THE MARGOLIS HOMOLOGY OF THE DICKSON ALGEBRA AND PENGELLEY-SINHA'S CONJECTURE

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Abstract. We completely compute the Margolis homology of the Dickson algebra D_n, i.e. the homology of D_n with the differential to be the Milnor operation Q_j, for every n and j. The motivation for this problem is that, the Margolis homology of the Dickson algebra plays a key role in study of the Morava K-theory $K(j)^*(BS_m)$ of the symmetric group on m letters S_m.

We show that Pengelley-Sinha's conjecture on $H_*(D_n; Q_j)$ for $n \leq j$ is true if and only if $n = 1$ or 2. For $3 \leq n \leq j$, our result proves that this conjecture turns out to be false since the occurrence of some "critical elements" $h_{s_1,\ldots,s_k}$'s of degree $(2^{j+1} - 2^n) + \sum_{i=1}^{k}(2^n - 2^{s_i})$ in this homology for $0 < s_1 < \cdots < s_k < n$ and $k > 1$.

Let $\mathcal{A}$ be the mod 2 Steenrod algebra, genenated by the cohomology operations Sq^j with $j \geq 0$ and subject to the Adem relation with $Sq^0 = 1$. Further $\mathcal{A}$ is a Hopf algebra, whose coproduct is given by the formula $\Delta(Sq^j) = \sum_{i=0}^{n} Sq^i \otimes Sq^{j-i}$.

Let $\mathcal{A}_*$ be the Hopf algebra, which is dual to $\mathcal{A}$. Let $\xi_j = (Sq^{2^j} \cdots Sq^2 Sq^1)^*$ be the Milnor element of degree $2^{j+1} - 1$ in $\mathcal{A}_*$, for $j \geq 0$, where the duality is taken with respect to the admissible basis of $\mathcal{A}$. According to Milnor [4], as an algebra, $\mathcal{A}_* \cong \mathbb{F}_2[\xi_0, \xi_1, \ldots, \xi_j, \ldots]$, the polynomial algebra in infinitely many generators $\xi_0, \xi_1, \ldots, \xi_j, \ldots$.

Let Q_j, for $j \geq 0$, be the Milnor operation (see [4]) of degree $(2^{j+1} - 1)$ in $\mathcal{A}$, which is dual to ξ_j with respect to the basis of $\mathcal{A}_*$ consisting of all monomials in the generators $\xi_0, \xi_1, \ldots, \xi_j, \ldots$. Remarkably, Q_j is a differential, that is $Q_j^2 = 0$ for every j. In fact, $Q_0 = Sq^1$, $Q_j = [Q_{j-1}, Sq^{2^j}]$, the commutator of Q_{j-1} and Sq^{2^j} in the Steenrod algebra $\mathcal{A}$, for $j > 0$.

In the article, we compute the Margolis homology of the Dickson algebra D_n, i.e. the homology of D_n with the differential to be the Milnor operation Q_j.

The real goal that we persue is to compute the Morava K-theory $K(j)^*(BS_m)$ of the symmetric group on m letters S_m. It was well known that, the Milnor operation is the first non-zero differential, $Q_j = d_{2^{j+1}-1}$, in the Atiyah-Hirzebruch spectral sequence for computing $K(j)^*(X)$, the Morava K-theory of a space X. So, the Q_j-homology of $H^*(X)$ is the $E_{2^{j+1}}$-page in the Atiyah-Hirzebruch spectral sequence for $K(j)^*(X)$. (See e.g. Yagita [10, §2], although the fact was well known before this article.)

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[1]This research is funded by the National Foundation for Science and Technology Development (NAFOSTED) of Vietnam, under grant number 101.04-2019.300.

[2]2010 *Mathematics Subject Classification.* Primary 55S05, 55S10, 55N99.

[3]*Key words and phrases.* Steenrod algebra, Milnor operation, Margolis homology, Invariant theory, Dickson algebra.

