Best n-term approximation of diagonal operators and application to function spaces with mixed smoothness

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Abstract

In this paper we give exact values of the best n-term approximation widths of diagonal operators between $\ell_p(\mathbb{N})$ and $\ell_q(\mathbb{N})$ with $0 < p, q \le \infty$. The result will be applied to obtain the asymptotic constants of best n-term approximation widths of embeddings of function spaces with mixed smoothness by trigonometric system.

Keywords and Phrases: diagonal operator, best n-term approximation, mixed smoothness, asymptotic constant, dimensional dependence

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1 Introduction

Nowadays, it is well understood that nonlinear methods of approximation and numerical methods derived from them often produce superior performance when compared with linear methods. In the last three decades there has been a great success in studying nonlinear approximation which was motivated by numerous applications such as numerical analysis, image processing, statistical learning as well as in the design of neural networks. We refer the reader to [16, 17, 18] for the development of nonlinear approximation and its application.

In the present paper we concentrate on a particular nonlinear method, the so-called best n-term approximation. Our particular interest is exact values of best n-term approximation of diagonal linear operators from $\ell_p(\mathbb{N})$ to $\ell_q(\mathbb{N})$. The exact values of approximation quantities of diagonal operators play an important role in high-dimensional approximation and particularly in studying tractability, see, e.g., [8, 30, 31, 33]. In this paper, the exact values of best n-term approximation of diagonal operators will be applied to get the asymptotic constants of best n-term approximation of function spaces with mixed smoothness by trigonometric system.

Let X, Y be Banach spaces and T a continuous linear operator from X to Y. Let $\mathcal{D}$ be a given countable set in Y, called dictionary. For given $x \in X$ we consider the algorithm to approximate Tx by a finite linear combination of elements contained in this dictionary. The error of this approximation is

$$\sigma_n(Tx;\mathcal{D}) := \inf_{\substack{(a_j)_{j=1}^n \subset \mathbb{C}, \\ (y_j)_{j=1}^n \subset \mathcal{D}}} \Big\| Tx - \sum_{j=1}^n a_j y_j \Big\|_Y , \qquad n \in \mathbb{N}.$$

We wish to approximate Tx for all x in the closed unit ball of X with respect to the dictionary $\mathcal{D}$. This can be measured by the following benchmark quantity

$$\sigma_n(T;\mathcal{D}) := \sup_{x\in X,\, \|x\|_X\le 1} \sigma_n(Tx;\mathcal{D}), \qquad n\in\mathbb{N}.$$

In what follows, we shall call this quantity the best n-term approximation width.

Let $\ell_p(\mathbb{N})$, $0<p\le\infty$, be the classical complex sequence space with the usual (quasi) norm. For $0<p,q\le\infty$ and positive non-increasing sequence $\lambda=(\lambda_k)_{k\in\mathbb{N}}$, consider the diagonal linear operator

$$T_\lambda:\ (\xi_k)_{k\in\mathbb{N}} \mapsto (\lambda_k\xi_k)_{k\in\mathbb{N}} \tag{1.1}$$

from $\ell_p(\mathbb{N})$ to $\ell_q(\mathbb{N})$ and $\mathcal{E}=\{e_k: k\in\mathbb{N}\}$ where $e_k=(\delta_{k,j})_{j\in\mathbb{N}}$ and $\delta_{k,j}$ denotes the Kronecker delta. We are concerned with the exact value of $\sigma_n(T_\lambda,\mathcal{E})$. The first result in this direction was given by Stepanets [42] in the case $p=q$ with the condition $\lim_{k\to\infty}\lambda_k=0$. Later Stepanets generalized his result to the case $0<p\le q<\infty$ in [43] and $0<q<p<\infty$ in [44, Theorem 6.1], see also [45]. Under the same condition $\lim_{k\to\infty}\lambda_k=0$ but by different approach, Gensun and Lixin [20] also obtained exact value of $\sigma_n(T_\lambda,\mathcal{E})$ in the case $p=q$. The results of Stepanets were extended to the Orlicz sequence spaces by Schidlich and Chaichenko [41]. In the case of the finite dimensional sequence spaces, for all $0<p,q\le\infty$, exact values of the quantity σ_n of the diagonal operator from ℓ_p^M to ℓ_q^M with respect to the standard basis of $\mathbb{R}^M$ were obtained by Gao, see [19]. Analogous results for best approximation of integrals by integrals of finite rank for functions on $\mathbb{R}^d$ were given in [46].

In this paper we give exact values of the best n-term approximation widths $\sigma_n(T_\lambda,\mathcal{E})$, $n\in\mathbb{N}$ for all $0<p,q\le\infty$. We also show that the condition $\lim_{k\to\infty}\lambda_k=0$ in the case $0<p<q<\infty$ is not necessary. The proof is based on the exact values of best n-term approximation widths of the diagonal operators between finite dimensional sequence spaces obtained by Gao [19]. Our result reads as follows. If $0<p<q<\infty$, then we have

$$\sigma_n(T_\lambda,\mathcal{E}) = \frac{(n^*-n)^{1/q}}{\big(\sum_{k=1}^{n^*}\lambda_k^{-p}\big)^{1/p}},$$

where n^* is the smallest integer $m>n$ such that

$$\frac{(m-n)^{1/q}}{\big(\sum_{k=1}^{m}\lambda_k^{-p}\big)^{1/p}} \ge \frac{(m+1-n)^{1/q}}{\big(\sum_{k=1}^{m+1}\lambda_k^{-p}\big)^{1/p}}.$$

