ARTICLE TEMPLATE

Smooth approximation of quaternionic plurisubharmonic functions

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ABSTRACT
In this paper, we are interested in giving sufficient conditions of a quaternionic plurisubharmonic function defined on a bounded quaternionic hyperconvex domain such that it can be approximated by a decreasing sequence of smooth functions. As an application, we study the geometric property of quaternionic B-regular domains.

KEYWORDS
approximation; subharmonic; plurisubharmonic; quaternion; hyperonvex; B-regular

1. Introduction

The classical theory of plurisubharmonic functions is one of the important and central problems of complex variables. In recent years these functions has been generalized in several directions, e.g., q-plurisubharmonic functions, m-subharmonic functions, plurifinely plurisubharmonic functions, etc.

In 2003, S. Alesker [1] have introduced and studied a class of plurisubharmonic functions of quaternionic variables on the flat quaternionic space $\mathbb{H}^n$, which extends the notion of plurisubharmonic functions on open subsets of $\mathbb{C}^n$ to open subsets of $\mathbb{H}^n$ in a natural way. Recall that a function defined on open subsets of $\mathbb{H}^n$ is quaternionic plurisubharmonic if it is upper semi-continuous and its restriction to any right quaternionic line is subharmonic (in the usual sense). Later on, S. Alesker [2] studied the plurisubharmonic functions on arbitrary quaternionic manifolds. The class of the plurisubharmonic functions of quaternionic variables has most of the properties of usual plurisubharmonic functions, e.g. these are closed under addition, satisfy the maximum principle, standard smoothing techniques are available, etc. However, this class nevertheless reflect a rather different geometry behind, which is often essential when studying a given problem in analysis.

The purpose of this paper is to investigate the geometry of the domains that admit a quaternionic plurisubharmonic, exhaustion function. Firstly let us introduce the following definition.

Definition 1.1. Let Ω be a bounded domain in $\mathbb{H}^n$. We say that Ω is quaternionic hyperconvex if it admits a negative, quaternionic plurisubharmonic, exhaustion function φ on Ω.

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From the definition of quaternionic plurisubharmonic functions, it is easy to see that every plurisuharmonic functions (in the usual sense) defined on a subset of $\mathbb{C}^{2n}$ are quaternionic plurisubharmonic, and hence, every bounded hyperconvex domain (in the usual sense) in $\mathbb{C}^{2n}$ is quaternionic hyperconvex. However, the converse is not true (for example $\{q \in \mathbb{H} : 1 < \|q\| < 2\}$ is quaternionic hyperconvex in $\mathbb{H}$ but is not hyperconvex).

The aim of this study is to consider the approximation of a quaternionic plurisubharmonic function u by a decreasing sequence of smooth, strictly quaternionic plurisubharmonic, exhaustion functions $\{u_s\}$. Our first result is the following theorem.

Theorem 1.2. *Let $\Omega \Subset \mathbb{H}^n$ be a bounded quaternionic hyperconvex domain and let u be a negative, quaternionic plurisubharmonic, exhaustion function in Ω. Assume that the function*

$$u(q) - c\|q\|^2, \ q \in \Omega,$$

is quaternionic plurisubharmonic in Ω, for some $c \geq 0$. Then, there exists a decreasing sequence of negative, smooth functions $\{u_s\}$ defined on Ω such that:

(i) u_s are negative, quaternionic plurisubharmonic, exhaustion functions in Ω;

(ii) $u_s(q) - (1 - s^{-1})c\|q\|^2$ are strictly quaternionic plurisubharmonic in Ω;

(iii) $\{u_s\}$ converges pointwise to u in Ω.

Here, we say that a function φ is called strictly quaternionic plurisubharmonic if for every open set $U \Subset \Omega$, there exists a constant $c_U > 0$ such that $\varphi(q) - c_U\|q\|^2$ is quaternionic plurisubharmonic on U.

