MULTI-TERM FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN POWER GROWTH FUNCTION SPACES IN "FCAA" JOURNAL

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Abstract

In this paper we characterize the Laplace transform of functions with power growth square averages and study several multi-term Caputo and Riemann-Liouville fractional integro-differential equations in this space of functions.

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1. Introduction

Denote by $\mathcal{L}$ and $\mathcal{L}^{-1}$ the Laplace transform and its inverse transform [9]

$$\begin{aligned} F(s) = (\mathcal{L}f)(s) &:= \int_0^\infty e^{-st} f(t)\, dt, \\ f(t) = (\mathcal{L}^{-1}F)(t) &:= \frac{1}{2\pi i} \int_{Res=d} F(s)\, e^{st}\, ds. \end{aligned} \tag{1.1}$$

The Laplace transform of functions with bounded growth averages, introduced in [10], has been characterized in [8]

THEOREM 1.1. [8] *A function $F(s)$ is the Laplace transform of f such that*

$$f \in BSA(\mathbb{R}_+) \quad \Longleftrightarrow \quad \sup_{T>0} \frac{1}{T+1} \int_0^T |f(t)|^2\, dt < \infty, \tag{1.2}$$

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if and only if $F(s)$ *is holomorphic in the right-half plane* $\operatorname{Re} s > 0$, *and*

$$\sup_{x>0} \frac{x}{x+1} \int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy < \infty. \tag{1.3}$$

For the following Caputo and Riemann-Liouville fractional integro-differential equations

$${}^{\mathcal{C}}\partial_t^{\alpha} f(t) + kf(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \qquad f(0) = f_0, \tag{1.4}$$

$${}^{\alpha}_{0+} f(t) + kf(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \quad I_{0+}^{1-\alpha} f(0+) = f_0, \quad \frac{1}{2} < \alpha \leq 1, \tag{1.5}$$

where ${}^{\mathcal{C}}\partial_t^{\alpha}$, D_{0+}^{α}, and $I_{0+}^{1-\alpha}$ are the Caputo and Riemann-Liouville fractional derivatives and the Riemann-Liouville fractional integral [4], it was shown [8] that if $g, h \in L^1(\mathbb{R}_+)$, and $\|g\|_1 < k$, then the Caputo fractional integro-differential equation (1.4) and the Riemann-Liouville fractional integro-differential equation (1.5) have unique solutions f from $BSA(\mathbb{R}_+)$.

In this paper we will study multi-term Caputo and Riemann-Liouville fractional integro-differential equations. The solutions as it turns out will have some power growth at infinity. It is well known [9] that if $f(t)$ is locally integrable and has a power growth, then $F(s)$ exists and is holomorphic in the right-half plane $\operatorname{Re} s > 0$. The Tauberian theorem for the Laplace transform [9]

$$f(t) \sim \frac{At^{p-1}}{\Gamma(p)}, \quad t \to \infty \quad \Longrightarrow \quad F(s) \sim \frac{A}{s^p} \quad s \to 0_+ \quad p > 0, \tag{1.6}$$

says that, moreover, if $f(t)$ grows as t^{p-1} at infinity, then $F(s)$ grows as s^{-p} at 0.

The converse question is if $F(s)$ is holomorphic in the right-half plane $\operatorname{Re} s > 0$, and has a power growth at 0, whether it is the Laplace transform of a power growth function. The answer turns out affirmative if we consider functions of square average power growth instead of functions with pointwise power growth.

2. **Functions with Square Average Power Growth**

We now generalize the class of functions investigated in [8] to functions with square average power growth on $\mathbb{R}_+ = (0; \infty)$.

Definition 2.1. By $BSA_p(\mathbb{R}_+)$, the linear space of functions with square average power growth of order $p \geq 0$, we denote the set of locally integrable functions f on $\mathbb{R}_+$ such that

$$\sup_{T>0} \frac{1}{(T+1)^p} \int_0^T |f(t)|^2 \, dt < \infty, \tag{2.7}$$

and

$$BSA_\infty(\mathbb{R}_+) = \bigcup_{p>0} BSA_p(\mathbb{R}_+). \tag{2.8}$$

We say $f \in BSA_p^m(\mathbb{R}_+)$ if $f, f', ..., f^{(m)} \in BSA_p(\mathbb{R}_+)$.