In the case $0<q<p<\infty$ and the series $\sum_{k=1}^\infty \lambda_k^{pq/(p-q)}$ converges, we get

$$\sigma_n(T_\lambda,\mathcal{E}) = \left(\frac{(n_*-n)^{\frac{p}{p-q}}}{\big(\sum_{k=1}^{n_*}\lambda_k^{-p}\big)^{\frac{q}{p-q}}} + \sum_{k=n_*+1}^{\infty}\lambda_k^{\frac{pq}{p-q}}\right)^{\frac1q-\frac1p},$$

where n_* is the largest integer $m>n$ such that

$$(m-n)\lambda_m^{-p} \le \sum_{k=1}^{m}\lambda_k^{-p}.$$

The limiting cases $p=q$ or $p=\infty$ and/or $q=\infty$ are also obtained, see Theorem 2.1.

The above results will be applied to study best n-term approximation of embedding of function spaces with mixed smoothness by trigonometric system $\mathcal{T}^d := \{e^{ikx}: k\in\mathbb{Z}^d\}$ on the torus $\mathbb{T}^d$ of dimension d. Our motivation comes from high-dimensional approximation which has been the object of an intensive study recently. In many high-dimensional approximation problems when the high-dimensional signals or functions have appropriate mixed smoothness, one can apply efficiently approximation methods and sampling algorithms constructed on sparse grids to obtain tractability for algorithms or numerical methods. We refer the reader to the monographs [34, 35] for concepts of computation complexity and results on high dimensional problems. Survey on various aspects of high-dimensional approximation of functions having mixed smoothness can be found in the recent book [15].

The original justification for considering the d-dependence of approximative characteristics stems from certain needs of numerical analysis on high-dimensional approximation. Concerning the asymptotic constants as well as the preasymptotic bounds explicitly in d of the approximation numbers, we refer the reader to Dinh Dũng, Ullrich [12], Chernov, Dinh Dũng [7]; Cobos, Kühn, Sickel [8, 9]; Krieg [27]; Kühn [28]; Kühn, Mayer, Ullrich [29]; and Kühn, Sickel, Ullrich [30, 31, 32]. In all these quoted papers, the embedding of a weighted Hilbert space $F_\omega(\mathbb{T}^d)$ either into $L_2(\mathbb{T}^d)$ or into $L_\infty(\mathbb{T}^d)$ were considered. In a recent preprint [33] asymptotic constants of approximation numbers as well as Bernstein, Kolmogorov and Weyl numbers of embeddings of a weighted Wiener algebra either into Wiener algebra $\mathcal{A}(\mathbb{T}^d)$ or $L_2(\mathbb{T}^d)$ has been investigated.

There has been a numerous papers working on best n-term approximation of embeddings of function spaces with mixed smoothness by different dictionaries. For instance, Bazarkhanov [2], Dinh Dũng [10, 13, 14], Kashin and Temlyakov [26], Romanyuk [38, 39], Romanyuk and Romanyuk [40], Temlyakov [47, 49, 48, 50] worked on trigonometric system; Hansen and Sickel [22, 23], Balgimbayeva and Smirnow [1], Dinh Dũng [11] on wavelet system. For some recent contributions in this direction we refer to [3, 5, 51, 53]. Historical comments and further references for studies of best n-term approximation of function spaces with mixed smoothness can be found in the two recent books [15, Chapter 7] and [52, Chapter 9].

Let $0 < s < \infty$ and $0 < r \le \infty$. This paper considers the best n-term approximation of embedding of Sobolev space with mixed smoothness $H^{s,r}_{\text{mix}}(\mathbb{T}^d)$ on the torus $\mathbb{T}^d$ into either $L_2(\mathbb{T}^d)$ or Wiener space $\mathcal{A}(\mathbb{T}^d)$. In this context we will not only investigate the optimal order of the decay of the best n-term approximation widths but we will determine the asymptotic constant as well. This sheds some light not only on the dependence on n, but also on the dependence on s, r and in particular on d. We have

$$\lim_{n\to\infty} \frac{\sigma_n\big(id: H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s}(\ln n)^{s(d-1)}} = \frac{s^s}{(s+\frac{1}{2})^s}\left(\frac{2^d}{(d-1)!}\right)^s$$

and if $s > 1/2$

$$\lim_{n\to\infty} \frac{\sigma_n\big(id: H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s+\frac{1}{2}}(\ln n)^{s(d-1)}} = \left(\frac{s}{s+\frac{1}{2}}\right)^s\left(\frac{1}{s-\frac{1}{2}}\right)^{\frac{1}{2}}\left(\frac{2^d}{(d-1)!}\right)^s.$$

In this paper we also obtain the asymptotic constants of best n-term approximation widths of embeddings of Sobolev spaces with mixed smoothness $H^{s,2}_{\text{mix}}(\mathbb{T}^d)$ into the energy norm space $H^1(\mathbb{T}^d)$. Those embeddings are of particular importance with respect to the numerical solution of the Poisson equation, see [4]. In this case, with $s > 1$ we get

$$\lim_{n\to\infty} \frac{\sigma_n\big(id: H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s+1}} = \frac{(s-1)^{s-1}}{(s-\frac{1}{2})^{s-1}}(2d)^{s-1}(2S+1)^{(s-1)(d-1)},$$

where

$$S := \sum_{k=1}^{+\infty} \frac{1}{(k^2+1)^{\frac{s}{2(s-1)}}}.$$

The paper is organized as follows. In Section 2 we collect some properties of best n-term approximation widths and give exact values of best n-term approximation widths of diagonal operators. The next Section 3 is devoted to the study of asymptotic constants of best n-term approximation widths of embeddings of weighted classes $F_{\omega,p}(\mathbb{T}^d)$. These results will be used in final Section 4, where we deal with the particular family of weights associated to function spaces of dominating mixed smoothness.