A key step in the determination of the symmetric group's cohomology is to apply the Quillen restiction from this cohomology to the cohomologies of all elementary abelian subgroups of the symmetric group. For $m = 2^n$ and the "generic" elementary abelian 2-subgroup $(\mathbb{Z}/2)^n$ of the symmetric group S_{2^n}, the image of the restriction $H^*(BS_{2^n}) \to H^*(B(\mathbb{Z}/2)^n)$ is exactly the Dickson algebra D_n (see Mùi [5, Thm II.6.2]). So, the $E_{2^{j+1}}$-page in the Atiyah-Hirzebruch spectral sequence for $K(j)^*(BS_{2^n})$ maps to the Margolis homology $H_*(D_n; Q_j)$. This is why the Margolis homology of the Dickson algebra is taken into account.

Let us study the range n Dickson algebra of invariants

$$D_n = \mathbb{F}_2[x_1, \dots, x_n]^{GL(n,\mathbb{F}_2)},$$

where each generator x_i is of degree 1, and the general linear group $GL(n, \mathbb{F}_2)$ acts canonically on $\mathbb{F}_2[x_1, \dots, x_n]$. Following Dickson [1], let us consider the determinant

$$[e_1, \dots, e_n] = \det \begin{pmatrix} x_1^{2^{e_1}} & \dots & x_n^{2^{e_1}} \\ . & \ddots & . \\ x_1^{2^{e_n}} & \dots & x_n^{2^{e_n}} \end{pmatrix}$$

for non-negative integers $e_1, \dots, e_n$. Then $\omega[e_1, \dots, e_n] = \det(\omega)[e_1, \dots, e_n]$, for $\omega \in GL(n, \mathbb{F}_2)$ (see [1]). Set

$$L_{n,s} = [0, 1, \dots, \widehat{s}, \dots, n], \quad (0 \le s \le n),$$

where $\widehat{s}$ means s being omitted. The Dickson invariant $c_{n,s}$ of degree $2^n - 2^s$ is originally defined as follows:

$$c_{n,s} = L_{n,s}/L_{n,n}, \quad (0 \le s < n).$$

Dickson proved in [1] that D_n is a polynomial algebra

$$D_n = \mathbb{F}_2[c_{n,0}, \dots, c_{n,n-1}].$$

To be explicit, the Dickson invariant can be expressed as in Hưng-Peterson [3, §2]:

$$c_{n,s} = \sum_{i_1 + \cdots i_n = 2^n - 2^s} x_1^{i_1} \cdots x_n^{i_n},$$

where $0 \le s < n$ and the sum is over all sequences $i_1, \dots, i_n$ with i_j either 0 or a power of 2.

For $j \ge n$, we are interested in the following elements of the Dickson algebra D_n:

$$A_{j,n,s} = \begin{cases} [0, \dots, \widehat{s-1} \dots, n-1, j]/L_{n,n}, & \text{for } 0 < s \le n, \\ 0, & \text{for } s = 0. \end{cases}$$

In particular, $A_{j,n,n} = [0, \dots, \dots, n-2, j]/L_{n,n}$.

In this article, when j and n are fixed, the elements $c_{n,s}$ and $A_{j,n,s}$ will respectively be denoted by c_s and A_s for abbreviation.

Lemma 1. *For* $0 \le j$, $0 \le s < n$,

$$Q_j(c_s) = \begin{cases} c_0, & 0 \le j < n-1, \ j = s-1, \\ 0, & 0 \le j < n-1, \ j \ne s-1, \\ c_0 c_s, & j = n-1, \ 0 \le s < n, \\ c_0\big(c_s A_n^2 + A_s^2\big), & 0 \le s < n \le j. \end{cases}$$

The action of the Steenrod algebra on the Dickson one is basically computed in [2]. Related and partial results concerning the lemma can be seen in [7], [9], [8].

The next two theorems are stated in Sinha [6]. Their proofs are straightforward from Lemma 1.

