REGULAR PROJECTION IN O-MINIMAL STRUCTURES

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Abstract. In this paper, we give a proof for Mostowski's regular projection theorem for definable sets in o-minimal structures, which is a positive answer to the question of Parusiński about a definable version of the regular projection theorem. A consequence of this result is the existence of definable regular covers for definable sets. Our result holds for arbitrary o-minimal structures over real closed fields.

1. Introduction

The regular projection theorem was introduced by Mostowski in his famous paper [5] on Lipschitz stratification for complex analytic sets. It was extended to subanalytic sets by Parusiński [6] (see also [8]). The theorem has many applications, in particular in proving the existence of Lipschitz stratifications (see [5], [7], [8]). Recently, Parusiński [9] used it as the key to the proof of the regular cover theorem for subanalytic sets. In the same paper, he asked whether the regular projection theorem is still valid in the o-minimal setting ([9], Question 2.1). He also mentioned that his arguments cannot apply to o-minimal structures because they so much rely on Pawłucki's Puiseux Theorem [10]. In fact, a positive answer to Parusiński's question would provide a definable version of the regular cover theorem (see Section 4). We would like to remark further that the regular cover theorem was used as a principal tool in constructing sheaves on subanalytic site by Guillermou–Schapira [2] and in constructing what is called Sobolev sheaves by Lebeau [3].

In this paper we prove that the regular projection theorem holds for definable sets in o-minimal structures. The arguments used in our proof are elementary, they work for any o-minimal structure over real closed fields.

Let us recall the definition of regular projections and state the main result.

Let $\lambda \in \mathbb{R}^{n-1}$. We denote by $\pi_\lambda : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection parallel to the vector $(\lambda, 1) \in \mathbb{R}^n$. Let $X \subset \mathbb{R}^n$ and let ε, C be positive constant and $p \in \mathbb{N} \cup \{\infty, \omega\}$. The projection π_λ is said to be (ε, C, p)-*regular* at a point $x \in \mathbb{R}^n$ (with respect to X) if

(1) $\pi_\lambda|_X$ is finite, i.e., $\{\pi_\lambda^{-1}(x') \cap X\}$ is a finite set for every $x' \in \mathbb{R}^{n-1}$,

(2) the intersection of X with the open cone

$$C_\varepsilon(x, \lambda) = \{x + t(\lambda', 1), t \in \mathbb{R}^*, |\lambda' - \lambda| < \varepsilon\}$$

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is either empty or a disjoint union of sets of the form

$$\Gamma = \{x + \alpha_i(\lambda')(\lambda', 1), |\lambda' - \lambda| < \varepsilon\}$$

where α_i is a C^p nowhere vanishing function defined on $|\lambda' - \lambda| < \varepsilon$,

(3) the functions α_i in (2) satisfy for all $|\lambda' - \lambda| < \varepsilon$

$$|\text{grad}\alpha_i(\lambda')| \leq C|\alpha_i(\lambda')|.$$

Theorem 1.1 (Main Theorem). *Let $p \in \mathbb{N}$. Let $X \subset \mathbb{R}^n$ be a definable set of dimension $< n$. Then there exist $\varepsilon, C > 0$ and $\Lambda = \{\lambda_1, \ldots, \lambda_k\} \subset \mathbb{R}^{n-1}$ such that for every $x \in \mathbb{R}^n$ there is $\lambda_i \in \Lambda$ such that π_{λ_i} is (ε, C, p)-regular at x.*

Throughout the paper, we denote by $\mathbb{N}$ the set the natural numbers and by $\mathbb{R}^*$ the set of nonzero real numbers. We denote by $B_\varepsilon(x)$ the open ball of radius ε centered at x. For a set $A \subset \mathbb{R}^n$, $\overline{A}$ denotes the closure of A in $\mathbb{R}^n$. By "definable" we mean definable in o-minimal structures on the field of the real numbers $(\mathbb{R}, +, .)$.

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2. Preliminaries

In this section we will recall the definition of o-minimal structures and known properties of definable sets that will be used in the next section. We first fix some notations.

