DEPENDENCE OF HILBERT COEFFICIENTS

LE XUAN DUNG
Department of Natural Science, Hong Duc University
307 Le Lai, Thanh Hoa, Vietnam
E-mail: lxdung27@gmail.com

and

LE TUAN HOA
Vietnam Institute for Advanced Study in Mathematics
7th Floor, Ta Quang Buu Library, in HUST's campus
No. 1, Dai Co Viet Str., Hanoi, Vietnam
E-mail: lthoa@math.ac.vn

Abstract. Let M be a finitely generated module of dimension d and depth t over a Noetherian local ring $(A, \mathfrak{m})$ and I an $\mathfrak{m}$-primary ideal. In the main result it is showed that the last t Hilbert coefficients $e_{d-t+1}(I,M), ..., e_d(I,M)$ are bounded below and above in terms of the first $d-t+1$ Hilbert coefficients $e_0(I,M), ..., e_{d-t}(I,M)$ and d.

Introduction

Let M be a finitely generated module of dimension d over a Noetherian local ring $(A, \mathfrak{m})$ and I an $\mathfrak{m}$-primary ideal. The Hilbert-Samuel function $H_{I,M}(n) = \ell(M/I^{n+1}M)$ agrees with the Hilbert-Samuel polynomial $P_{I,M}(n)$ for $n \gg 0$ and we may write

$$P_{I,M}(n) = e_0(I,M)\binom{n+d}{d} - e_1(I,M)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I,M).$$

The numbers $e_0(I,M), e_1(I,M), ..., e_d(I,M)$ are called Hilbert coefficients of M with respect to I.

The Hilbert-Samuel function and the Hilbert-Samuel polynomial give a lot of information on M. Therefore it is of interest to study properties of Hilbert coefficients. Assume that A is a Cohen-Macaulay ring and M is a Cohen-Macaulay A-module. Then Northcott [11] and Narita [10] showed that $e_1(I,A) \geq 0$ and $e_2(I,A) \geq 0$, respectively. Note that already $e_3(I,A)$ maybe negative. Later, Rhodes [12] showed that the above results also hold for good I-filtrations of submodules of M. Moreover, Kirby and Mehran [7] were able to show that $e_1(I,M) \leq \binom{e_0(I,M)}{2}$ and $e_2(I,M) \leq \binom{e_1(I,M)}{2}$. Subsequently these results were improved by several authors.

Both authors were partially supported by NAFOSTED (Vietnam)
2000 Mathematics Subject Classification: Primary 13D45, 13A30
Key words and phrases: Castelnuovo-Mumford regularity, associated graded module, good filtration, reduction number.

How about other coefficients? In 1997, Srinivas and Trivedi [16] and Trivedi [17] obtained a surprising result by showing that all $|e_i(I,A)|, i \geq 1$, are bounded by a function depending only on $e_0(I,A)$ and d.

What happens for non-Cohen-Macaulay modules? Inspired by the above mentioned result of Srinivas and Trivedi and of Trivedi [18], Rossi-Trung-Valla [14] showed that all $|e_i(I,A)|$, are bounded by functions depending on the so-called extended degree $\mathrm{Deg}(I,A)$ and d. These results were extended to modules in [9] and [6]. However from these results one can not deduce further relations between Hilbert coefficients.

Using a bound on the Castelnuovo-Mumford regularity in terms of Hilbert coefficients given in [17, Theorem 2] one can immediately see that $(-1)^{i-1}e_i(I,A)$ is bounded above by a (complicated and not explicit) function depending only on $e_0(I,A), ..., e_{i-1}(I,A)$ and i for all i. An explicit bound will be given in Theorem 3.1. However, even in the case $d=2$ and $t=1$ it was shown in [15] that $|e_1(I,A)|$ is in general not bounded in terms of $e_0(I,A)$. So it is natural to ask how many Hilbert coefficients are enough to take such that they completely bound the absolute values of all other? The main result of this paper is to show that the first $d-t+1$ Hilbert-coefficients have this property, where $t = \operatorname{depth} M$ (see Theorem 3.8 and Corollary 3.9). From that we can show that there is only a finite number of Hilbert-Samuel functions such that $e_0(I,M), e_1(I,M), ..., e_{d-t}(I,M)$ and d are fixed, see Theorem 3.10.

In fact, we will do with a more general situation, namely with good I-filtrations $\mathbb{M}$. In this case our bounds also involve the so-called reduction number $r(\mathbb{M})$. Our approach is somewhat similar to that of [16, 17] and [14] in the sense that we use the Castelnuovo-Mumford regularity $\mathrm{reg}(G(\mathbb{M}))$ of the associated module $G(\mathbb{M})$ of $\mathbb{M}$ to bound the Hilbert coefficients, see Proposition 2.7. Then one has to bound $\mathrm{reg}(G(\mathbb{M}))$ in terms of the first $d-t+1$ Hilbert coefficients. In order to do that, in Section 1 we first give a bound on $\mathrm{reg}(G(\mathbb{M}))$ which depends on the length of certain cohomology modules, see Theorem 1.2. Then in Section 2, using [17, Theorem 2] we will give a bound for $\mathrm{reg}(G(\mathbb{M}))$ in terms of all Hilbert coefficients, see Theorem 2.6. Combining the two bounds in Section 1 and Section 2, we will show in the last section that already the first $d-t+1$ Hilbert coefficients are enough to bound $\mathrm{reg}(G(\mathbb{M}))$, see Theorem 3.7. Then one can quickly deduce the main results (Theorem 3.8 and Corollary 3.9). Finally, we would like to remark that bounds established in this paper are huge functions. Therefore instead of seeking better bounds we are looking for more compact formulas. In any case the main meaning of the bounds is not their values, but the fact that the last t Hilbert coefficients are bounded by the first $d-t+1$ ones.

1. Castelnuovo-Mumford regularity and local cohomology modules

First let us recall some notations and definitions. Let $(A,\mathfrak{m})$ be a Noetherian local ring with an infinite residue field $K := A/\mathfrak{m}$ and M a finitely generated A-module. (Although the assumption K being infinite is not essential, because we can tensor A with $K(t)$.) Given a proper ideal I. A chain of submodules

$$\mathbb{M}: \ M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

is called an I-*filtration* of M if $IM_i \subseteq M_{i+1}$ for all i, and a *good I-filtration* if $IM_i = M_{i+1}$ for all sufficiently large i. A module M with a filtration is called a *filtered module* (see [2, III 2.1]). If N is a submodule of M, then by the Artin-Rees Lemma, the sequence $\{N \cap M_n\}$ is a good I-filtration of N and we will denote it by $\mathbb{M} \cap N$. The sequence $\{M_n + N/N\}$ is a good I-filtration of M/N and will be denoted by $\mathbb{M}/N$.