Clearly, $BSA_0(\mathbb{R}_+) = L^2(\mathbb{R}_+)$, and $BSA_p(\mathbb{R}_+) \subset BSA_{p'}(\mathbb{R}_+)$ if $p < p'$. It is readily seen that $L^2(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+) \subset BSA_p(\mathbb{R}_+), p \geq 1$, and by Hölder's inequality $L^q(\mathbb{R}_+) \subset BSA_p(\mathbb{R}_+)$, for $2 \leq q \leq \infty, p \geq 1$. However, note that, for $p \geq 0$, we have $f(t) = t^p \in BSA_{2p+1}(\mathbb{R}_+)$, and yet $f(t) \notin L^q((\mathbb{R}_+)), 0 < q < \infty$.

Functions with bounded square averages on the whole real line have been studied first in [10]. The special case $p = 1$ has been considered in [7, 8].

Now we characterize the Laplace transform of functions from $BSA_p(\mathbb{R}_+)$.

THEOREM 2.1. *A function $F(s)$ is the Laplace transform of $f \in BSA_p(\mathbb{R}_+)$ if and only if $F(s)$ is holomorphic in the right-half plane* $\operatorname{Re} s > 0$, *and*

$$\sup_{x>0} \left(\frac{x}{x+1}\right)^p \int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy < \infty. \tag{2.9}$$

P r o o f. The case $p = 0$ is the Paley-Wiener theorem for the Laplace transform [6, 9]

$$f(t) \in L^2(\mathbb{R}_+) \iff F(s) \text{ is holomorphic in} \operatorname{Re} > 0, \quad \sup_{x>0} \int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy < \infty. \tag{2.10}$$

For $p > 0$ we follow the proof in [8]. Let $f \in BSA_p(\mathbb{R}_+)$. Denote $\tilde{f}(T) = \int_0^T f(t) \, dt$. Integration by parts gives

$$F(s) := \int_0^\infty e^{-st} f(t) \, dt = e^{-sT} \tilde{f}(T)\Big|_{T=0}^{T=\infty} + s \int_0^\infty e^{-st} \tilde{f}(t) \, dt, \quad \operatorname{Re} > 0.$$

By the Hölder inequality we have, for $T > 0$,

$$|\tilde{f}(T)| \le \int_0^T 1.|f(t)|dt \le \sqrt{\int_0^T dt \int_0^T |f(t)|^2 dt} = \sqrt{T}\sqrt{\int_0^T |f(t)|^2 dt}$$
$$\le C\sqrt{T}(T+1)^{\frac{p}{2}}.$$

Here and throughout the paper C denotes a universal constant that can be distinct in different places. Hence

$$e^{-sT}\tilde{f}(T)\Big|_{T=0}^{T=\infty} = 0, \quad \mathrm{Re} > 0,$$

and

$$F(s) = s\int_0^\infty e^{-st}\tilde{f}(t)\,dt, \quad \mathrm{Re}\, s > 0.$$

Since $|\tilde{f}(t)| \le C\sqrt{t(t+1)^p}$, the Laplace transform of $\tilde{f}(t)$, i.e. $\frac{F(s)}{s}$, exists and is holomorphic in the right half plane $\mathrm{Re}\, s > 0$.

Integration by parts yields

$$\int_0^\infty e^{-2xt}|f(t)|^2 dt = e^{-2xT}\int_0^T |f(t)|^2\,dt\Big|_{T=0}^{T=\infty}$$
$$+2x\int_0^\infty e^{-2xT}\int_0^T |f(t)|^2\,dt\,dT \le Cx\int_0^\infty (T+1)^p e^{-2xT}\,dT$$
$$= \frac{Ce^{2x}}{2^{p+1}x^p}\int_{2x}^\infty \tau^p e^{-\tau} d\tau = \frac{Ce^{2x}}{2^{p+1}x^p}\Gamma(p+1, 2x) \tag{2.11}$$

where $\Gamma(p+1; 2x)$ is the upper incomplete Gamma function [1]. Using the asymptotics of the upper incomplete Gamma function [1]

$$\Gamma(p,x) \sim \quad x^{p-1}e^{-x}, \quad x \to \infty,$$
$$\Gamma(p,x) \sim \quad \Gamma(p), \quad x \to 0,$$

we see that the last expression of (2.11) is bounded at infinity and $\sim x^{-p}$ at 0. Consequently,

$$\int_0^\infty e^{-2xt}|f(t)|^2 dt \le C\Big(\frac{x+1}{x}\Big)^p. \tag{2.12}$$

Hence, $e^{-xt}f(t) \in L^2(\mathbb{R}_+)$ for any $x > 0$. Consequently, $F(s)$ with $\mathrm{Re}\, s > x_0 > 0$ is the Laplace transform of $e^{-x_0 t}f(t) \in L^2(\mathbb{R}_+)$ at the point $s - x_0$. The Parseval formula for the Laplace transform in $L^2(\mathbb{R}_+)$, see [9], gives

$$\int_0^\infty e^{-2xt}|f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty |F(x+iy)|^2\,dy, \quad x > x_0 > 0. \tag{2.13}$$