Now, assume that u is a negative quaternionic plurisubharmonic function defined in bounded hyperconvex domain Ω and φ is a negative, quaternionic plurisubharmonic exhaustion function of Ω. Observe that the functions $\max(u, s\varphi)$ are negative, quaternionic plurisubharmonic, exhaustion in Ω and the sequence $\{\max(u, s\varphi)\}$ decreases to u as $s \to +\infty$. Hence, from Theorem 1.2 (in the case $c = 0$), we easily deduce the following.

Corollary 1.3. *Let $\Omega \Subset \mathbb{H}^n$ be a bounded quaternionic hyperconvex domain. Then, for every negative quaternionic plurisubharmonic function u in Ω, there exists a decreasing sequence of negative, smooth, strictly quaternionic plurisubharmonic, exhaustion functions $\{u_s\}$ defined on Ω that converges pointwise to u on Ω.*

In particular, Ω admits a negative, smooth, strictly quaternionic plurisubharmonic, exhaustion function.

Our second result is the theorem about the geometric properties of quaternionic B-regular domains.

Theorem 1.4. *Assume that Ω is a bounded domain in $\mathbb{H}^n$. Then, the following assertions are equivalent:*

(i) Ω is quaternionic B-regular, i.e. for every continuous functions f on $\partial\Omega$, there exists a quaternionic plurisubharmonic function u in Ω such that

$$\lim_{\Omega \ni q \to p} u(q) = f(p), \ \forall p \in \partial\Omega;$$

(ii) Ω admits a negative, smooth, exhaustion function φ such that $\varphi(q) - \|q\|^2$ is quaternionic plurisubharmonic;

(iii) For every $p \in \partial\Omega$, there exist $r > 0$ and a negative function φ defined on $\Omega \cap \mathbb{B}(p, r)$ such that $\lim_{q\to p} \varphi(q) = 0$ and $\varphi(q) - \|q\|^2$ is quaternionic plurisubharmonic on $\Omega \cap \mathbb{B}(p, r)$.

Remark 1. Since every plurisuharmonic function is quaternionic plurisubharmonic, Theorem 1.2 and Theorem 1.4 are generalizations from [6] and [9] in the case quaternionic plurisuharmonic functions and quaternionic hyperconvex domains.

2. Proof of theorem 1.2

Some elements of pluripotential theory (quaternionic potential theory) that are used in the following were given by [1]-[10]. Firstly, we need the following result.

Lemma 2.1. *Let $\varepsilon > 0$ and let $\chi : \mathbb{R} \to (0, +\infty)$ be a smooth convex function such that $\chi(t) = |t|$, $\forall |t| \geq \varepsilon$. Assume that $\Omega \subset \mathbb{H}^n$ is an open set and u, v are smooth quaternionic plurisubharmonic functions in Ω. Then, the function*

$$w := \frac{1}{2}\left[u + v + \chi(u - v)\right]$$

satisfies:
(i) w is smooth quaternionic plurisubharmonic in Ω;
(ii) $w \geq \max(u, v)$ on Ω;
(iii) $w = u$ on $\Omega \cap \{u - v > \varepsilon\}$;
(iv) $w = v$ on $\Omega \cap \{u - v < -\varepsilon\}$;
(v) if $u(q) - c\|q\|^2$ and $v(q) - c\|q\|^2$ are quaternionic plurisubharmonic in Ω then $w(q) - c\|q\|^2$ is also quaternionic plurisubharmonic.

Proof. The proof is almost the same as the one given in [[6], Lemma 2.2]. For the convenience of the reader, we sketch the proof of the lemma.