In this paper we always assume that I is an $\mathfrak{m}$-primary ideal and $\mathbb{M}$ is a good I-filtration. The *associated graded module* to the filtration $\mathbb{M}$ is defined by

$$G(\mathbb{M}) = \bigoplus_{n\geq 0} M_n/M_{n+1}.$$

We also say that $G(\mathbb{M})$ is the associated ring of the filtered module M. This is a finitely generated graded module over the standard graded ring $G := G_I(A) := \oplus_{n\geq 0} I^n/I^{n+1}$ (see [2, Proposition III 3.3]). In the particular case, when $\mathbb{M}$ is the I-adic filtration $\{I^nM\}$, $G(\mathbb{M})$ is just the usual associated graded module $G_I(M)$.

Let $R = \oplus_{n\geq 0} R_n$ be a Noetherian standard graded ring over a local Artinian ring $(R_0, \mathfrak{m}_0)$. Let E be a finitely generated graded R-module of dimension d. For $0 \leq i \leq d$, put

$$a_i(E) = \sup\{n|\ H^i_{R_+}(E)_n \neq 0\},$$

where $R_+ = \oplus_{n>0} R_n$. The *Castelnuovo-Mumford regularity* of E is defined by

$$\operatorname{reg}(E) = \max\{a_i(E) + i \mid 0 \leq i \leq d\},$$

and the *Castelnuovo-Mumford regularity of E at and above level l*, $0 \leq l \leq d$, is defined by

$$\operatorname{reg}^l(E) = \max\{a_i(E) + i \mid l \leq i \leq d\}.$$

We call

$$r(\mathbb{M}) = \min\{r \geq 0 \mid M_{n+1} = IM_n \ \text{ for all } \ n \geq r\}$$

the reduction number of $\mathbb{M}$ (w.r.t. I). In the sequel we denote $M/H^0_{\mathfrak{m}}(M)$ by $\overline{M}$ and the filtration $\mathbb{M}/H^0_{\mathfrak{m}}(M)$ of $\overline{M}$ by $\overline{\mathbb{M}}$. Then we have

Lemma 1.1. ([6, Lemma 1.9]) $\operatorname{reg}(G(\mathbb{M})) \leq \max\{\operatorname{reg}(G(\overline{\mathbb{M}}));\ r_I(\mathbb{M})\} + \ell(H^0_{\mathfrak{m}}(M))$.

An element $x \in I$ is called $\mathbb{M}$-superficial element for I if there exists a non-negative integer c such that $(M_{n+1} :_M a) \cap M_c = M_n$ for every $n \geq c$ and we say that a sequence of elements $x_1, ..., x_r$ is an $\mathbb{M}$-superficial sequence for I if, for $i = 1, 2, ..., r$, x_i is an $\mathbb{M}/(x_1, ..., x_{i-1})M$-superficial sequence for I (see [13, Section 1.2]).

For a finitely generated module M, let

$$h^0(M) = \ell(H^0_{\mathfrak{m}}(M)).$$

The following theorem is similar to [6, Theorem 1.5]. The new point here is that we use the length of local homology modules instead of the so-called "extended degree". Its proof is similar to that of [6, Theorem 1.5] and [8, Theorem 4.4]. Hence we will omit some details.

Theorem 1.2. *Let M be a finitely generated A-module with $\dim M = d \geq 1$, $\mathbb{M} = \{M_n\}_{n\geq 0}$ a good I-filtration of M and $x_1, ..., x_d$ an $\mathbb{M}$-superficial sequence for I. Set $B(I, M) = \ell(M/(x_1, ..., x_d)M)$ and*

$$\kappa(I, M) = \max\{h^0(M/(x_1, ..., x_i)M) | 0 \leq i \leq d - 1\}.$$

Then

(i) $\operatorname{reg}(G(\mathbb{M})) \le B(I,M)+\kappa(I,M)+r(\mathbb{M})-1$ if $d=1$,

(ii) $\operatorname{reg}(G(\mathbb{M})) \le [B(I,M)+\kappa(I,M)+r(\mathbb{M})+1]^{3(d-1)!-1}-d$ if $d\ge 2$.

Proof. Let $B := B(I,M), \kappa := \kappa(I,M)$ and $r := r(\mathbb{M})$.

If $d=1$, then $\overline{M}$ is a Cohen-Macaulay module. By [8, Lemma 2.2]

$$\begin{aligned}\operatorname{reg}(G(\overline{\mathbb{M}})) &\le e_0(G(\overline{\mathbb{M}}))+r(\overline{\mathbb{M}})-1\\ &\le e_0(I,M)+r-1 \le B+r-1.\end{aligned}$$

Hence, by Lemma 1.1,

$$\operatorname{reg}(G(\mathbb{M})) \le \max\{\operatorname{reg}(G(\overline{\mathbb{M}}));\ r\}+h^0(M) \le B+\kappa+r-1.$$

Assume that $d\ge 2$. Set $N=\overline{M}/x_1\overline{M}$ and $\mathbb{N}=\overline{\mathbb{M}}/x_1\overline{M}$. Let $m\ge \max\{\operatorname{reg}(G(\mathbb{N})), r\}$. Then

$$\operatorname{reg}^1(G(\overline{\mathbb{M}})/x_1^*G(\overline{\mathbb{M}})) = \operatorname{reg}^1(G(\mathbb{N})),$$

and by [8, Theorem 2.7]

$$\operatorname{reg}^1(G(\overline{\mathbb{M}})) \le m+P_{G(\overline{\mathbb{M}})}(m).$$

By [6, Lemma 1.6] and [6, Lemma 1.7(i)]

$$\begin{aligned}P_{G(\overline{\mathbb{M}})(m)} &\le H_{I,N}(m) \le \tbinom{m+d-1}{d-1}\ell(N/(x_2,...,x_d)N)\\ &\le B\tbinom{m+d-1}{d-1}.\end{aligned}$$