Since x_0 is an arbitrary positive constant, formula (2.13) holds for any $x>0$. Combining formulas (2.12) and (2.13) we obtain

$$\int_{-\infty}^{\infty} |F(x+iy)|^2\, dy \le \frac{C(x+1)^p}{x^p},$$

that yields (2.9).

Conversely, assume that $F(s)$ is holomorphic in the right-half plane $\operatorname{Re} s > 0$ and formula (2.9) holds. Then

$$\sup_{x>x_0} \int_{-\infty}^{\infty} |F(x+iy)|^2 dy < \infty, \quad x_0 > 0.$$

By the Paley-Wiener theorem [6, 9] function $F(x_0+s)$ is the Laplace transform of a function, say, $f_{x_0}(t) \in L^2(\mathbb{R}_+)$

$$F(x_0+s) = \int_0^{\infty} e^{-st} f_{x_0}(t)\, dt, \quad \operatorname{Re} s > 0.$$

Thus

$$\begin{aligned} F(x_0+x_1+s) &= \int_0^{\infty} e^{(-x_1-s)t} f_{x_0}(t) dt \\ &= \int_0^{\infty} e^{(-x_0-s)t} f_{x_1}(t) dt, \quad \operatorname{Re} s, x_0, x_1 > 0. \end{aligned}$$

Consequently, $e^{-x_1 t} f_{x_0}(t) = e^{-x_0 t} f_{x_1}(t)$. Denote $f(t) = e^{x_0 t} f_{x_0}(t)$. It is clear that $f(t)$ is independent of $x_0 > 0$ and F is the Laplace transform of f

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt, \quad \operatorname{Re} s > x_0 + x_1.$$

As $e^{-x_0 t} f(t) = f_{x_0}(t) \in L^2(\mathbb{R}_+)$, the Parseval formula for the Laplace transform [9] yields

$$\int_0^{\infty} e^{-2x_0 t} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(x_0+iy)|^2\, dy \le \frac{C(x_0+1)^p}{x_0^p}, \quad x_0 > 0.$$

Let g be a bounded function on $\mathbb{R}_+$. Then

$$\begin{aligned} \int_0^{\infty} e^{-2xt} g(e^{-2xt}) |f(t)|^2\, dt &\le \|g\|_\infty \int_0^{\infty} e^{-2xt} |f(t)|^2\, dt \\ &\le \frac{C(x+1)^p}{x^p} \|g\|_\infty, \quad x > 0. \end{aligned} \tag{2.14}$$

Take

$$g(t) = \begin{cases} \frac{1}{t}, & t > e^{-2} \\ 0, & 0 < t \le e^{-2} \end{cases}.$$

Then $\|g\|_\infty = e^2$, and (2.14) becomes

$$\int_0^{1/x} |f(t)|^2\,dt \le \frac{C(x+1)^p}{x^p}, \quad x > 0.$$

Replacing x by $\frac{1}{T}$ we arrive at

$$\frac{1}{(T+1)^p}\int_0^T |f(t)|^2\,dt \le C, \quad T > 0.$$

Thus $f \in BSA_p(\mathbb{R}_+)$, and Theorem 2.1 is proved. □

3. **Special cases**

Corollary 3.1. *Let $F(s)$ be holomorphic in the right half plane* $\operatorname{Re} s > 0$ *and* $|F(s)| \le C|s|^{-\alpha}$, $\alpha > \frac{1}{2}$. *Then F is the Laplace transform of a function* $f \in BSA_{2\alpha-1}(\mathbb{R}_+)$.

P r o o f. Because $\alpha > \frac{1}{2}$, $F(x+i\bullet) \in L^2(\mathbb{R})$, and

$$\frac{x^{2\alpha-1}}{(x+1)^{2\alpha-1}}\int_{-\infty}^{\infty} |F(x+iy)|^2 dy \le \frac{Cx^{2\alpha-1}}{(x+1)^{2\alpha-1}}\int_{-\infty}^{\infty}(x^2+y^2)^{-\alpha}dy$$
$$= \frac{C\sqrt{\pi}\,\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma(\alpha)(x+1)^{2\alpha-1}} < \infty,$$

hence, formula (2.9) holds, i.e., $f \in BSA_{2\alpha-1}(\mathbb{R}_+)$. □

The following result explains the importance of $BSA_\alpha(\mathbb{R}_+)$ in studying fractional calculus.