(i) Since u, v are smooth on Ω and χ is smooth in $\mathbb{R}$ so w is smooth on Ω. Let $\mathcal{M} \subset \mathbb{R}^2$ be defined by

$$\mathcal{M} := \{(a, b) \in \mathbb{R}^2 : -1 \leq a \leq 1,\ at + b \leq \chi(t),\ \forall t \in \mathbb{R}\}.$$

Since χ is convex so $-1 \leq \chi'(t) \leq 1$, $\forall t \in \mathbb{R}$, and therefore,

$$\chi(t) = \sup\{at + b : (a, b) \in \mathcal{M}\}.$$

This implies that

$$\begin{aligned} w &= \frac{1}{2}\left[\sup_{(a,b)\in\mathcal{M}} (u + v + a(u - v) + b)\right] \\ &= \frac{1}{2}\left[\sup_{(a,b)\in\mathcal{M}} ((1 + a)u + (1 - a)v + b)\right]. \end{aligned} \tag{1}$$

Because the functions $(1 + a)u + (1 - a)v + b$ are quaternionic plurisubharmonic on Ω, for all real numbers a, b with $-1 \leq a \leq 1$, we conclude by (1) that the function w is quaternionic plurisubharmonic on Ω.

(ii) Let $t \in \mathbb{R}$ be such that $0 \leq t \leq \varepsilon$. Since χ is convex and $\varepsilon = t/2 + (2\varepsilon - t)/2$ so

$$\chi(\varepsilon) \leq \chi(t)/2 + \chi(2\varepsilon - t)/2,$$

and hence,

$$\chi(t) \geq 2\chi(\varepsilon) - \chi(2\varepsilon - t) = 2\varepsilon - (2\varepsilon - t) = t, \ \forall t \geq 0.$$

Similarly, we also have $\chi(t) \geq -t$, $\forall t \leq 0$. This implies that $\chi(t) \geq |t|$, $\forall t \in \mathbb{R}$. Therefore,

$$\begin{aligned} w &= \frac{1}{2}\left[u + v + \chi(u - v)\right] \\ &\geq \frac{1}{2}\left[u + v + |u - v|\right] = \max(u, v) \end{aligned}$$

in Ω.

(iii) and (iv) are obvious.

(v) Since $u(q) - c\|q\|^2$ and $v(q) - c\|q\|^2$ are quaternionic plurisubharmonic in Ω, we infer by (i) that the function

$$\begin{aligned} w(q) - c\|q\|^2 =& \frac{1}{2}\left\{\left[u(q) - c\|q\|^2\right] + \left[v(q) - c\|q\|^2\right]\right. \\ & \left. + \chi\left(\left[u(q) - c\|q\|^2\right] - \left[v(q) - c\|q\|^2\right]\right)\right\} \end{aligned}$$

is quaternionic plurisubharmonic in Ω. The proof is complete. □

Now, we recall the definition of the convolution. Let $\mathbb{B}(0,1) \subset \mathbb{H}^n$ be the unit ball and let $\rho \in \mathcal{C}_0^\infty(\mathbb{B}(0,1))$ be such that $0 \leq \rho \leq 1$, $\rho(q)$ depends only on $\|q\|$ and

$$\int_{\{\|q\|<1\}} \rho(q) dV_{4n}(q) = 1,$$

where dV_{4n} is the Lebesgue measure in $\mathbb{H}^n$. Assume that $\varepsilon > 0$ and u is a subharmonic function in an open subset Ω of $\mathbb{H}^n$. Define

$$u * \rho_\varepsilon(q) := \begin{cases} \int_{\{\|p\|<1\}} u\left(q - \varepsilon p\right) \rho(p) dV_{4n}(p) & \text{if } q \in \Omega_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\Omega_\varepsilon := \{q \in \Omega : \ dist(q, \partial\Omega) > \varepsilon\}$. We have the following.

Lemma 2.2. *Let Ω be an open set in $\mathbb{H}^n$ and let u be a negative function defined on Ω such that the function*

$$v(q) := u(q) - c\|q\|^2, \ q \in \Omega$$

is quaternionic plurisubharmonic in Ω, for some constant $c \geq 0$. Then, $u * \rho_\varepsilon \searrow u$ on Ω as $\varepsilon \searrow 0$. Moreover, for all $\varepsilon > 0$, the functions

$$u * \rho_\varepsilon(q) - c\|q\|^2$$

are negative, smooth quaternionic plurisubharmonic in Ω_ε.