Notation. As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0$ the non-negative integers, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers. We denote by $\mathbb{T}$ the torus, represented by the interval $[0, 2\pi]$, where the end points of the interval are identified. For a real number a we denote by $\lfloor a \rfloor$ the greatest integer not larger than a. The letter d is always reserved for the dimension in $\mathbb{N}^d$, $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{C}^d$, and $\mathbb{T}^d$. For two Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the set of continuous linear operators from X to Y. If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences, the symbol $a_n \sim b_n, n \to \infty$, indicates that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. The equivalence $a_n \asymp b_n$ means that there are constants $0 < c_1 \le c_2 < \infty$ such that $c_1\, a_n \le b_n \le c_2\, a_n$ for all $n \in \mathbb{N}$.

2 Best n-term approximation widths of diagonal operators

This section is devoted to give exact values of the best n-term approximation widths of the diagonal operator defined in (1.1). Let X, Y be Banach spaces, $T \in \mathcal{L}(X, Y)$, and $\mathcal{D} \subset Y$ a dictionary. By definition, it is clear that $(\sigma_n(T, \mathcal{D}))_{n\in\mathbb{N}}$ is a non-increasing sequence. If W, Z are Banach spaces and $A \in \mathcal{L}(W, X)$, $B \in \mathcal{L}(Y, Z)$ then we have

$$\sigma_n(BTA, B(\mathcal{D})) \le \|B\| \cdot \sigma_n(T, \mathcal{D}) \cdot \|A\|. \tag{2.1}$$

A proof of this fact can be found in [5, Lemma 6.1]. For further properties of the best n-term approximation widths such as additivity, interpolation we refer the reader to [21, 5, 53]. In fact the best n-term approximation widths belong to the notion of pseudo s-numbers introduced by Pietsch, see [53].

Let T_λ be the diagonal operator from $\ell_p(\mathbb{N})$ to $\ell_q(\mathbb{N})$ defined in (1.1). By definition we have

$$\sigma_n(T_\lambda, \mathcal{E}) = \begin{cases} \sup\limits_{(\xi_k)_{k\in\mathbb{N}}\in B_p} \inf\limits_{\Gamma_n} \Big(\sum\limits_{k\notin\Gamma_n} |\lambda_k \xi_k|^q \Big)^{1/q} & \text{if } 0 < q < \infty \\ \sup\limits_{(\xi_k)_{k\in\mathbb{N}}\in B_p} \inf\limits_{\Gamma_n} \sup\limits_{k\notin\Gamma_n} |\lambda_k \xi_k| & \text{if } q = \infty, \end{cases} \tag{2.2}$$

where B_p is the closed unit ball of $\ell_p(\mathbb{N})$ and Γ_n is an arbitrary subset of $\mathbb{N}$ with n elements. In the following we give exact value of $\sigma_n(T_\lambda, \mathcal{E})$ with $0 < p, q \le \infty$. The proof is mainly based on the exact values of n-term approximation widths of the diagonal operators from ℓ_p^M to ℓ_q^M obtained by Gao in [19]. Here ℓ_p^M stands for $\mathbb{C}^M$ equipped with the usual norm $\|\cdot\|_{\ell_p^M}$. For a vector $\lambda = (\lambda_j)_{j=1}^M$ with $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_M > 0$ the diagonal operator T_λ^M from ℓ_p^M to ℓ_q^M is defined by $(\xi_j)_{j=1}^M \mapsto (\lambda_j \xi_j)_{j=1}^M$. Let $\mathcal{E}_M = \{e_1, \ldots, e_M\}$ be the standard basis of $\mathbb{R}^M$. If $n \in \mathbb{N}$ and $n \le M$ we have

$$\sigma_n(T_\lambda^M, \mathcal{E}_M) = \sup_{(\xi_k)_{k=1}^M\in B_p^M} \inf_{\Gamma_n^M} \Big(\sum_{k\notin\Gamma_n^M} |\lambda_k \xi_k|^q \Big)^{1/q}, \quad 0 < q < \infty, \tag{2.3}$$

where B_p^M is the closed unit ball of ℓ_p^M and Γ_n^M is an arbitrary subset of $\{1, \ldots, M\}$ with n elements. For $\lambda = (\lambda_j)_{j\in\mathbb{N}}$ we define $T_\lambda^M := T_{\tilde\lambda}^M$ where $\tilde\lambda = (\lambda_j)_{j=1}^M$. When $q = \infty$ the summation in (2.3) is replaced by supremum.

Note that if $\lambda = (\lambda_j)_{j\in\mathbb{N}}$ satisfying $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_M > 0$ and $\lambda_j = 0$ for $j \ge M+1$, then

$$\sigma_n(T_\lambda, \mathcal{E}) = \sigma_n(T_\lambda^M, \mathcal{E}_M), \quad n \in \mathbb{N}$$

which were obtained in [19]. Therefore, we only consider the operator T_λ where $\lambda = (\lambda_j)_{j\in\mathbb{N}}$ is a positive sequence. Our main result in this section reads as follows.

