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RESEARCH ARTICLE

Positivity and stability of mixed fractional-order systems with unbounded delays: Necessary and sufficient conditions

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Abstract

This paper provides a comprehensive study on quantitative properties of linear mixed fractional-order systems with multiple time-varying delays. The delays can be bounded or unbounded. We first obtain a result on existence and uniqueness of solutions to these systems. Then, we prove a necessary and sufficient condition for their positivity. Finally, we provide a necessary and sufficient criterion to characterize asymptotic stability of positive linear mixed fractional-order systems with multiple time-varying delays.

KEYWORDS:
Fractional differential equations, Linear mixed fractional-order systems, Time-varying delays, Positive systems, Asymptotic stability

1 | INTRODUCTION

Fractional differential equations are widely used to describe memory and hereditary properties of materials and processes. For details, see[1,2,3] and the references therein. On the other hand, time-delay systems have received considerable attention due to the fact that many processes include after-effect phenomena in their inner dynamics, see, e.g.,[4,5,6]. While positive systems play a key role in understanding many processes in biological and medical sciences, see e.g.,[7,8]. Recently, positive system theory has gained renewed interest from the viewpoint of convex optimization. We refer the reader to an interesting paper on L_q/L_p Hankel norm[9] and a paper on geometric programming for optimal positive linear systems[10]. As such, the qualitative theory of positive fractional-order systems with delays is an important research topic, which is the main focus of this paper.

One of the important problems in the dynamical system theory of time-delay fractional-order systems is stability analysis. Using the characteristic polynomial, in[11,12], the authors obtained conditions depending on magnitude of the delay for asymptotic stability of fractional-order systems with the linear part comprises a pure delay. In[13,14], by using Lyapunov-candidate-functions, the authors proposed some results on stability of fractional systems with delays. An analytical approach based on the Laplace transform and 'inf-sup' method for studying finite-time stability of singular fractional-order switched systems with delay was presented in[15]. By using the Lyapunov method combined with the concept of uniformly positive definite matrix functions and Hamilton–Jacobi–Riccati inequalities, robust stability of the almost periodic solution to uncertain impulsive functional differential systems of fractional order was investigated in[16]. In[17] the authors studied robust stability of a fractional-order time-delay system in the frequency domain based on finite spectrum assignment.

Up to now, in our view, an important contribution to the study of asymptotic behavior of solutions to positive mixed fractional-order systems with delays is the paper by Shen and Lam[18]. In that paper, the authors reported a criterion for positivity of a linear mixed fractional-order systems with a time-varying delay. They also obtained a result on asymptotic stability of a positive linear mixed fractional-order system with a bounded time-varying delay.

Let $d \in \mathbb{N}$, $\hat{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0, 1] \times \cdots \times (0, 1]$, $r > 0$, $m \in \mathbb{N}$. Motivated by [18], in this paper, we consider the following linear mixed fractional-order systems with multiple *unbounded* time-varying delays

$$ {}^C D^{\hat{\alpha}}_{0+} x(t) = Ax(t) + \sum_{1}^{m} B_k x(t - h_k(t)), \ t > 0, \tag{1} $$

with the initial condition $x(\cdot) = \phi(\cdot) \in C([-r, 0]; \mathbb{R}^d)$ on $[-r, 0]$, where

$$ {}^C D^{\hat{\alpha}}_{0+} x(t) = ({}^C D^{\alpha_1}_{0+} x_1(t), \ldots, {}^C D^{\alpha_k}_{0+} x_k(t), \ldots, {}^C D^{\alpha_d}_{0+} x_d(t))^{\mathrm{T}} $$

is a column vector in which ${}^C D^{\alpha_k}_{0+}$ is the Caputo derivative operator of the order α_k, $A = (a_{ij})_{1 \le i,j \le d}$, $B_k = (b^k_{ij})_{1 \le i,j \le d}$, $h_k : [0, \infty) \to \mathbb{R}_{\ge 0}$ is continuous and satisfies the growth rate as in [19]. Our main aim is to study asymptotic stability of system (1) for the case it is positive. It is worth noting that the approaches as in [11,12] (based on the eigenvalues of the characteristic polynomials) and [18] (based on comparing the trajectory of the time-varying delay system with that of the constant delay system) cannot be applied for (1) where the delays $h_k(\cdot)$ $(1 \le k \le m)$ are time-varying and *unbounded*.

This paper is organized as follows. In Section 2, we first introduce a result on existence and uniqueness of global solutions to linear mixed fractional-order with multiple time-varying delays. Then, we give a necessary and sufficient condition to characterize positivity of these systems. The main result of the paper is given in Section 3. In particular, in Theorem 1, we provide a necessary and sufficient criterion to ensure asymptotic stability of positive linear mixed fractional-order systems with bounded and unbounded time-varying delays.

Before concluding this section, we introduce some notations which are used throughout this paper. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}_{\ge 0}$ be the set of nonnegative integers, $\mathbb{R}$ ($\mathbb{R}_{\ge 0}$) be the set of real numbers (nonnegative real numbers, respectively), and $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with a norm $\|\cdot\|$. Without loss of generality, in this paper we use the symbol $\|\cdot\|$ to denote the max norm of Euclidean spaces. For any $[a, b] \subset \mathbb{R}$, let $C([a, b]; \mathbb{R}^d)$ be the space of continuous functions $\xi : [a, b] \to \mathbb{R}^d$. A matrix $A = (a_{ij})_{1 \le i,j \le d} \in \mathbb{R}^{d \times d}$ is called Metzler if $a_{ij} \ge 0$ for all $1 \le i \ne j \le d$. A matrix $A \in \mathbb{R}^{d \times d}$ is said to be Hurwitz if its spectrum $\sigma(A)$ satisfies

$$ \sigma(A) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}. $$

Let $n, m \in \mathbb{N}$ and $A = (a_{ij})^{1 \le j \le m}_{1 \le i \le n}$, $B = (B_{ij})^{1 \le j \le m}_{1 \le i \le n} \in \mathbb{R}^{n \times m}$. We write $A \succ B$ ($A \succeq B$) if $a_{ij} > b_{ij}$ ($a_{ij} \ge b_{ij}$, respectively) for all $1 \le i \le n, \ 1 \le j \le m$. The matrix A is said to be nonnegative if $a_{ij} \ge 0$ for all $1 \le i \le n$, $1 \le j \le m$. For $\alpha \in (0, 1)$ and an integrable function $x : [a, b] \to \mathbb{R}$, the Riemann–Liouville integral operator of $x(\cdot)$ with the order α is defined by

$$ (I^{\alpha}_{a+} x)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} x(\tau) \, d\tau, \quad t \in (a, b], $$

where $\Gamma(\cdot)$ is the Gamma function. The *Caputo fractional derivative* ${}^C D^{\alpha}_{a+} x$ of a function $x \in AC([a, b]; \mathbb{R})$ is defined by

$$ ({}^C D^{\alpha}_{a+} x)(t) := (I^{1-\alpha}_{a+} Dx)(t), \quad t \in (a, b], $$

where $AC([a, b]; \mathbb{R})$ denotes the space of absolutely continuous functions and D is the classical derivative.

