SOME REMARKS ON THE CEGRELL'S CLASS $\mathcal{F}$

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ABSTRACT. In this paper, we study the near-boundary behavior of functions $u \in \mathcal{F}(\Omega)$ in the case where Ω is strictly pseudoconvex. We also introduce a sufficient condition for belonging to $\mathcal{F}$ in the case where Ω is the unit ball.

INTRODUCTION

Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. By [Ceg04], the class $\mathcal{F}(\Omega)$ is defined as the following: $u \in \mathcal{F}(\Omega)$ iff there exists a sequence of functions $u_j \in \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ as $j \to \infty$ and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$. Here

$$\mathcal{E}_0(\Omega) = \{u \in PSH(\Omega) \cap L^\infty(\Omega) : \lim_{z \to \partial\Omega} u(z) = 0, \int\limits_\Omega (dd^c u)^n < \infty\}.$$

The class $\mathcal{F}(\Omega)$ has many nice properties. This is a subclass of the domain of definition of Monge-Ampère operator [Ceg04, Blo06]. Moreover, by [Ceg04], for each sequence of functions $u_j \in \mathcal{E}_0(\Omega)$ such that $u_j \searrow u \in \mathcal{F}(\Omega)$ as $j \to \infty$, we have

$$\lim_{j \to \infty} \int\limits_\Omega (dd^c u_j)^n = \int\limits_\Omega (dd^c u)^n.$$

By [Ceg98, Ceg04], for every pluripolar set $E \subset \Omega$, there exists $u \in \mathcal{F}(\Omega)$ such that $E \subset \{u = -\infty\}$. In [Ceg04], Cegrell also proved some inequalities, a generalized comparison principle and a decomposition of $(dd^c u)^n, u \in \mathcal{F}(\Omega)$. In [NP09], Nguyen and Pham proved a strong version of comparison principle in the class $\mathcal{F}(\Omega)$.

The class $\mathcal{F}(\Omega)$ has been used to characterize the boundary behavior in the Dirichlet problem for Monge-Ampère equation [Ceg04, Aha07]. For every $u \in \mathcal{F}(\Omega)$, for each $z \in \partial\Omega$, we have $\limsup\limits_{\Omega \ni \xi \to z} u(\xi) = 0$ (see [Aha07]). Moreover, if we define by $\mathcal{N}$ the set of functions in the domain of definition of Monge-Ampère operator with the smallest maximal plurisubharmonic majorant identically zero then, by the comparison principles in $\mathcal{F}$ and in $\mathcal{N}$ (see [NP09] and [ACCP09]) and by Cegrell's approximation theorem [Ceg04], we have

$$\mathcal{F}(\Omega) = \{u \in \mathcal{N}(\Omega) : \int\limits_\Omega (dd^c u)^n < \infty\}.$$

In this paper, we study the near-boundary behavior of functions $u \in \mathcal{F}(\Omega)$ in the case where Ω is a bounded strictly pseudoconvex domain, i.e., there exists $\rho \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $\rho|_{\partial\Omega} = 0$, $D\rho|_{\partial\Omega} \neq 0$ and $dd^c\rho \geq c\omega := cdd^c|z|^2$ for some $c > 0$.

Our first main result is the following:

Theorem 1. *Assume that Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}(\Omega)$. Then, there exists $C > 0$ depending only on Ω, n and u such that*

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$$Vol_{2n}(\{z \in \Omega | d(z, \partial\Omega) < d, u(z) < -\epsilon\}) \leq \frac{C.d^{n+1-na}}{(1-a)a^n\epsilon^n}, \tag{1}$$

for any $0 < \epsilon, a < 1$ *and* $d > 0$.

For the convenience, we denote $W_d = \{z \in \Omega | d(z, \partial\Omega) < d\}$. By Theorem 1, we have

$$\lim_{d\to 0} \frac{Vol_{2n}(\{z \in W_d | u(z) < -\epsilon\})}{d^t} = 0,$$

for every $0 < t < n+1$. It helps us to estimate the "density" of the the set $\{u < -\epsilon\}$ near the boundary.