Theorem 2. *For $0 \le j < n-1$,*

$$H_*(D_n, Q_j) \cong \mathbb{F}_2[c_{j+1}^2] \otimes \mathbb{F}_2[c_1, \dots, \widehat{c}_{j+1}, \dots, c_{n-1}],$$

where $\widehat{c}_{j+1}$ means c_{j+1} being omitted.

Let $\mathbb{F}_2[c_1, \dots, c_{n-1}]_{\text{ev}}$ be the $\mathbb{F}_2$-submodule of $\mathbb{F}_2[c_1, \dots, c_{n-1}]$ generated by all the monomials $c_1^{i_1} \cdots c_{n-1}^{i_{n-1}}$ with $i_1 + \cdots + i_{n-1}$ even.

Theorem 3.

$$H_*(D_n; Q_{n-1}) \cong \mathbb{F}_2[c_1, \dots, c_{n-1}]_{ev}.$$

Proposition 4. *For $0 \le s_1, \dots, s_k < n \le j$,*

$$Q_j(c_{s_1} \cdots c_{s_k}) = \begin{cases} c_0\big(\sum_{i=1}^k (c_{s_1} \dots \widehat{c}_{s_i} \dots c_{s_k}) A_{s_i}^2 \big), & k \text{ even,} \\ c_0\big(c_{s_1} \cdots c_{s_k} A_n^2 + \sum_{i=1}^k (c_{s_1} \dots \widehat{c}_{s_i} \dots c_{s_k}) A_{s_i}^2 \big), & k \text{ odd.} \end{cases}$$

Here, $\widehat{c}_{s_i}$ means c_{s_i} being omitted.

Conjecture 5. (D. Pengelley - D. Sinha, see [6]) For $n \le j$,

$$H_*(D_n; Q_j) \cong D_n^2 / \big(Q_j(c_0), Q_j(c_0 c_1) \dots, Q_j(c_0 c_{n-1}) \big).$$

Let D_n^{odd} be the $\mathbb{F}_2$-submodule of D_n spanned by all monomials $c_0^{i_0} \cdots c_{n-1}^{i_{n-1}}$ with at least one of the powers $i_0, \dots, i_{n-1}$ odd. Note clearly that D_n^{odd} is not a Q_j-submodule of D_n, but $\text{Im} Q_j \cap D_n^{\text{odd}}$ is, since Q_j vanishes on this module.

Pengelley-Sinha's conjecture is equivalent to the equality:

$$\text{Ker} Q_j = \big(\text{Im} Q_j \cap D_n^{\text{odd}} \big) \oplus D_n^2.$$

In other words, there is no class in $H_*(D_n; Q_j)$ represented by an element in D_n^{odd}.

The following two theorems show that Pengelley-Sinha's conjecture is true for $n = 1$ or 2 and every j.

Theorem 6. *For $n = 1$, $0 \le j$,*

$$H_*(D_1; Q_j) \cong \mathbb{F}_2[c_0^2]/(c_0^{2^{j+1}}).$$

In particular, $H_*(D_1; Q_0) = \mathbb{F}_2$ (this is also a special case of Theorem 3), $H_*(D_1; Q_1) = \Lambda(c_0^2)$, where $\Lambda(c_0^2)$ denotes the $\mathbb{F}_2$-exterior algebra on c_0^2.

Set $\overline{\Lambda(c_0^2)} = \Lambda(c_0^2)/(\mathbb{F}_2 \cdot 1)$.

Theorem 7. *For $n = 2$,*

$$H_*(D_2; Q_j) \cong \begin{cases} \mathbb{F}_2[c_1^2], & \text{for } j = 0, 1, \\ \overline{\Lambda(c_0^2)} \oplus \mathbb{F}_2[c_1^2], & \text{for } j = 2, \\ \mathbb{F}_2[c_0^2, c_1^2]/(c_0^2 A_1^2, c_0^2 A_2^2), & \text{for } j > 2, \end{cases}$$

where $A_1 = (x_1^2 x_2^{2^j} + x_1^{2^j} x_2^2)/(x_1 x_2^2 + x_1^2 x_2)$, $A_2 = (x_1 x_2^{2^j} + x_1^{2^j} x_2)/(x_1 x_2^2 + x_1^2 x_2)$.

