New results on finite-time stability of fractional-order neural networks with time-varying delay

Nguyen T. Thanh[a], Vu N. Phat[b]

[a]*Department of Mathematics, Hanoi University of Mining and Geology, Hanoi, Vietnam*

[b] *ICRTM, Institute of Mathematics, VAST, 18 Hoang Quoc Viet Road, Hanoi, Vietnam*

Abstract

Finite-time stability problem of fractional-order neural networks with time-varying delay is considered in this paper. We first propose some important results on the existence of solutions and estimating the Caputo derivative of some quadratic functions. Then, we present improved delay-dependent sufficient conditions for finite-time stability, which are formulated in terms of tractable linear matrix inequalities and Mittag-Leffler functions. A numerical example is given to illustrate the effectiveness of the proposed method.

1. Introduction

In the real world, neural networks have been found everywhere such as in weather forecasting and business processes because neural networks can create simulations and predictions for complex systems and relationships. Because of its promising potential for applications, neural networks have been investigated intensively in optimization, image processing and so on [1-4] . Some of important neural networks are intelligent transportation and signal analysis [5-7].

In recent decades, surveys have shown that fractional calculus describe real world phenomena better and more accurate than integer-order models can do.Therefore, fractional calculus is applied to physics, control theory, applied mathematics, and engineering [8,9]. Motivated by the above trend, fractional calculus has been incorporated into various neural networks as artificial neural networks, Hopfield neural networks, etc.[10,11]. However, the study of the stability of of fractional-order systems (FOSs) is not easy since the fractional derivative has the non-local property and weakly singular kernels. On the other hand, time delays, which appear in real systems, cause instability and serious deterioration in the performance of systems.

In realistic applications, the key problems are the behavior of the systems over a fixed finite - time interval [12-14]. Therefore, investigation of finite-time stability (FTS) for fractional-order neural networks with delay has become a very hot research topic. Two main approaches in studying FTS for differential equations are using some inequalities (Gronwall inequalities, Holder Inequality, etc.) or Lyapunov function method. The authors [15] studied FTS of linear FOS and [16] studied FTS of nonlinear FOSs by using a Generalized Gronwall inequality approach [17]. In [18],

Email address: `Corresponding author: vnphat@math.ac.vn` (Vu N. Phat)

Yang et al. studied FTS of fractional-order neural networks with constant delay based on the Gronwall inequalitity. Chen et al. [19] used some Holder-type inequalities to derive delay-dependent sufficient conditions for FTS. Combining the Holder inequality and Gronwall inequality, Wu et al. [20] obtained sufficient conditions for FTS of fractional-order time-varying neural networks with constant delay. Based on this approach Changjin Xu and Peiluan Li in [21] developed the similar results for the systems with proportional delays. Du and Lu [22] proposed a generalization of the existing Gronwall inequality to obtain two new criteria for the FTS of FONND (2.1) with constant delays. It is well known that the Lyapunov-Krasovskii function method is a powerful in studying stability of differential delay equations [23,24]. However, it is not easy to apply the method to FOSs with time delay because the use of Lyapunov-Krasovskii function and estimating its fractional derivative are still difficult. The authors [25-27] attempt to construct appropriate Lyapunov-Krasovskii functions for FOSs with time delay, unfortunately, the main proofs in these papers contains some gaps. Therefore, finite-time stability problem for fractional-order neural networks with time-varying delay still remains open, which motivates our study.

In this paper, we study FTS of fractional-order neural networks with time-varying delay. Based on the Lyapunov function and Laplace transform without using generalized fractional Gronwall or Holder inequalities, we propose new delay-dependent conditions for FTS. The main contributions of our paper are summarized as follows. (i) We consider fractional-order delay neural networks, where the time-varying delay is assumed to be a bounded continuous function. New auxiliary results on the existence of solutions and the estimating the Caputo derivative of some quadratic functions are proposed. (ii) Using the Laplace transform and "inf-sup" method combining with LMI technique, delay-dependent sufficient conditions for FTS are established in terms of a tractable linear matrix inequality and Mittag-Leffler functions. A numerical example is provided to illustrate the effectiveness of the obtained results.

The paper is organized as follows. In Section 2, we give some preliminaries on some basic definitions from fractional calculus as Captuto derivatives, Laplace transforms, Mittag-Leffler functions and auxiliary technical lemmas needed in next section. Section 3 presents main result on finite-time stability of the systems. The validity and effectiveness of the proposed method are illustrated by a numerical example with simulations.