Hence, by Lemma 1.1

$$\begin{aligned}\operatorname{reg}(G(\mathbb{M})) &\le m+h^0(M)+B\binom{m+d-1}{d-1}\\ &\le m+\kappa+B\binom{m+d-1}{d-1}. \qquad (1)\end{aligned}$$

If $d=2$, by the induction hypothesis one can take $m=B+\kappa+r-1$ and get

$$\begin{aligned}\operatorname{reg}(G(\mathbb{M})) &\le m+\kappa+B(m+1)\\ &= B(m+1)+B+2\kappa+r-1\\ &\le B(B+\kappa+r)+B+2\kappa+r-1\\ &< (B+\kappa+r+1)^2-1.\end{aligned}$$

Assume $d\ge 3$. Then for all $m>1$

$$m+B\binom{m+d-2}{d-1} < B(m+1)^{d-1}. \qquad (2)$$

By the induction hypothesis one can take

$$m=(B+\kappa+r+1)^{3(d-2)!-1}-d+1>1.$$

From (1) and (2) we get

$$\begin{aligned}\operatorname{reg}(G(\mathbb{M})) &\le \kappa+B(m+1)^{d-1}-1\\ &\le \kappa+B[(B+\kappa+r+1)^{3(d-2)!-1}-d+2]^{d-1}-1\\ &\le (B+\kappa)(B+\kappa+r+1)^{3(d-1)!-(d-1)}-d\\ &\le (B+\kappa+r+1)^{3(d-1)!-1}-d.\end{aligned}$$

$\square$

2. Castelnuovo-Mumford regularity and Hilbert coefficients

In this section we provide a bound for $\operatorname{reg}(G(\mathbb{M}))$ in terms of the Hilbert coefficients $e_i(\mathbb{M})$.

Let $R = \oplus_{n\geq 0} R_n$ be a Noetherian standard graded ring over a local Artinian ring $(R_0, \mathfrak{m}_0)$ such that $R_0/\mathfrak{m}_0$ is an infinite field. Let E be a finitely generated graded R-module of dimension d. We denote the Hilbert function $\ell_{R_0}(E_t)$ and the Hilbert polynomial of E by $h_E(t)$ and $p_E(t)$, respectively. Writing $p_E(t)$ in the form:

$$p_E(t) = \sum_{i=0}^{d-1} (-1)^i e_i(E) \binom{t+d-1-i}{d-1-i},$$

we call the numbers $e_i(M)$ *Hilbert coefficients* of E. Let $\Delta(E)$ denote the maximal generating degree of E. Easy examples show that one cannot bound $\operatorname{reg}(E)$ in terms of $\Delta(E), e_0(E), ..., e_{d-1}(E)$. However these invariants bound $\operatorname{reg}^1(E)$, as shown in [3, Theorem 17.2.7] and [17, Theorem 2]. Below we recall the bound by Trivedi which does not depend on the number of generators of E as the one in [3]. Let

$$\Delta'(E) = \max\{\Delta(E),\ 0\}.$$

We inductively define a sequence of integers as follows: $m_1 = e_0(E) + \Delta'(E)$, and for all $i \geq 2$,

$$m_i = m_{i-1} + e_0 \binom{m_{i-1}+i-2}{i-1} - e_1 \binom{m_{i-1}+i-3}{i-2} + \cdots + (-1)^{i-1} e_{i-1}.$$

Then

Lemma 2.1. ([17, Theorem 2]) *Assume that $d \geq 1$. Then* $\operatorname{reg}^1(E) \leq m_d - 1$.

In fact the above result was formulated in [17] for $G_I(M)$, which corresponds to the case E being generated by elements of degree zero. But this assumption is not essential. The proof was eventually given in [16, Lemma 4]. For a more algebraic proof one can use [8, Theorem 1.7].

From the above bound we can derive an explicit bound for $\operatorname{reg}^1(E)$ in terms of $e_i(E)$ and $\Delta(E)$. However this bound is weaker.

Lemma 2.2. *Let E be a finitely generated graded R-module of dimension $d \geq 1$. Put*

$$\xi(M) = \max\{e_0(M), |e_1(M)|, ..., |e_{d-1}(M)|\}.$$

and

$$\Delta^*(E) = \max\{\Delta(E), 1\}.$$

Then we have

$$\operatorname{reg}^1(E) \leq (\xi(E) + \Delta^*(E))^{d!} - 1.$$

Proof. For short, we put $e_i := e_i(E)$, $\xi := \xi(E)$ and $\Delta^* := \Delta^*(E)$. By Lemma 2.1 it suffices to show that $m_d \leq (\xi + \Delta^*)^{d!}$. This is trivial for $d = 1$. By the induction hypothesis we may assume

$$m_{d-1} \leq (\xi + \Delta^*)^{(d-1)!} - 1 =: \alpha.$$

Note that
$$\sum_{i=0}^{d-1}(-1)^i e_i\binom{\alpha+d-2-i}{d-1-i} \le \xi \sum_{i=0}^{d-1}\binom{\alpha+d-2-i}{d-1-i} = \xi\binom{\alpha+d-1}{d-1}.$$
Hence
$$m_d \le \alpha + \xi\binom{\alpha+d-1}{d-1}.$$
If $d = 2$, then $\alpha = \xi + \Delta^* - 1$, and
$$m_d \le \xi + \Delta^* - 1 + \xi(\xi+\Delta^*) = (\xi+\Delta^*)(\xi+1) - 1 \le (\xi+\Delta^*)^2 - 1.$$
Assume $d \ge 3$. Observing that $\binom{\alpha+d-1}{d-1} \le (\alpha+1)^{d-1}$ for all $\alpha \ge 1$, we get
$$\begin{aligned} m_d &\le \alpha + \xi(\alpha+1)^{d-1} \\ &= (\xi+\Delta^*)^{(d-1)!} - 1 + \xi[(\xi+\Delta^*)^{(d-1)!}]^{d-1} \\ &< (\xi+\Delta^*)^{(d-1)!} + (\xi+\Delta^*)^{d!-1} - 1 \\ &< (\xi+\Delta^*)^{d!}(\tfrac{1}{\xi+\Delta^*} + \tfrac{1}{\xi+\Delta^*}) - 1 \\ &< (\xi+\Delta^*)^{d!} - 1. \end{aligned}$$