Theorem 3.1. *Let* $g \in L^1(\mathbb{R}_+)$ *with* $\|g\|_1 < k$, $0 < \alpha \le 1$. *Then*

$$\frac{1}{\left|s^\alpha + k + G(s)\right|} \le \frac{4k}{k - \|g\|_1}|s|^{-\alpha}, \quad \operatorname{Re} s > 0. \tag{3.15}$$

If, moreover, $\frac{1}{2} < \alpha \le 1$, *then the inverse Laplace transform*

$$f := \mathcal{L}^{-1}\left(\frac{1}{s^\alpha + k + G(s)}\right) \tag{3.16}$$

is from $BSA_{2\alpha-1}(\mathbb{R}_+)$.

P r o o f. Since $g \in L^1(\mathbb{R}_+)$, its Laplace transform $G(s)$ is holomorphic in the right half-plane, and from

$$|G(s)| \le \int_0^\infty e^{-(\operatorname{Re} s)\,t}\,|g(t)|\,dt \le \|g\|_1 < k, \ \operatorname{Re}(s^\alpha) \ge 0, \quad \text{for} \quad \operatorname{Re} s > 0,$$

we deduce that $\mathrm{Re}\,(s^\alpha + k + G(s)) > 0$ when $\mathrm{Re}\,s > 0$. Consequently, $\dfrac{1}{s^\alpha + k + G(s)}$ is also holomorphic in the right half-plane. Let us denote by

$$h(s) = k + G(s),$$

then $h(s)$ is clearly holomorphic in the right half-plane, and for $\mathrm{Re}\,s > 0$,

$$0 < k - \|g\|_1 \leq \mathrm{Re}\,h(s) \tag{3.17}$$

and also $|h(s)| < 2k$.
For $|s|^\alpha > 4k, \quad \mathrm{Re}\,s > 0$, we have

$$|s^\alpha + h(s)| \geq |s|^\alpha - |h(s)| > |s|^\alpha - 2k > \frac{1}{2}|s|^\alpha > \frac{k - \|g\|_1}{4k}|s|^\alpha. \tag{3.18}$$

For $|s|^\alpha \leq 4k, \quad \mathrm{Re}\,s > 0$, we have

$$|s^\alpha + h(s)| \geq \mathrm{Re}(s^\alpha + h(s)) > k - \|g\|_1 > \frac{k - \|g\|_1}{4k}|s|^\alpha. \tag{3.19}$$

Combining (3.18) and (3.19) we obtain (3.15). Statement (3.16) follows from Corollary 3.1. □

Combining Corollary 3.1 and Theory 3.1 we arrive at

Corollary 3.2. *Let* $\|g\|_1 < k, 0 < \alpha \leq 1, \beta < \alpha - \frac{1}{2}$, *then*

$$\mathcal{L}^{-1}\left(\frac{s^\beta}{s^\alpha + k + G(s)}\right) \in BSA_{2(\alpha-\beta)-1}(\mathbb{R}_+). \tag{3.20}$$

As an example consider the two-parametric Mittag-Leffler function [3]

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha > 0.$$

We have [3]

$$\mathcal{L}(t^{\beta-1}E_{\alpha,\beta}(-kt^\alpha))(s) = \frac{s^{\alpha-\beta}}{s^\alpha + k}.$$

Consequently, by Corollary 3.2 if $k > 0, 0 < \alpha \leq 1$, and $\beta > \frac{1}{2}$, then $t^{\beta-1}E_{\alpha,\beta}(-kt^\alpha) \in BSA_{2\beta-1}(\mathbb{R}_+)$.

Theorem 3.2. *Let* $g \in L^1(\mathbb{R}_+)$ *with* $\|g\|_1 < k$, $0 < \alpha_n < ... < \alpha_1 < \alpha_0 \leq 1$, *and* $a_1, a_2, ..., a_n > 0$. *Then*

$$\frac{1}{\left|s^{\alpha_0}+\sum_{j=1}^n a_j s^{\alpha_j}+k+G(s)\right|}\leq \frac{C_1^{\alpha_0}}{k-\|g\|_1}|s|^{-\alpha_0},\quad \operatorname{Re} s>0, \tag{3.21}$$

where

$$C_1=\max_{1\leq j\leq n}\left\{[(n+2)a_j]^{\frac{1}{\alpha_0-\alpha_j}},[2k(n+2)]^{\frac{1}{\alpha_0}}\right\}. \tag{3.22}$$