2. Preliminaries

Throughout this paper we use the following notations. $R^{n\times r}$ stands for the space of all $(n\times r)-$ matrices; R^n stands for the space of all $(n\times 1)-$ matrices; $(x,y)=x^\top y,\ \|x\|=\sqrt{(x,x)},\ \forall x,y\in\mathbb{R}^n$; $C([-h_2,0],R^n)$ stands for the set of all R^n-valued continuously functions on $[h_2,0]$; $C[a,b]$ stands for the set of all continuous functions on $[a,b]$; $L^1[a,b]$ stands for the space of all integrable functions on $[a,b]$; For $\alpha\in(0,\ 1)$, the standard Holder space $H^\alpha[0,T]$ stands for

$$H^\alpha[0,T]=\{x(t)\in C[0,T]\Big|\,|x|_{H^\alpha}:=\max_{t\in[0,T]}|x(t)|+\sup_{0\le s<t\le T}\frac{|x(t)-x(s)|}{(t-s)^\alpha}<\infty\};$$

$$H_0^\alpha[0,T]=\{x(t)\in H^\alpha[0,T]\Big|\sup_{0\le s<t\le T,\ t-s\le\varepsilon}\frac{|x(t)-x(s)|}{(t-s)^\alpha}\to 0,\ \text{ as } \varepsilon\to 0\}.$$

$\mathbb{C}$ stands for the complex space. $\lambda(A)$ stands for the set of all eigenvalues of A; $\lambda_{max}(A)=max\{Re(\lambda):\ \lambda\in\lambda(A)\}$; $\lambda_{min}(A)=min\{Re(\lambda):\ \lambda\in\lambda(A)\}$; $\|A\|=\sqrt{\lambda_{\max}(A^\top A)}$; $[a]$ denotes the integer part of number a.

First of all, we mention to definitions and auxiliary results of fractional calculus from [1,2].

Definition 1. For $\alpha \in (0,1)$ and $f \in L^1[0,T]$, the Riemann-Liouville integral, the Riemann-Liouville derivative, and the Caputo fractional derivative of order α are respectively defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$
$$D_R^{\alpha} f(t) = \frac{d}{dt}(I^{1-\alpha} f(t)),\ D^{\alpha} f(t) = D_R^{\alpha}(f(t) - f(0)), \quad t \in [0,T],$$

where the gamma function $\Gamma(s) = \int\limits_0^{\infty} e^{-t} t^{s-1} dt, s > 0$.

The Mittag-Lefller function with two parameters $\alpha > 0,\ \beta > 0$, which is important in the theory of the fractional calculus, is defined by $E_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \dfrac{z^n}{\Gamma(n\alpha + \beta)}, \forall z \in \mathbb{C}$. Noting that $E_{\alpha}(z)$ is increasing on R^+ and $E_{\alpha}(z) \geq 1,\ \forall z \in \mathbb{R}^+$, where $E_{\alpha}(z) := E_{\alpha,1}(z),\ \alpha > 0$.

The Laplace transform of the integrable function $f(.)$ is defined by $\mathbb{L}[f(t)](s) = \int\limits_0^{\infty} e^{-st} f(t) dt$.

Lemma 1. [1] *Assume that $f(.), g(.)$ are exponentially bounded integrable functions on R^+, $f * g(t) = \int\limits_0^t f(t-\tau) g(\tau) d\tau$, and $0 < \alpha < 1$. Then*

(i)

$$\mathbb{L}[D^{\alpha} f(t)](s) = s^{\alpha} \mathbb{L}[f(t)](s) - s^{\alpha-1} f(0),$$

(ii) For $\beta > 0$,

$$\mathbb{L}[t^{\alpha-1} E_{\alpha,\alpha}(\beta t^{\alpha})](s) = \frac{1}{s^{\alpha} - \beta},$$
$$\mathbb{L}[E_{\alpha}(\beta t^{\alpha})](s) = \frac{s^{\alpha-1}}{s^{\alpha} - \beta},$$

(iii)

$$\mathbb{L}[f * g(t)](s) = \mathbb{L}[f(t)](s) \cdot \mathbb{L}[g(t)](s).$$

Consider the following fractional-order neural networks with time varying delay:

$$\begin{cases} D^{\alpha} x_i(t) = -m_i x_i(t) + \sum\limits_{j=1}^{n} a_{ij} f_j(x_i(t)) + \sum\limits_{j=1}^{n} b_{ij} g_j(x_j(t - h(t))), \\ x_i(\theta) = \varphi_i(\theta), \quad \forall \theta \in [-h_2, 0],\ \forall i = \overline{1,n}, \end{cases} \tag{1}$$

or in the following form:

$$D^{\alpha} x(t) = -Mx(t) + Af(x(t)) + Bg(x(t - h(t))), \tag{2}$$

where $0 < \alpha < 1$; $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ is the state vector; the time varying delay $h(t)$ satisfies

$$0 < h_1 \le h(t) \le h_2,\ \forall t \ge 0;$$

the initial function

$$\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))^\top,$$

with the norm $\|\varphi\| = \sup\limits_{s\in[-h_2,0]} \|\varphi(s)\|$, where $\varphi_i \in C([-h_2,0],\mathbb{R})$; the neuron activation functions $f, g:\ \mathbb{R}^n \to \mathbb{R}^n:$