□

Now we apply the above result to derive a bound for $\operatorname{reg}(G(\mathbb{M}))$ in terms of Hilbert coefficients. We call
$$H_{\mathbb{M}}(n) = \ell(M/M_{n+1})$$
the Hilbert-Samuel function of M w.r.t $\mathbb{M}$. This function agrees with a polynomial - called the Hilbert-Samuel polynomial and denoted by $P_{\mathbb{M}}(n)$ - for $n \gg 0$. If we write
$$P_{\mathbb{M}}(t) = \sum_{i=0}^{d}(-1)^i e_i(\mathbb{M})\binom{t+d-i}{d-i},$$
then the integers $e_i(\mathbb{M})$ are called *Hilbert coefficients* of $\mathbb{M}$ (see [13, Section 1]). When $\mathbb{M} = \{I^n M\}$, $H_{\mathbb{M}}(n)$, $P_{\mathbb{M}}(n)$ are usually denoted by $H_{I,M}(n)$ and $P_{I,M}(n)$, respectively, and $e_i(\mathbb{M}) = e_i(I, M)$. Note that $e_i(\mathbb{M}) = e_i(G(\mathbb{M}))$ for $0 \le i \le d-1$. Using the above results we can already bound $\operatorname{reg}^1(G(\mathbb{M}))$ in terms of $e_0(\mathbb{M}), ..., e_{d-1}(\mathbb{M})$. If $\operatorname{depth}(M) > 0$, by [6, Lemma 1.8], $\operatorname{reg}(G(\mathbb{M})) = \operatorname{reg}^1(G(\mathbb{M}))$, and so it is bounded in terms of $e_i(\mathbb{M})$, $i < d$. The following example shows that this is not the case if $\operatorname{depth}(M) = 0$.

Example. Let $A = k[[x,y]]/(x^2, xy^s)$, $s \ge 1$. Then $G_{\mathfrak{m}}(A) \cong k[x,y]/(x^2, xy^s)$. Since (x^2, xy^s) is a so-called stable ideal, $\operatorname{reg}(G_{\mathfrak{m}}(A)) = s$ can be arbitrarily large, while $e_0(A) = 1$.

The main aim of this section is to show that also using $e_d(\mathbb{M})$ we can bound $\operatorname{reg}(G(\mathbb{M}))$. For that we need

Lemma 2.3. ([13, Proposition 2.3]) *For all n we have*
$$\ell(H^0_{\mathfrak{m}}(M)) = P_{\mathbb{M}}(n) - P_{\overline{\mathbb{M}}}(n) = (-1)^d[e_d(\mathbb{M}) - e_d(\overline{\mathbb{M}})].$$

Using the Grothendieck-Serre formula to $G(\mathbb{M})$ and the arguments in the proof of [8, Lemma 3.4], we get

Lemma 2.4. *$P_{\mathbb{M}}(n) = H_{\mathbb{M}}(n)$ for all $n \ge \operatorname{reg}(G(\mathbb{M}))$.*

Lemma 2.5. *$\ell(H^0_{\mathfrak{m}}(M)) \le P_{\mathbb{M}}(n)$ for all $n \ge \operatorname{reg}(G(\overline{\mathbb{M}}))$.*

Proof. By Lemma 2.4, $P_{\overline{\mathbb{M}}}(n) = H_{\overline{\mathbb{M}}}(n)$ for all $n \ge \operatorname{reg}(G(\overline{\mathbb{M}}))$. Hence, by Lemma 2.3, $\ell(H^0_{\mathfrak{m}}(M)) = P_{\mathbb{M}}(n) - P_{\overline{\mathbb{M}}}(n) = P_{\mathbb{M}}(n) - H_{\overline{\mathbb{M}}}(n) \le P_{\mathbb{M}}(n)$ for all $n \ge \operatorname{reg}(G(\overline{\mathbb{M}}))$. □

Now we can state and prove the main theorem of this section.

Theorem 2.6. *Let $\mathbb{M}$ be a good I-filtration of M of dimension $d \ge 1$. Put $r'(\mathbb{M}) = \max\{1, r(\mathbb{M})\}$ and*

$$\xi(\mathbb{M}) = \max\{e_0(\mathbb{M}), |e_1(\mathbb{M})|, ..., |e_d(\mathbb{M})|\}.$$

Then

$$\operatorname{reg}(G(\mathbb{M})) \le (\xi(\mathbb{M}) + r'(\mathbb{M}))^{d!} + \xi(\mathbb{M})\binom{(\xi(\mathbb{M}) + r'(\mathbb{M}))^{d!} + d}{d} - 1.$$

Proof. Let $r = r(\mathbb{M})$, $r' = r'(\mathbb{M})$ and $e_i = e_i(\mathbb{M})$. By [6, Lemma 1.8] we have $\operatorname{reg}(G(\overline{\mathbb{M}})) = \operatorname{reg}^1(G(\overline{\mathbb{M}}))$. By Lemma 1.1,

$$\operatorname{reg}(G(\mathbb{M})) \le \max\{\operatorname{reg}^1(G(\overline{\mathbb{M}})),\ r\} + \ell(H^0_{\mathfrak{m}}(M)). \tag{3}$$

Set $\alpha := (\xi + r')^{d!} - 1 \ge r$. By Lemma 2.3, $e_i(G(\overline{\mathbb{M}})) = e_i(\overline{\mathbb{M}}) = e_i$ for all $i \le d-1$. As mentioned above $G(\overline{\mathbb{M}})$ is generated by elements of degrees at most $r(\mathbb{M}) \ge 0$. Therefore, by Lemma 2.2, $\operatorname{reg}^1(G(\overline{\mathbb{M}})) \le \alpha$. Using (3) and Lemma 2.5 we then get

$$\begin{aligned}\operatorname{reg}(G(\mathbb{M})) &\le \alpha + P_{\mathbb{M}}(\alpha) \le \alpha + \xi \textstyle\sum_{i=0}^{d} \binom{\alpha+d-i}{d-i} \\ &= \alpha + \xi\binom{\alpha+d+1}{d} = (\xi + r')^{d!} - 1 + \xi\binom{(\xi+r')^{d!}+d}{d}.\end{aligned}$$

□

In the next section we also need a result about bounding Hilbert coefficients in terms of the Castelnuovo-Mumford regularity.