Moreover, by using Theorem 1 for $\epsilon = d^\alpha$ and $0 < a < 1 - \alpha$, we have

Corollary 2. *Assume that* Ω *is a strictly pseudoconvex domain in* $\mathbb{C}^n$ *and* $u \in \mathcal{F}(\Omega)$. *Then, for every* $0 < \alpha < 1$,

$$\lim_{d\to 0} \frac{Vol_{2n}(\{z \in W_d | u(z) < -d^\alpha\})}{d} = 0.$$

When Ω is the unit ball, this result can be improved as following:

Theorem 3. *If* $u \in \mathcal{F}(\mathbb{B}^{2n})$ *then*

$$\lim_{r\to 1^-} \frac{\int_{\{|z|=r\}} |u(z)| d\sigma(z)}{1-r} < \infty.$$

In particular, there exists $C > 0$ *such that*

$$\limsup_{d\to 0^+} \frac{Vol_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1-d, u(z) < -Ad\})}{d} < \frac{C}{A},$$

for every $A > 0$.

Our second purpose is to find a sharp sufficient condition for u to belong to $\mathcal{F}(\Omega)$ based on the near-boundary behavior of u. We are interested in the following question:

Question 4. *Let* Ω *be a bounded strictly pseudoconvex domain. Assume that* u *is a negative plurisubharmonic function in* Ω *satisfying*

$$\lim_{d\to 0^+} \frac{Vol_{2n}(\{z \in W_d : u(z) < -Ad\})}{d} = 0,$$

for some $A > 0$. *Then, do we have* $u \in \mathcal{F}(\Omega)$*?*

In this paper, we answer this question for the case where Ω is the unit ball.

Theorem 5. *Let* $u \in PSH^-(\mathbb{B}^{2n})$. *Assume that there exists* $A > 0$ *such that*

$$\lim_{d\to 0^+} \frac{Vol_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1-d, u(z) < -Ad\})}{d} = 0. \tag{2}$$

Then $u \in \mathcal{F}(\mathbb{B}^{2n})$.

Corollary 6. *Let* $u \in \mathcal{N}(\mathbb{B}^{2n})$ *such that* $\int_{\mathbb{B}^{2n}} (dd^c u)^n = \infty$. *Then, for every* $A > 0$,

$$\limsup_{d\to 0^+} \frac{Vol_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1-d, u(z) < -Ad\})}{d} > 0.$$

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1. Proof of Theorem 1

Since Ω is bounded strictly pseudoconvex, there exists $\rho \in C^2(\bar{\Omega}, [-1, 0])$ such that $\Omega = \{z : \rho(z) < 0\}$ and

$$|D\rho| > C_1 \text{ in } \bar{\Omega}, \tag{3}$$

and

$$dd^c\rho \geq C_2 dd^c|z|^2 = C_2\omega, \tag{4}$$

where $C_1, C_2 > 0$ are constants.

By (3), there exist $C_3, C_4 > 0$ depending only on Ω and ρ such that

$$C_3 d(z, \partial\Omega) \leq -\rho(z) \leq C_4 d(z, \partial\Omega), \tag{5}$$

for every $z \in \Omega$.

For every $a \in (0, 1)$ and $z \in \Omega$, we have

$$dd^c\rho_a(z) := dd^c(-(-\rho(z))^a) = a(1-a)(-\rho)^{a-2} d\rho \wedge d^c\rho + a(-\rho)^{a-1} dd^c\rho.$$

Then

$$(dd^c\rho_a)^n \geq a^n(1-a)(-\rho)^{na-n-1} d\rho \wedge d^c\rho \wedge (dd^c\rho)^{n-1}. \tag{6}$$

Hence, by (3), (4) and (5), there exists $1 \gg d_0 > 0$ depending only on Ω and ρ such that, for every $0 < d < d_0$ and $z \in W_d := \{\xi \in \Omega : d(\xi, \partial\Omega) < d\}$,

$$(dd^c\rho_a)^n \geq C_5(1-a)a^n d^{na-n-1}\omega^n, \tag{7}$$

where $C_5 > 0$ depends only on n and ρ.