Theorem 2.1. *Let $0 < p, q \le \infty$ and $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ be a positive non-increasing sequence. Let T_λ be defined in (1.1) and $n \in \mathbb{N}$.*

(i) *If $0 < p \le q < \infty$ we have*

$$\sigma_n(T_\lambda, \mathcal{E}) = \sup_{m>n} \frac{(m-n)^{1/q}}{\left(\sum_{k=1}^m \lambda_k^{-p}\right)^{1/p}}.$$

Moreover, if either $0 < p < q < \infty$ or $\lim\limits_{k\to\infty} \lambda_k = 0$ then

$$\sigma_n(T_\lambda, \mathcal{E}) = \frac{(n^*-n)^{1/q}}{\left(\sum_{k=1}^{n^*} \lambda_k^{-p}\right)^{1/p}},$$

where n^* is the smallest integer $m > n$ such that

$$\frac{(m-n)^{1/q}}{\left(\sum_{k=1}^m \lambda_k^{-p}\right)^{1/p}} \ge \frac{(m+1-n)^{1/q}}{\left(\sum_{k=1}^{m+1} \lambda_k^{-p}\right)^{1/p}}.$$

(ii) *If $0 < q < p < \infty$ and the series $\sum_{k=1}^\infty \lambda_k^{pq/(p-q)}$ converges, then we have*

$$\sigma_n(T_\lambda, \mathcal{E}) = \left(\frac{(n_*-n)^{\frac{p}{p-q}}}{\left(\sum_{k=1}^{n_*} \lambda_k^{-p}\right)^{\frac{q}{p-q}}} + \sum_{k=n_*+1}^\infty \lambda_k^{\frac{pq}{p-q}} \right)^{\frac{1}{q}-\frac{1}{p}}, \tag{2.4}$$

where n_* is the largest integer $m > n$ such that

$$(m-n)\lambda_m^{-p} \le \sum_{k=1}^m \lambda_k^{-p}. \tag{2.5}$$

(iii) *If $0 < p < q = \infty$ then*

$$\sigma_n(T_\lambda, \mathcal{E}) = \left(\sum_{k=1}^{n+1} \lambda_k^{-p}\right)^{-1/p}.$$

(iv) *If $0 < q < p = \infty$ and the series $\sum_{k=1}^\infty \lambda_k^q$ converges then*

$$\sigma_n(T_\lambda, \mathcal{E}) = \left(\sum_{k=n+1}^\infty \lambda_k^q\right)^{1/q}.$$

(v) *If $p = q = \infty$ then*

$$\sigma_n(T_\lambda, \mathcal{E}) = \lambda_{n+1}.$$

As mentioned in Introduction, the exact values of $\sigma_n(T_\lambda, \mathcal{E})$, $n \in \mathbb{N}$, in the case $0 < p \le q \le \infty$ were obtained in [42, 43, 20] under the condition $\lim_{k\to\infty} \lambda_k = 0$. To prove the above theorem, we need some auxiliary results.

Lemma 2.2. *Let $0 < p < q < \infty$ and $(\lambda_k)_{k=1}^{\infty}$ be a positive non-increasing sequence. Then for $n \in \mathbb{N}$, there is $n^* = n^*(n) \in \mathbb{N}$ such that*

$$\sup_{m>n} \frac{(m-n)^{1/q}}{\left(\sum_{k=1}^{m} \lambda_k^{-p}\right)^{1/p}} = \frac{(n^*-n)^{1/q}}{\left(\sum_{k=1}^{n^*} \lambda_k^{-p}\right)^{1/p}}. \tag{2.6}$$

Moreover, n^* can be chosen as the smallest integer $m > n$ such that

$$\frac{(m-n)^{1/q}}{\left(\sum_{k=1}^{m} \lambda_k^{-p}\right)^{1/p}} \geq \frac{(m+1-n)^{1/q}}{\left(\sum_{k=1}^{m+1} \lambda_k^{-p}\right)^{1/p}}.$$

Proof. We first show that n^* exists. The case $\lim_{k\to\infty} \lambda_k = 0$ was already considered in [43]. We prove the case $\lim_{k\to\infty} \lambda_k = K > 0$. We will show that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{(m-n)^{1/q}}{\left(\sum_{k=1}^{m} \lambda_k^{-p}\right)^{1/p}} < \frac{(2n-n)^{1/q}}{\left(\sum_{k=1}^{2n} \lambda_k^{-p}\right)^{1/p}} \tag{2.7}$$

for $m > n_0$ and as a consequence we obtain (2.6) for some $n^* \in \{n+1, \dots, n_0\}$. Observe that if $m \in \{jn+1, \dots, (j+1)n\}$ for some $j \in \mathbb{N}$ we have

$$\frac{(m-n)^{1/q}}{\left(\sum_{k=1}^{m} \lambda_k^{-p}\right)^{1/p}} < \frac{(jn)^{1/q}}{\left(\sum_{k=1}^{jn} \lambda_k^{-p}\right)^{1/p}} \leq \frac{(jn)^{1/q}}{\lambda_1^{-1}(jn)^{1/p}}.$$

We also have

$$\frac{n^{1/q}}{K^{-1}(2n)^{1/p}} \leq \frac{(2n-n)^{1/q}}{\left(\sum_{k=1}^{2n} \lambda_k^{-p}\right)^{1/p}}.$$

Therefore

$$\sup_{m\in\{jn+1,\dots,(j+1)n\}} \frac{(m-n)^{1/q}}{\left(\sum_{k=1}^{m} \lambda_k^{-p}\right)^{1/p}} < \frac{(2n-n)^{1/q}}{\left(\sum_{k=1}^{2n} \lambda_k^{-p}\right)^{1/p}}$$

if

$$\frac{(jn)^{1/q}}{\lambda_1^{-1}(jn)^{1/p}} < \frac{n^{1/q}}{K^{-1}(2n)^{1/p}} \iff j > \left(\frac{\lambda_1 2^{1/p}}{K}\right)^{\frac{pq}{q-p}}.$$

Denoting $n_0 = \lceil \left(\frac{\lambda_1 2^{1/p}}{K}\right)^{\frac{pq}{q-p}} \rceil n$ we obtain (2.7) for $m > n_0$ and (2.6) follows.