2 | POSITIVITY OF LINEAR MIXED-ORDER FRACTIONAL SYSTEMS WITH TIME-VARYING DELAYS

Let $\hat{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0, 1] \times \cdots \times (0, 1] \subset \mathbb{R}^d$, $T, r > 0$, $m \in \mathbb{N}$. Consider the following system on $(0, T]$

$$ {}^C D^{\hat{\alpha}}_{0+} x(t) = Ax(t) + \sum_{1}^{m} B_k x(t - h_k(t)) + Uw(t), \tag{2} $$

and $x(\cdot) = \phi(\cdot) \in C([-r, 0]; \mathbb{R}^d)$ on $[-r, 0]$, where $A = (a_{ij})_{1 \le i,j \le d}$, $B_k = (b^k_{ij})_{1 \le i,j \le d}$ $(1 \le k \le m)$, $U = (u_{ij})_{1 \le i,j \le d} \in \mathbb{R}^{d \times d}$ and $w(\cdot) \in C([0, T]; \mathbb{R}^d)$. Assume that $h_k : [0, T] \to \mathbb{R}_{\ge 0}$ $(1 \le k \le m)$ is continuous such that

(F1) $h_k(0) > 0$;

(F2) $t - h_k(t) \ge -r$ for all $t \in [0, T]$;

(F3) $h_k(0) \ne h_l(0)$ for any $1 \le k \ne l \le m$.

Using the same arguments as in the proof of[20, Lemma 6.2, pp. 86], we see that a vector valued function $\varphi(\cdot,\phi) \in C([-r,T];\mathbb{R}^d)$ is a solution of (2) with $x(\cdot) = \phi(\cdot)$ on $[-r,T]$ if and only if it satisfies the time-delay integral system on $(0,T]$,

$$x_i(t) = \phi_i(0) + \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} \sum_{1\le j\le d} \Big(a_{ij}x_j(s) + \sum_{1\le k\le m} b^k_{ij} x_j(s-h_k(s)) + u_{ij}w_j(s)\Big)ds,\ 1\le i\le d,$$

and $x(\cdot) = \phi(\cdot)$ on $[-r,0]$.

Surprisingly, up to now, there has been no result reported in the literature on existence and uniqueness of solutions to mixed fractional-order systems with multiple time-varying delays. Hence, we first introduce here a rigorous proof for existence and uniqueness of global solutions to the system in (2).

Lemma 1 (Existence and uniqueness of linear mixed fractional-order with time-varying delays)**.** Assume that $h_k : [0,T] \to \mathbb{R}_{\ge 0}$ $(1 \le k \le m)$ is continuous such that condition (F2) holds. Then, for any $\phi(\cdot) \in C([-r,0];\mathbb{R}^d)$ and $w(\cdot) \in C([0,T];\mathbb{R}^d)$, system (2) with initial condition $x(t) = \phi(t),\ t \in [-r,0]$ has a unique solution $\varphi(\cdot,\phi)$ on $[-r,T]$.

Proof. Let

$$C_\phi := \big\{\xi \in C([-r,T];\mathbb{R}^d) : \xi(t) = \phi(t),\ t\in[-r,0]\big\}$$

and define a functional $\|\cdot\|_\gamma$ on C_ϕ by

$$\|\xi\|_\gamma = \max_{t\in[0,T]} \frac{\xi^*(t)}{\exp(\gamma t)},$$

where $\gamma > 0$ is fixed and chosen later and $\xi^*(t) = \max_{-r\le\theta\le t} \|\xi(\theta)\|$. Notice that $\|\cdot\|_\gamma$ is a norm and $(C_\phi, \|\cdot\|_\gamma)$ is a Banach space. On this space, we establish an operator $\mathcal{T}_\phi : C_\phi \to C_\phi$ as follows.

$$(\mathcal{T}_\phi\xi)^i(t) = \phi_i(0) + \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} \Big(\sum_{1\le j\le d} a_{ij}\xi_j(s) + \sum_{1\le k\le m} b^k_{ij}\xi_j(s-h_k(s)) + u_{ij}w_j(s)\Big)d\tau,$$

for $t \in (0,T]$, $1\le i\le d$, and $(\mathcal{T}_\phi\xi)(t) = \phi(t)$ on $[-r,0]$. To complete the proof of this lemma, we only have to show that $\mathcal{T}_\phi$ is contractive. For that, for any $\xi(\cdot), \hat{\xi}(\cdot) \in C_\phi$, $t\in[0,T]$, $1\le i\le d$, we have

$$\begin{aligned}
I(t) &= |(\mathcal{T}_\phi\xi)^i(t) - (\mathcal{T}_\phi\hat{\xi})^i(t)| \\
&\le \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} \sum_{1\le j\le d} \Big(|a_{ij}||\xi_j(s) - \hat{\xi}_j(s)| \\
&\qquad + \sum_{1\le k\le m} |b^k_{ij}||\xi_j(s-h_k(s)) - \hat{\xi}_j(s-h_k(s))|\Big)ds \\
&\le \frac{\max_{1\le i\le d}\big(\sum_{j=1}^d (|a_{ij}| + \sum_{1\le k\le m}|b^k_{ij}|)\big)\exp(\gamma t)}{\Gamma(\alpha_i)} \\
&\qquad \times \int_0^t (t-s)^{\alpha_i-1}\exp(-\gamma(t-s))\frac{(\xi-\hat{\xi})^*(s)}{\exp(\gamma s)}ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
I(t) &\le \frac{\max_{1\le i\le d}\big(\sum_{1\le j\le d}(|a_{ij}| + \sum_{1\le k\le m}|b^k_{ij}|)\big)\exp(\gamma t)}{\Gamma(\alpha_i)} \int_0^t v^{\alpha_i-1}\exp(-\gamma v)dv\|\xi-\hat{\xi}\|_\gamma \\
&\le \frac{\max_{\le i\le d}\big(\sum_{1\le j\le d}(|a_{ij}| + \sum_{1\le k\le m}|b^k_{ij}|)\big)\exp(\gamma t)}{\Gamma(\alpha_i)\lambda^{\alpha_i}} \int_0^{\gamma t} u^{\alpha_i-1}\exp(-u)du\|\xi-\hat{\xi}\|_\gamma,
\end{aligned}$$