The cases $j = 0, 1$ in the previous theorem are special cases of Theorems 2 and 3.

Proposition 8. *Pengelley-Sinha's Conjecture for $n \le j$ is true if and only if $1 \le n \le 2$.*

How can we adjust Pengelley-Sinha's conjecture to make a correct one in the problem for $3 \leq n \leq j$?

The critical elements $h_{s_1,\dots,s_k}$'s defined below in the Margolis homology of the Dickson algebra D_n, for $0 < s_1 < \cdots < s_k < n$ and $1 < k$, are the main ingredient in our correction of Pengelley-Sinha's conjecture for $3 \leq n \leq j$.

Definition 9. The critical element is defined as follows

$$h_{s_1,\dots,s_k} = \begin{cases} \sum_{i=1}^k (c_{s_1} \dots \widehat{c}_{s_i} \dots c_{s_k}) A_{s_i}^2, & k \text{ odd}, \\ c_{s_1} \cdots c_{s_k} A_n^2 + \sum_{i=1}^k (c_{s_1} \dots \widehat{c}_{s_i} \dots c_{s_k}) A_{s_i}^2, & k \text{ even}, \end{cases}$$

for $s_1, \dots, s_k$ pairwise distinct, with $0 \leq s_1, \dots, s_k < n$, $3 \leq n \leq j$.

It should be noted that, $h_{s_1,\dots,s_k} \in D_n^{\mathrm{odd}}$ if $k > 1$, for $s_1, \dots, s_k$ pairwise distinct, with $0 \leq s_1, \dots, s_k < n$, and that $h_{s_1,\dots,s_k}$ depends also on n and j. Further, if $s_1, \dots, s_k$ are non-zero, then c_0^2 divides $Q_j(c_0 c_{s_1} \cdots c_{s_k})$ in D_n, and

$$h_{s_1,\dots,s_k} = \frac{1}{c_0^2} Q_j(c_0 c_{s_1} \cdots c_{s_k}).$$

Q_j is a (total) derivation, that is $Q_j(ab) = Q_j(a)b + aQ_j(b)$. We study the s-th partial derivation for $0 \leq s \leq n$, and its "inverse", the so-called integral on a direction. These notions will play key roles in the remaining part of the article.

Definition 10. Let $s_1, \dots, s_k$ be pairwise distinct, with $0 \leq s_1, \dots, s_k < n$, and $R \in D_n$. The s-th partial derivation is defined for $0 \leq s \leq n$ as follows:

$$\partial_s(c_{s_1} \cdots c_{s_k} R^2) = \begin{cases} c_0 c_{s_1} \cdots c_{s_k} A_n^2 R^2, & k \text{ odd}, s = n, \\ c_0 c_{s_1} \cdots \widehat{c}_{s_i} \cdots c_{s_k} A_{s_i}^2 R^2, & s = s_i, \\ 0, & \text{otherwise}. \end{cases}$$

From definition, $A_0 = 0$, it implies $\partial_0 = 0$. Note that, if $\partial_s(c_{s_1} \cdots c_{s_k}) \neq 0$, then s should be one of the indices $s_1, \dots, s_k$ or n.

By Proposition 4, the following is true, not depending on whether k odd or even.

Lemma 11. *Let $s_1, \dots, s_k$ be pairwise distinct, with $0 \leq s_1, \dots, s_k < n$, and $R \in D_n$. Then*

$$Q_j(c_{s_1} \cdots c_{s_k} R^2) = \sum_{s=1}^n \partial_s(c_{s_1} \cdots c_{s_k}) R^2.$$

Definition 12. Let $s_1, \dots, s_k$ be pairwise distinct, with $0 \leq s_1, \dots, s_k < n$. The integral on the r-th direction is defined for $0 < r \leq n$ and $R \in D_n$ as follows:

(i) $$I_r(c_0 c_{s_1} \cdots c_{s_k} R^2) = \begin{cases} c_{s_1} \cdots c_{s_k} \frac{R^2}{A_n^2}, & k \text{ odd}, r = n, \ A_n^2 | R^2, \\ c_{s_1} \cdots c_{s_k} c_r \frac{R^2}{A_r^2}, & r \neq s_1, \dots, s_k, n, \ A_r^2 | R^2, \\ 0, & \text{otherwise}. \end{cases}$$

(ii) For $\min\{s_1, \dots, s_k\} > 0$,

$$I_r(c_{s_1} \cdots c_{s_k} R^2) = 0.$$

Lemma 13. *Let $s_1, \dots, s_k$ be pairwise distinct, with $0 \leq s_1, \dots, s_k < n$, $0 < s \leq n$, and $R \in D_n$. Then*

(i) $$I_s \partial_s(c_{s_1} \cdots c_{s_k} R^2) = \begin{cases} c_{s_1} \cdots c_{s_k} R^2, & \textit{either } k \textit{ odd, } s = n, \textit{ or } s \in \{s_1, \dots, s_l\}, \\ 0, & \textit{otherwise}. \end{cases}$$

(ii) $\partial_s I_s(c_0 c_{s_1}\cdots c_{s_k}R^2) = \begin{cases} c_0c_{s_1}\cdots c_{s_k}R^2, & \textit{if } I_s(c_0s_1\cdots c_{s_k}R^2)\neq 0,\\ 0, & \textit{otherwise.}\end{cases}$

Theorem 14. *For* $3 \le n \le j$,

$$KerQ_j \cap D_n^{odd} = \big(ImQ_j \cap D_n^{odd}\big) \bigoplus_{\substack{0<s_1<\cdots<s_k<n\\ 1<k}} h_{s_1,\ldots,s_k}\mathbb{F}_2[c_1^2,\ldots,c_{n-1}^2].$$

Here is an explanation for the occurrence of the critical elements in the homology.

If $P \in D_n^{\rm odd}$, then P is a sum of some terms of the form $c_{t_1}\cdots c_{t_\ell}R^2$, for $\ell > 0$, $t_1,\ldots,t_\ell$ are pairwise distinct, $0 \le t_1,...,t_\ell < n$, and R is a Dickson monomial.

Suppose further $P \in D_n^{\rm odd}$ and $Q_j(P) = 0$. Let $c_{t_1}\cdots c_{t_\ell}R^2$ be a Dickson monomial of P, where $t_1,\ldots,t_\ell$ are pairwise distinct. In other words,

$$P = c_{t_1}\cdots c_{t_\ell}R^2 + \text{others},$$

where others mean a sum of some other Dickson monomials. So

$$Q_j(P) \quad = \quad Q_j(c_{t_1}\cdots c_{t_\ell}R^2) + Q_j(\text{others}).$$

To kill $Q_j(c_{t_1}\cdots c_{t_\ell}R^2)$ so that $Q_j(P) = 0$, the polynomial P should also contain $I_rQ_j(c_{t_1}\cdots c_{t_\ell})R^2 = \sum_{s=1}^n I_r\partial_s(c_{t_1}\cdots c_{t_\ell})R^2$ for some r, such that

$$I_r\partial_s(c_{t_1}\cdots c_{t_\ell})R^2 \neq c_{t_1}\cdots c_{t_\ell}R^2,$$

for every s. (See Definitions 10 and 12.) Recall that, if $\partial_s(c_{t_1}\cdots c_{t_\ell}) \neq 0$, then s should be one of the indices $t_1,\ldots,t_\ell$ or n (by Definition 10). The inequality $I_r\partial_s(c_{t_1}\cdots c_{t_\ell})R^2 \neq c_{t_1}\cdots c_{t_\ell}R^2$, for every s, means that $r \neq t_1,\ldots,t_\ell$ and n (by Lemma 13).