We turn to the second statement. Assume

$$\frac{m_0-n}{\left(\sum_{k=1}^{m_0} \lambda_k^{-p}\right)^{q/p}} \geq \frac{m_0+1-n}{\left(\sum_{k=1}^{m_0+1} \lambda_k^{-p}\right)^{q/p}} \tag{2.8}$$

for some $m_0 > n$, $m_0 \in \mathbb{N}$. Such m_0 exists by the first statement. We will prove

$$\frac{m_0+1-n}{\left(\sum_{k=1}^{m_0+1} \lambda_k^{-p}\right)^{q/p}} \geq \frac{m_0+2-n}{\left(\sum_{k=1}^{m_0+2} \lambda_k^{-p}\right)^{q/p}} \tag{2.9}$$

by showing that

$$\frac{m_0+1-n}{\left(\sum_{k=1}^{m_0+1} \lambda_k^{-p}\right)^{q/p}} \geq \frac{m_0+2-n}{\left(\sum_{k=1}^{m_0+1} \lambda_k^{-p} + \lambda_{m_0+1}^{-p}\right)^{q/p}}. \tag{2.10}$$

Putting $A = \sum_{k=1}^{m_0} \lambda_k^{-p}$ and $a = \lambda_{m_0+1}^{-p}$ we consider the function $g(h) = \frac{m_0-n+h}{(A+ha)^{q/p}}$. We have

$$g'(h) = \frac{(A+ha) - \frac{q}{p}a(m_0-n+h)}{(A+ha)^{\frac{q}{p}+1}}$$

and $g'(h) \le 0$ if

$$(A + ha) - \frac{q}{p}a(m_0 - n + h) \le 0 \iff h \ge \frac{A - \frac{q}{p}a(m_0 - n)}{a(\frac{q}{p} - 1)}.$$

Assume

$$\frac{A - \frac{q}{p}a(m_0 - n)}{a(\frac{q}{p} - 1)} \ge 1 \iff \frac{a}{A} \le \frac{1}{\frac{q}{p}(m_0 - n) + \frac{q}{p} - 1}. \tag{2.11}$$

Observe that the condition (2.8) implies $\left(1 + \frac{a}{A}\right)^{q/p} \ge 1 + \frac{1}{m_0 - n}$. From this and (2.11) we get

$$\left(1 + \frac{1}{\frac{q}{p}(m_0 - n) + \frac{q}{p} - 1}\right)^{q/p} \ge 1 + \frac{1}{m_0 - n}.$$

But this is a contradiction since $\varphi(t) = (1 + \frac{1}{t(m_0 - n) + t - 1})^t$ is a strictly decreasing function on $[1, +\infty)$ and $\varphi(1) = 1 + \frac{1}{m_0 - n}$. Consequently, g is decreasing on $[1, +\infty)$. This proves (2.10) and (2.9) follows. $\square$

Lemma 2.3. *Let $(\delta_k)_{k \in \mathbb{N}}$ be a positive increasing sequence and $\lim_{k \to \infty} \delta_k = +\infty$. Let $n \in \mathbb{N}$ and $n_* = n_*(n)$ be the largest integer $m > n$ such that*

$$(m - n)\delta_m \le \sum_{k=1}^{m} \delta_k. \tag{2.12}$$

Then n_* is finite and for any $m \in \{n + 1, \dots, n_*\}$ the inequality (2.12) holds true.

Proof. First of all, observe that $m = n + 1$ satisfies (2.12). If $m \ge n + 1$ and m satisfies (2.12) we can write

$$\begin{aligned} m &\le n + \frac{\delta_1 + \ldots + \delta_{n+1}}{\delta_m} + \frac{\delta_{n+2} + \ldots + \delta_m}{\delta_m} \\ &\le n + \frac{\delta_1 + \ldots + \delta_{n+1}}{\delta_m} + m - n - 1 = m - 1 + \frac{\delta_1 + \ldots + \delta_{n+1}}{\delta_m}. \end{aligned}$$

Observe that the term $\frac{\delta_1 + \ldots + \delta_{n+1}}{\delta_m}$ tends to zero when $m \to \infty$. Consequently n_* is finite. Assume

$$(m_0 - n)\delta_{m_0} > \sum_{k=1}^{m_0} \delta_k \tag{2.13}$$

for some $m_0 \in \mathbb{N}$, $m_0 > n$. Then we have

$$m_0 + 1 > \frac{1}{\delta_{m_0}} \sum_{k=1}^{m_0} \delta_k + n + 1 > \frac{1}{\delta_{m_0+1}} \sum_{k=1}^{m_0} \delta_k + 1 + n = \frac{1}{\delta_{m_0+1}} \sum_{k=1}^{m_0+1} \delta_k + n.$$

This shows that the inequality (2.13) is satisfied with m_0 being replaced by $m_0 + 1$ and therefore is satisfied with any $m > m_0$, $m \in \mathbb{N}$. As a consequence we conclude that for any $m \in \{n + 1, \dots, n_*\}$ the inequality (2.12) holds true. $\square$

We are now in position to prove Theorem 2.1.