Since $u \in \mathcal{F}(\Omega)$, there exists $\{u_j\}_{j=1}^\infty \subset \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ and

$$\int\limits_\Omega (dd^c u_j)^n < C_6, \tag{8}$$

for every $j \in \mathbb{Z}^+$, where $C_6 > 0$ depends only on u. By using (7), (8) and the Bedford-Taylor comparison principle [BT76, BT82] (see also [Kli91]), we have, for every $j \in \mathbb{Z}^+$, $\epsilon, d > 0$ and $a \in (0, 1)$,

$$\begin{aligned} C_6 > \int\limits_{\{u_j < \epsilon\rho_a\}} (dd^c u_j)^n &\geq \int\limits_{\{u_j < \epsilon\rho_a\}} (dd^c \epsilon\rho_a)^n \\ &\geq \frac{C_5(1-a)a^n\epsilon^n}{d^{n+1-na}} \int\limits_{\{u_j < \epsilon\rho_a\} \cap W_d} \omega^n. \end{aligned}$$

Hence, for every $0 < d < d_0$,

$$Vol_{2n}(\{z \in W_d | u_j(z) < -\epsilon\}) \leq \frac{C_7.d^{n+1-na}}{(1-a)a^n\epsilon^n},$$

where $C_7 > 0$ depends only on Ω, ρ, n and u. Letting $j \to \infty$, we get

$$Vol_{2n}(\{z \in W_d | u(z) < -\epsilon\}) \leq \frac{C_7.d^{n+1-na}}{(1-a)a^n\epsilon^n},$$

for every $0 < d < d_0$. By setting

$$C = \max\{C_7, \frac{Vol_{2n}(\Omega)}{d_0^{n+1}}\},$$

we have

$$Vol_{2n}(\{z \in W_d | u(z) < -\epsilon\}) \leq \frac{C.d^{n+1-na}}{(1-a)a^n\epsilon^n},$$

for every $d > 0$. This completes the proof of Theorem 1.

2. Proof of Theorem 3

In order to prove Theorem 3, we need the following lemma:

Lemma 7. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and (X, d, μ) be a totally bounded metric probability space. Let $u : \Omega \times X \to [-\infty, 0)$ such that*

(i) *For every $a \in X$, $u(\cdot, a) \in \mathcal{F}(\Omega)$ and*
$$\int_\Omega (dd^c u(z, a))^n < M,$$
where $M > 0$ is a constant.

(ii) *For every $z \in \Omega$, the function $u(z, \cdot)$ is upper semicontinuous in X.*

Then $\tilde{u}(z) = \int_X u(z, a) d\mu(a) \in \mathcal{F}(\Omega)$. Moreover

$$\int_\Omega (dd^c \tilde{u})^n \leq M.$$

Proof. It is well known that either $\tilde{u} \in PSH^-(\Omega)$ or $\tilde{u} \equiv -\infty$ (see, for example, [Kli91, Theorem 2.6.5]). We need to find a sequence of functions $\tilde{u}_j \in \mathcal{F}(\Omega)$ such that $\tilde{u}_j$ is decreasing to $\tilde{u}$ as $j \to \infty$ and $\sup_j \int_\Omega (dd^c \tilde{u}_j)^n \leq M$.