$$f(x) = (f_1(x_1)), \ldots, f_n(x_n)))^\top, \quad g(x) = (g_1(x_1)), \ldots, g_n(x_n)))^\top$$

satisfy $f(0) = 0,\ g(0) = 0$, and for all $u, v \in \mathbb{R},\ i = \overline{1,n}:$

$$\begin{aligned} &\exists l_i > 0 : |f_i(u) - f_i(v)| \le l_i|u - v|, \\ &\exists k_i > 0 : |g_i(u) - g_i(v)| \le k_i|u - v|; \end{aligned} \tag{3}$$

the rate with which the i^{th} neuron m_i $(m_i > 0)$, will reset its potential to the resting state inisolation when disconnection from the networks and the external inputs; $M = diag(m_1, m_2, \ldots, m_n)$; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ referred to the connection of the j^{th} neuron to the i^{th} neuron at time t and $t - h(t)$, respectively.

Definition 2. For give positive numbers $c_1, c_2, T,\ c_1 \le c_2$, the system (1) is said to be finite - time stable with respect to (c_1, c_2, T) if any solution $x(t)$ of (1),

$$\|\varphi\|^2 \le c_1 \Rightarrow \|x(t)\|^2 \le c_2,\ \forall t \in [0,T].$$

Lemma 2. *Assume that the initial function $\varphi \in C([-h_2,0],\mathbb{R}^n)$, system (1) under the assumption (3) has a unique solution $x \in C([-h_2,T),\mathbb{R}^n)$.*

Proof. Let

$$v_y(t) = -My(t) + Af(y(t)) + Bg(y(t - h(t))).$$

It is easy to see that under (3) and $y \in C([-h_2,T],\mathbb{R}^n)$, the function $v_y(t)$ is continuous on [0,T]. From Volterra integtal form of system (2),

$$x(t) = x(0) + I^\alpha[-Mx(t) + Af(x(t)) + Bg(x(t - h(t)))],$$

we use the function

$$H(y)(t) = \begin{cases} \varphi(0) + I^\alpha[v_y(t)] & \text{if } t \ge 0, \\ \varphi(t) & \text{if } t \in [-h_2, 0). \end{cases}$$

to prove the existence of the solution of system (2), where $y \in C([-h_2,T],\mathbb{R}^n)$.

First, the function $H(\cdot)$ is from $C([-h_2,T],\mathbb{R}^n)$ into $C([-h_2,T],\mathbb{R}^n)$. In fact, from the uniform continuity of $v_y(t)$ on $[0,T]$, there is a $\delta > 0$ such that for all $t_1, t_2 \in [0,T]$ and

$$|t_1 - t_2| \le \delta \Rightarrow |v_y(t_1) - v_y(t_2)| \le \varepsilon,$$

hence

$$
\begin{aligned}
|H(y)(t_1)-H(y)(t_2)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}v_y(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}v_y(s)ds\Big| \\
&= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}s^{\alpha-1}v_y(t_1-s)ds-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}s^{\alpha-1}v_y(t_2-s)ds\Big| \\
&\le \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}s^{\alpha-1}[v_y(t_1-s)-v_y(t_2-s)]ds\Big|+\Big|\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}s^{\alpha-1}v_y(t_2-s)ds\Big| \\
&\le \frac{\varepsilon}{\Gamma(\alpha)}\Big|\int_0^{t_2}s^{\alpha-1}ds\Big|+\frac{1}{\Gamma(\alpha)}\sup_{s\in[0,T]}|v_y(s)|\Big|\int_{t_2}^{t_1}s^{\alpha-1}ds\Big| \\
&= \frac{\varepsilon}{\Gamma(\alpha)}\frac{t_2^\alpha}{\alpha}+\frac{1}{\Gamma(\alpha)}\sup_{s\in[0,T]}|v(s)|\Big|\frac{t_2^\alpha}{\alpha}-\frac{t_1^\alpha}{\alpha}\Big| \\
&\le \frac{\varepsilon}{\Gamma(\alpha)}\frac{T^\alpha}{\alpha}+\frac{1}{\Gamma(\alpha)}\sup_{s\in[0,T]}|v(s)|\Big|\frac{t_2^\alpha}{\alpha}-\frac{t_1^\alpha}{\alpha}\Big|,
\end{aligned}
$$

which also shows the continuity of $H(y)(t)$ on $[-h_2,T]$.