Proof. *Step 1.* Proof of (i). Given $\varepsilon > 0$. For any $(\xi_k)_{k\in\mathbb{N}} \in B_p$ we take $M \in \mathbb{N}$ (depending on $(\xi_k)_{k\in\mathbb{N}}$) such that

$$\sum_{k=M+1}^{\infty} |\xi_k|^p < \varepsilon.$$

Then we have

$$\inf_{\Gamma_n}\Big(\sum_{k\notin\Gamma_n} |\lambda_k\xi_k|^q\Big) \le \inf_{\Gamma_n^M}\Big(\sum_{k\notin\Gamma_n^M} |\lambda_k\xi_k|^q\Big) = \inf_{\Gamma_n^M}\Big(\sum_{k\in\{1,\dots,M\}\setminus\Gamma_n^M} |\lambda_k\xi_k|^q\Big) + \sum_{k=M+1}^{\infty} |\lambda_k\xi_k|^q. \tag{2.14}$$

The first term on the right side can be estimated as follows

$$\inf_{\Gamma_n^M}\Big(\sum_{k\in\{1,\dots,M\}\setminus\Gamma_n^M} |\lambda_k\xi_k|^q\Big) \le \sup_{(\gamma_k)_{k=1}^M\in B_p^M}\inf_{\Gamma_n^M}\Big(\sum_{k\in\{1,\dots,M\}\setminus\Gamma_n^M} |\lambda_k\gamma_k|^q\Big) = \sigma_n(T_\lambda^M,\mathcal{E}_M)^q, \tag{2.15}$$

see (2.3). It has been proved in [19] that

$$\sigma_n(T_\lambda^M,\mathcal{E}_M) = \sup_{n<m\le M}\frac{(m-n)^{1/q}}{\big(\sum_{k=1}^m \lambda_k^{-p}\big)^{1/p}}. \tag{2.16}$$

Hence we get

$$\inf_{\Gamma_n^M}\Big(\sum_{k\in\{1,\dots,M\}\setminus\Gamma_n^M} |\lambda_k\xi_k|^q\Big) \le \sup_{n<m\le M}\frac{m-n}{\big(\sum_{k=1}^m \lambda_k^{-p}\big)^{q/p}} \le \sup_{n<m}\frac{m-n}{\big(\sum_{k=1}^m \lambda_k^{-p}\big)^{q/p}}.$$

Since $0 < p \le q \le \infty$, for the second term we have

$$\sum_{k=M+1}^{\infty} |\lambda_k\xi_k|^q \le \Big(\sum_{k=M+1}^{\infty} |\lambda_k\xi_k|^p\Big)^{q/p} \le \sup_{k>M}|\lambda_k|^q\Big(\sum_{k=M+1}^{\infty}|\xi_k|^p\Big)^{q/p} \le \lambda_1^q\varepsilon^{q/p}.$$

Consequently we obtain

$$\inf_{\Gamma_n}\Big(\sum_{k\notin\Gamma_n} |\lambda_k\xi_k|^q\Big) \le \sup_{m>n}\frac{m-n}{\big(\sum_{k=1}^m \lambda_k^{-p}\big)^{q/p}} + \lambda_1^q\varepsilon^{q/p}.$$

Observe that the right-hand side is independent of $(\xi_k)_{k\in\mathbb{N}} \in B_p$ and $\varepsilon > 0$ is arbitrarily small. In view of (2.2) we obtain the upper bound.

We now give a proof for the lower bound. Take $M \in \mathbb{N}$ arbitrarily large and consider the following diagram

$$\begin{array}{ccc} \ell_p^M & \xrightarrow{T_\lambda^M} & \ell_q^M \\ \big\downarrow J & & \big\uparrow Q \\ \ell_p(\mathbb{N}) & \xrightarrow{T_\lambda} & \ell_q(\mathbb{N}) \end{array},$$

where

$$J(\xi_1,\dots,\xi_M) = (\xi_1,\dots,\xi_M,0,0,\dots)$$
$$Q(\xi_1,\dots,\xi_M,\xi_{M+1},\dots) = (\xi_1,\dots,\xi_M).$$

We have $T_\lambda^M = QT_\lambda J$ and $\|J\| = \|Q\| = 1$ which by property (2.1) implies

$$\sigma_n(T_\lambda^M,\mathcal{E}_M) \le \|J\|\cdot\sigma_n(T_\lambda,\mathcal{E})\cdot\|Q\| = \sigma_n(T_\lambda,\mathcal{E}).$$

Using (2.16) again we deduce

$$\sup_{n<m\le M} \frac{(m-n)^{1/q}}{\left(\sum_{k=1}^{m} \lambda_k^{-p}\right)^{1/p}} \le \sigma_n(T_\lambda, \mathcal{E}).$$

Since M is arbitrarily large, we obtain the lower bound. The second statement follows from Lemma 2.2.