Given $A \subset \mathbb{R}^{n+m}$ and a map $f : A \to \mathbb{R}^k$. For $x \in \mathbb{R}^n$ we define

$$A_x = \{y \in \mathbb{R}^m : (x, y) \in A\}, \qquad f_x : A_x \to \mathbb{R}^k, f_x(y) = f(x, y).$$

Suppose $\mathcal{A} = \{A_i\}_{i=1,\ldots,k}$ is a collection of subsets of $\mathbb{R}^{n+m}$. By $f(\mathcal{A})$ we mean the collection $\{f(A_i)\}_{i=1,\ldots,k}$ and by $\mathcal{A}_x$ we mean the collection $\{(A_i)_x\}_{i=1,\ldots,k}$.

For $m, n \in \mathbb{N}$, $m > n$, we denote by $\pi^m_n : \mathbb{R}^m \to \mathbb{R}^n$ the projection on the first n coordinates.

2.1. **O-minimal structures.** An o-minimal structure over the real field $(\mathbb{R}, +, .)$ is a family $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ such that

(1) each $\mathcal{D}_n$ is an Boolean algebra of subsets of $\mathbb{R}^n$,

(2) if $A \in \mathcal{D}_n$ then $\mathbb{R} \times A$ and $A \times \mathbb{R}$ belong to $\mathcal{D}_{n+1}$,

(3) $\mathcal{D}_n$ contains all the zero sets of polynomials of n variables,

(4) if $A \in \mathcal{D}_n$ then $\pi^n_{n-1}(A) \in \mathcal{D}_{n-1}$,

(5) every element of $\mathcal{D}_1$ is a finite union of open intervals and points.

Elements of $\mathcal{D}_n$ f are called **definable sets**. A map is called definable if its graph is a definable set.

The class of semi-algebraic sets and the class of global subanalytic sets are typical examples of o-minimal structures. We refer the reader to Coste [1] and van den Dries [11] for more details about o-minimal structures.

2.2. **Cell decompositions.** Let $p \in \mathbb{N}$. A C^p definable **cell** in $\mathbb{R}$ is either a point or an open interval. A C^p definable cell in $\mathbb{R}^n$ is a set of the following forms $\Gamma_\xi = \{(x,y) \in C \times \mathbb{R}, y = \xi(x)\}$ (graph); $(\xi_1, \xi_2) = \{(x,y) \in C \times \mathbb{R}, \xi_1(x) < y < \xi_2(x)\}$ (band) where $\xi, \xi_1, \xi_2 : C \to \mathbb{R}$ are C^p definable functions. It allows $\xi_1 = -\infty$ and $\xi_2 = +\infty$.

A C^p definable cell decomposition of $\mathbb{R}^n$ is defined by induction as follows: a cell decomposition of $\mathbb{R}$ is finite collection of intervals and points

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}.$$

A C^p definable cell decomposition of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$ into C^p definable cells such that the collection of all images of these cells under the projection π^n_{n-1} forms a C^p definable cell decomposition of $\mathbb{R}^{n-1}$.

Let $\mathcal{D}$ be a cell decomposition of $\mathbb{R}^n$. Suppose that C is a cell of $\pi^n_{n-1}(\mathcal{D})$. Then $(\pi^n_{n-1})^{-1}(C)$ is a union of cells in $\mathcal{D}$ formed by functions $-\infty = \xi_0 < \xi_1 < \ldots < \xi_\sigma = \infty$ defined on C. We call such ξ_i functions **associated** to C.

The cell decomposition $\mathcal{D}$ is called **compatible** with $\mathcal{A} = \{A_i\}_{i=1,\ldots,k}$, a collection of subsets of $\mathbb{R}^n$, if each A_i is a union of some cells in $\mathcal{D}$.

Theorem 2.1 ([1], [11], Cell Decomposition)**.** *Let $\mathcal{A} = \{A_i\}_{i=1,\ldots,k}$ be a collection of definable subsets of $\mathbb{R}^n$. For any $p \in \mathbb{N}$, there exists a C^p definable cell decomposition compatible with $\mathcal{A}$.*

2.3. **Transversality.** A C^1 map $f : M \to N$ between two C^1 manifolds is said to be **transverse** to a C^1 submanifold $S \subset N$ at $p \in M$ if either $f(p) \notin S$ or $D_pf(T_pM) + T_{f(p)}S = T_pN$, then we write $f \pitchfork_p S$. If f is transverse at all points in M we say f is transverse to M and write $f \pitchfork S$. Suppose $\mathcal{S}$ be a finite collection of C^1 submanifolds of N. Then f is said to be transverse to $\mathcal{S}$, denoted by $f \pitchfork \mathcal{S}$, if f is transverse to each element of $\mathcal{S}$. In the sequel by a C^p **definable manifold** we mean a definable subset which is also a C^p submanifold of $\mathbb{R}^n$ for some n.