which implies

$$\frac{|(\mathcal{T}_\phi\xi)^i(t) - (\mathcal{T}_\phi\hat{\xi})^i(t)|}{\exp(\gamma t)} \le \max_{1\le i\le d} \frac{\sum_{1\le j\le d}(|a_{ij}| + \sum_{1\le k\le m}|b^k_{ij}|)}{\gamma^{\alpha_i}}\|\xi-\hat{\xi}\|_\gamma, \tag{3}$$

where we used the fact that

$$\int_0^\infty u^{\alpha_i-1}\exp(-u)du = \Gamma(\alpha_i)$$

and the estimates

$$\|\xi(s)-\hat{\xi}(s)\|,\ \|\xi(s-h_k(s))-\hat{\xi}(s-h_k(s))\| \le (\xi-\hat{\xi})^*(s)$$

for $s\in[0,T]$, $1\le k\le m$. From (3), we obtain

$$\frac{(\mathcal{T}_\phi\xi-\mathcal{T}_\phi\hat{\xi})^*(t)}{\exp(\gamma t)} \le \max_{1\le l\le d}\frac{\max_{1\le i\le d}\sum_{j=1}^d(|a_{ij}|+\sum_{1\le k\le m}|b_{ij}^k|)}{\gamma^{\alpha_l}}\|\xi-\hat{\xi}\|_\gamma$$

for all $t\in[0,T]$. Thus,

$$\|\mathcal{T}_\phi\xi-\mathcal{T}_\phi\hat{\xi}\|_\gamma \le \max_{1\le l\le d}\frac{\max_{1\le i\le d}\sum_{1\le j\le d}(|a_{ij}|+\sum_{1\le k\le m}|b_{ij}^k|)}{\gamma^{\alpha_l}}\|\xi-\hat{\xi}\|_\gamma.$$

By choosing $\gamma>0$ such that

$$\max_{1\le l\le d}\frac{\max_{1\le i\le d}\sum_{1\le j\le d}(|a_{ij}|+\sum_{1\le k\le m}|b_{ij}^k|)}{\gamma^{\alpha_l}} < 1,$$

then $\mathcal{T}_\phi$ is contractive. Banach fixed point theorem implies that this operator has a fixed point in $(C_\phi,\|\cdot\|_\gamma)$ which is also the unique solution to initial value problem (2) with the initial condition $x(t)=\phi(t)$, $t\in[-r,0]$. The proof is complete. □

Our main aim in this section is to introduce a criterion to characterize positivity of linear mixed-order fractional systems with time-varying delays.

Definition 1. System (2) is positive if for any $\phi(t)\ge 0$ on $[-r,0]$ and $w(t)\ge 0$ on $[0,T]$, its solution $\varphi(\cdot,\phi)$ satisfies $\varphi(t,\phi)\ge 0$ on $[0,T]$.

The main result in this section is the following proposition.

Proposition 1 (Necessary and sufficient condition for positivity)**.** Let $h_k:[0,T]\to\mathbb{R}_{\ge 0}$ $(1\le k\le m)$ be continuous such that conditions (F1), (F2) and (F3) hold. Then, system (2) is positive if and only if A is Metzler, B_k $(1\le k\le m)$ and U are nonnegative.

Proof. **Necessity:** Let system (2) be positive. We first show that $U=(u_{ij})_{1\le i,j\le d}$ is nonnegative. To do this, assume that there is an element $u_{i_0j_0}<0$. By choosing $\phi(t)=0$ on $[-r,0]$ and $w(t)=e_{j_0}$ on $[0,T]$, we have the representation of the i_0-component of $\varphi(\cdot,\phi)$ as

$$\begin{aligned}\varphi_{i_0}(t,\phi) =& \frac{1}{\Gamma(\alpha_{i_0})}\int_0^t (t-s)^{i_0-1}\sum_{1\le j\le d}a_{i_0j}\varphi_j(s,\phi)ds + \frac{1}{\Gamma(\alpha_{i_0})}\int_0^t (t-s)^{i_0-1}\sum_{1\le k\le m}\sum_{1\le j\le d}b_{i_0j}^k\varphi_j(s-h_k(s),\phi)ds\\ &+\frac{1}{\Gamma(\alpha_{i_0})}\int_0^t (t-s)^{i_0-1}u_{i_0j_0}ds,\quad t\in[0,T],\end{aligned}$$

where $e_{j_0}=(0,\dots,1,\dots,0)^{\mathrm{T}}$ denotes the unit vector in $\mathbb{R}^d$ with the j_0-coordinate equals to 1. Hence, for $t_0>0$ small enough, for example, for all $t\in[0,t_0]$,

$$t-h_k(t) < -\max_{1\le k\le m}h_k(0)/2,\quad \sum_{1\le j\le d}a_{i_0j}\varphi_j(s,\phi) < |u_{i_0j_0}|,$$

then

$$\begin{aligned}\varphi_{i_0}(t_0,\phi) &= \frac{1}{\Gamma(\alpha_{i_0})}\int_0^{t_0}(t_0-s)^{i_0-1}\sum_{1\le j\le d}a_{i_0j}\varphi_j(s,\phi)ds + \frac{1}{\Gamma(\alpha_{i_0})}\int_0^{t_0}(t_0-s)^{i_0-1}u_{i_0j_0}ds\\ &< 0,\end{aligned}$$

a contradiction. Next, assume, ad absurdum, $A = (a_{ij})_{1\le i,j\le d}$ is not Metzler, that is, there exist indexes $1 \le i_0 \ne j_0 \le d$ such that $a_{i_0 j_0} < 0$. Let $\phi(\cdot) \in C([-r,0];\mathbb{R}^d)$ be a vector valued function with

$$\phi(t) = \begin{cases} e_{j_0}, & \text{if} \quad t = 0, \\ 0, & \text{if} \quad t \in [-r, -\max_{1\le k\le m} h_k(0)/2], \end{cases}$$