Next, we see that for $y,z\in C([-h_2,T],\mathbb{R}^n)$, for $t\ge 0$,

$$
\begin{aligned}
|v_y(t)-v_z(t)| &\le |M||y(t)-z(t)|+|A||f(y(t))-f(z(t))|+|B||g(y(t-h(t)))-g(z(t-h(t)))| \\
&\le (|M|+|A|\max_i l_i+|B|\max_i k_i)\sup_{s\in[-h_2,T]}|y(s)-z(s)|,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
|H(y)(t)-H(z)(t)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}[v_y(s)-v_z(s)]ds\Big| \\
&\le \frac{\gamma}{\Gamma(\alpha)}\frac{t^\alpha}{\alpha}\sup_{s\in[-h_2,T]}|y(s)-z(s)|,
\end{aligned}
$$

where $\gamma=|M|+|A|\max_i l_i+|B|\max_i k_i$. Similarly, by induction, we have

$$
|H^m(y)(t)-H^m(z)(t)|\le\gamma\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\sup_{s\in[-h_2,T]}|y(s)-z(s)|,\ m=1,2,\ldots,
$$

where $H^{m+1}(y)=H(H^m(y)),\ H^1(y)=H(y)$. Moreover,

$$
\sup_{s\in[-h_2,T]}|H^m(y)(s)-H^m(z)(s)|\le\gamma\frac{T^{m\alpha}}{\Gamma(m\alpha+1)}\sup_{s\in[-h_2,T]}|y(s)-z(s)|.
$$

Besides, the space $C([-h_2,T],\mathbb{R}^n)$ with the norm $\|y\|=\sup_{s\in[-h_2,T]}|y(s)|$ is a Banach space. Hence, $H^m(\cdot):C([-h_2,T],\mathbb{R}^n)\to C([-h_2,T],\mathbb{R}^n)$ is a contraction map with this sup norm as m enough large. Using Fixed-point Theorem, we show the existence of a unique solution $x \in C([-h_2,T],\mathbb{R}^n)$. $\square$

Lemma 3. [29]. *For $0<\alpha<1,\ T>0$, and $x\in C[0,T]$, the following conditions are equivalent :*

(i) $\exists D^\alpha x$ *is continuous on* $[0,T]$,

(ii)

$$\exists \lim_{t\to 0}\frac{x(t)-x(0)}{t^\alpha}=\gamma,$$

$$\sup_{0<t\le T}\Big|\int_{\xi t}^{t}\frac{x(t)-x(s)}{(t-s)^{1+\alpha}}ds\Big|\to 0 \text{ when } \xi\to 1^-;$$

(iii) *x has the structure $x-x(0)=\gamma t^\alpha+x_0$, where γ is a constant, $x_0\in H_0^\alpha[0,T]$, and $\int\limits_0^t(t-s)^{-\alpha-1}(x(t)-x(s))ds=:w(t)$ converges for every $t\in(0,T]$ defining a function $w\in C(0,T]$ which has a finite limit $\lim\limits_{t\to 0}w(t):=w(0)$.*

For x and $D^\alpha x\in C[0,T]$, we have $(D^\alpha x)(0)=\Gamma(\alpha+1)\gamma$, and

$$(D^\alpha x)(t)=\frac{1}{\Gamma(1-\alpha)}\Big(\frac{x(t)-x(0)}{t^\alpha}+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^t\frac{x(t)-x(s)}{(t-s)^{1+\alpha}}ds\Big),\ t\in(0,T].$$

Lemma 4. [28] *Let $T>0,\ h>0,\ a>1,\ b\ge 0$, and $H:[-h,T]\to\mathbb{R}^+$ be a non-decreasing function satisfying*

$$H(t)\le aH(0)+bH(t-h),\ \forall t\ge 0,$$

we have

$$H(t)\le H(0)a\sum_{j=0}^{[T/h]+1}b^j,\ \forall t\in[0,T].$$

3. Main result

Before introducing the main result, the following notations are defined for simplicity:

$\gamma_2=\dfrac{h_2}{2\max\limits_i k_i^2},\ \mathbb{I}_n=diag\{1,\dots,1\}\in\mathbb{R}^{n\times n}$,

$E_{11}=-2PM-h_2P+\gamma_1\max\limits_i l_i^2\mathbb{I}_n,\ E_{12}=PA,\ E_{21}=[PA]^\top,\ E_{13}=PB,\ E_{31}=[PB]^\top$,

$E_{22}=-\gamma_1\mathbb{I}_n,\ E_{33}=-\gamma_2P,\ E_{44}=\mathbb{I}_n-P,\ E_{55}=P-2\mathbb{I}_n$,

and all the other elements of the matrix $\mathbf{E}:=\Big[E_{ij}\Big]_{5\times 5}$ are zero.

Theorem 5. *For given positive numbers $c_1, c_2, T,\ c_1 \leq c_2$. Assume that there exist a positive scalar γ_1, a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that the following conditions hold*

$$\boldsymbol{E} = \Big[E_{ij}\Big]_{5\times 5} < 0, \tag{4}$$

$$\frac{\lambda_{max}(P)}{\lambda_{min}(P)} E_\alpha(h_2 T^\alpha) \sum_{j=0}^{[T/h_1]+1} (E_\alpha(h_2 T^\alpha) - 1)^j < \frac{c_2}{c_1}, \tag{5}$$

then the system (1) is finite-time stable with respect to (c_1, c_2, T).