One of the most basic properties of Transversality Theory is that if V is a C^1 submanifold of N then $f^{-1}(V)$ is a C^1 submanifold of M and $\dim M - \dim(f^{-1}(V)) = \dim N - \dim V$. Suppose that M, N are C^1 definable manifolds and f is a C^1 definable map. It follows from the above fact and Cell Decomposition that for any definable subset A of N, $f^{-1}(A)$ is a definable subset of M of the same codimension as that of A, i.e., $\dim M - \dim f^{-1}(A) = \dim N - \dim A$. Another property is that

Proposition 2.2 ([4], Lemma 3)**.** *Let M, J and N be C^1 definable manifolds, and $f : M \times J \to N$ be a C^1 definable submersion. Let $\mathcal{S}$ be a finite collection of C^1 definable submanifolds of N. Then*

$$\tau(f, \mathcal{S}) = \{s \in J : f_s = f(., s) \pitchfork \mathcal{S}\}$$

is a definable set and $\dim(J \setminus \tau(f, \mathcal{S})) < \dim J$.

3. Proof of Theorem 1.1

We need the following lemmas to prove Theorem 1.1.

Lemma 3.1. *Let $-\infty = \xi_0 < \xi_1 < \ldots < \xi_\sigma = +\infty$ be continuous definable functions defined on a definable set $C \subset \mathbb{R}^n$. Then there is a finite definable partition $\mathcal{C}$ of C such that for each element E of $\mathcal{C}$ there is an interval $(a, b) \subset \mathbb{R}$ such that $E \times (a, b) \subset (\xi_{i-1}, \xi_i)$ for some $1 \leq i \leq \sigma$.*

Proof. Let $a_1 < \ldots < a_\sigma$ be real numbers. Set

$$\xi_1^+ = \{(x, y) \in \Gamma_{\xi_1}, y \geq a_1\}, \qquad \xi_1^- = \{(x, y) \in \Gamma_{\xi_1}, y < a_1\},$$
$$H_1^+ = \{(x, y), x \in \pi_n^{n+1}(\xi_1^+), y < a_1\}, \qquad H_1^- = \{(x, y), x \in \pi_n^{n+1}(\xi_1^-), y > a_1\},$$
$$C_1^+ = \pi_n^{n+1}(\xi_1^+), \qquad C_1^- = \pi_n^{n+1}(\xi_1^-).$$

and for $1 < i \leq \sigma$,

$$\xi_i^+ = \{(x, y) \in \Gamma_{\xi_i}, y \geq a_i\} \cap H_{i-1}^-, \qquad \xi_i^- = \{(x, y) \in \Gamma_{\xi_i}, y < a_i\} \cap H_{i-1}^-,$$
$$H_i^+ = \{(x, y), x \in \pi_n^{n+1}(\xi_i^+), a_{i-1} < y < a_i\}, \qquad H_i^- = \{(x, y), x \in \pi_n^{n+1}(\xi_i^-), y > a_i\},$$
$$C_i^+ = \pi_n^{n+1}(\xi_i^+), \qquad C_i^- = \pi_n^{n+1}(\xi_i^-),$$

where Γ_{ξ_i} is the graph of ξ_i.

It is easy to see from the construction that $C = C_1^+ \cup C_1^-$ and $C_i^- = C_{i+1}^+ \cup C_{i+1}^-$. This implies that $C = \bigcup_i C_i^+$. Moreover, $H_i^+ = C_i^+ \times (a_{i-1}, a_i)$ which is contained in the band (ξ_{i-1}, ξ_i). This shows that $\{C_i^+\}_{i=1}^\sigma$ is the desired partition. □

Lemma 3.2. *($P_{m,n}$) - Let $m > n$ be positive integers. Let $\mathcal{S}$ be a definable cell decomposition of $\mathbb{R}^m$ and $\mathcal{A}$ be a finite collection of definable subsets of $\mathbb{R}^n$. Then there is a definable cell decomposition $\mathcal{S}^*$ of $\mathbb{R}^n$ compatible with $\mathcal{A}$ and for each cell C of $\mathcal{S}^*$, there is a box B in $\mathbb{R}^{m-n}$ such that $C \times B$ is contained in a band of $\mathcal{S}$.*