and $w(t) = 0$ on $[0,T]$. Due to continuity of solutions and the delay $h_k(\cdot)$ and that $h_k(0) > 0$ $(1 \le k \le m)$, we can find $t_0 > 0$ (small enough) such that $t - h_k(t) \le -\max_{1\le k\le m} h_k(0)/2$, $\varphi_{j_0}(t,\phi) > 1/2$, and $\sum_{1\le j\le d, j\ne j_0} a_{i_0 j}\varphi_j(t,\phi) < \frac{|a_{i_0 j_0}|}{2}$ for all $t \in [0,t_0]$. Then, the i_0-component of $\varphi(t_0,\phi)$ satisfies

$$\varphi_{i_0}(t_0,\phi) = \frac{1}{\Gamma(\alpha_{i_0})}\int_0^{t_0} (t_0 - s)^{i_0 - 1} a_{i_0 j_0}\varphi_{j_0}(s,\phi)ds + \frac{1}{\Gamma(\alpha_{i_0})}\int_0^{t_0} (t_0 - s)^{i_0-1} \sum_{1\le j\le d, j\ne j_0} a_{i_0 j}\varphi_j(s,\phi)ds < 0,$$

a contradiction. We now prove that B_k is nonnegative for any $1 \le k \le m$. From (F1) and (F3), without loss of generality, let $0 < h_1(0) < \cdots < h_m(0)$. First, we show that B_1 is nonnegative. Suppose, ad absurdum, $B_1 = (b^1_{ij})_{1\le i,j\le d}$ is not nonnegative. That is, there is $b^1_{i_0 j_0} < 0$. Choose $\phi(\cdot) \in C([-r,0];\mathbb{R}^d)$ such that

$$\phi(t) = \begin{cases} 0, & \text{if} \quad t = 0, \\ e_{j_0}, & \text{if} \quad t \in [\frac{-2h_1(0) - h_2(0)}{3}, \frac{-h_1(0)}{2}], \\ 0, & \text{if} \quad t \in [-r, \frac{-2h_2(0) - h_1(0)}{3}], \end{cases}$$

and $w(t) = 0$ on $[0,T]$. Then, for $t_0 > 0$ small enough so that on the interval $[0,t_0]$:

- $\frac{-2h_1(0) - h_2(0)}{3} \le t - h_1(t) \le \frac{-h_1(0)}{2}$;
- $-r \le t - h_k(t) \le \frac{-2h_2(0) - h_1(0)}{3}$, $2 \le k \le m$;
- $\sum_{1\le j\le d} a_{i_0 j}\varphi_j(t,\phi) < |b^1_{i_0 j_0}|$.

Then, the i_0-component of the solution $\varphi(\cdot,\phi)$ at $t = t_0$ verifies

$$\varphi_{i_0}(t_0,\phi) = \frac{1}{\Gamma(\alpha_{i_0})}\int_0^{t_0} (t_0 - s)^{i_0-1} b^1_{i_0 j_0} ds + \frac{1}{\Gamma(\alpha_{i_0})}\int_0^{t_0} (t_0 - s)^{i_0 - 1}\sum_{1\le j\le d} a_{i_0 j}\varphi_j(s,\phi)ds < 0,$$

which implies a contradiction. By similar arguments, we also see B_k, $2 \le k \le m$, is nonnegative. Thus, B_k $(1 \le k \le m)$ are nonnegative.

Sufficiency: Let $A = (a_{ij})_{1\le i,j\le d}$ be Metzler and $B_k = (b^k_{ij})_{1\le i,j\le d}$, $U = (u_{ij})_{1\le i,j\le d}$ be nonnegative. We first show that if $\phi(t) \succ 0$ on $[-r,0]$ and $w(t) \succeq 0$ on $[0,T]$, then $\varphi(t,\phi) \succeq 0$ on $[0,T]$. Indeed, due to the fact that A is Metzler, there exists a positive constant $\rho > 0$ such that

$$A = -\rho I_d + (\rho I_d + A),$$

where $\rho I_d + A$ is nonnegative. Then, system (2) is rewritten as

$$^C D^{\hat{\alpha}}_{0+} x(t) = \rho I_d x(t) + (\rho I_d + A)x(t) + \sum_{1\le k\le m} B_k x(t - h_k(t)) + U w(t), \quad t \in (0,T].$$

By virtue of the variation of constants formula (see, e.g.,[21, Lemma 3.1]), the solution $\varphi(\cdot,\phi) = (\varphi_1(\cdot,\phi),\ldots,\varphi_d(\cdot,\phi))^{\mathrm{T}}$ of (2) with $\varphi(\cdot,\phi) = \phi(\cdot)$ on $[-r,0]$ has the following form:

$$\begin{aligned} \varphi_i(t,\phi) = & E_{\alpha_i}(-\rho t^{\alpha_i})\phi_i(0) + \int_0^t (t-s)^{\alpha_i - 1} E_{\alpha_i,\alpha_i}(-\rho(t-s)^{\alpha_i}) \sum_{1\le j\le d} \Big((a_{ij} + \rho\delta_{ij})\varphi_j(s,\phi) \\ & + \sum_{1\le k\le m} b^k_{ij}\varphi_j(s - h_k(s),\phi) + u_{ij}w_j(s)\Big)ds \end{aligned} \tag{4}$$

for $t \in [0,T]$, $1 \le i \le d$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \ne j, \end{cases}$$

and

$$E_{\alpha_i}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha_i + 1)}, \quad E_{\alpha_i,\alpha_i}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha_i + \alpha_i)}$$

are Mittag-Leffler functions. Suppose that there exists $t_0 > 0$ so that $\varphi(t_0, \phi) \not\succeq 0$. From this, we can find an index $i_0 \in \{1, \ldots, d\}$ satisfying $\varphi_{i_0}(t_0, \phi) = 0$. Take

$$t^* = \inf\{t \in [0, T] : \varphi_{i_0}(t, \phi) = 0\}.$$

Then $t^* > 0$, $\varphi_{i_0}(t^*, \phi) = 0$ and $\varphi_{i_0}(t, \phi) > 0$ for all $t \in [0, t^*)$. However, from (4),

$$\begin{aligned}\varphi_{i_0}(t^*, \phi) = {} & E_{\alpha_{i_0}}(-\rho t^{*\alpha_{i_0}})\phi_{i_0}(0) + \int_0^{t^*} (t^* - s)^{\alpha_{i_0}-1} E_{\alpha_{i_0},\alpha_{i_0}}(-\rho(t^* - s)^{\alpha_{i_0}}) \sum_{1\le j\le d} \Big((a_{i_0 j} + \rho\delta_{i_0 j})\varphi_j(s, \phi) \\ & + \sum_{1\le k\le m} b^k_{i_0 j}\varphi_j(s - h_k(s), \phi) + u_{i_0 j} w_j(s)\Big) ds \\ \ge {} & E_{\alpha_{i_0}}(-\rho t^{*\alpha_{i_0}})\phi_{i_0}(0) > 0,\end{aligned}$$