Proof. Consider the following quadratic non-negative function

$$V(x(t)) = x(t)^\top P x(t).$$

Since the solution $x(t)$ may not be non-differentiable, we can not use the approach of [30] to estimate the fractional derivative of $V(x(t))$. To overcome this difficulty, we propose the following technical lemma.

Lemma 6. If the solution $x(t) \in C[-h_2, T]$, then the Caputo derivative $D^\alpha(V(x(t))) \in C[0,T]$ exists and $D^\alpha[V(x(t))] \leq 2x(t)^\top P D^\alpha x(t), \quad t \geq 0$.

To prove this lemma, we note that $x \in C[-h_2, T]$ (by Lemma 3), the function

$$v(t) = -Mx(t) + Af(x(t)) + Bg(x(t-h(t))),$$

is continuous on $[0,T]$. Hence, we get

$$\begin{aligned}\Big|\frac{x(t)-x(0)}{t^\alpha} - \frac{v(0)}{\Gamma(\alpha+1)}\Big| &= \Big|\frac{\int_0^t (t-s)^{\alpha-1}(v(s)-v(0))ds}{t^\alpha \Gamma(\alpha)}\Big| \\ &\leq \sup_{s\in[0,t]} \Big|v(s)-v(0)\Big| \Big|\frac{\int_0^t (t-s)^{\alpha-1}ds}{t^\alpha\Gamma(\alpha)}\Big| = \frac{1}{\Gamma(\alpha+1)} \sup_{s\in[0,t]}\Big|v(s)-v(0)\Big| \to 0,\end{aligned}$$

as $t \to 0$. In the other words,

$$\lim_{t\to 0} \frac{x(t)-x(0)}{t^\alpha} = \frac{v(0)}{\Gamma(\alpha+1)}. \tag{6}$$

Consequently,

$$\begin{aligned}\lim_{t\to 0} \frac{V(x(t)) - V(x(0))}{t^\alpha} &= \lim_{t\to 0}\Big[\Big(\frac{x(t)-x(0)}{t^\alpha}, Px(t)\Big) + \Big(x(0), P\frac{x(t)-x(0)}{t^\alpha}\Big)\Big] \\ &= 2\Big(x(0), \frac{Pv(0)}{\Gamma(\alpha+1)}\Big).\end{aligned} \tag{7}$$

It is easy to calculate the following integral

$$\begin{aligned}\int_{\xi t}^t \frac{V(x(t)) - V(x(s))}{(t-s)^{\alpha+1}} ds = & \int_{\xi t}^t \frac{(x(t)-x(s), 2Px(t))}{(t-s)^{\alpha+1}} ds \\ & - \int_{\xi t}^t \frac{(x(t)-x(s), P[x(t)-x(s)])}{(t-s)^{\alpha+1}} ds \\ = & I_1(t,\xi) - I_2(t,\xi).\end{aligned} \tag{8}$$

From Lemma 3 (i) - (ii) and $\exists D^{\alpha}x = v \in C[0,T]$, we derive that when $\xi \to 1^-$,

$$
\begin{aligned}
|I_1(t,\xi)| =& \Big|\Big(\int_{\xi t}^{t} \frac{x(t)-x(s)}{(t-s)^{\alpha+1}}ds,\ 2Px(t)\Big)\Big| \leq \Big|\int_{\xi t}^{t} \frac{x(t)-x(s)}{(t-s)^{\alpha+1}}ds\Big| 2|Px(t)| \\
&\leq \sup_{0<t\leq T}\Big|\int_{\xi t}^{t} \frac{x(t)-x(s)}{(t-s)^{\alpha+1}}ds\Big| 2\sup_{t\in[0,T]}|Px(t)| \to 0.
\end{aligned}
\tag{9}
$$

Moreover, applying Lemma 3 (i) and (iii) gives

$$x = x(0) + \gamma t^{\alpha} + x_0,\ x_0 \in H_0^{\alpha}[0,T],\ t \in (0,T].$$

Hence, for $0 \leq \xi t \leq s < t \leq T,\ \xi \in (0,1]$, we have

$$
\begin{aligned}
\Big|\frac{x(t)-x(s)}{(t-s)^{\alpha}}\Big| \leq& \Big|\gamma\frac{t^{\alpha}-s^{\alpha}}{(t-s)^{\alpha}}\Big| + \Big|\frac{x_0(t)-x_0(s)}{(t-s)^{\alpha}}\Big| = \gamma\frac{(t-s)\alpha c^{\alpha-1}}{(t-s)^{\alpha}} + \Big|\frac{x_0(t)-x_0(s)}{(t-s)^{\alpha}}\Big|, \\
\leq& h(\xi) := \gamma\alpha[1/\xi-1]^{1-\alpha} + \sup_{0\leq s<t\leq T, |t-s|\leq T(1-\xi)}\Big|\frac{x_0(t)-x_0(s)}{(t-s)^{\alpha}}\Big|,
\end{aligned}
$$