Since X is totally bounded, there exists a finite cover $\{X_k\}_{k=1}^{m_1}$ of X such that the diameter of each X_k is at most $1/2$. Denote

$$U_{1,1} = X_1, U_{1,2} = X_2 \setminus X_1, ..., U_{1,m_1} = X_{m_1} \setminus (\cup_{l=1}^{m_1 - 1} X_l).$$

Then $\{U_{1,k}\}_{k=1}^{m_1}$ is a finite cover of X such that its elements are pairwise disjoint and of diameter at most $1/2$. Moreover, $U_{1,k}$ is totally bounded for every k. By using induction, for every $j \in \mathbb{Z}^+$, we can divide X into a finite pairwise disjoint collection $\{U_{j,k}\}_{k=1}^{m_j}$ of sets of diameter at most 2^{-j} satisfying: for every $1 \leq k \leq m_{j+1}$, there exists $1 \leq l \leq m_j$ such that $U_{j+1,k} \subset U_{j,l}$.

For every $j \in \mathbb{Z}^+$, we define

$$u_j(z) = \sum_{k=1}^{m_j} \mu(U_{j,k}) \sup_{a \in U_{j,k}} u(z, a) \qquad \text{and} \qquad \tilde{u}_j = (u_j)^*.$$

Then $\tilde{u}_j \in \mathcal{F}(\Omega)$. Moreover, by using the comparison principle [NP09, Proposition 3.1] for $\tilde{u}_j$ and $\sum_{k=1}^{m_j} \mu(U_{j,k}) u(z, a_k)$ (with $a_k \in U_{j,k}$) and by applying [Ceg04, Corollary 5.6], we have

$$\begin{aligned}
&\int\limits_\Omega (dd^c\tilde{u}_j)^n \le \int\limits_\Omega (dd^c(\sum_{k=1}^{m_j}\mu(U_{j,k})u(z,a_k)))^n \\
&= \sum_{k_1+...+k_{m_j}=n} \frac{n!}{k_1!...k_{m_j}!}\prod_{l=1}^{m_j}\mu(U_{j,l})^{k_l}\int\limits_\Omega (dd^c u(z,a_1))^{k_1}\wedge ... \wedge (dd^c u(z,a_{m_j}))^{k_{m_j}} \\
&\le \sum_{k_1+...+k_{m_j}=n} \frac{n!}{k_1!...k_{m_j}!}\prod_{l=1}^{m_j}\mu(U_{j,l})^{k_l}\prod_{l=1}^{m_j}(\int\limits_\Omega (dd^c u(z,a_l))^n)^{k_l/n} \\
&\le M\sum_{k_1+...+k_{m_j}=n} \frac{n!}{k_1!...k_{m_j}!}\prod_{l=1}^{m_j}\mu(U_{j,l})^{k_l} \\
&= M(\mu(U_{j,1})+...+\mu(U_{j,k_{m_j}}))^n \\
&= M,
\end{aligned}$$

for all $j\in\mathbb{Z}^+$.

We will show that $\tilde{u}_j$ is decreasing to $\tilde{u}$ and use Lemma 8 to prove that $\tilde{u}\in\mathcal{F}(\Omega)$.

For every $z\in\Omega, a\in X$ and $j\in\mathbb{Z}^+$, we define

$$\phi_j(z,a)=\sum_{k=1}^{m_j}\chi_{U_{j,k}}(a)\sup_{a\in U_{j,k}}u(z,a)=\sup_{\xi\in U_{j,k(j,a)}}u(z,\xi),$$

where $\chi_{U_{j,k}}$ is the characteristic function of $U_{j,k}$ and $k(j,a)$ is given by $a\in U_{j,k(j,a)}$. Then, we have

$$u_j(z)=\int\limits_X \phi_j(z,a)d\mu(a)\ge\int\limits_X u(z,a)d\mu(a)=\tilde{u}(z), \tag{9}$$

for every $z\in\Omega$ and $j\in\mathbb{Z}^+$.