Proof. Since $u<0$ in Ω and $u(q)=v(q)+c\|q\|^2$, $q\in\Omega$ so u is negative quaternionic plurisubharmonic in Ω. By the classical results for subharmonic functions, it is easy to see that $u*\rho_\varepsilon$ are negative, smooth quaternionic plurisubharmonic functions in Ω_ε and $u*\rho_\varepsilon\searrow u$ on Ω as $\varepsilon\searrow 0$.

Now, fix $\varepsilon>0$. For $q\in\Omega_\varepsilon$, we have

$$\begin{aligned} v*\rho_\varepsilon(q) &= \int_{\{\|p\|<1\}} v\left(q-\varepsilon p\right)\rho(p)dV_{4n}(p)\\ &= \int_{\{\|p\|<1\}} \left[u\left(q-\varepsilon p\right)-c\|q-\varepsilon p\|^2\right]\rho(p)dV_{4n}(p)\\ &= u*\rho_\varepsilon(q)-c\|q\|^2+w(q). \end{aligned} \tag{2}$$

Here,

$$w(q):=c\int_{\{\|p\|<1\}}\left(\|q\|^2-\|q-\varepsilon p\|^2\right)\rho(p)dV_{4n}(p).$$

We claim that w is quaternionic plurisubharmonic in q. Indeed, we can write $q=(x_1,x_2,\dots,x_{4n})$ and $p=(y_1,y_2,\dots,y_{4n})$. Put

$$a_s:=\int_{\{\|p\|<1\}} y_s\rho(p)dV_{4n}(p),\ \ s=1,\dots,4n,$$

and

$$b:=\int_{\{\|p\|<1\}}\|p\|^2\rho(p)dV_{4n}(p).$$

It is easy to see that

$$\begin{aligned} \|q\|^2-\|q-\varepsilon p\|^2 &= \sum_{s=1}^{4n}[x_s^2-(x_s-\varepsilon y_s)^2]\\ &= 2\varepsilon\sum_{s=1}^{4n}x_sy_s-\varepsilon^2\sum_{s=1}^{4n}y_s^2 = 2\varepsilon\sum_{s=1}^{4n}x_sy_s-\varepsilon^2\|p\|^2. \end{aligned}$$

This implies that

$$w(q)=2\varepsilon c\sum_{s=1}^{4n}a_sx_s-\varepsilon^2bc.$$

Hence, $w|_H$ is harmonic, for every subspace $H\subset\mathbb{R}^{4n}$. In particular, the restriction of w to any right quaternionic line is subharmonic. Thus, w is quaternionic plurisubharmonic. This proves the claim, and therefore, we conclude by (2) that $u*\rho_\varepsilon(q)-c\|q\|^2$ is negative, smooth quaternionic plurisubharmonic in Ω_ε. The proof is complete. □

We now able to give the proof of theorem 1.2.

Proof of theorem 1.2. Without loss of generality we can assume that $\Omega \Subset \mathbb{B}(0,1)$. Let $\{\delta_s\}$ be a decreasing sequence of positive real numbers such that

$$0 < \delta_s < \frac{\delta_{s-1}}{2} < \frac{1}{2s^2} \tag{3}$$

and

$$\{u < -\delta_s\} \Subset \{u < -6\sqrt{\delta_{s+1}}\}. \tag{4}$$

Observe that $\delta_{s+1} < 6\sqrt{\delta_{s+1}}$ because $\delta_{s+1} \in (0,1)$. Hence,

$$\{u < -6\sqrt{\delta_{s+1}}\} \subset \{u < -\delta_{s+1}\} \Subset \Omega.$$

Therefore, we reduce by (4) that there exists a decreasing sequence of positive real numbers $\{\varepsilon_s\}$ such that $0 < \varepsilon_s < \delta_s$ and

$$\{u < -\delta_s\} + \mathbb{B}(0, 2\varepsilon_s) \Subset \{u < -6\sqrt{\delta_{s+1}}\} \Subset \Omega_{\varepsilon_s}. \tag{5}$$