where $c \in (s,t)$. Thus, as $\xi \to 1^-$, we get

$$
\begin{aligned}
|I_2(t,\xi)| =& \int_{\xi t}^{t} \frac{(x(t)-x(s), P[x(t)-x(s)])}{(t-s)^{\alpha+1}}ds \leq \int_{\xi t}^{t} \frac{(t-s)^{2\alpha}}{(t-s)^{\alpha+1}}ds\|P\|h(\xi)^2 \\
=& \int_{\xi t}^{t}(t-s)^{\alpha-1}ds\ \|P\|h(\xi)^2 = \frac{(t-t\xi)^{\alpha}}{\alpha}\|P\|h(\xi)^2 \leq \frac{T^{\alpha}(1-\xi)^{\alpha}}{\alpha}\|P\|h(\xi)^2 \to 0,
\end{aligned}
\tag{10}
$$

because $h(\xi)$ is independent on s,t, $\lim_{\xi\to 1^-} h(\xi) = 0$ and $y_0 \in H_0^{\alpha}[0,T]$. From (8), (9), (10), we obtain

$$\sup_{0<t\leq T}\Big|\int_{\xi t}^{t}(t-s)^{-\alpha-1}(V(x(t))-V(x(s)))ds\Big| \to 0, \text{ as } \xi \to 1^-, \tag{11}$$

Lemma 3, (7), and (11) show the existence of $D^{\alpha}V(x(t)) \in C[0,T]$ and

$$
\begin{aligned}
&D^{\alpha}(V(x(t)))(0) = 2\Big(x(0), Pv(0)\Big), \\
&D^{\alpha}(V(x(t))) = \frac{V(x(t))-V(x(0))}{t^{\alpha}\Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{V(x(t))-V(x(s))}{(t-s)^{\alpha+1}}ds,\ t \in (0,T].
\end{aligned}
\tag{12}
$$

Besides, from $x, D^{\alpha}x \in C[0,T]$, and (6), Lemma 3 also gives the following:

$$
\begin{aligned}
(D^{\alpha}x)(0) =& \Gamma(\alpha+1)\lim_{t\to 0}\frac{x(t)-x(0)}{t^{\alpha}} = \Gamma(\alpha+1)\frac{v(0)}{\Gamma(\alpha+1)} = v(0), \\
(D^{\alpha}x)(t) =& \frac{1}{\Gamma(1-\alpha)}\Big(\frac{x(t)-x(0)}{t^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{x(t)-x(s)}{(t-s)^{1+\alpha}}ds\Big).
\end{aligned}
\tag{13}
$$

The identities (12) and (13) lead to

for $t = 0$,

$$D^{\alpha}(V(x(t))) - 2(x(t), PD^{\alpha}x(t)) = 0,$$

for $t \in (0, T]$,

$$D^{\alpha}(V(x(t))) - 2(x(t), PD^{\alpha}x(t)) = -\frac{V(x(t)-x(0))}{t^{\alpha}\Gamma(1-\alpha)} - \frac{\alpha}{\Gamma(1-\alpha)}\int\limits_0^t \frac{V(x(t)-x(\tau))}{(t-\tau)^{\alpha+1}}d\tau \leq 0.$$

This completes the proof of the Lemma 6.

Next, we use Lemma 6 to evaluate the fractional derivative of $V(x(t))$ as follows. From (4), we have $\overline{\mathbf{E}} := [E_{ij}]_{3\times 3} < 0$. Besides,

$$f(\cdot)^{\top}f(\cdot) = \sum_{i=1}^{n}|f_i(x_i(t))|^2 \leq \max_i l_i^2 \sum_{i=1}^{n}|x_i(t)|^2 = \max_i l_i^2\, x(t)^{\top}x(t).$$

Hence,

$$\begin{aligned}
D^{\alpha}V(x(t)) &\leq 2x(t)^{\top}PD^{\alpha}x(t) = 2x(t)^{\top}P\Big(-Mx(t)+Af(x(t))+Bg(x(t-h(t)))\Big)\\
&= 2x(t)^{\top}P\Big(-Mx(t)+Af(x(t))+Bg(x(t-h(t)))\Big)\\
&\quad -\gamma_1 f(\cdot)^{\top}f(\cdot)+\gamma_1 f(\cdot)^{\top}f(\cdot)\\
&\quad -\gamma_2 g(\cdot)^{\top}Pg(\cdot)+\gamma_2 g(\cdot)^{\top}Pg(\cdot)-h_2x(t)^{\top}Px(t)+h_2V(x(t))\\
&\leq 2x(t)^{\top}P\Big(-Mx(t)+Af(x(t))+Bg(x(t-h(t)))\Big)-\gamma_1 f(\cdot)^{\top}f(\cdot)-\gamma_2 g(\cdot)^{\top}Pg(\cdot)\\
&\quad +\gamma_1\max_i l_i^2\, x(t)^{\top}x(t)-h_2x(t)^{\top}Px(t)+h_2V(x(t))+\gamma_2 g(\cdot)^{\top}Pg(\cdot)\\
&= \xi(t)^{\top}\overline{\mathbf{E}}\xi(t)+h_2V(x(t))+\gamma_2 g(\cdot)^{\top}Pg(\cdot)\leq h_2V(x(t))+\gamma_2 g(\cdot)^{\top}Pg(\cdot),
\end{aligned} \tag{14}$$

where

$$\xi(t)^{\top} = [x(t)^{\top},\ f(\cdot)^{\top},\ g(\cdot)^{\top}],\ f(\cdot) = f(x(t)),\ g(\cdot) = g(x(t-h(t))).$$