Proposition 2.7. *Let $l_1, \dots, l_d \in I$ such that their initial forms in $G_I(A)$ form a filter regular sequence on $G(\mathbb{M})$ and $B = \ell(M/(l_1, ..., l_d)M)$. Then*

(a) *For all $1 \le i \le d-1$, $|e_i(\mathbb{M})| \le B(\operatorname{reg}^1(G(\mathbb{M})) + 1)^i$;*

(b) *$|e_d(\mathbb{M})| \le B(d+1)(\operatorname{reg}(G(\mathbb{M})) + 1)^d$.*

Proof. The inequalities in (a) immediately follow from [4, Theorem 4.6] by noticing that $\operatorname{reg}(\overline{G(\mathbb{M})}) = \operatorname{reg}^1(G(\mathbb{M}))$ and that $G(\mathbb{M})$ is generated in non-negative degrees. In fact, the proof of [4, Theorem 4.6] is based on [4, Theorem 4.2] and the local duality. [4, Theorem 4.2] is formulated for graded modules over a polynomial ring over a field. However its proof works for any polynomial ring over an Artinian local ring. Moreover, by taking completion we may assume that this is a Gorenstein polynomial ring, and so we can still use the local duality to derive an estimation on cohomological Hilbert functions from that on Ext modules as done in [4, Theorem 4.5]. One can also rewrite the proof of [4, Theorem 4.2] in terms of local cohomology modules, and in this case we do not need the local duality.

(b) Let $a = \operatorname{reg}(G(\mathbb{M}))$ and $e_i = e_i(\mathbb{M})$. By Lemma 2.4, $H_{\mathbb{M}}(a) = P_{\mathbb{M}}(a)$. By [6, Lemma 1.7],

$$H_{\mathbb{M}}(a) = \ell(M/M_{a+1}) \le \ell(M/I^{a+1}M) \le B\binom{a+d}{d}.$$

Since $\binom{a+j}{j} \le (a+1)^j$ and $\sum_{i=0}^{d}(-1)^i e_i\binom{a+d-i}{d-i} = H_{\mathbb{M}}(a)$, by (a) we get

$$\begin{aligned}|e_d| &\le H_{\mathbb{M}}(a) + \textstyle\sum_{i=0}^{d-1}|e_i|\binom{a+d-i}{d-i}\\ &\le B\binom{a+d}{d} + B\textstyle\sum_{i=0}^{d-1}\binom{a+d-i}{d-i}(a+1)^i\\ &\le B(a+1)^d + B\textstyle\sum_{i=0}^{d-1}(a+1)^{d-i}(a+1)^i\\ &= B(d+1)(a+1)^d.\end{aligned}$$

□

3. Relationship between Hilbert coefficients

In this section we always assume that M is an A-module of positive dimension d and $\mathbb{M}$ is a good I-filtration of M, where I is an $\mathfrak{m}$-primary ideal. First we give an upper bound for $(-1)^{i-1}e_i(\mathbb{M})$ in terms of its preceding Hilbert coefficients. The first statement of the following results is explicitly contained in [13].

Theorem 3.1. (i) $e_1(\mathbb{M}) \le \binom{e_0(\mathbb{M})}{2}$.

(ii) *Let* $\varsigma_{i-1} = \max\{e_0(\mathbb{M}), |e_1(\mathbb{M})|, ..., |e_{i-1}(\mathbb{M})|\}$ *and* $r' = \max\{1, r(\mathbb{M})\}$. *For* $i \ge 2$ *we have*

$$(-1)^{i-1}e_i(\mathbb{M}) \le \varsigma_{i-1}\binom{(\varsigma_{i-1}+r')^{i!}+i}{i}.$$

Proof. We do induction on d. Let $d = 1$. Then the inequality $e_1(\mathbb{M}) \le \binom{e_0(\mathbb{M})}{2}$ follows from [13, Proposition 2.8 and Lemma 2.3].

Assume that $d \ge 2$. First we prove the statement for $i \le d-1$. Let $\overline{\mathbb{M}} = M/H^0_{\mathfrak{m}}(M)$. Since $e_j(\mathbb{M}) = e_j(\overline{\mathbb{M}})$ for all $j \le d-1$, we may assume that depth $M > 0$. Let x be an $\mathbb{M}$-superficial element for I. Then $\dim(M/xM) = d-1$ and by [13, Proposition 1.2], $e_j(\mathbb{M}) = e_j(\mathbb{M}/xM)$ for all $j \le d-1$. Hence the inequalities follow from the induction hypothesis applied to $\mathbb{M}/xM$.

Finally let $i = d$. Since $G(\overline{\mathbb{M}})$ is generated by elements of degrees at most $r(\mathbb{M}) \ge 0$, by [6, Lemma 1.8] and Lemma 2.2 we have

$$\operatorname{reg}(G(\overline{\mathbb{M}})) = \operatorname{reg}^1(G(\overline{\mathbb{M}})) \le (\varsigma_{d-1}+r')^{d!} - 1 =: \alpha.$$

By Lemma 1.1 and Lemma 2.5 we then get

$$\begin{aligned}\operatorname{reg}(G(\mathbb{M})) &\le \max\{\operatorname{reg}^1(G(\overline{\mathbb{M}})),\ r\} + \ell(H^0_{\mathfrak{m}}(M))\\ &\le \max\{\operatorname{reg}^1(G(\overline{\mathbb{M}})),\ r\} + P_{\mathbb{M}}(\alpha)\\ &\le \alpha + \textstyle\sum_{i=0}^{d-1} e_i(\mathbb{M})\binom{\alpha+d-i}{d-i} + (-1)^d e_d(\mathbb{M})\\ &\le \varsigma_{d-1}\binom{\alpha+d+1}{d} + (-1)^d e_d(\mathbb{M}).\end{aligned}$$

Since $\operatorname{reg}(G(\mathbb{M})) \ge 0$, we then get

$$(-1)^{d-1}e_d(\mathbb{M}) \le \varsigma_{d-1}\binom{\alpha+d+1}{d} = \varsigma_{d-1}\binom{(\varsigma_{d-1}+r')^{d!}+d}{d},$$

as required. □

Remark 3.2. Using Lemma 2.1 and induction one can derive a better bound for $(-1)^{i-1}e_i(\mathbb{M})$, $i \le d-1$. Since this bound is of almost the same complexity as the one in the above theorem, we do not give it here. The fact that $(-1)^{i-1}e_i(I, A)$ is bounded above by a function depending on $e_0(I, A), ..., e_{i-1}(I, A)$ for $i \le d-1$ was mentioned in [1, Remark 3.10], provided that A is an equicharacteristic local ring. Also no explicit bound was given there.