Now, we set $\psi(q) := \|q\|^2 - 3,\ q \in \mathbb{H}^n$ and

$$v_s := (1 - \sqrt{\delta_s})u * \rho_{\varepsilon_s} + \delta_s \psi.$$

Since $u \leq u * \rho_{\varepsilon_s} \leq 0$ on Ω and $0 < \delta_s < 1$ so

$$(1 - \sqrt{\delta_s})u * \rho_{\varepsilon_s} - (1 - \sqrt{\delta_{s+1}})u * \rho_{\varepsilon_{s+1}} \leq -(1 - \sqrt{\delta_{s+1}})u \quad \text{on } \Omega.$$

This implies that

$$\begin{aligned} v_s - v_{s+1} &= (\delta_s - \delta_{s+1})\psi + [(1 - \sqrt{\delta_s})u * \rho_{\varepsilon_s} - (1 - \sqrt{\delta_{s+1}})u * \rho_{\varepsilon_{s+1}}] \\ &\leq (\delta_s - \delta_{s+1})\psi - (1 - \sqrt{\delta_{s+1}})u \\ &< (\delta_s - \delta_{s+1})\psi - u \end{aligned}$$

on Ω because $u < 0$. Moreover, since $-3 \leq \psi \leq -2$ in Ω, we infer from (3) that

$$\begin{aligned} -\delta_s - u &> (\delta_s - \delta_{s+1})\psi - u \\ &> v_s - v_{s+1} \\ &\geq (\sqrt{\delta_{s+1}} - \sqrt{\delta_s})u * \rho_{\varepsilon_s} + (\delta_s - \delta_{s+1})\psi \\ &\geq (\sqrt{\delta_{s+1}} - \sqrt{\delta_s})(u * \rho_{\varepsilon_s} + 6\sqrt{\delta_s}) \end{aligned} \tag{6}$$

on Ω_{ε_s}. Now, since $0 < \varepsilon_s \leq \varepsilon_{s-1}$, we obtain by (5) that

$$u * \rho_{\varepsilon_s}(q) < -6\sqrt{\delta_s},\ \forall q \in \{u < -\delta_{s-1}\} + \mathbb{B}(0, \varepsilon_s).$$

Combining this with (3), (5) and (6) we arrive at

$$\begin{aligned}\{u < -\delta_{s-1}\} &\Subset \Omega_{\varepsilon_s} \cap \{u * \rho_{\varepsilon_s} < -6\sqrt{\delta_s}\} \\ &\subset \Omega_{\varepsilon_s} \cap \{v_{s+1} < v_s\} \subset \Omega_{\varepsilon_s} \cap \{v_{s+1} \le v_s\} \\ &\subset \Omega_{\varepsilon_s} \cap \{u < -\delta_s\} \Subset \Omega_{\varepsilon_{s-1}}.\end{aligned}$$

Since v_s, v_{s+1} are continuous on Ω_{ε_s}, there exists $\gamma_s > 0$ such that

$$\begin{aligned}\{u < -\delta_{s-1}\} &\Subset \Omega_{\varepsilon_s} \cap \{v_s - v_{s+1} > \gamma_s\} \\ &\Subset \Omega_{\varepsilon_s} \cap \{v_s - v_{s+1} \ge -\gamma_s\} \\ &\Subset \{u < -\delta_s\} \Subset \Omega_{\varepsilon_{s-1}}.\end{aligned} \tag{7}$$