Let

$$M(t) = D^{\alpha}V(x(t)) - h_2V(x(t)),\ t \geq 0. \tag{15}$$

Applying the Laplace transform (by Lemma 1-(i)) to the both sides of (15) gives

$$\mathbb{L}[M(t)](s) = s^{\alpha}\mathbb{L}[V(x(t))](s) - s^{\alpha-1}V(x(0)) - h_2\mathbb{L}[V(x(t))](s),$$

equivalently

$$\mathbb{L}[V(x(t))](s) = (s^{\alpha}-h_2)^{-1}\Big(s^{\alpha-1}V(x(0)) + \mathbb{L}[M(t)](s)\Big).$$

Using Lemma 3 (ii)-(iii), we obtain that

$$\mathbb{L}[V(x(t))](s) = (s^{\alpha}-h_2)^{-1}s^{\alpha-1}V(x(0)) + (s^{\alpha}-h_2)^{-1}\mathbb{L}[M(t)](s)$$

$$= V(x(0))\mathbb{L}[E_\alpha(h_2t^\alpha)](s) + \mathbb{L}[t^{\alpha-1}E_{\alpha,\alpha}(h_2t^\alpha)](s)\mathbb{L}[M(t)](s)$$
$$= \mathbb{L}\Big[V(x(0))E_\alpha(h_2t^\alpha) + t^{\alpha-1}E_{\alpha,\alpha}(h_2t^\alpha) * M(t)\Big](s)$$

Therefore, taking the inverse Laplace transform to both sides of the above equation, we get

$$V(x(t)) = V(x(0))E_\alpha(h_2t^\alpha) + \int_0^t \frac{M(\tau)}{(t-\tau)^{1-\alpha}} E_{\alpha,\alpha}(h_2(t-\tau)^\alpha)d\tau. \tag{16}$$

From (4) and (14), we have $\mathbb{I}_n \le P \le 2\mathbb{I}_n$, and

$$\begin{aligned} M(t) &\le \gamma_2 g(\cdot)^\top P g(\cdot) \le 2\gamma_2 g(\cdot)^\top g(\cdot) = 2\gamma_2 \sum_{i=1}^n |g_i(x_i(t-h(t)))|^2 \\ &\le 2\gamma_2 \max_i [k_i]^2 \sum_{i=1}^n |x_i(t-h(t))|^2 \\ &\le 2\gamma_2 \max_i [k_i]^2 [x(t-h(t))]^\top P[x(t-h(t))] \\ &= h_2 x(t-h(t))^\top P x(t-h(t)) = h_2 V(x(t-h(t))), \end{aligned}$$

then

$$\sup_{\tau\in[0,t]} M(\tau) \le h_2 \sup_{\theta\in[-h_2,t-h_1]} V(x(\theta)). \tag{17}$$

Combining (16) and (17), we obtain

$$\begin{aligned} V(x(t)) &\le V(x(0))E_\alpha(h_2t^\alpha) + \sup_{\tau\in[0,t]} M(\tau) \int_0^t \frac{E_{\alpha,\alpha}(h_2(t-\tau)^\alpha)}{(t-\tau)^{1-\alpha}} d\tau \\ &\le V(x(0))E_\alpha(h_2t^\alpha) + (E_\alpha(h_2t^\alpha) - 1) \sup_{\theta\in[-h_2,t-h_1]} V(x(\theta)), \end{aligned}$$

Moreover

$$\sup_{\theta\in[-h_2,t]} V(y(\theta)) \le E_\alpha(h_2T^\alpha)V(x(0)) + [E_\alpha(h_2T^\alpha) - 1] \sup_{\theta\in[-h_2,t-h_1]} V(x(\theta)),\ \forall t \in [0,T]. \tag{18}$$

Applying Lemma 4 with

$$H(t) = \sup_{\theta\in[-h_2,t]} V(y(\theta)),\ a = E_\alpha(h_2T^\alpha),\ b = E_\alpha(h_2T^\alpha) - 1,$$

we have

$$\sup_{\theta\in[-h_2,t]} V(x(\theta)) \le q \sup_{\theta\in[-h_2,0]} V(x(\theta)) = q \sup_{\theta\in[-h_2,0]} (\varphi(\theta))^\top P(\varphi(\theta)) \le q\lambda_{max}(P)\|\varphi\|^2, \tag{19}$$

where

$$q = E_\alpha(h_2T^\alpha) \sum_{j=0}^{[T/h_1]+1} (E_\alpha(h_2T^\alpha) - 1)^j.$$