Next we show that the last t Hilbert coefficients completely depend on the first $d+1-t$ Hilbert coefficients, where $t = \operatorname{depth} M$. For that we need some more auxiliary results.

Lemma 3.3. *Let x be an $\mathbb{M}$-superficial element for I. Then*

$$\operatorname{reg}(G(\mathbb{M}/xM)) \leq \operatorname{reg}(G(\mathbb{M})).$$

Proof. We have the following exact sequence

$$0 \longrightarrow \bigoplus \frac{xM \cap M_n}{xM_{n-1} + xM \cap M_{n+1}} \longrightarrow G(\mathbb{M})/x^*G(\mathbb{M}) \longrightarrow G(\mathbb{M}/xM) \longrightarrow 0.$$

By [6, Lemma 1.3(ii)] (see also [19, Lemma 4.4]), $xM \cap M_n = xM_{n-1}$ for $n \gg 0$. Hence

$$\operatorname{reg}(G(\mathbb{M}/xM)) \leq \operatorname{reg}(G(\mathbb{M})/x^*G(\mathbb{M})) \leq \operatorname{reg}(G(\mathbb{M})).$$

□

Lemma 3.4. *Let $\xi(\mathbb{M}) = \max\{e_0(\mathbb{M}), |e_1(\mathbb{M})|, ..., |e_d(\mathbb{M})|\}$. Then*

$$\operatorname{reg}(G(\mathbb{M})) < [\xi(\mathbb{M}) + r(\mathbb{M}) + 2]^{(d+1)!}.$$

Proof. For short, let $\xi = \xi(\mathbb{M})$ and $r := r(\mathbb{M})$. By Theorem 2.6,

$$\begin{aligned}\operatorname{reg}(G(\mathbb{M})) &\leq (\xi + r + 1)^{d!} + \xi\binom{(\xi + r + 1)^{d!} + d}{d} - 1 \\ &< (\xi + r + 2)^{(d+1)!}\end{aligned}$$

□

Lemma 3.5. *Let $x_1, x_2, ..., x_d$ be an $\mathbb{M}$-superficial sequence for I. Set $\mathbb{M}_i = \mathbb{M}/(x_1, ..., x_i)M$ and $M_i = M/(x_1, ..., x_i)M$, where $\mathbb{M}_0 = \mathbb{M}$ and $M_0 = M$. Then for all $0 \leq i \leq d-1$, we have*

$$h^0(M_i) \leq (i+1)\xi(\mathbb{M})(\operatorname{reg}(G(\mathbb{M})) + 2)^d.$$

Proof. Set $a := \operatorname{reg}(G(\mathbb{M}))$ and $\xi := \xi(\mathbb{M})$. Induction by i. Note by Lemma 3.3 that $\operatorname{reg}(G(\overline{\mathbb{M}_i})) \leq \operatorname{reg}(G(\mathbb{M}_i)) \leq \operatorname{reg}(G(\mathbb{M})) = a$.

For $i = 0$, by Lemma 2.5, we have

$$h^0(M_0) = h^0(M) \leq P_{\mathbb{M}}(a) \leq \xi \sum_{j=0}^{d} \binom{d + a - j}{a - j} = \xi \binom{a + d + 1}{d} \leq \xi(a+2)^d.$$

For $0 < i \leq d-1$, by [13, Proposition 1.2], we have $e_j(\mathbb{M}_i) = e_j(\mathbb{M}_{i-1})$ for all $0 \leq j \leq d-i-1$ and

$$\begin{aligned}|e_{d-i}(\mathbb{M}_i)| &= |e_{d-i}(\mathbb{M}_{i-1}) + (-1)^{d-i}\ell(0 :_{M_{i-1}} x_i)| \\ &\leq |e_{d-i}(\mathbb{M}_{i-1})| + h^0(M_{i-1}) \leq \xi + h^0(M_{i-1}).\end{aligned}$$

Hence, by Lemma 2.5 and the induction hypothesis we get

$$\begin{aligned} h^0(M_i) &\le P_{\mathbb{M}_i}(a) \\ &\le \xi \sum_{j=0}^{d-i-1} \binom{d-i+a-j}{d-i-j} + |e_{d-i}(\mathbb{M}_i)| \\ &= \xi\binom{a+d-i+1}{d-i} - \xi + |e_{d-i}(\mathbb{M}_i)| \\ &\le \xi(a+2)^{d-i} - \xi + \xi + h^0(M_{i-1}) \\ &\le \xi(a+2)^{d-i} + i\xi(a+2)^d \le (i+1)\xi(a+2)^d. \end{aligned}$$

□

Lemma 3.6. *In the notation of Lemma 3.5, let* $B = \ell(M/(x_1, x_2, ..., x_d)M)$. *Then*

$$B \le (d+1)\xi(\mathbb{M})(\mathrm{reg}(G(\mathbb{M})) + 2)^d.$$

Proof. Keep the notation in the proof of the previous lemma. Since $\dim(M_{d-1}) = 1$, M_{d-1} is a generalized Cohen-Macaulay module. By [5, Lemma 1.5]

$$B - e_0(x_d; M_{d-1}) = \ell(M_{d-1}/x_d M_{d-1}) - e_0(x_d; M_{d-1}) \le h^0(M_{d-1}).$$

Since $e_0(x_d; M_{d-1}) = e_0(x_1, ..., x_d; M) = e_0(\mathbb{M}) = e_0$, we get

$$B \le e_0 + h^0(M_{d-1}) \le \xi + h^0(M_{d-1}).$$

By Lemma 3.5, $h^0(M_{d-1}) \le d\xi(a+2)^d$. From this estimation we immediately get $B \le (d+1)\xi(a+2)^d$. □

In the sequel we use the following notation:

$$\xi_t(\mathbb{M}) = \max\{e_0(\mathbb{M}), |e_1(\mathbb{M})|, ..., |e_{d-t}(\mathbb{M})|\},$$

where $0 \le t \le d$. Thus $\xi_0(\mathbb{M}) = \xi(\mathbb{M})$. In Lemma 3.4, $\mathrm{reg}(G(\mathbb{M}))$ is bounded by d, $\xi(\mathbb{M})$ and $r(\mathbb{M})$. Now we show that instead of $\xi(\mathbb{M})$ one can use $\xi_t(\mathbb{M})$ ($t = \mathrm{depth}(M)$).