Proof. Proof of ($P_{n+1,n}$). Let $\mathcal{S}' = \pi_n^{n+1}(\mathcal{S})$ and let $\mathcal{S}''$ be a definable cell decomposition of $\mathbb{R}^n$ compatible with $\{\mathcal{S}', \mathcal{A}\}$. For each $C \in \mathcal{S}''$, there is $E \in \mathcal{S}'$ such that $C \subset E$. By applying Lemma 3.1 to C and the restriction of $\mathcal{F}_E$ to C where $\mathcal{F}_E$ is the set of the functions associated to E (with respect to $\mathcal{S}$), we obtain a partition $\mathcal{P}$ of $\mathbb{R}^n$.

Take $\mathcal{S}^*$ to be a definable cell decomposition of $\mathbb{R}^n$ compatible with $\mathcal{P}$. We claim that $\mathcal{S}^*$ is the desired cell decomposition. Suppose that V is a cell of $\mathcal{S}^*$. Then $V \subset V'$ for some element $V' \in \mathcal{P}$. Note that there is an open interval (a, b) in $\mathbb{R}$ such that $V' \times (a, b)$

is contained in some band in $\mathcal{S}$. Hence $V \times (a, b)$ is also contained in the same band. Therefore, $(P_{n+1,n})$ is proved.

Assume that $(P_{m,n})$ holds true for any $m > n$. We now give a proof for $(P_{m+1,n})$. Suppose that $\mathcal{C}$ is a cell decomposition of $\mathbb{R}^m$ obtained by applying $(P_{m+1,m})$ to $(\mathcal{S}, \mathbb{R}^m)$. Put $\mathcal{C}'$ to be a definable cell decomposition of $\mathbb{R}^n$ by applying $(P_{m,n})$ to $(\mathcal{C}, \mathcal{A})$. We will prove that $\mathcal{C}'$ is the desired decomposition. Assume that $U \in \mathcal{C}'$. Then there is a box B in $\mathbb{R}^{m-n}$ such that $U \times B$ is contained in a band M of $\mathcal{C}$. Since M is a cell of $\mathcal{C}$, there is an interval (a, b) in $\mathbb{R}$ such that $M \times (a, b)$ contained in a band of $\mathcal{S}$. Therefore, $U \times B \times (a, b)$ is contained in the same band. The proof completes. $\square$

Now we consider a set $X \subset \mathbb{R}^n$ as given in Theorem 1.1. Denote by $X^{p,reg}$ the set of p-regular points of X i.e., the set of points in X at which X is a $(n-1)$-dimensional C^p submanifold of $\mathbb{R}^n$. Set $\Sigma(X) = X \setminus X^{p,reg}$. For $x \in \mathbb{R}^n$, we define

$$V(X, x) = \{y \in X^{p,reg} : y - x \in T_y X^{p,reg}\},$$
$$S(X, x) = V(X, x) \cup \Sigma(X),$$
$$R(X, x) = X \setminus S(X, x) = X^{p,reg} \setminus V(X, x).$$

Set

$$\Delta = \{(x, \lambda, t) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^* : x + t(\lambda, 1) \in S(X, x)\}.$$

Consider the following mappings

$$\eta : \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^* \to \mathbb{R}^n, (x, \lambda, t) \mapsto x + t(\lambda, 1)$$

$$\pi' : \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^{n-1}, (x, \lambda, t) \mapsto (x, \lambda),$$

Set $Y = \eta^{-1}(X)$, $Y' = \pi'(Y)$ and $\Delta' = \pi'(\Delta)$.

Lemma 3.3. $\dim \Delta'_x < n - 1$.

Proof. We have $\Delta'_x = \pi'_x(\Delta_x) = \pi'_x(\eta_x^{-1}(S(X, x))) = \pi'_x(\eta_x^{-1}(V(X, x))) \cup \pi'_x(\eta_x^{-1}(\Sigma(X)))$. Set $A = \pi'_x(\eta_x^{-1}(V(X, x)))$ and $B = \pi'_x(\eta_x^{-1}(\Sigma(X)))$. It suffices to show that $\dim A$ and $\dim B$ both are less than $< n - 1$.