Let $\chi_s : \mathbb{R} \to \mathbb{R}^+$ be a smooth convex function such that $\chi_s(t) = |t|$, $\forall |t| > \gamma_s$. Since $1 - \sqrt{\delta_s} > 1 - s^{-1}$, Lemma 2.2 states that the function

$$v_s(q) - [(1 - s^{-1})c + \delta_s]\|q\|^2$$

is negative, smooth, quaternionic plurisubharmonic on Ω_{ε_s}. Because

$$\min\left\{(1 - s^{-1})c + \delta_s, (1 - (s+1)^{-1})c + \delta_{s+1}\right\} \ge (1 - s^{-1})c + \delta_{s+1},$$

Lemma 2.1 implies that the function

$$w_s := \frac{1}{2}\left[v_s + v_{s+1} + \chi_s(v_s - v_{s+1})\right]$$

satisfies:
(i) $w_s(q) - [(1 - s^{-1})c + \delta_{s+1}]\|q\|^2$ is smooth, quaternionic plurisubharmonic in Ω_{ε_s};
(ii) $w_s \ge \max(v_s, v_{s+1})$ on Ω_{ε_s};
(iii) $w_s = v_s$ on $\Omega_{\varepsilon_s} \cap \{v_s - v_{s+1} > \gamma_s\}$;
(iv) $w_s = v_{s+1}$ on $\Omega_{\varepsilon_s} \cap \{v_s - v_{s+1} < -\gamma_s\}$.
From (7) we have

$$\partial\{u < -\delta_s\} \Subset \Omega_{\varepsilon_s} \cap \{v_{s+1} - v_{s+2} > \gamma_{s+1}\} \cap \{v_s - v_{s+1} < -\gamma_s\}.$$

Hence, (iii) and (iv) imply that

$$w_{s+1} = v_{s+1} = w_s \text{ on an open neighborhood of } \partial\{u < -\delta_s\}.$$

Therefore, the function

$$u_s := \begin{cases} w_s & \text{on } \{u < -\delta_s\} \\ w_r & \text{on } \{u < -\delta_r\} \setminus \{u < -\delta_{r-1}\},\ r = s+1, s+2, \dots \end{cases}$$

is negative, smooth, strictly quaternionic plurisubharmonic function in Ω. It is easy to see that

$$u_{s+1} = u_s \text{ on } \{u \ge -\delta_s\}.$$

From (ii) and (7) we have

$$u_{s+1} = w_{s+1} = v_{s+1} \le w_s = u_s \text{ on } \{u < -\delta_s\}.$$

Hence, $u_{s+1} \le u_s$ in Ω. Since $\{\delta_s\}$ is decreasing sequence, by (iii) and (7) we arrive at

$$u_s = v_s \text{ on } \{u < \delta_{r-1}\},\ \forall s \ge r.$$

This implies that $u_s \searrow u$ in Ω as $s \nearrow +\infty$. Since

$$(1 - r^{-1})c + \delta_{r+1} > (1 - s^{-1})c,\ \forall r \ge s,$$

we deduce by (i) that $w_r(q) - (1 - s^{-1})c\|q\|^2$ is smooth, strictly quaternionic plurisubharmonic in Ω_{ε_s}, and therefore, $u_s(q) - (1 - s^{-1})c\|q\|^2$ is strictly quaternionic plurisubharmonic in Ω. The proof is complete. □

3. Proof of theorem 1.4

Proof. (ii) $\Rightarrow$ (iii) is obvious. (i) $\Rightarrow$ (ii): The idea of the proof is to use Sibony's argument from [9]. Let f be a continuous function in $\mathbb{H}^n$ defined by

$$f(q) := -2\|q\|^2,\ q \in \mathbb{H}^n.$$

Since Ω is quaternionic B-regular, there exists a quaternionic plurisubharmonic function u in Ω such that

$$\lim_{\Omega \ni q \to p} u(q) = f(p),\ \forall p \in \partial\Omega.$$

It is clear that $u(q) + 2\|q\|^2 < 0$ on Ω. According to Theorem 1.2 (in the case $c = 2$), we can find a negative, smooth, strictly quaternionic plurisubharmonic, exhaustion functions φ on Ω such that $\varphi(q) - \|q\|^2$ is quaternionic plurisubharmonic in Ω. This proves (i) $\Rightarrow$ (ii).