If, moreover, $\frac{1}{2}<\alpha_0\leq 1$, *then the inverse Laplace transform*

$$\mathcal{L}^{-1}\left(\frac{1}{s^{\alpha_0}+\sum_{j=1}^n a_j s^{\alpha_j}+k+G(s)}\right)\in BSA_{2\alpha_0-1}(\mathbb{R}_+). \tag{3.23}$$

P r o o f. As in the proof of Theory 3.1, $G(s)$ is holomorphic in the right half-plane $\operatorname{Re} s>0$, and there $\operatorname{Re}(k+G(s))\geq k-\|g\|_1>0$. Since $0<\alpha_j\leq 1$, then $\operatorname{Re}(s^{\alpha_j})>0, j=0,1,2,...,n,$ in $\operatorname{Re} s>0$. Together with $a_1,a_2,...a_n>0$ we arrive at

$$\operatorname{Re}\left(s^{\alpha_0}+\sum_{j=1}^n a_j s^{\alpha_j}+k+G(s)\right)>0,$$

in $\operatorname{Re} s>0$. Consequently, $\frac{1}{s^{\alpha_0}+\sum_{j=1}^n a_j s^{\alpha_j}+k+G(s)}$ is also holomorphic in the right half-plane. For $|s|>C_1, \operatorname{Re} s>0$, we have

$$\frac{|s|^{\alpha_0}}{n+2}-a_j|s|^{\alpha_j}\geq 0,\quad j=1,2,3,...,n,\quad \frac{|s|^{\alpha_0}}{n+2}-k-|G(s)|\geq 0.$$

Consequently,

$$\left|s^{\alpha_0}+\sum_{j=1}^n a_j s^{\alpha_j}+k+G(s)\right|\geq |s|^{\alpha_0}-\sum_{j=1}^n a_j|s|^{\alpha_j}-k-|G(s)| \tag{3.24}$$

$$\geq \frac{|s|^{\alpha_0}}{n+2}+\sum_{j=1}^n\left(\frac{|s|^{\alpha_0}}{n+2}-a_j|s|^{\alpha_j}\right)+\left(\frac{|s|^{\alpha_0}}{n+2}-k-|G(s)|\right)>\frac{|s|^{\alpha_0}}{n+2}.$$

For $|s|\leq C_1,\ \operatorname{Re} s>0$,

$$\left|s^{\alpha_0}+\sum_{j=1}^n a_j s^{\alpha_j}+k+G(s)\right|\geq \operatorname{Re}\left(s^{\alpha_0}+\sum_{j=1}^n a_j s^{\alpha_j}+k+G(s)\right) \tag{3.25}$$

$$>\operatorname{Re}(k+G(s))\geq k-\|g\|_1\geq \frac{k-\|g\|_1}{C_1^{\alpha_0}}|s|^{\alpha_0}.$$

Since $C_1^{\alpha_0} \geq 2k(n+2)$ we have

$$\frac{k - \|g\|_1}{C_1^{\alpha_0}} \leq \frac{k}{2k(n+2)} < \frac{1}{n+2}.$$

Combining (3.24) and (3.25) we obtain (3.21). Statement (3.23) follows from Corollary 3.1. □

LEMMA 3.1. *Let $f \in BSA_p(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+)$. Then their Laplace convolution*

$$h(t) = (f * g)(t) := \int_0^t f(t-\tau)\, g(\tau)\, d\tau \tag{3.26}$$

belongs to $BSA_p(\mathbb{R}_+)$.

P r o o f. In fact, applying the Laplace transform to (3.26), we obtain $H(s) = F(s)G(s)$, therefore, $|H(s)| \leq |F(s)|\, \|g\|_1$, and

$$\begin{aligned} &\sup_{x>0} \left(\frac{x}{x+1}\right)^p \int_{-\infty}^{\infty} |H(x+iy)|^2\, dy \\ &\leq \|g\|_1^2 \sup_{x>0} \left(\frac{x}{x+1}\right)^p \int_{-\infty}^{\infty} |F(x+iy)|^2\, dy < \infty. \end{aligned} \tag{3.27}$$