Let us take an index $r \notin \{t_1,\ldots,t_\ell,n\}$. By Proposition 4, we have

$$\begin{aligned} & I_rQ_j(c_{t_1}\cdots c_{t_\ell})R^2 \\ = \quad & \begin{cases} I_r\big\{c_0\big(c_{t_1}\cdots c_{t_\ell}A_n^2 + \sum_{i=1}^\ell (c_{t_1}\ldots\widehat{c}_{t_i}\ldots c_{t_\ell})A_{t_i}^2\big)\big\}R^2, & \ell \text{ odd},\\ I_r\big\{c_0\big(\sum_{i=1}^\ell (c_{t_1}\ldots\widehat{c}_{t_i}\ldots c_{t_\ell})A_{t_i}^2\big)\big\}R^2, & \ell \text{ even}.\end{cases}\end{aligned}$$

Consider the 2 cases of either ℓ odd or ℓ even, and we get

$$\big(id + I_rQ_j\big)(c_{t_1}\cdots c_{t_\ell})R^2 = h_{r,t_1,\ldots,t_\ell}\frac{R^2}{A_r^2}.$$

Now the indices $r, t_1,\ldots,t_\ell$ are re-denoted and ordered by $s_1,\ldots,s_k$ with $k = \ell+1$ and $0 \le s_1 < \cdots < s_k < n$. In the two cases of either ℓ being odd or even, we have

$$\big(id + I_rQ_j\big)(c_{t_1}\cdots c_{t_\ell})R^2 = h_{r,t_1,\ldots,t_\ell}\frac{R^2}{A_r^2} = h_{s_1,\ldots,s_k}\frac{R^2}{A_r^2}.$$

If $s_1 = 0$, then we get

$$\begin{aligned}\big(id + I_rQ_j\big)(c_{t_1}\cdots c_{t_\ell})R^2 \quad &= \quad h_{0,s_2\ldots,s_k}\frac{R^2}{A_r^2}\\ &= \quad Q_j(c_{s_2}\cdots c_{s_k})\frac{R^2}{A_r^2} \in \text{Im}Q_j.\end{aligned}$$

So, it suffices to consider the case of $0 < s_1 < \cdots < s_k < n$.

If R is divisible by c_0 in D_n, then R^2 is divisible by c_0^2 in D_n. We have

$$\begin{aligned} h_{s_1,\ldots,s_k}\frac{R^2}{A_r^2} &= c_0^2 h_{s_1,\ldots,s_k}\frac{R^2}{c_0^2A_r^2} \\ &= Q_j(c_0c_{s_1}\cdots c_{s_k})\frac{R^2}{c_0^2A_r^2} \in \mathrm{Im}Q_j. \end{aligned}$$

If R is not divisible by c_0 in D_n, then so is $h_{s_1,\ldots,s_k}\frac{R^2}{A_r^2}$. By Proposition 4, the latter element is not in the image of Q_j.

The above argument has shown that if $P \in \mathrm{Ker}Q_j \cap D_n^{\mathrm{odd}}$, then either $P \in \mathrm{Im}Q_j \cap D_n^{\mathrm{odd}}$ or P is in the space spanned by $h_{s_1,\ldots,s_k}\mathbb{F}_2[c_1^2,\ldots,c_{n-1}^2]$, for $0 < s_1 < \cdots < s_k < n$ and $k = \ell+1 > 1$.

The following theorem is a consequence of the preceding one and the equalities:

$$\begin{aligned} Q_j(c_0) &= c_0^2A_n^2, \\ Q_j(c_0c_s) &= c_0^2A_s^2, \ \ (0 < s < n). \end{aligned}$$

Theorem 15. *For* $3 \le n \le j$,

$$H_*(D_n;Q_j) = \frac{D_n^2}{(c_0^2A_1^2,\ldots,c_0^2A_n^2)} \bigoplus_{\substack{0<s_1<\cdots<s_k<n \\ 1<k}} h_{s_1,\ldots,s_k}\mathbb{F}_2[c_1^2,\ldots,c_{n-1}^2].$$