Step 2. Proof of (ii). First note that n_* is finite by Lemma 2.3. Let $\varepsilon > 0$. We choose $M > n_*$ such that

$$\left(\sum_{k=M+1}^{\infty} \lambda_k^{\frac{pq}{p-q}}\right)^{\frac{p-q}{p}} < \varepsilon.$$

For $(\xi_k)_{k\in\mathbb{N}} \in B_p$, we use the estimate (2.14). Applying Hölder's inequality we get

$$\sum_{k=M+1}^{\infty} |\lambda_k \xi_k|^q \le \left(\sum_{k=M+1}^{\infty} \lambda_k^{\frac{pq}{p-q}}\right)^{\frac{p-q}{p}} \left(\sum_{k=M+1}^{\infty} |\xi_k|^p\right)^{\frac{q}{p}} < \varepsilon$$

which by (2.14) implies

$$\inf_{\Gamma_n} \left(\sum_{k\notin\Gamma_n} |\lambda_k \xi_k|^q\right) \le \inf_{\Gamma_n^M} \left(\sum_{k\in\{1,\dots,M\}\setminus\Gamma_n^M} |\lambda_k \xi_k|^q\right) + \varepsilon = \sigma_n(T_\lambda^M, \mathcal{E}_M)^q + \varepsilon,$$

see (2.15). Using the result in [19] for the case $0 < q < p < \infty$

$$\sigma_n(T_\lambda^M, \mathcal{E}_M) = \left(\frac{(n_*-n)^{\frac{p}{p-q}}}{\left(\sum_{k=1}^{n_*} \lambda_k^{-p}\right)^{\frac{q}{p-q}}} + \sum_{k=n_*+1}^{M} \lambda_k^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \le \left(\frac{(n_*-n)^{\frac{p}{p-q}}}{\left(\sum_{k=1}^{n_*} \lambda_k^{-p}\right)^{\frac{q}{p-q}}} + \sum_{k=n_*+1}^{\infty} \lambda_k^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}$$

and following the argument as in Step 1 we obtain the upper bound. The lower bound is carried out similarly as Step 1 with $M > n_*$. The other cases are proved similarly with a slight modification. $\square$

3 Best n-term approximation of function classes $F_{\omega,p}(\mathbb{T}^d)$

Let $\mathbb{T}^d$ be the $d-$dimensional torus. We equip $\mathbb{T}^d$ with the probability measure $(2\pi)^{-d}\mathrm{d}x$. In this section we study the asymptotic constants of best n-term approximation widths of embeddings of the weighted function classes $F_{\omega,p}(\mathbb{T}^d)$ by trigonometric system $\mathcal{T}^d$. For a function $f \in L_1(\mathbb{T}^d)$, its Fourier coefficients are defined as

$$\hat{f}(k) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-\mathrm{i}kx} \mathrm{d}x, \qquad k \in \mathbb{Z}^d.$$

Hence, it holds for any $f \in L_2(\mathbb{T}^d)$ that

$$\|f\|_{L_2(\mathbb{T}^d)}^2 = (2\pi)^{-d} \int_{\mathbb{T}^d} |f(x)|^2 \mathrm{d}x = \sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2 .$$

Let $\omega = (\omega(k))_{k\in\mathbb{Z}^d}$ be a sequence of positive numbers. Those sequences we will call a weight in what follows. For $0 < p \le \infty$ we introduce the class $F_{\omega,p}(\mathbb{T}^d)$ as the collection of all functions $f \in L_1(\mathbb{T}^d)$ such that

$$\|f\|_{F_{\omega,p}(\mathbb{T}^d)} := \left(\sum_{k\in\mathbb{Z}^d} |\omega(k) \hat{f}(k)|^p\right)^{1/p} < \infty .$$

When $\omega(k) = 1$ for all $k \in \mathbb{Z}^d$ we use the notation $F_p(\mathbb{T}^d)$ instead of $F_{\omega,p}(\mathbb{T}^d)$. In this case we get back the space $L_2(\mathbb{T}^d)$ when $p = 2$ and the classical Wiener algebra $\mathcal{A}(\mathbb{T}^d)$ when $p = 1$.

We suppose that

$$\lim_{|k_1|+\ldots+|k_d|\to\infty} \omega(k) = +\infty, \quad k = (k_1, \ldots, k_d). \tag{3.1}$$

In what follows we denote the non-increasing rearrangement of the sequence $(1/\omega(k))_{k\in\mathbb{Z}^d}$ by $\lambda = (\lambda_n)_{n\in\mathbb{N}}$. Observe that $id : F_{\omega,2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$. In fact we have

$$\lambda_n = a_n(id : F_{\omega,2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)),$$

where $a_n(id : F_{\omega,2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ is the n-th approximation number (linear width) of the operator $id : F_{\omega,2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$, see [31]. Recall that for two Banach spaces X, Y and $T \in \mathcal{L}(X, Y)$, the n-th approximation number of T is defined as

$$a_n(T) := \inf\big\{\|T - A : X \to Y\| : \ A \in \mathcal{L}(X, Y), \ \ \operatorname{rank}(A) < n\big\}, \quad n \in \mathbb{N}.$$

Basic properties of this quantity can be found in [36, 37].

We have the following embedding property of the class $F_{\omega,p}(\mathbb{T}^d)$.