Note that $U_{j+1,k(j+1,a)}\cap U_{j,k(j,a)}\ne\emptyset$. Then, it follows from the construction of the sets $U_{j,k}$ that $U_{j+1,k(j+1,a)}\subset U_{j,k(j,a)}$. Hence

$$u_j(z)=\int\limits_X \phi_j(z,a)d\mu(a)\ge\int\limits_X \phi_{j+1}(z,a)d\mu(a)=u_{j+1}(z), \tag{10}$$

for every $z\in$ and $j\in\mathbb{Z}^+$.

By the semicontinuity of $u(z,\cdot)$, we have,

$$u(z,a)\ge\lim_{j\to\infty}(\sup\{u(z,\xi):|\xi-a|\le 2^{-j}\})\ge\lim_{j\to\infty}\phi_j(z,a), \tag{11}$$

for every $z\in\Omega$ and $a\in X$. By integrating the sides of (11) with respect to a and using Fatou's lemma, we get

$$\tilde{u}(z)\ge\lim_{j\to\infty}u_j(z), \tag{12}$$

for every $z\in\Omega$.

Combining (9), (10) and (12), we get that u_j is decreasing to $\tilde{u}$ as $j\to\infty$. Note that $u_j=\tilde{u}_j$ almost everywhere [Kli91, Proposition 2.6.2], and then $\lim\limits_{j\to\infty}\tilde{u}_j=\tilde{u}$ almost everywhere. Since $\lim\limits_{j\to\infty}\tilde{u}_j$ is either plurisubharmonic or identically $-\infty$, we have $\lim\limits_{j\to\infty}\tilde{u}_j=\tilde{u}$ everywhere. Therefore, $\tilde{u}_j$ is decreasing to $\tilde{u}$ as $j\to\infty$.

By Lemma 8, $\max\{\tilde{u}, -k\} \in \mathcal{F}(\Omega)$ for $k > 0$ and it implies that $\tilde{u}$ is not identically $-\infty$. Then, by using Lemma 8 for $\tilde{u}$, we get that $\tilde{u} \in \mathcal{F}(\Omega)$. Moreover, since the sequence $\tilde{u}_j$ is decreasing, we have

$$\int_\Omega (dd^c\tilde{u})^n \leq \liminf_{j\to\infty} \int_\Omega (dd^c\tilde{u}_j)^n \leq M.$$

□

Lemma 8. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and $u \in PSH^-(\Omega)$. Assume that there are $u_j \in \mathcal{F}(\Omega)$, $j \in \mathbb{N}$, such that u_j converges almost everywhere to u as $j \to \infty$. If $\sup_{j>0} \int_\Omega (dd^c u_j)^n < \infty$ then $u \in \mathcal{F}(\Omega)$.*

Lemma 8 is an immediate corollary of [NP09, Theorem 3.7]. It also can be proved by using [Ceg04, Proposition 5.1].

Recall that if u is a radial plurisubharmonic function then $u(z) = \chi(\log|z|)$ for some convex, increasing function χ. We have the following lemma:

Lemma 9. *Let $u = \chi(\log|z|)$ be a radial plurisubharmonic function in $\mathbb{B}^{2n}$. Then, $u \in \mathcal{F}(\mathbb{B}^{2n})$ iff the following conditions hold*

(i) $\lim\limits_{t\to 0^-} \chi(t) = 0$;

(ii) $\lim\limits_{t\to 0^-} \dfrac{\chi(t)}{t} < \infty$.

Proof. By Theorem 1, the condition (i) is a necessary condition for $u \in \mathcal{F}(\mathbb{B}^{2n})$. We need to show that, when (i) is satisfied, the condition $u \in \mathcal{F}(\mathbb{B}^{2n})$ is equivalent to (ii).

If (ii) is satisfied then there exists $k_0 \gg 1$ such that $k_0 t < \chi(t)$. Hence $u(z) > k_0 \log|z| \in \mathcal{F}(\mathbb{B}^{2n})$. Thus, $u \in \mathcal{F}(\mathbb{B}^{2n})$.