Theorem 3.7. *Let* $\mathbb{M}$ *be a good* I*-filtration of* M *with* $\dim(M) = d \ge 1$. *Assume that* $\mathrm{depth}(M) = t$. *Then*

$$\mathrm{reg}(G(\mathbb{M})) < [2(d+1)\xi_t(\mathbb{M})]^{3(d-1)!}[\xi_t(\mathbb{M}) + r(\mathbb{M}) + 4]^{3d!(d-t+1)!},$$

where $0! = 1$.

Proof. Let $x_1, ..., x_d$ be an $\mathbb{M}$-superficial sequence for I. Let $M_i = M/(x_1, ..., x_i)M$ and $\mathbb{M}_i = \mathbb{M}/(x_1, ..., x_i)M$. For short we write $\xi_t := \xi_t(\mathbb{M})$ and $r := r(\mathbb{M})$.

If $\mathrm{depth}(M) = d$, i.e. if M is a Cohen-Macaulay module, then in the notation of Theorem 1.2 we have $B(I, M) = e_0(I, M) = \xi_d(\mathbb{M})$ and $\kappa(I, M) = 0$, which yield

$$\mathrm{reg}(G(\mathbb{M})) < (\xi_d + r + 1)^{3(d-1)!-1}.$$

Assume that $\mathrm{depth}(M) = t < d$. Then $B(I, M) = B(I, M_t)$ and $\kappa(I, M) = \kappa(I, M_t)$. Let $a_t := \mathrm{reg}(G(\mathbb{M}_t))$. Note by [13, Proposition 1.2] that $e_i(\mathbb{M}_t) = e_i(\mathbb{M})$ for all $i \le t$. Hence, $\xi(\mathbb{M}_t) = \xi_t$. Applying Lemma 3.5 to M_t we get

$$\kappa(I, M_t) \le (d-t)\xi_t(a_t + 2)^{d-t}.$$

By Lemma 3.6
$$B(I, M_t) \leq (d-t+1)\xi_t(a_t+2)^{d-t} \leq (d+1)\xi_t(a_t+2)^{d-t}. \tag{4}$$
Hence, by Theorem 1.2
$$\begin{aligned}\operatorname{reg}(G(\mathbb{M})) &< [B(I, M) + \kappa(I, M) + r(\mathbb{M}) + 1]^{3(d-1)!} \\ &= [B(I, M_t) + \kappa(I, M_t) + r + 1]^{3(d-1)!} \\ &\leq [(2d+1)\xi_t(a_t+2)^d + r + 1]^{3(d-1)!}.\end{aligned}$$
By Lemma 3.4,
$$a_t \leq [\xi_t(\mathbb{M}_t) + r(\mathbb{M}_t) + 2]^{(d-t+1)!}. \tag{5}$$
Therefore
$$\begin{aligned}\operatorname{reg}(G(\mathbb{M})) &\leq [(2d+2)\xi_t((\xi_t + r + 2)^{(d-t+1)!} + 2)^d]^{3(d-1)!} \\ &\leq [2(d+1)\xi_t((\xi_t + r + 4)^{(d-t+1)!})^d]^{3(d-1)!} \\ &\leq [2(d+1)\xi_t]^{3(d-1)!}(\xi_t + r + 4)^{3d!(d-t+1)!}. \end{aligned} \tag{6}$$
□

The following theorem is the main result of this paper:

Theorem 3.8. *Let* $\mathbb{M}$ *be a good* I*-filtration of* M*. Assume that* $\dim(M) \geq 1$ *and* $\operatorname{depth}(M) = t \geq 1$*. Then* $|e_d(\mathbb{M})|, |e_{d-1}(\mathbb{M})|, ..., |e_{d-t+1}(\mathbb{M})|$ *are bounded by functions depending only on* $d, e_0(\mathbb{M}), e_1(\mathbb{M}), ..., e_{d-t}(\mathbb{M})$ *and* $r(\mathbb{M})$*. Namely, for all* $j \geq d-t+1$ *we have*
$$|e_j(\mathbb{M})| < [2(j+1)\xi_t(\mathbb{M})]^{3j!+2}[\xi_t(\mathbb{M}) + r(\mathbb{M}) + 4)]^{3(j+1)!(j+1-t)!}.$$

Proof. First we prove for $i = d$. Let $x_1, ..., x_d$ be an $\mathbb{M}$-superficial sequence for I. Keep the notation in the proof of Theorem 3.7. Then by (4) and (5) we get
$$\begin{aligned}B = B(I, M_t) &\leq (d+1)\xi_t[(\xi_t + r + 2)^{(d-t+1)!} + 2]^d \\ &\leq (d+1)\xi_t(\xi_t + r + 4)^{d(d-t+1)!}.\end{aligned}$$
Hence, by Proposition 2.7 and Theorem 3.7 we have
$$\begin{aligned}|e_d(\mathbb{M})| &\leq B(d+1)(\operatorname{reg}(G(\mathbb{M})) + 1)^d \\ &\leq (d+1)^2\xi_t(\xi_t + r + 4)^{d(d-t+1)!}[2(d+1)\xi]^{3d!}(\xi_t + r + 4)^{3d\cdot d!(d-t+1)!} \\ &< [2(d+1)\xi_t]^{3d!+2}(\xi_t + r + 4)^{3(d+1)!(d-t+1)!}. \end{aligned} \tag{7}$$
Now let $d-t+1 \leq j \leq d-1$. Since $\operatorname{depth}(M) = t$, by [13, Proposition 1.2], $e_j(\mathbb{M}) = e_j(\mathbb{M}_{d-j})$. Note that $\dim(M_{d-j}) = j$, $\operatorname{depth}(M_{d-j}) = t + j - d \geq 1$ and $r(\mathbb{M}_{d-j}) \leq r(\mathbb{M})$. Therefore
$$\xi_{t+j-d}(\mathbb{M}_{d-j}) = \xi_t(\mathbb{M}) = \xi_t.$$
Applying (7) to $\mathbb{M}_{d-j}$, we then get
$$|e_j(\mathbb{M})| = |e_j(\mathbb{M}_{d-j})| \leq [2(j+1)\xi_t]^{3j!+2}(\xi_t + r + 4)^{3(j+1)!(j+1-t)!}.$$
□

For the I-adic filtration $\mathbb{M} = \{I^n M\}_{n\geq 0}$, $r(\mathbb{M}) = 0$. Hence as an immediate of consequence of Theorem 3.8 we get the following extension of [16, Theorem 1] to the non-Cohen-Macaulay case. In the Cohen-Macaulay case our bound is much bigger than that of [16, Theorem 1] (see also [14, Theorem 4.1] and [6, Theorem 1.10]).

