove part (i), while part (ii) is similar to the one in Theorem 4.1 so we omitted here.

By (4.48) with $N = 1$

$$e^{\nu t}|f(t) - F_1(t)|_{\rho^*,\sigma_0} = \mathcal{O}(e^{-\delta_{1,\rho^*}t}), \tag{4.50}$$

and

$$|f(t)|_{\rho^*,\sigma_0} \leq |f_1(t)|_{\rho^*,\sigma_0}e^{-\nu t} + |f(t) - f_1(t)e^{-\nu t}|_{\rho^*,\sigma_0} = \mathcal{O}(e^{-\nu\lambda t}), \quad \forall \lambda \in (0,1).$$

Then, by applying Corollary 3.1, we have

$$|\mathcal{B}(u(t), v(t))|_{\rho^*,\sigma_0} = \mathcal{O}(e^{-2\nu(1-\delta)t}), \quad \forall \delta \in (0,1). \tag{4.51}$$

Base case $N = 1$: By using the same arguments in case $N = 1$ of proof of Theorem 4.1 with $\rho = \rho^*$ and $\theta = \theta^*$, we deduce the existence and definition of $q_1(t)$ are similar to (4.19), (4.21) and (4.22). Moreover, since $f_1 \in \mathcal{P}^{\theta,\sigma_0}$, the same proof (see (4.23) and (4.24)) yields $q_1 \in \mathcal{P}^{\theta+2,\sigma_0}$.

If $N^*=1$, then the proof is finished here. We now consider $N^*\ge 2$.

Recursive step: Let $1\le N\le N^*-1$. Assume there already exist $q_n\in\mathcal{P}^{\theta_n,\sigma_0}$ for $1\le n\le N$ such that

$$\begin{aligned}|e_N(t)|_{\theta_N,\sigma_0} &:= \left|u(t)-\sum_{n=1}^N q_n(t)e^{-\nu nt}\right|_{\theta_N,\sigma_0} = \mathcal{O}(e^{-\nu(N+\varepsilon)t}),\quad \forall\varepsilon\in(0,\delta_N^*),\\ |r_N(t)|_{\theta_N-1,\sigma_0} &:= \left|v(t)-\sum_{n=1}^N v_n(t)\right|_{\theta_N-1,\sigma_0} = \mathcal{O}(e^{-\nu(N+\varepsilon)t}),\quad \forall\varepsilon\in(0,\delta_N^*),\end{aligned} \tag{4.52}$$

where $v_n=u_n+\alpha^2Au_n$, $v=u+\alpha^2Au$ and the ODEs (4.6) hold in G_{θ_N,σ_0} for $n=1,2,\cdots,N$.

Put

$$\rho=\rho_{N+1}=\rho_N-3/2\ge 1/2,\quad\text{and}\quad \theta=\theta_{N+1}=\theta_N-3/2\ge 1/2. \tag{4.53}$$

Notice that for $n=1,2,\cdots,N$, we have $\theta_n\ge\rho_n\ge 1/2$ and both θ_n,ρ_n are decreasing. hence

$$u_n(t),q_n(t)\in G_{\theta_n,\sigma_0}\subset G_{\theta_N,\sigma_0}=G_{\theta+3/2,\sigma_0}\subset G_{\rho_N,\sigma_0}=G_{\rho+3/2,\sigma_0},\quad \forall t\in\mathbb{R}. \tag{4.54}$$

Then, from (4.52), we have

$$\begin{aligned}|e_N(t)|_{\rho+3/2,\sigma_0} &= \mathcal{O}(e^{-\nu(N+\varepsilon)t}),\quad \forall\varepsilon\in(0,\delta_N^*),\\ |r_N(t)|_{\rho+1/2,\sigma_0} &= \mathcal{O}(e^{-\nu(N+\varepsilon)t}),\quad \forall\varepsilon\in(0,\delta_N^*).\end{aligned} \tag{4.55}$$

We now construct a polynomial $q_{N+1}\in\mathcal{P}^{\theta+2,\sigma_0}$ such that (4.49) holds true with $n=N+1$ and the ODE (4.6) holds in $G_{\theta_{N+1},\sigma_0}=G_{\theta,\sigma_0}$.