a contradiction. Thus, $\varphi(t, \phi) \succ 0$ on $[0, T]$. We now consider the case where inputs $\phi(t) \succeq 0$ on $[-r, 0]$ and $w(t) \succeq 0$ on $[0, T]$. Using arguments as in[22, Proposition 1], we get the initial conditions $\phi^n(\cdot) = \phi(\cdot) + \frac{1}{n}\mathbf{1}$ on $[-r, 0]$ with $n \in \mathbb{N}$ and $\mathbf{1} = (1, \ldots, 1)^{\mathrm{T}}$. It is obvious to see that $\{\varphi(\cdot, \phi^n)\}_{n=1}^{\infty}$ is a decreasing sequence of continuous positive functions on $[-r, T]$. Define $\varphi^*(t) := \lim_{n\to\infty} \varphi(t, \phi_n)$ for each $t \in [-r, T]$. By Dini's theorem (see, e.g.,[23, Theorem 7.13, pp. 150]), the sequence $\{\varphi(\cdot, \phi^n)\}_{n=1}^{\infty}$ converges uniformly to $\varphi^*(\cdot)$ and this function is also continuous and nonnegative on $[-r, T]$. Notice that for each $n \in \mathbb{N}$, $\varphi(\cdot, \phi^n)$ verifies

$$\begin{aligned}\varphi_i(t, \phi^n) = {} & E_{\alpha_i}(-\rho t^{\alpha_i})(\phi^n(0))_i + \int_0^t (t - s)^{\alpha_i-1} E_{\alpha_i,\alpha_i}(-\rho(t - s)^{\alpha_i}) \sum_{1\le j\le d} \Big((a_{ij} + \rho\delta_{ij})\varphi_j(s, \phi^n) \\ & + \sum_{1\le k\le m} b^k_{ij}\varphi_j(s - h_k(s), \phi^n) + u_{ij} w_j(s)\Big) ds,\end{aligned}$$

for $1 \le i \le d$, $t \in [0, T]$ and $\varphi(t, \phi^n) = \phi^n(t)$ on $[-r, 0]$. Let $n \to \infty$, we obtain

$$\begin{aligned}\varphi^*_i(t) = {} & E_{\alpha_i}(-\rho t^{\alpha_i})\phi_i(0) + \int_0^t (t - s)^{\alpha_i-1} E_{\alpha_i,\alpha_i}(-\rho(t - s)^{\alpha_i}) \sum_{1\le j\le d} \Big((a_{ij} + \rho\delta_{ij})\varphi^*_j(s) \\ & + \sum_{1\le k\le m} b^k_{ij}\varphi^*_j(s - h_k(s)) + u_{ij} w_j(s)\Big) ds,\end{aligned}$$

for $1 \le i \le d$, $t \in [0, T]$ and $\varphi^*(t) = \phi(t)$ on $[-r, 0]$. Since the original system has a unique solution (see Lemma 1) and it has the form as in (4), $\varphi^*(\cdot)$ is the unique solution of this system. On the other hand, as shown above, $\varphi^*(t) \succeq 0$ on $[-r, T]$, which implies that $\varphi(\cdot, \phi)$ is nonnegative on the existence interval $[0, T]$. The proof is complete. □

Remark 1. In the classical case, to prove the positivity of the time-delay system

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + \sum_{1\le k\le m} B_k x(t - h_k), & t \ge 0, \\ x(t) = \phi(t) \in \mathbb{R}^d, & t \in [-r, 0], \end{cases}$$

one usually adopts the following representation for its solution on $[0, \infty)$

$$x(t) = \exp(tA)\phi(0) + \int_0^t \exp((t - s)A) \sum_1^m B_k x(s - h_k)) ds,$$

see, e.g.,[24, Proposition 3.1]. In our opinion, this approach is also true for time-delay systems with a non-integer derivative. However, it does not work for mixed fractional-order systems because there is not a similar variation of constants formula for solution to these systems.

3 | ASYMPTOTIC STABILITY OF POSITIVITY OF POSITIVE LINEAR MIXED-ORDER SYSTEMS WITH TIME-VARYING DELAYS

Let $\hat{\alpha} = (\alpha_1, \dots, \alpha_d) \in (0,1] \times \cdots \times (0,1] \subset \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, $r > 0$, $m \in \mathbb{N}$. In this section, we consider the following linear mixed-order fractional system on $(0, \infty)$

$$^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + \sum_{1\le k\le m} B_k x(t - h_k(t)) \tag{5}$$

with $x(\cdot) = \phi(\cdot) \in C([-r,0];\mathbb{R}^d)$ on $[-r,0]$, where $A, B_k \in \mathbb{R}^{d\times d}$, $h_k : [0,\infty) \to \mathbb{R}_+$ $(1 \le k \le m)$ is continuous and satisfies the following conditions

(G1) $h_k(0) > 0$;

(G2) $t - h_k(t) \ge -r$ for all $t \in [0,\infty)$;

(G3) $h_k(0) \ne h_l(0)$ for any $1 \le k \ne l \le m$

(G4) $\lim_{t\to\infty} t - h_k(t) = \infty$ $(1 \le k \le m)$.

For linear systems, asymptotic stability in the Lyapunov sense and attractivity are equivalent, see e.g.,[25, Theorem 6]. Hence, in this paper, we use the following definition for asymptotic stability of system (5).

Definition 2. System (5) is said to be asymptotically stable if for any $\phi(\cdot) \in C([-r,0];\mathbb{R}^d)$, its solution $\varphi(\cdot,\phi)$ converges to the origin as $t \to \infty$.

Based on Proposition 1 about positivity of time-delay linear fractional-order systems, we obtain a necessary and sufficient condition for asymptotic stability of positive linear mixed-order fractional systems with unbounded time-varying delays in the following theorem.

Theorem 1 (A characterization of the asymptotic stability)**.** Assume that system (5) is positive. Then, it is asymptotically stable if and only if $A + \sum_{1\le k\le m} B_k$ is Hurwitz.