□

4. Multi-Term Riemann-Liouville Fractional integro-differential equation

Consider now the following multi-term Riemann-Liouville fractional integro-differential equation

$$D_{0+}^{\alpha_0} f(t) + \sum_{j=1}^{n} a_j D_{0+}^{\alpha_j} f(t) + k f(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \quad I_{0+}^{1-\alpha_0} f(0+) = f_0, \tag{4.28}$$

$$\frac{1}{2} < \alpha_0 \leq 1,\ 0 < \alpha_n < ... < \alpha_1 < \alpha_0,$$

where $k, a_1, a_2, ..., a_n \in \mathbb{R}_+,\ g, h \in L^1(\mathbb{R}_+)$ are given, and f is the unknown. Here D_{0+}^{α} is the Riemann-Liouville fractional derivative [4]

$$D_{0+}^{\alpha} f(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} f(t), \quad I_{0+}^{n-\alpha} f(t) = \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(\tau)\, d\tau, \quad \alpha < n. \tag{4.29}$$

Special cases of (4.28) have been considered in [4].
It is well known [4] that

$$\mathcal{L}\left(D_{0+}^{\alpha} f\right)(s) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} D_{0+}^{\alpha+k-n} f(0+), \quad n-1 \le \alpha < n. \tag{4.30}$$

THEOREM 4.1. *Let $k > 0$, $f_0 \in \mathbb{R}$, $g, h \in L^1(\mathbb{R}_+)$, be given, and $\|g\|_1 < k$. Then the multi-term Riemann-Liouville fractional integro-differential equation* (4.28) *has a unique solution f from $BSA_{2\alpha_0-1}(\mathbb{R}_+)$.*

P r o o f. Since $I_{0+}^{1-\alpha_0} f(0+) = f_0$, and $1-\alpha_0 < 1-\alpha_j$, then $I_{0+}^{1-\alpha_j} f(0+) = 0, j = 1, 2, ..., n$. Applying the Laplace transform to equation (4.28) and taking into account (4.30) we obtain

$$s^{\alpha_0} F(s) - f_0 + \sum_{j=1}^{n} a_j s^{\alpha_j} F(s) + kF(s) + G(s)F(s) = H(s). \tag{4.31}$$

Solving for $F(s)$ yields

$$F(s) = \frac{f_0 + H(s)}{s^{\alpha_0} + \sum_{j=1}^{n} a_j s^{\alpha_j} + k + G(s)}. \tag{4.32}$$

Denote

$$M(s) = \frac{1}{s^{\alpha_0} + \sum_{j=1}^{n} a_j s^{\alpha_j} + k + G(s)}, \tag{4.33}$$

then according to Theory 3.2, its inverse Laplace transform, namely $m(t)$, belongs to $BSA_{2\alpha_0-1}(\mathbb{R}_+)$, and

$$f(t) = f_0 m(t) + \int_0^t m(t-\tau) h(\tau)\, d\tau. \tag{4.34}$$

Since $m \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$ and $h \in L^1(\mathbb{R}_+)$, by Lemma 3.1, their Laplace convolution $m*h$ belongs to $BSA_{2\alpha_0-1}(\mathbb{R}_+)$. Hence, f, defined by (4.34), is from $BSA_{2\alpha_0-1}(\mathbb{R}_+)$. Using the Tauberian theorem for the Laplace transform [9] we have

$$M(s) \sim \frac{1}{s^{\alpha_0}}, \quad s \to \infty \quad \Longrightarrow \quad m(t) \sim \frac{t^{\alpha_0 - 1}}{\Gamma(\alpha_0)}, \quad t \to 0+. \tag{4.35}$$

Consequently, $I_{0+}^{1-\alpha_0} m(t) \sim 1, \quad t \to 0+$. Together with (4.34) it yields $I_{0+}^{1-\alpha_0} f(0+) = f_0$.

Conversely, let f be given by (4.34), where m is defined as the Laplace inverse of (4.33). Then $f \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$ and $I_{0+}^{1-\alpha_0} f(0+) = f_0$. Applying the Laplace transform to (4.34) and taking into account (4.33) we arrive at (4.32). Hence, (4.31) holds. The Laplace inverse of (4.31) yields (4.28). □