Note that the Jacobian matrix of the map $\eta_x : \mathbb{R}^{n-1} \times \mathbb{R}^* \to \mathbb{R}^n, (\lambda, t) \to x + t(\lambda, 1)$ has rank n at every point in the domain. So it is a submersion. Since $\Sigma(X)$ is a definable set of codimension > 1, $\eta_x^{-1}(\Sigma(X))$ is a definable set of the same codimension. Thus $\dim \eta_x^{-1}(\Sigma(X)) < n - 1$. In other words, $\dim B < n - 1$.

We show now that $\dim A < n - 1$. Since the map η_x is a submersion, by Proposition 2.2, the set

$$\Omega := \{\lambda \in \mathbb{R}^{n-1} : \eta_{x,\lambda} \pitchfork X^{p,reg}\}$$

is a definable dense set in $\mathbb{R}^n$.

Note that

$$A = \{\lambda \in \mathbb{R}^{n-1}, \exists t \in \mathbb{R}^* : x + t(\lambda, 1) \in V(X, x)\}$$

$$= \{\lambda \in \mathbb{R}^{n-1}, \exists t \in \mathbb{R}^* : y = x + t(\lambda, 1) \in X^{p,reg}, x - y = t(\lambda, 1) \in T_y X^{p,reg}\}$$
$$= \{\lambda \in \mathbb{R}^{n-1}, \eta_{x,\lambda} \not\pitchfork X^{p,reg}\}.$$

It follows that $A = \mathbb{R}^{n-1} \setminus \Omega$. Then $\dim A < n - 1$. □

Lemma 3.4. *The map $\eta_x|_{(Y\setminus\Delta)_x} : (Y \setminus \Delta)_x \to R(X, x)$ is a C^p-diffeomorphism.*

Proof. First we prove that $R(X, x)$ is a C^p submanifold of $X^{p,reg}$. Note that $R(X, x) = X^{p,reg} \setminus V(X, x)$. It suffices to show that $V(X, x)$ is closed in $X^{p,reg}$. Let $y_* \in \overline{V(X, x)} \cap X^{p,reg}$. Then there is a sequence of points $\{y_k\}$ in $V(X, x)$ tending to y_*. Since $(y_k - x) \in T_{y_k} X^{p,reg}$ and $\lim_{k\to\infty} T_{y_k} X^{p,reg} = T_{y_*} X^{p,reg}$, $(y_k - x) \to (y_* - x) \in T_{y_*} X^{p,reg}$. This implies that $y_* \in V(X, x)$, and hence $V(X, x)$ is closed in $X^{p,reg}$.

Since $R(X, x)$ is a C^p submanifold of $\mathbb{R}^n$ and η_x is a submersion, $\eta_x^{-1}(R(X, x))$ is a C^p submanifold of $\mathbb{R}^{n-1} \times \mathbb{R}^*$. It is clear from the definitions that $\eta_x^{-1}(R(X, x)) = (Y \setminus \Delta)_x$. To prove the lemma it is enough to show that $\eta_x|_{(Y\setminus\Delta)_x} : (Y \setminus \Delta)_x \to R(X, x)$ is injective. Suppose (λ_1, t_1) and (λ_2, t_2) are points in $(Y \setminus \Delta)_x$ such that $\eta_x(\lambda_1, t_1) = \eta_x(\lambda_2, t_2)$, i.e., $x+t_1(\lambda_1, 1) = x+t_2(\lambda_2, 1)$. It is clear that $(\lambda_1, t_1) = (\lambda_2, t_2)$. Thus, the map is injective. □

Let $\mathcal{D}$ be a C^p definable cell decomposition of $\mathbb{R}^{2n}$ compatible with $\{Y, \Delta\}$. It follows that $\mathcal{D}' = \pi'(\mathcal{D})$ is a C^p definable cell decomposition of $\mathbb{R}^{2n-1}$ compatible with $\{Y', \Delta'\}$. Set $\mathcal{A} = \{C \in \mathcal{D}' : \dim C_x < n - 1, \forall x \in \mathbb{R}^n\}$ and put $\tilde{\Delta} = \bigcup_{C_i\in\mathcal{A}} C_i$. It is obvious that $\dim \tilde{\Delta}_x < n - 1$. By Lemma 3.3, we have $\Delta' \subset \tilde{\Delta}$.