Proof. **Necessity:** Let the positive system (5) be asymptotically stable. Suppose, ad absurdum, $A + \sum_{1\le k\le m} B_k$ is not Hurwitz. Notice that A is Metzler and B_k $(1 \le k \le m)$ is nonnegative and thus $A + \sum_{1\le k\le m} B_k$ is also Metzler. From[26, Theorem 2.5.3, p. 114], we have $(A + \sum_{1\le k\le m} B_k)\lambda \ge 0$ for any $\lambda \succ 0$. Choose and fix such a positive vector $\lambda \in \mathbb{R}^d$, and put $e_0(t) := \varphi(t,\lambda) - \lambda$ for all $t \in [-r,\infty)$. Then, $e_0(\cdot)$ is the unique solution to the system

$$\begin{aligned} ^{C}D^{\hat{\alpha}}_{0+}x(t) &= Ax(t) + \sum_{1\le k\le m} B_k x(t - h_k(t)) + (A + \sum_{1\le k\le m} B_k)\lambda, \quad t > 0, \\ x(\cdot) &= 0 \quad \text{on } [-r,0]. \end{aligned} \tag{6}$$

On the other hand, by virtue Proposition 1, system (6) is positive. Hence, $e_0(t) \ge 0$ on $[0,\infty)$. This implies that $\varphi(t,\lambda) \ge \lambda \succ 0$, $\forall t \in [0,\infty)$. It is a contradiction because from the original assumption, $\lim_{t\to\infty} \varphi(t,\lambda) = 0$.

Sufficiency: Let $A + \sum_{1\le k\le m} B_k$ be Hurwitz. By virtue[26, Theorem 2.5.3, p. 114], we can find a vector $\lambda \succ 0$ such that

$$(A + \sum_{1\le k\le m} B_k)\lambda \prec 0. \tag{7}$$

First step: In this step, we will prove that there exists $t_1 > 0$ and $\nu \in (0,1)$ such that

$$\varphi(t,\lambda) \prec \nu\lambda, \quad \forall t \ge t_1. \tag{8}$$

For that, at first, let $u_0(t) = \lambda - \varphi(t,\lambda)$, $t \ge -r$. Then, $u_0(\cdot)$ is the unique solution of the system

$$\begin{aligned} ^{C}D^{\hat{\alpha}}_{0+}u_0(t) &= Au_0(t) + \sum_{1\le k\le m} B_k u_0(t - h_k(t)) - (A + \sum_{1\le k\le m} B_k)\lambda, \quad t > 0, \\ u_0(t) &= 0, \quad t \in [-r,0]. \end{aligned}$$

This system is positive, hence, $u_0(t) \ge 0$ on $[0,\infty)$, which implies that $\varphi(t,\lambda) \preceq \lambda$ for all $t \ge 0$. Next, let $y(\cdot)$ is the unique solution of the system

$$\begin{cases} ^{C}D^{\hat{\alpha}}_{0+}y(t) = Ay(t) + \sum_{1\le k\le m} B_k\lambda, \quad t > 0, \\ y(0) = \lambda. \end{cases}$$

Using the same arguments as above, we see that $0 \prec y(t) \preceq \lambda$ for all $t \geq 0$. Moreover,

$$0 \preceq \varphi(t, \lambda) \preceq y(t), \quad t \geq 0. \tag{9}$$

Now, for any $c > 0$, define $u_1(t) = y(t) - y(t + c)$, $t \geq 0$. This vector valued function satisfies the system

$$\begin{cases} {}^C D_{0+}^{\hat{\alpha}} u_1(t) = Au_1(t), \quad t > 0, \\ u_1(0) \succeq 0. \end{cases} \tag{10}$$

Due to the fact that system (10) is positive, $u_1(t) \succeq 0$ for all $t \geq 0$, that is, $y(t) \succeq y(t + c)$, for all $t \geq 0$. In particular,

(S1) $0 \prec y(t) \preceq \lambda$ for all $t \geq 0$;

(S2) $y(\cdot)$ is decreasing on $[0, \infty)$.

From (S1) and (S2), the limit $\lim_{t\to\infty} y(t)$ exists. Put $y^* = \lim_{t\to\infty} y(t)$ and denote by $\mathcal{L}$ the Laplace transform. In light of the Final value theorem (see, e.g.,[20, Theorem D13]), we obtain

$$\begin{aligned} \lim_{s\to+0} s\mathcal{L}\{{}^C D_{0+}^{\hat{\alpha}} y(\cdot)\} &= \lim_{t\to\infty} {}^C D_{0+}^{\hat{\alpha}} y(t) \\ &= \lim_{t\to\infty}(Ay(t) + \sum_{1\leq k\leq m} B_k \lambda) \\ &= Ay^* + \sum_{1\leq k\leq m} B_k \lambda. \end{aligned}$$

Furthermore,

$$\begin{aligned} \lim_{s\to+0} s\mathcal{L}\{{}^C D_{0+}^{\hat{\alpha}} y(\cdot)\} &= \lim_{s\to+0} s[s^{\alpha_1}\mathcal{L}\{y_1(\cdot)\}(s) - s^{\alpha_1-1}\lambda_1, \dots, s^{\alpha_d}\mathcal{L}\{y_d(\cdot)\}(s) - s^{\alpha_d-1}\lambda_d] \\ &= \lim_{s\to+0}[s^{\alpha_1}(s\mathcal{L}\{y_1(\cdot)\}(s) - \lambda_1), \dots, s^{\alpha_d}(s\mathcal{L}\{y_d(\cdot)\}(s) - \lambda_d)] \\ &= 0 \end{aligned}$$

due to the fact that, for all $1 \leq j \leq d$,

$$\lim_{s\to+0} s\mathcal{L}\{y_j(\cdot)\}(s) = \lim_{t\to\infty} y_j(t) = y_j^*.$$

This leads to that $y^* = \lim_{t\to\infty} y(t) = -A^{-1}\sum_{1\leq k\leq m} B_k\lambda$. Note that A is Metzler and Hurwitz. From[26, Theorem 2.5.3, p. 114], $-A^{-1} \succeq 0$ which together with (7) implies that

$$\lim_{t\to\infty} y(t) = -A^{-1}\sum_{1\leq k\leq m} B_k\lambda \prec \lambda. \tag{11}$$

By combining (9) and (11), we can find $t_1 > 0$ and $\nu \in (0, 1)$ such that the estimate (8) holds.