Conversely, if (ii) is not satisfied, we consider the functions $u_k = \max\{u, k\log|z|\}$. Then, for every k, $u_k > u$ near $\partial\mathbb{B}^{2n}$. Hence

$$\int_\Omega (dd^c u)^n \geq \int_\Omega (dd^c u_k)^n = k^n \int_\Omega (dd^c \log|z|)^n \stackrel{k\to\infty}{\longrightarrow} \infty.$$

Thus $u \notin \mathcal{F}(\mathbb{B}^{2n})$.

The proof is completed. □

Proof of Theorem 3. Denote by μ the unique invariant probability measure on the unitary group $U(n)$. For every $z \in \mathbb{B}^{2n}$, we define

$$\tilde{u}(z) = \int_{U(n)} u(\phi(z))d\mu(\phi) = \frac{1}{c_{2n-1}|z|^{2n-1}} \int_{\{|w|=|z|\}} u(w)d\sigma(w),$$

where c_{2n-1} is the $(2n-1)$-dimensional volume of $\partial\mathbb{B}^{2n}$. By Lemma 7, we have $\tilde{u} \in \mathcal{F}(\mathbb{B}^{2n})$. Since $\tilde{u}$ is radial, we have, by Lemma 9,

$$\lim_{|z|\to 1^-} \frac{\tilde{u}(z)}{|z|-1} = \lim_{|z|\to 1^-} \frac{\tilde{u}(z)}{\log|z|} < \infty.$$

Hence

$$\lim_{r\to 1^-} \frac{\int_{\{|z|=r\}} |u(z)|d\sigma(z)}{1-r} = M < \infty.$$

Consequently, we have, for $0 < d \ll 1$,

$$Vol_{2n-1}(\{z \in \mathbb{B}^{2n} : \|z\| = 1 - d, u(z) < -Ad\}) \leq \frac{M+1}{A}, \tag{13}$$

for all $A > 0$. Note that

$$Vol_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1 - d, u(z) < -Ad\}) = \int_0^d Vol_{2n-1}(\{z \in \mathbb{B}^{2n} : \|z\| = 1 - t, u(z) < -Ad\})dt.$$

Hence, by (13), we have, for $0 < d \ll 1$,

$$Vol_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1 - d, u(z) < -Ad\}) \leq \int_0^d \frac{(M+1)}{Ad/t} dt = \frac{(M+1)d}{2A}.$$

Thus we get the last assertion of Theorem 3.

The proof is completed. □

3. Proof of Theorem 5

We will find a sequence of functions $u_j \in \mathcal{F}(\mathbb{B}^{2n})$ such that $\sup_{j\geq 0} \int_\Omega (dd^c u_j)^n < \infty$ and u_j converges almost everywhere to u as $j \to \infty$. Then, by using Lemma 8, we will obtain $u \in \mathcal{F}(\mathbb{B}^{2n})$.

For every $0 < a < 1$, we denote $S_a = \{\phi \in U(n) : \|\phi - Id\| < a\}$.

For every $0 < \epsilon, a < 1$ and $z \in \mathbb{B}^{2n}_{1-\epsilon} := \{w \in \mathbb{C}^n : \|w\| < 1 - \epsilon\}$, we define

$$u_{a,\epsilon}(z) = (\sup\{u((1+r)\phi(z)) : \phi \in S_a, 0 \leq r \leq \epsilon\})^*.$$