5. Multi-Term Caputo Fractional integro-differential equation

Consider the following Caputo fractional integro-differential equation

$$ {}^{\mathcal{C}}\partial_t^{\alpha_0} f(t) + \sum_{j=1}^{n} a_j {}^{\mathcal{C}}\partial_t^{\alpha_j} f(t) + k f(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \quad f(0+) = f_0, \tag{5.36} $$

$$ \frac{1}{2} < \alpha_0 \le 1, \; 0 < \alpha_n < ... < \alpha_1 < \alpha_0, $$

where $a_j, k \in \mathbb{R}_+, j = 1, 2, ..., n$, and $g, h \in L^1(\mathbb{R}_+)$ are given, and f is the unknown. Here ${}^{\mathcal{C}}\partial_t^{\alpha}$ is the Caputo fractional derivative [4]

$$ {}^{\mathcal{C}}\partial_t^{\alpha} f(t) = \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(\tau)\, d\tau, \quad n-1 < \alpha < n, \qquad {}^{\mathcal{C}}\partial_t^{n} f(t) = f^{(n)}(t). \tag{5.37} $$

It is well known [4] that

$$ \mathcal{L}\left({}^{\mathcal{C}}\partial_t^{\alpha} f\right)(s) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha \le n. \tag{5.38} $$

THEOREM 5.1. *Let $k > 0$, $f_0 \in \mathbb{R}$, $g, h \in L^1(\mathbb{R}_+)$, be given, and $\|g\|_1 < k$. Then the Caputo fractional integro-differential equation* (5.36) *has a unique solution f from $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$.*

P r o o f. Applying the Laplace transform to equation (5.36) and taking into account (5.38) we obtain

$$ (s^{\alpha_0} F(s) - s^{\alpha_0-1} f_0) + \sum_{j=1}^{n} a_j (s^{\alpha_j} F(s) - s^{\alpha_j-1} f_0) + kF(s) + G(s)F(s) = H(s). \tag{5.39} $$

Solving for $F(s)$ yields

$$ F(s) = \frac{s^{\alpha_0-1} f_0 + f_0 \sum_{j=1}^{n} a_j s^{\alpha_j-1} + H(s)}{s^{\alpha_0} + \sum_{j=1}^{n} a_j s^{\alpha_j} + k + G(s)}. \tag{5.40} $$

Denote

$$ \begin{aligned} L_j(s) &= \frac{s^{\alpha_j-1}}{s^{\alpha_0} + \sum_{j=1}^{n} a_j s^{\alpha_j} + k + G(s)}, \; j = 0, 1, \cdots, n, \\ M(s) &= \frac{1}{s^{\alpha_0} + \sum_{j=1}^{n} a_j s^{\alpha_j} + k + G(s)}, \end{aligned} \tag{5.41} $$

then according to Theorem 3.2, their inverse Laplace transforms, namely $l_j(t)$ and $m(t)$, belong to $BSA_{2(\alpha_0-\alpha_j)+1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$, and $BSA_{2\alpha_0-1}(\mathbb{R}_+)$, respectively. Moreover,

$$f(t) = f_0\, l_0(t) + f_0 \sum_{j=1}^{n} a_j l_j(t) + \int_0^t m(t-\tau)h(\tau)\, d\tau. \tag{5.42}$$

Since $m \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$ and $h \in L^1(\mathbb{R}_+)$, by Lemma 3.1, their Laplace convolution $m{*}h$ belongs to $BSA_{2\alpha_0-1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$. Hence, f, defined by (5.42), is from $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$. Using the Tauberian theorem for the Laplace transform [9] we have

$$L_0(s) \sim \frac{1}{s}, \quad s \to \infty \quad \Longrightarrow \quad l_0(t) \sim 1, \quad t \to 0+,$$

and

$$L_j(s) \sim \frac{1}{s^{\alpha_0-\alpha_j+1}}, \quad s \to \infty \quad \Longrightarrow \quad l_j(t) \sim \frac{t^{\alpha_0-\alpha_j}}{\Gamma(\alpha_0-\alpha_j+1)}, \quad t \to 0+,$$
$$j = 1, \cdots, n.$$

Consequently, $f(0+) = f_0$.

Conversely, let f be given by (5.42), where l_j, m are defined as the Laplace inverse transforms of (5.41). Then $f \in BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$ and $f(0+) = f_0$. Applying the Laplace transform to (5.42) and taking into account (5.41) we arrive at (5.40). Hence, (5.39) holds. The Laplace inverse transform of (5.39) yields (5.36). □

Remark 5.1. If $f_0 = 0$, then $f \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$.

Remark 5.2. Although $g, h \in L^1(\mathbb{R}_+)$, in general $f \notin L^1(\mathbb{R}_+)$. In fact, if $f \in L^1(\mathbb{R}_+)$, then $|F(s)| \le \|f\|_1$ for $\operatorname{Re} s \ge 0$. But if $f_0 \ne 0$, then from (5.40) we have

$$F(s) \sim C s^{\alpha_n - 1} \to \infty,$$

as $s \to 0+$. Consequently, $f \notin L^1(\mathbb{R}_+)$.