Second step: In this step, we will show that there exists an increasing sequence $\{T_n\}_{n=0}^{\infty}$ with $T_0 = 0$ and $\lim_{n\to\infty} T_n = \infty$ such that for any $n \in \mathbb{Z}_{\geq 0}$,

$$\varphi(t, \lambda) \preceq \nu^n \lambda, \quad \forall t \in [T_n, T_{n+1}]. \tag{12}$$

To do this, we use a proof by induction. From (G4), there exists $\hat{t}_1 > t_1$ such that $t - h_k(t) \geq t_1$ for all $t \geq \hat{t}_1$, $1 \leq k \leq m$. Put $T_1 := \hat{t}_1$. Then, (12) holds for $n = 0$ and $\varphi(t, \lambda) \preceq \nu\lambda$ for all $t \geq T_1$.

Next, define $y_1(t) = \varphi(t + T_1, \lambda)$, $t \geq 0$. Then, $y_1(\cdot)$ satisfies the system

$$\begin{cases} {}^C D_{0+}^{\hat{\alpha}} y_1(t) = Ay_1(t) + \sum_{1\leq k\leq m} B_k f_k(t), \ t > 0, \\ y_1(0) = \varphi(T_1, \lambda), \end{cases} \tag{13}$$

where $f_k(t) = \varphi(t + T_1 - h_k(t + T_1), \lambda)$, $t \geq 0$. Thus, $0 \preceq f_k(t) \preceq \nu\lambda$ for all $t \geq 0$. Now, consider the system

$$\begin{cases} {}^C D_{0+}^{\hat{\alpha}} z_1(t) = Az_1(t) + \sum_1^m B_k \nu\lambda, \ t > 0, \\ z_1(0) = \nu\lambda. \end{cases} \tag{14}$$

By the comparison principle for solutions of (13) and (14) and arguments as shown above, we obtain

- $0 \preceq y_1(t) \preceq z_1(t) \preceq \nu\lambda$ for all $t \geq 0$;
- $\lim_{t\to\infty} z_1(t) = -A^{-1}\sum_{1\leq k\leq m} B_k\nu\lambda$.

Hence, there exists $t_2 > 0$ such that $\varphi(t+T_1,\lambda) = y_1(t) \preceq v^2\lambda$ for all $t \geq t_2$. Take $\hat{t}_2 = T_1 + t_2$, then $\varphi(t,\lambda) \preceq v^2\lambda$ for all $t \geq \hat{t}_2$. Using (G4) again, we have $T_2 > \hat{t}_2$ so that $t - h_k(t) \geq \hat{t}_2$ for all $t \geq T_2$, $1 \leq k \leq m$. Thus, (12) holds for $n = 1$ and $\varphi(t,\lambda) \preceq v^2\lambda$ for all $t \geq T_2$. By a similar procedure, we also see that (12) holds for $n = 2, 3, \ldots$, and thus the proof of Second step is complete.

Third step: From (12), we see that $\lim_{t\to\infty}\varphi(t,\lambda) = 0$. Let $\phi(\cdot) \in C([-r,0];\mathbb{R}^d_+)$ be arbitrary. There is a positive constant γ such that

$$\phi(t) \preceq \gamma\lambda, \quad t \in [-r,0].$$

Due to positivity, linearity and existence and uniqueness of solutions of system (5), we have

$$\varphi(t,\phi) \preceq \varphi(t,\gamma\lambda) = \gamma\varphi(t,\lambda), \quad t \geq 0.$$

Thus,

$$0 \preceq \lim_{t\to\infty}\varphi(t,\phi) \preceq \gamma\lim_{t\to\infty}\varphi(t,\lambda) = 0.$$

This shows that system (5) is asymptotically stable. □

Remark 2. In [18, Theorem 2], the authors studied asymptotic stability of linear mixed fractional-order with a bounded time-varying delay

$$\begin{cases} {}^C D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + Bx(t-\tau(t)), & t \geq 0, \\ x(t) = \varphi(t) \in \mathbb{R}^d, & t \in [-r,0], \end{cases} \tag{15}$$

where $0 \leq \tau(t) \leq r$ for all $t \geq 0$. Assume that $\lambda \succ 0$ satisfying $(A+B)\lambda \prec 0$. Their approach is to compare solution $\varphi(\cdot,\lambda)$ of system (15) with the one of the following system

$$\begin{cases} {}^C D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + Bx(t-r), & t \geq 0, \\ x(t) = \lambda, & t \in [-r,0]. \end{cases}$$

It is easy to see that this method cannot be applied for the case where the delay $\tau(\cdot)$ is not bounded which is the main objective in our research.

Remark 3. For the case where $\alpha_1 = \cdots = \alpha_d = \alpha \in (0,1]$ and $h_k = 0$ $(1 \leq k \leq m)$, system (5) becomes

$${}^C D^{\alpha}_{0+}x(t) = (A + \sum_{1\leq k\leq m} B_k)x(t), \quad t > 0. \tag{16}$$

When A and B_k $(1 \leq k \leq m)$ in (16) are not required to be Metzler and nonnegative, respectively. Then, by a well-known result from fractional calculus field (see, e.g., [20, Theorem 7.20, p. 158]), system (16) is asymptotically stable if and only if the eigenvalues λ of $A + \sum_{1\leq k\leq m} B_k$ satisfy

$$|\arg(\lambda)| > \frac{\alpha\pi}{2}. \tag{17}$$

In our study, we deal with positive systems and thus $A + \sum_{k=1}^m B_k$ is a Metzler matrix. From [27, Theorem 4], the eigenvalue λ_{mr} of $A + \sum_{k=1}^m B_k$ with the largest real part must be real. This together with the stability condition (17) implies that $\lambda_{mr} < 0$. Therefore, all the eigenvalues of this matrix have negative real parts, that is, $A + \sum_{k=1}^m B_k$ is Hurwitz.

4 | NUMERICAL EXAMPLES

In this section, we give two numerical examples to illustrate effectiveness of our proposed results.