For $t \in [0,T]$, (5) and (19) show that

$$\|x(t)\|^2 \le \frac{x(t)^\top P x(t)}{\lambda_{min}(P)} \le \frac{\sup\limits_{\theta\in[-h_2,t]} V(x(\theta))}{\lambda_{min}(P)} \le q\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\|\varphi\|^2 \le q\frac{\lambda_{max}(P)}{\lambda_{min}(P)}c_1 \le c_2.$$

In the other word, the system (1) is finite - time stable w.r.t (c_1, c_2, T).

Remark 1. Note that the parameters c_1, c_2, do not involve in the conditions (4). Hence, we first determine solutions P, γ_1 from the condition (4) and then verify condition (5).

In the sequel, we give a numerical example to show the effectiveness of the main result.

Example 1. Consider system (1) , where

$$\alpha = 0.5,\ h(t) = 0.1 + 0.05\sin^2(t),$$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

the neuron activation functions $f, g : R^2 \to R^2$ defined by

$$f(x) = (f_1(x_1), f_2(x_2))^\top,\ g(x) = (g_1(x_1), g_2(x_2))^\top,\ \forall (x_1, x_2)^\top \in \mathbb{R}^2,$$

$$f_1(t) = f_2(t) = g_1(t) = g_2(t) = 0.08\frac{t}{1+t^2},\ \forall t \in \mathbb{R}.$$

It can be shown that

$$0 < h_1 = 0.1 \le h(t) \le h_2 = 0.15,$$

$f(0) = g(0) = 0$, and the neuron activation functions satisfying the Lipschitz conditions (3) with

$$l_1 = l_2 = k_1 = k_2 = 0.1.$$

By using LMI Toolbox in Matlab, LMI (4) is feasible with

$$P = \begin{bmatrix} 1.7413 & 0.1105 \\ 0.1105 & 1.7544 \end{bmatrix},\ \gamma_1 = 5.8115.$$

In this case, it can be computed that

$$\gamma_2 = 7.5,\ \lambda_{max}(P) = 1.8586,\ \lambda_{min}(P) = 1.6371,$$

For $c_1 = 1,\ c_2 = 4,\ T = 10$, we can verify the condition (5) as

$$E_\alpha(h_2 T^\alpha) \sum_{j=0}^{[T/h_1]+1} (E_\alpha(h_2 T^\alpha) - 1)^j \frac{\lambda_{max}(P)}{\lambda_{min}(P)} c_1 = 3.9939 < c_2 = 4.$$

Hence, by Theorem 5 the system (1) is finite-time stable w.r.t. (1, 4, 10). Fig. 1 shows the trajectories of $\|x(t)\|^2$ of the system with the initial condition $\varphi(t) = \begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix}$, $t \in [-0.15,\ 0]$.

□

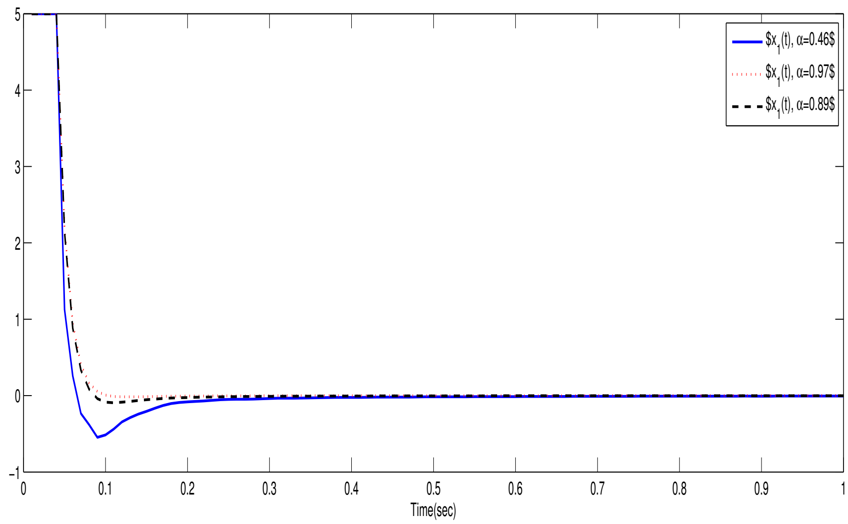

Figure 1: State response of the system

4. Conclusions

The problem of finite-time stability for fractional-order neural networks with time-varying delay has been investigated. We have proposed an analytical approach based on the Laplace transform and "inf-sup" method to derive delay-dependent sufficient conditions for FTS. The conditions have been established in the form of a tractable linear matrix inequality and Mittag-Leffler functions. An example with simulations is presented to verify the effectiveness of the proposed results.

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