Lemma 3.5. *Let $x \in \mathbb{R}^n$ be fixed. Suppose that $\overline{B}_\varepsilon(\lambda^*) \subset \mathbb{R}^{n-1} \setminus \tilde{\Delta}_x$. Then the intersection $X \cap C_\varepsilon(x, \lambda^*)$ satisfies the properties (2) and (3) in Theorem 1.1.*

Proof. By the definition, $\tilde{\Delta}_x$ contains all cells of dimension $< n - 1$ of the decomposition $\mathcal{D}'_x$. Because the ball $\overline{B}_\varepsilon(\lambda^*) \cap \tilde{\Delta}_x = \emptyset$, it must be contained in a cell C in $\mathcal{D}'_x$ which has dimension $n - 1$. Let D be the cell in $\mathcal{D}'$ such that $C = D_x$. Notice that D is a cell not contained in $\tilde{\Delta}$ and $\Delta' \subset \tilde{\Delta}$, hence D is outside Δ'. Since the restriction of π' to $Y \setminus \Delta$ is finite, the intersection $\pi'^{-1}(D) \cap Y$ is either empty or a disjoint union of the graphs of C^p definable functions defined on D. These functions are nowhere vanishing since $\{Y_{x,\lambda} \subset \mathbb{R}^*\}$. It induces that $\pi_x'^{-1}(\overline{B}_\varepsilon(\lambda^*)) \cap Y_x$ is either empty or a disjoint union of finitely many graphs of nowhere vanishing C^p functions $\xi_i : \overline{B}_\varepsilon(\lambda^*) \to \mathbb{R}$. Note that $C_\varepsilon(x, \lambda^*) \cap X = \eta_x(\pi'^{-1}(B_\varepsilon(\lambda^*)) \cap Y_x)$. It is nothing to do with the empty case because $C_\varepsilon(x, \lambda^*) \cap X = \emptyset$.

For the non-empty case, consider the restriction of ξ_i to $B_\varepsilon(\lambda^*)$. It follows from Lemma 3.4 that $\eta_x(\Gamma_{\xi_i})$ are disjoint C^p submanifolds of $\mathbb{R}^n$ of the form $\{y = \xi_i(\lambda)(\lambda, 1), \lambda \in B_\varepsilon(\lambda^*)\}$. The property (2) in Theorem 1.1 is then satisfied.

Set $c_i(\lambda) = \frac{|\mathrm{grad}\xi_i(\lambda)|}{|\xi_i(\lambda)|}$. The functions c_i are continuous on $\overline{B}_\varepsilon(\lambda^*)$ since $\xi(\lambda) \neq 0, \forall \lambda \in \overline{B}_\varepsilon(\lambda^*)$. Put $C = \max_i\{\max c_i(\lambda), \lambda \in \overline{B}_\varepsilon(\lambda^*)\}$. Then the constant C satisfies the property (3) in Theorem 1.1. □

Proof of Theorem 1.1. It is known that the set $\{\lambda \in \mathbb{R}^{n-1} : \pi_\lambda|_X \text{ is finite}\}$ is definable and dense in $\mathbb{R}^{n-1}$ (see, [8], Lemma 5.6). Therefore, it contains a definable set, denoted Q, which is open and dense in $\mathbb{R}^{n-1}$. By Lemma 3.5, to prove the theorem it suffices to prove that there is a finite definable partition of $\mathbb{R}^n$ such that for each element C of the partition there is a closed ball B in Q such that $(C \times B) \cap \tilde{\Delta} = \emptyset$.

Note that $\mathcal{D}'$ is a cell decomposition of $\mathbb{R}^{2n-1}$ compatible with $\tilde{\Delta}$. Set $\tilde{\Delta}' = \pi_n^{2n-1}(\tilde{\Delta})$. Applying $(P_{2n-1,n})$ to $(\mathcal{D}', \tilde{\Delta}')$ we get a cell decomposition $\mathcal{B}$ of $\mathbb{R}^n$ compatible with $\tilde{\Delta}'$. Let $C \in \mathcal{B}$. There is a box $I \subset \mathbb{R}^{n-1}$ such that $C \times I$ is contained in a band H of $\mathcal{D}'$. There are two cases: (i) $H \subset \tilde{\Delta}$ and (ii) $H \cap \tilde{\Delta} = \emptyset$. We claim that first case cannot happen. Indeed, if (i) happens, $(C \times I)_x \subset \tilde{\Delta}_x, \forall x \in C$. This is impossible because $\dim(C \times I)_x = \dim I = n-1$ while $\dim \tilde{\Delta}_x < n-1$ (see the definition of $\tilde{\Delta}$). Since Q is a definable, open and dense set in $\mathbb{R}^{n-1}$, $I \cap Q$ is a definable set of dimension $n-1$. Choose B to be a closed ball contained in $I \cap Q$. It is obvious that $(C \times B) \cap \tilde{\Delta} = \emptyset$. The theorem is proved.