Example 1: Let $\hat{\alpha} = (\alpha_1,\alpha_2,\alpha_3) \in (0,1]\times(0,1]\times(0,1]$, and continuous function $h : [0,\infty) \to \mathbb{R}_{\geq 0}$ be defined by $h(t) = \frac{t\sin^2 t}{2} + 1$ for all $t \geq 0$. Consider the following positive linear mixed-order fractional system with the unbounded time-varying delay

$${}^C D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + Bx(t-h(t)), \quad t \geq 0, \tag{18}$$

where

$$A = \begin{pmatrix} -5 & 1 & 0 \\ 0.5 & -4 & 0.5 \\ 1 & 0 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We see that the delay $h(t)$ satisfies assumptions (G1), (G2) and (G4), A is Metzler, B is nonnegative and $A + B$ is Hurwitz. Since the considered system is positive, by Theorem 1, system (18) is asymptotically stable, that is, for any $\phi(\cdot) \in C([-1,0];\mathbb{R}^3)$,

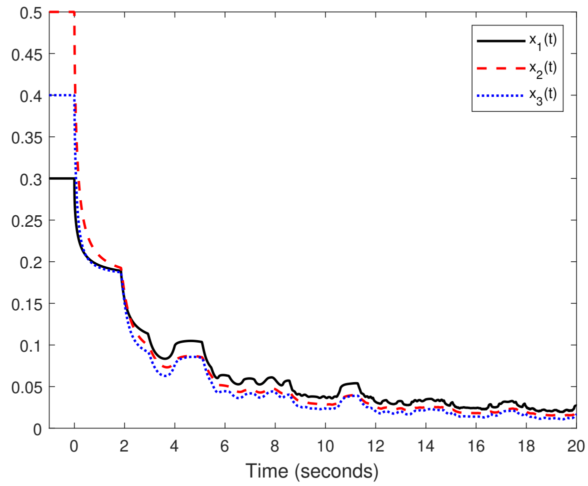

FIGURE 1 Trajectories of the solution $\varphi(\cdot, \phi)$ to system (18) when $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\alpha_3 = 0.8$.

the solution $\varphi(t, \phi) \to 0$ as $t \to \infty$. In Figure 1, we simulate trajectories of the solution $\varphi(\cdot, \phi)$ to system (18) when $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\alpha_3 = 0.8$ and the initial condition as $\phi(t) = (0.3, 0.5, 0.4)^{\mathrm{T}}$ on the interval $[-1, 0]$.

Example 2: This example is used to demonstrate the necessary conditions of both Proposition 1, Theorem 1, and Remark 3. Let $\hat{\alpha} = (\alpha_1, \alpha_2) \in (0, 1] \times (0, 1]$, and continuous function $h : [0, \infty) \to \mathbb{R}_{\geq 0}$ be defined by $h(t) = \frac{t \sin^2 t}{2} + 1$ for all $t \geq 0$. Consider the following linear mixed-order fractional system with the unbounded time-varying delay

$$ {}^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + Bx(t - h(t)), \quad t \geq 0, \tag{19} $$

where

$$ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

We see that A is Metzler, however, B is not nonnegative and $A + B$ is not Hurwitz as its eigenvalues are at $\{j\sqrt{2}, -j\sqrt{2}\}$. By Proposition 1, system (19) is not positive and thus Theorem 1 which is based on the assumption that system (19) is a positive system cannot be applied. In Figure 2, we simulate trajectories of the solution $\varphi(\cdot, \phi)$ to system (19) when $\alpha_1 = 0.5$, $\alpha_2 = 0.7$ and the initial condition as $\phi(t) = (1, 2)^{\mathrm{T}}$ on the interval $[-1, 0]$. It is clear from Figure 2 that the trajectories of the solution are not always nonnegative, and furthermore, they are unbounded, i.e., the system is not stable with the considered unbounded time-varying delay.

To further illustrate Remark 3, we also simulate trajectories of the solution to system (19) for the case where the time-varying delay is zero (i.e., $h(t) = 0$), $\alpha_1 = \alpha_2 = 0.8$ and with initial condition as $\phi(0) = (1, 2)^{\mathrm{T}}$. Note that for this case, even though the eigenvalues of the matrix $A + B$ satisfy the stability condition for non-positive systems, i.e., $|\arg(\lambda)| > \frac{\alpha\pi}{2}$, matrix B, however, does not satisfy the positivity condition as stated in Proposition 1. Figure 3 shows the trajectories of the solution and it is clear that they are not always nonnegative. Hence, positivity and asymptotic stability cannot be met.

5 | CONCLUSION

In this paper, by using a new weighted type norm which is adaptive with time-delay systems, we have obtained a result on existence and uniqueness of solutions to linear mixed fractional-order systems with time-varying delays. Then, by using the integral representation of solutions, we have derived a necessary and sufficient condition for positivity of these systems. Finally, by comparing trajectories of solutions of a time-delay system with that of inhomogeneous systems having the inhomogeneous parts constant and decreasing on time and the inductive principle, we have established a necessary and sufficient criterion to guarantee asymptotic stability of positive linear mixed fractional-order systems with both multiple bounded and unbounded time-varying delays.

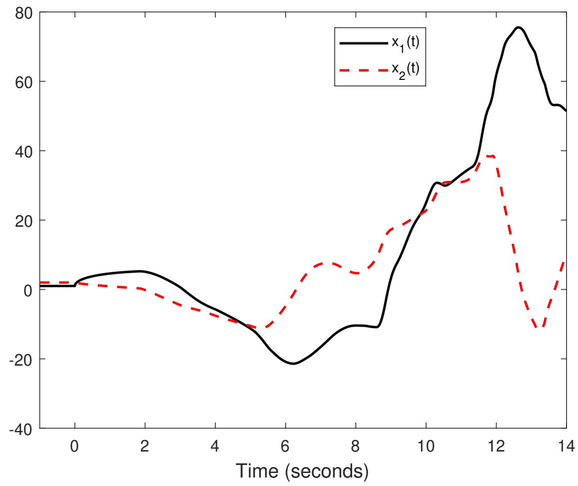

FIGURE 2 Trajectories of the solution $\varphi(\cdot, \phi)$ to system (19) when $\alpha_1 = 0.5$, $\alpha_2 = 0.7$.

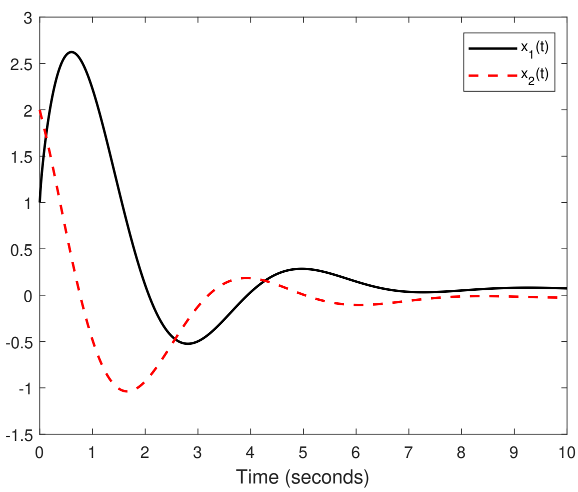

FIGURE 3 Trajectories of the solution to system (19) when $h(t) = 0$, $\alpha_1 = \alpha_2 = 0.8$.

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