Then $u_{a,\epsilon}$ is plurisubharmonic in $\mathbb{B}^{2n}_{1-\epsilon}$ (see [Kli91, Corollary 2.9.5] and [Kli91, Theorem 2.9.14]) and, by the semicontinuity of u, we have

$$\lim_{\max a,\epsilon \to 0^+} u_{a,\epsilon}(z) = u(z), \tag{14}$$

for every $z \in \mathbb{B}^{2n}$. Moreover, for $z \neq 0$,

$$u_{a,\epsilon}(z) = (\sup\{u(\xi) : \xi \in B_{a,\epsilon,z}\})^*, \tag{15}$$

where

$$\begin{aligned} B_{a,\epsilon,z} &= \{\xi \in \mathbb{C}^n : \|\frac{z}{\|z\|} - \frac{\xi}{\|\xi\|}\| < a, \|z\| \leq \|\xi\| \leq (1+\epsilon)\|z\|\} \\ &= \{t\xi : t \in [\|z\|, (1+\epsilon)\|z\|], \xi \in \partial\mathbb{B}^{2n}, \|\xi - \frac{z}{\|z\|}\| < a\}. \end{aligned}$$

Denote

$$S_{z/\|z\|,a} = \{\xi \in \mathbb{C}^n : \|\xi\| = 1, \|\xi - \frac{z}{\|z\|}\| < a\}.$$

We have

$$\begin{aligned} Vol_{2n}(B_{a,\epsilon,z}) = \int_{S_{z/\|z\|,a}} \int_{\|z\|}^{(1+\epsilon)\|z\|} t dt dS(\xi) &= \frac{(2\epsilon + \epsilon^2)\|z\|^2}{2} \int_{S_{z/\|z\|,a}} dS(\xi) \\ &= \frac{(2\epsilon + \epsilon^2)\|z\|^2}{2} \int_{S_{(0,\dots,0,1),a}} dS(\xi), \end{aligned}$$

the last equality holds since the volume of hypersurfaces are preserved under rotations.

We will show that, for every $\epsilon_a \geq 3\epsilon \geq 1 - \|z\| \geq \epsilon > 0$,

$$u_{a,\epsilon}(z) \geq -3A\epsilon. \tag{16}$$

Consider the parameterization

$$p : \mathbb{B}^{2n-1} \to \partial\mathbb{B}^{2n} \cap \{z \in \mathbb{C}^n = \mathbb{R}^{2n} : y_n > 0\}$$
$$s = (s_1, ..., s_{2n-1}) \longmapsto p(s) = (s, \sqrt{1 - s^2}).$$

For each $s \in \mathbb{B}^{2n-1}$, we consider the angle α between the vectors $e_{2n} = (0, ..., 0, 1)$ and $p(s)$. We have

$$\sin(\frac{\alpha}{2}) = \frac{\|e_{2n} - p(s)\|}{2} \qquad \text{and} \qquad \sin(\alpha) = \|s\|.$$

Hence,

$$\|s\| = \|e_{2n} - p(s)\| \sqrt{1 - \frac{\|e_{2n} - p(s)\|^2}{4}}.$$

Then $p(\mathbb{B}^{2n-1}_{a\sqrt{1-a^2/4}}) = S_{e_{2n},a}$ and we have

$$\begin{aligned} Vol_{2n}(B_{a,\epsilon,z}) &= \frac{(2\epsilon + \epsilon^2)\|z\|^2}{2} \int\limits_{S_{e_{2n},a}} dS(\xi) \\ &= \frac{(2\epsilon + \epsilon^2)\|z\|^2}{2} \int\limits_{\mathbb{B}^{2n-1}_{a\sqrt{1-a^2/4}}} \sqrt{1 + \|\nabla\sqrt{1 - \|\xi\|^2}\|^2} d\xi \\ &= \frac{(2\epsilon + \epsilon^2)\|z\|^2}{2} \int\limits_{\mathbb{B}^{2n-1}_{a\sqrt{1-a^2/4}}} \frac{d\xi}{\sqrt{1 - \|\xi\|^2}}. \end{aligned}$$

Therefore, there exist $C_1, C_2 > 0$ such that

$$C_1 a^{2n-1}\epsilon < Vol_{2n}(B_{a,\epsilon,z}) < C_2 a^{2n-1}\epsilon, \tag{17}$$

for every $0 < \epsilon, a < 1/2$ and $1/2 < \|z\| \leq 1 - \epsilon$.