6. Mixed Caputo Riemann-Liouville Fractional integro-differential equation

Now we consider the following mixed Caputo Riemann-Liouville fractional integro-differential equation with a dominant Caputo fractional derivative

$$
\begin{aligned}
&{}^{\mathcal{C}}\partial_t^{\alpha_0} f(t) + \sum_{j=1}^{n} a_j {}^{\mathcal{C}}\partial_t^{\alpha_j} f(t) + \sum_{j=1}^{m} b_j D_{0+}^{\beta_j} f(t) + k f(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t),\\
&f(0+) = f_0,\\
&\frac{1}{2} < \alpha_0 \le 1, \quad 0 < \alpha_n < ... < \alpha_1 < \alpha_0, \quad 0 < \beta_m < ... < \beta_1 < \alpha_0, \quad (6.43)
\end{aligned}
$$

where $g, h \in L^1(\mathbb{R}_+), a_1, \cdots, a_n, b_1, \cdots, b_m, k \in \mathbb{R}_+$, are given, and f is the unknown.

THEOREM 6.1. *Let $k > 0$, $f_0 \in \mathbb{R}$, $g, h \in L^1(\mathbb{R}_+)$, be given, and $\|g\|_1 < k$. Then the mixed Caputo Riemann-Liouville fractional integro-differential equation* (6.43) *has a unique solution f from $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$.*

P r o o f. Since $f(0+) = f_0$, then $I_{0+}^{1-\beta_j}(0+) = 0$, $j = 1, \cdots, m$, and applying the Laplace transform to equation (6.43) and taking into account (4.28) and (5.38), we obtain

$$
\begin{aligned}
(s^{\alpha_0} F(s) - s^{\alpha_0-1} f_0) &+ \sum_{j=1}^{n} a_j (s^{\alpha_j} F(s) - s^{\alpha_j-1} f_0)\\
&+ \sum_{j=1}^{m} b_j s^{\beta_j} F(s) + kF(s) + G(s)F(s) = H(s).
\end{aligned} \tag{6.44}
$$

Solving for $F(s)$ yields

$$
F(s) = \frac{f_0 s^{\alpha_0-1} + f_0 \sum_{j=1}^{n} a_j s^{\alpha_j-1} + H(s)}{s^{\alpha_0} + \sum_{j=1}^{n} a_j s^{\alpha_j} + \sum_{j=1}^{m} b_j s^{\beta_j} + k + G(s)}. \tag{6.45}
$$

Denote

$$
\begin{aligned}
L_j(s) &= \frac{s^{\alpha_j-1}}{s^{\alpha_0} + \sum_{j=1}^{n} a_j s^{\alpha_j} + \sum_{j=1}^{m} b_j s^{\beta_j} + k + G(s)}, \quad j = 0, 1, \cdots, n,\\
M(s) &= \frac{1}{s^{\alpha_0} + \sum_{j=1}^{n} a_j s^{\alpha_j} + \sum_{j=1}^{m} b_j s^{\beta_j} + k + G(s)},
\end{aligned} \tag{6.46}
$$

then according to Theorem 3.2, their inverse Laplace transforms, namely $l_j(t)$ and $m(t)$, belong to $BSA_{2(\alpha_0-\alpha_j)+1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$, and

$BSA_{2\alpha_0-1}(\mathbb{R}_+)$, respectively. Moreover,

$$f(t) = f_0\, l_0(t) + f_0 \sum_{j=1}^{n} a_j l_j(t) + \int_0^t m(t-\tau) h(\tau)\, d\tau. \tag{6.47}$$

Since $m \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$, and $h \in L^1(\mathbb{R}_+)$, by Lemma 3.1, their Laplace convolution $m*h$ belongs to $BSA_{2\alpha_0-1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$. Hence, f, defined by (6.47), is from $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$. From (6.45) we have

$$F(s) \sim \frac{f_0}{s}, \quad s \to \infty.$$

Using the Tauberian theorem for the Laplace transform [9] we obtain

$$f(t) \sim f_0, \quad t \to 0+.$$

Consequently, $f(0+) = f_0$.

Conversely, let f be given by (6.47), where l_j, m are defined as the inverse Laplace transforms of (6.47). Then $f \in BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$ and $f(0+) = f_0$. Applying the Laplace transform to (6.47) and taking into account (6.46) we arrive at (6.45). Hence, (6.44) holds. The inverse Laplace transform of (6.44) yields (6.43). □

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