Remark 3.6. Suppose that the o-minimal structure that we are considering allows C^∞ (resp. C^ω) cell decompositions. Then by the same arguments as above, the main theorem still holds if we replace $p \in \mathbb{N}$ with $p = \infty$ (resp. $p = \omega$).

4. Regular covers for definable sets

Let $U \subset \mathbb{R}^n$ be an open definable set. A **definable regular cover** for U is a finite family of open definable sets $\{U_i\}$ such that

(i) $U = \bigcup_i U_i$,

(ii) each U_i is definably homeomorphic to an open n-dimensional ball,

(iii) there is $C > 0$ such that for any $x \in U$,

$$\operatorname{dist}(x, \mathbb{R}^n \setminus U) \le C \max_i \{\operatorname{dist}(x, \mathbb{R}^n \setminus U_i)\}.$$

Parusiński [9] proved that every open bounded subanalytic subset of $\mathbb{R}^n$ has a subanalytic regular cover. By Theorem 1.1 and arguments as in the proof of Theorem 0.2, [9] we have

Theorem 4.1. *There always exists a definable regular cover for a given open bounded definable set.*

Remark 4.2. The boundedness in Theorem 4.1 is necessary. For example, consider $U = \mathbb{R}^n \setminus \{0\}$. We will show that there is no definable cover for U. Indeed, assume on the contrary that there is a definable regular cover $\mathcal{U} = \{U_i\}$ for U. Since U is unbounded, there are $U_i \in \mathcal{U}$ unbounded and a sequence $\{x_k\}$ in $U \setminus \cup_{j \neq i} U_j$ such that $\{u_k\} \to \infty$ and $\operatorname{dist}(u_k, \mathbb{R}^n \setminus U_i) \le 1$. Then,

$$\operatorname{dist}(x_k, \mathbb{R}^n \setminus U) = \|x_k\| \to \infty \text{ and } \max_m \{\operatorname{dist}(x_k, \mathbb{R}^n \setminus U_m)\} = \operatorname{dist}(x_k, \mathbb{R}^n \setminus U_i) \le 1$$

Therefore, there is no $C > 0$ such that the condition (iii) satisfies. This gives a contradiction.

References

[1] M. Coste. *An introduction to o-minimal geometry.* Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.

[2] S. Guillermou and P. Schapira. *Construction of sheaves on the subanalytic site.* Astérisque., No. 383, 1–60, 2016.

[3] G. Lebeau. *Sobolev spaces and Sobolev sheaves.* Astérisque., No. 383, 61–94, 2016.

[4] T. L. Loi. *Transversality theorem in o-minimal structures.* Compos. Math., 144(5):1227–1234, 2008.

[5] Mostowski. *Lipschitz equisingularity.* Dissertationes Math. (Rozprawy Mat.), 243 (46), 1985.

[6] A. Parusiński. *Regular projections for sub-analytic sets.* C. R. Acad. Sci. Paris., 307, 343–347, 1988.

[7] A. Parusiński. *Lipschitz Properties of semi-analytic sets.* Ann. Inst. Fourier, Grenoble., Vol. 38, No. 4, 1189–213, 1988

[8] A. Parusiński. *Lipschitz stratification of subanalytic sets.* Ann. Sci. Ecole Norm. Sup., 27, 661–696, 1994.

[9] A. Parusiński. *Regular subanalytic covers.* Astérisque., 383(1):95–102, 2016.

[10] W. Pawłucki. *Le theorème de Puiseux pour une application sous-analytique.* Bull. Acad. Polon. Sci. Ser. Math., Vol. 32, 555–560, 1984.

[11] L. van den Dries. Tame topology and o-minimal structures. Vol 248 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1998. Math., Vol. 32, 555–560, 1984.

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