By (2), for every $1/2 > a > 0$, there exists $a > \epsilon_a > 0$ such that, for every $\epsilon_a \geq 3\epsilon > 0$,

$$Vol\{\xi \in \mathbb{B}^{2n} : \|\xi\| > 1 - 3\epsilon, u(\xi) < -3A\epsilon\} < C_1 a^{2n-1}\epsilon,$$

and therefore, by (17), for every $3\epsilon \geq 1 - \|z\| \geq \epsilon$,

$$B_{a,\epsilon,z} \not\subseteq \{\xi \in \mathbb{B}^{2n} : \|\xi\| > 1 - 3\epsilon, u(\xi) < -3A\epsilon\}.$$

Then, by (15), for every $\epsilon_a \geq 3\epsilon \geq 1 - \|z\| \geq \epsilon > 0$, we have

$$u_{a,\epsilon}(z) \geq -3A\epsilon. \tag{18}$$

For each $1/2 > a > 0$ and $\epsilon_a \geq 3\epsilon > 0$, we consider the following function

$$\tilde{u}_{a,\epsilon}(z) = \begin{cases} 3A(-1 + |z|^2) & \text{if} \quad 1 - \epsilon \leq \|z\| \leq 1, \\ \max\{3A(-1 + |z|^2), u_{a,\epsilon}(z) - 6A\epsilon\} & \text{if} \quad 1 - 3\epsilon \leq \|z\| \leq 1 - \epsilon, \\ u_{a,\epsilon}(z) - 6A\epsilon & \text{if} \quad \|z\| \leq 1 - 3\epsilon. \end{cases}$$

By using the gluing theorem (see, for example, [Kli91, Corollary 2.9.15]), we have $\tilde{u}_{a,\epsilon} \in PSH(\mathbb{B}^{2n})$. For $m > 0$, we set $\tilde{u}^m_{a,\epsilon} = \max\{\tilde{u}_{a,\epsilon}, -m\}$. Then, we have $\tilde{u}^m_{a,\epsilon} \searrow \tilde{u}_{a,\epsilon}$, when $m \to \infty$. Moreover, since $\tilde{u}^m_{a,\epsilon} = 3A(-1+|z|^2)$ near $\partial\mathbb{B}^{2n}$, we have,

$$\int_{\mathbb{B}^{2n}} (dd^c \tilde{u}^m_{a,\epsilon})^n = \int_{\mathbb{B}^{2n}} (dd^c 3A(-1+|z|^2))^n < \infty,$$

for every $m > 0$. Then $\tilde{u}^m_{a,\epsilon} \in \mathcal{E}_0(\mathbb{B}^{2n})$. Therefore, $\tilde{u}_{a,\epsilon} \in \mathcal{F}(\mathbb{B}^{2n})$. Moreover, by [Ceg04, Proposition 5.1],

$$\int_{\mathbb{B}^{2n}} (dd^c \tilde{u}_{a,\epsilon})^n = \lim_{m\to\infty} \int_{\mathbb{B}^{2n}} (dd^c \tilde{u}^m_{a,\epsilon})^n < \infty, \tag{19}$$

for every $1/2 > a > 0$ and $\epsilon_a \geq 3\epsilon > 0$.

For every $j \in \mathbb{N}$, we denote $u_j = \tilde{u}_{2^{-j},3^{-1}\epsilon_{2^{-j}}}$. By (14), we have u_j converges pointwise to u as j tends to ∞. By (19), we have $\sup_j \int_{\mathbb{B}^{2n}} (dd^c u_j)^n < \infty$. Then, by using Lemma 8, we have $u \in \mathcal{F}(\mathbb{B}^{2n})$.

The proof is completed.

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