We use the same steps of construction $q_{N+1}(t)$ as in the proof of Theorem (4.1) with the use of ρ,θ in (4.53).

Estimate of function $h_N(t)$ defined by (4.32). Let $\varepsilon^*\in(0,\delta_{N+1}^*)$. By (4.48) with $n=N+1$, we have

$$\left|f(t)-\sum_{n=1}^{N+1}F_n(t)\right|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu(N+1+\delta_{N+1})t}) = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}). \tag{4.56}$$

Thanks to (4.54) and Corollary 3.1, we obtain

$$\begin{aligned}\sum_{\substack{1\le i,j\le N\\ i+j\le N+2}}|\mathcal{B}(u_i,v_j)|_{\rho,\sigma_0} &= \sum_{\substack{1\le i,j\le N\\ i+j\le N+2}} e^{-\nu(i+j)t}|\mathcal{B}(q_i,(I+\alpha^2A)q_j)|_{\rho,\sigma_0}\\ &= \mathcal{O}(e^{-\nu(N+2+2(1-\delta))t}) = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}).\end{aligned} \tag{4.57}$$

Take $\varepsilon\in(\varepsilon^*,\delta_{N+1}^*)\subset(0,\delta_N^*)$ in (4.55) and $\delta=\varepsilon-\varepsilon^*\in(0,1)$, we have

$$\left|\sum_{n=1}^N u_n(t)\right|_{\rho+3/2,\sigma_0} = \left|\sum_{n=1}^N u_n(t)\right|_{\rho_N,\sigma_0} = \mathcal{O}(e^{-\nu(1-\delta)t}), \tag{4.58}$$

and by applying Proposition 3.2, we also have

$$|u(t)|_{\rho+3/2,\sigma_0}\le |u(t)|_{\rho^*+3/2,\sigma_0} = \mathcal{O}(e^{-\nu(1-\delta)t}). \tag{4.59}$$

By Corollary 3.1 and the estimates (4.55), (4.58) and (4.59), we deduce

$$\begin{aligned}|\mathcal{B}(e_N,v)|_{\rho,\sigma_0} &= \mathcal{O}(e^{-\nu(N+\varepsilon+1-\delta)t}) = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}),\\ \left|\mathcal{B}\left(\sum_{n=1}^N u_n, r_N\right)\right|_{\rho,\sigma_0} &= \mathcal{O}(e^{-\nu(N+\varepsilon+1-\delta)t}) = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}),\end{aligned}$$

and then we obtain (4.37) again.

Since $f_{N+1}\in\mathcal{P}^{\theta,\sigma_0}$, from (4.54) and the fact that $q_i,q_j\in\mathcal{P}^{\theta+2,\sigma_0}$ for $1\le i,j\le N$, then we have from (3.5) that

$$f_{N+1}-\sum_{i+j=N+1}\mathcal{B}(q_i,(I+\alpha^2A)q_j)\in\mathcal{P}^{\theta,\sigma_0}.$$

Repeating the same proof as for q_1 implies that $q_{N+1} \in \mathcal{P}^{\theta+2,\sigma_0}$ and

$$\left| u(t) - \sum_{n=1}^{N+1} q_n(t) e^{-\nu n t} \right|_{\rho,\sigma_0} = |e_N(t) - e^{-\nu(N+1)t} q_{N+1}(t)|_{\rho,\sigma_0} = \mathcal{O}(e^{-\nu(N+1+\varepsilon^*)t}).$$

Since this holds for any $\varepsilon^* \in (0, \delta^*_{N+1})$, we deduce (4.49) with $n = N+1$.

We also have the same result for the ODE (4.6) with $n = N+1$, noting that the ODE now holds in G_{θ,σ_0}. And this finishes the recursive step or we have the completed proof. □

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