Example 16. For $j = n \ge 3$, we have $A_s = c_{s-1}$ for $0 < s < n$, $A_n = c_{n-1}$. So the critical element, which depends also on n and j, is explicitly given by

$$h_{s_1,\ldots,s_k} = \begin{cases} \sum_{i=1}^k (c_{s_1}\ldots\widehat{c}_{s_i}\ldots c_{s_k})c_{s_i-1}^2, & k \text{ odd}, \\ c_{s_1}\cdots c_{s_k}c_{n-1}^2 + \sum_{i=1}^k (c_{s_1}\ldots\widehat{c}_{s_i}\ldots c_{s_k})c_{s_i-1}^2, & k \text{ even}, \end{cases}$$

for $0 \le s_1 < \cdots < s_k < n$ and $1 < k$, where $c_{-1} = 0$ by convention. Therefore

$$\begin{aligned} H_*(D_n;Q_j) &= \frac{D_n^2}{(c_0^4, c_0^2c_1^2,\ldots,c_0^2c_{n-1}^2)} \bigoplus_{\substack{0<s_1<\cdots<s_k<n \\ 1<k}} h_{s_1,\ldots,s_k}\mathbb{F}_2[c_1^2,\ldots,c_{n-1}^2] \\ &= \overline{\Lambda(c_0^2)} \bigoplus \mathbb{F}_2[c_1^2,\ldots,c_{n-1}^2] \bigoplus_{\substack{0<s_1<\cdots<s_k<n \\ 1<k}} h_{s_1,\ldots,s_k}\mathbb{F}_2[c_1^2,\ldots,c_{n-1}^2]. \end{aligned}$$

The contains of this note will be published in detail elsewhere.

Acknowledgment. This research was carried out when the author visited the Vietnam Institute for Advanced Study in Mathematics (VIASM), Hanoi, in the academic year 2019 - 20. He would like to express his warmest thank to the VIASM for the hospitality and for the wonderful working condition.

The author hearty thanks Dev Sinha for an interesting discussion on the subject during the Vietnam-US Mathematical joint Metting, Quynhon June 10-13, 2019. He is deeply indebted to H. Miller, D. Ravenel, L. Schwartz, and S. Wilson for generous discussions on the goal of the problem and the references of this article.

The author was partially funded by the National Foundation for Science and Technology Development (NAFOSTED) of Vietnam under grant number 101.04-2019.300.

References

[1] L. E. Dickson, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. **12** (1911), 75-98.

[2] Nguyễn H. V. Hưng, *The action of the Steenrod squares on the modular invariants of linear groups*, Proc. Amer. Math. Soc. **113** (1991), 1097–1104. MR1064904

[3] Nguyễn H. V. Hưng and F. P. Peterson, *Spherical classes and the Dickson algebra*, Math. Proc. Camb. Phil. Soc. **124** (1998), 253-264.

[4] J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. **67** (1958), 150–171.

[5] Huỳnh Mùi, *Modular invariant theory and the cohomology algebras of symmetric group*, J. Frac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), 319-369.

[6] D. P. Sinha, *Cohomology of symmetric groups*, Lecture on the Vietnam-US Mathematical joint Metting, Quynhon June 10-13, 2019, https://pages.uoregon.edu/dps/VNUS2019/symmetric6.pdf

[7] L. Smith and R. Switzer, *Realizability and nonrealizability of Dickson algebras as cohomology rings*, Proc. Amer. Math. Soc. **89** (1983), 303-313.

[8] N. Sum, *The action of the primitive Steenrod-Milnor operations on the modular invariants*, Proc. of the Inter. School and Conf. in Alg. Top., Hanoi 2004, Geom. Topol. Monogr., Coventry, vol. **11**, 2007, pp. 349-367.

[9] C. Wilkerson, *A primer on the Dickson invariants*, Contemporary Mathematics, Amer. Math. Soc., Providence, R.I., vol. **19**, 1983, pp. 421-434.

[10] N. Yagita, *On the Steenrod algebra of Morava K-theory*, J. London Math. Soc. (2), **22** (1980), 423-438.

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