Corollary 3.9. *Assume that* $\dim(M) = d \geq 1$ *and* $\operatorname{depth}(M) = t \geq 1$. *Then for all* $d - t + 1 \leq j \leq d$ *we have*

$$|e_j(I, M)| \leq [2(j+1)\xi_t]^{3j!+2}(\xi_t + 4)^{3(j+1)!(j+1-t)!},$$

where

$$\xi_t = \max\{e_0(I, M), |e_1(I, M)|, ..., |e_{d-t}(I, M)|\}.$$

In other words, $|e_j(I, M)|$ *are bounded in terms of* $d, e_0(I, M), e_1(I, M), ..., e_{d-t}(I, M)$.

Finally we can state and prove a result about the finiteness of Hilbert-Samuel functions.

Theorem 3.10. *Let* $d \geq t \geq 0$, $e_0, ..., e_{d-t}$ *be integers. Then there exists only a finite number of Hilbert-Samuel functions associated to* d*-dimensional modules* M *and* $\mathfrak{m}$*-primary ideals* I *such that* $\operatorname{depth}(M) = t$ *and* $e_j(I, M) = e_j$ *for all* $0 \leq j \leq d - t$.

Proof. By Corollary 3.9 there exists only a finite number of Hilbert-Samuel polynomial $P_{I,M}(n)$ such that $e_j(I, M) = e_j$ for all $0 \leq j \leq d - t$. By [8, Lemma 3.4] $H_{I,M}(n) = P_{I,M}(n)$ for $n \geq \operatorname{reg}(G_I(M)) =: a$. By Theorem 3.7, a is bounded in terms of $e_0, e_1, ..., e_{d-t}$ and d. Since $H_{I,M}(n) = 0$ for $n < 0$ and $H_{I,M}(n)$ is an increasing function for $n \geq 0$, $H_{I,M}(n) \leq P_{I,M}(a)$ for all $n \leq a$. This implies that the number of such functions are bounded in terms of $e_0, e_1, ..., e_{d-t}$ and d. $\square$

Remark 3.11. In [15] there were constructed a complete regular local ring R and an infinite sequence $\mathfrak{p}_n$ of prime ideals of R such that $\dim(R/\mathfrak{p}_n) = 2$, $e_0(R/\mathfrak{p}_n) = 4$, but $e_1(R/\mathfrak{p}_n) = 8 - n$. This shows that one cannot reduce the number of "independent" coefficients in Theorem 3.8 and Theorem 3.10.

References

[1] C. Blancafort, *Hilbert functions of graded algebras over Artinian rings*, J. Pure Appl. Algebra **125** (1998), 55 - 78.

[2] N. Bourbaki, Algebèbre commutative, Hermann, Paris (1961 - 1965).

[3] M. P. Brodmann and R. Y. Sharp, Local cohomology: an algebraic introduction with geometric applications. Cambridge Studies in Advanced Mathematics, 60. Cambridge University Press, Cambridge, 1998;

[4] M. Chardin, D. T. Ha and L. T. Hoa, *Castelnuovo-Mumford regularity of* Ext *modules and homological degree*, Trans. Amer. Math. Soc. **363** (2011), 3439 - 3456.

[5] N. T. Cuong, P. Schenzel and N. V. Trung, *Über verallgemeinerte Cohen-Macaulay Modulen*, Math. Nachr. **85** (1978), 57 - 73.

[6] L. X. Dung and L. T. Hoa, *Castelnuovo-Mumford regularity of associated graded modules and fiber cones of filtered modules*, Comm. Algebra **40** (2012), 404 - 422.

[7] D. Kirby and H. A. Mehran, *A note on the coefficients of the Hilbert-Samuel polynomial for a Cohen-Macaulay module*, J. London Math. Soc. (2) **25** (1982), 449 - 457.

[8] C. H. Linh, *Upper bound for Castelnuovo-Mumford regularity of associated graded modules*, Comm. Algebra. **33** (2005), 1817 - 1831.

[9] C. H. Linh, *Castelnuovo-Mumford regularity and degree of nilpotency*, Math. Proc. Cambridge Philos. Soc. **142** (2007), 429 - 437.

[10] M. Narita, *A note on the coefficients of Hilbert characteristic functions in semi-regular local rings*, Proc. Cambridge Philos. Soc., **59** (1963), 269 - 275.

[11] N. G. Northcott, *A note on the coefficients of the abstract Hilbert function*, J. London Math. Soc. (1), **35** (1960), 209 - 214.

[12] C. P. L. Rhodes, *The Hilbert-Samuel polynomial in a filtered*, J. London Math. Soc. (1) **3** (1971), 73 - 85.

[13] M. E. Rossi and G. Valla, Hilbert functions of filtered modules. Lecture Notes of the Unione Matematica Italiana, **9**. Springer, Heidelberg, 2010.

[14] M. E. Rossi, N. V. Trung and G. Valla, *Castelnuovo-Mumford regularity and extended degree*, Trans. Amer. Math. Soc. **355** (2003), 1773 - 1786.

[15] V. Srinivas and V. Trivedi, *A finiteness theorem for the Hilbert functions of complete intersection local rings*. Math. Z. **225** (1997), 543 - 558.

[16] V. Srinivas and V. Trivedi, On the Hilbert function of a Cohen-Macaulay local ring, J. Algebraic Geom. **6** (1997), 733 - 751.

[17] V. Trivedi, *Hilbert functions, Castelnuovo-Mumford regularity and uniform Artin-Rees numbers*, Manuscripta Math. **94** (1997), 485 - 499.

[18] V. Trivedi, *Finiteness of Hilbert functions for generalized Cohen-Macaulay modules*. Comm. Algebra **29** (2001), 805 - 813.

[19] N. V. Trung, *Castelnuovo-Mumford regularity of the Rees algebra and the associated graded rings*, Trans. Amer. Math. Soc **350** (1998), 2813 - 2832.