It remains to prove (iii) $\Rightarrow$ (i). Fix $p \in \partial\Omega$. We claim that there exists a negative function u_p on Ω such that $u_p(q) - \|q\|^2$ is quaternionic plurisubharmonic in Ω and

$$\lim_{q \to p} u_p(q) = 0.$$

Indeed, by the hypotheses we can find a positive real number r_p and a negative function φ_p defined on $\Omega \cap \mathbb{B}(p, r_p)$ such that $\lim_{q\to p} \varphi_p(q) = 0$ and $\varphi_p(q) - \|q\|^2$ is quaternionic plurisubharmonic on $\Omega \cap \mathbb{B}(p, r_p)$. Let $a_p > 1$ be such that

$$\|q\|^2 < a_p r_p^2,\ \forall q \in \Omega.$$

Since the function $\|q\|^2 - \|q - p\|^2$, $q \in \mathbb{H}^n$, is quaternionic plurisubharmonic, so

$$\begin{aligned} &2a_p\varphi_p(q) - \|q\|^2 - a_p\|q - p\|^2 \\ &= 2a_p[\varphi_p(q) - \|q\|^2] + a_p(\|q\|^2 - \|q - p\|^2) + (a_p - 1)\|q\|^2 \end{aligned}$$

is quaternionic plurisubharmonic on $\Omega \cap \mathbb{B}(p, r_p)$. This implies that the function

$$\psi_p(q) := \begin{cases} \max(2a_p\varphi_p(q) - \|q\|^2 - a_p\|q-p\|^2, -a_p r_p^2) & \text{if } q \in \Omega \cap \mathbb{B}(p, r_p) \\ -a_p r_p^2 & \text{otherwise} \end{cases}$$

is quaternionic plurisubharmonic on Ω. Since $\psi_p(q) < -\|q\|^2$, $\forall q \in \Omega$, we infer that the function

$$u_p(q) := \psi_p(q) + \|q\|^2,\ q \in \Omega$$

is negative function on Ω which has the required properties. This proves the claim.

Now, assume that f is a continuous function on $\partial\Omega$. Let $\{f_s\}$ be a sequence of smooth functions in $\mathbb{H}^n$ such that

$$f_s - \frac{1}{s} < f < f_s \text{ on } \partial\Omega. \tag{8}$$

Let $b_{p,s} > 0$ be such that

$$-f_s + b_{p,s}u_p,\ f_s + b_{p,s}u_p \in \mathcal{QPSH}(\Omega). \tag{9}$$

We set

$$u := (\sup\{f_s - \frac{1}{s} + b_{p,s}u_p : p \in \partial\Omega,\ s \in \mathbb{N}^*\})^*.$$

Here, * denotes the upper semi-continuous regularization in $\overline{\Omega}$. It easy to see that u is quaternionic plurisubharmonic on Ω. Let $p, q \in \partial\Omega$ and $r, s \geq 1$. The inequality (8) tells us that

$$(f_s - \frac{1}{s} + b_{p,s}u_p) + (-f_r + b_{q,r}u_q) < 0 \text{ on } \partial\Omega.$$

Hence, we infer by (9) that

$$f_s - \frac{1}{s} + b_{p,s}u_p < f_r - b_{q,r}u_q \text{ on } \Omega.$$

Therefore,

$$f_s - \frac{1}{s} + b_{p,s}u_p \leq u \leq f_r - b_{q,r}u_q \text{ on } \Omega,\ \forall p, q \in \partial\Omega,\ \forall r, s \geq 1.$$

Combining this with (8) we conclude that

$$\lim_{\Omega \ni q \to p} u(q) = f(p),\ \forall p \in \partial\Omega.$$

The proof is complete. $\square$

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