Lemma 3.1. *Let $0 < p, q \leq \infty$ and $\omega = (\omega(k))_{k\in\mathbb{Z}^d}$ be a weight satisfying (3.1). Then the operator $id : F_{\omega,p}(\mathbb{T}^d) \hookrightarrow F_q(\mathbb{T}^d)$ is continuous if either $p \leq q$ or $q < p$ and the series $\sum_{k\in\mathbb{Z}^d} \omega(k)^{-\frac{pq}{p-q}}$ converges.*

Proof. If $q < p$ and $f \in F_{\omega,p}(\mathbb{T}^d)$, applying Hölder's inequality we get

$$\Big(\sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^q\Big)^{\frac{1}{q}} \leq \Big(\sum_{k\in\mathbb{Z}^d} \omega(k)^{-\frac{pq}{p-q}}\Big)^{\frac{p-q}{pq}} \Big(\sum_{k\in\mathbb{Z}^d} |\omega(k)\hat{f}(k)|^p\Big)^{\frac{1}{p}}.$$

This proves the case $q < p$. The case $p \leq q$ is obvious. $\square$

Our result for the best n-term approximation of the embedding $F_{\omega,p}(\mathbb{T}^d) \to F_q(\mathbb{T}^d)$ by the trigonometric system $\mathcal{T}^d$ reads as follows.

Theorem 3.2. *Let $0 < p, q \leq \infty$ and $\omega = (\omega(k))_{k\in\mathbb{Z}^d}$ be a weight satisfying conditions in Lemma 3.1. Then we have*

$$\sigma_n\big(id : F_{\omega,p}(\mathbb{T}^d) \to F_q(\mathbb{T}^d), \mathcal{T}^d\big) = \sigma_n\big(T_\lambda : \ell_p(\mathbb{N}) \to \ell_q(\mathbb{N}), \mathcal{E}\big), \quad n \in \mathbb{N},$$

where the value of $\sigma_n(T_\lambda, \mathcal{E})$ is given as in Theorem 2.1.

Proof. We consider the following commutative diagram

$$\begin{array}{ccc} F_{\omega,p}(\mathbb{T}^d) & \xrightarrow{id} & F_q(\mathbb{T}^d) \\ \big\downarrow A & & \big\uparrow B \\ \ell_p(\mathbb{Z}^d) & \xrightarrow{D_\omega} & \ell_q(\mathbb{Z}^d)\,, \end{array}$$

where the linear operators A, B and D_ω are defined as

$$\begin{aligned} Af &:= (\omega(k)\hat{f}(k))_{k\in\mathbb{Z}^d}\,, \\ D_\omega\xi &:= (\xi(k)/\omega(k))_{k\in\mathbb{Z}^d}\,, \qquad \xi = (\xi(k))_{k\in\mathbb{Z}^d} \\ (B\xi)(x) &:= \sum_{k\in\mathbb{Z}^d} \xi_k\, e^{ikx}\,, \qquad x \in \mathbb{T}^d\,. \end{aligned}$$

It is obvious that $\|A\| = \|B\| = 1$. Let $\mathcal{E}^d := \{e_k : k \in \mathbb{Z}^d\}$ where $e_k = (\delta_{k,l})_{l\in\mathbb{Z}^d}$. By the property (2.1) and the identity $id = B\,D_\omega\,A$ it follows

$$\sigma_n\big(id : F_{\omega,p}(\mathbb{T}^d) \to F_q(\mathbb{T}^d), \mathcal{T}^d\big) \le \sigma_n\big(D_\omega : \ell_p(\mathbb{Z}^d) \to \ell_q(\mathbb{Z}^d), \mathcal{E}^d\big), \qquad n \in \mathbb{N}\,.$$

From the fact that

$$\sigma_n\big(D_\omega : \ell_p(\mathbb{Z}^d) \to \ell_q(\mathbb{Z}^d), \mathcal{E}^d\big) = \sigma_n\big(T_\lambda : \ell_p(\mathbb{N}) \to \ell_q(\mathbb{N}), \mathcal{E}\big) \tag{3.2}$$

we obtain the estimate from above. Now we employ the same type of arguments with respect to the diagram

$$\begin{array}{ccc} \ell_p(\mathbb{Z}^d) & \xrightarrow{D_\omega} & \ell_q(\mathbb{Z}^d) \\ \big\downarrow{\scriptstyle A^{-1}} & & \big\uparrow{\scriptstyle B^{-1}} \\ F_{\omega,p}(\mathbb{T}^d) & \xrightarrow{id} & F_q(\mathbb{T}^d)\,. \end{array}$$

It is easy to see that the operators A and B are invertible and that $\|A^{-1}\| = \|B^{-1}\| = 1$. As above we conclude

$$\sigma_n\big(D_\omega : \ell_p(\mathbb{Z}^d) \to \ell_q(\mathbb{Z}^d), \mathcal{E}^d\big) \le \sigma_n\big(id : F_{\omega,p}(\mathbb{T}^d) \to F_q(\mathbb{T}^d), \mathcal{T}^d\big), \qquad n \in \mathbb{N}\,.$$

Now the estimate from below follows from (3.2). □

We need following auxiliary results.

Lemma 3.3. **(i)** *Let $s > 0$, $a > 1$, and $\beta \ge 0$. Then we have*

$$\lim_{n\to\infty} \int_{\frac{a}{n}}^{1} y^s \left(\frac{\ln n}{\ln(yn)}\right)^\beta \mathrm{d}y = \frac{1}{s+1}.$$

(ii) *Let $s > 1$, $\beta \ge 0$. Then we have*

$$\lim_{n\to\infty} \int_{1}^{+\infty} \frac{1}{t^s} \left(\frac{\ln(nt)}{\ln n}\right)^\beta \mathrm{d}t = \frac{1}{s-1}.$$

Proof. The first statement was proved in [33]. We prove the second one with concentration on the case $\beta > 0$ since the case $\beta = 0$ is obvious. We consider the sequence of functions

$$f_n(t) = \frac{1}{t^s}\left(\frac{\ln(nt)}{\ln n}\right)^\beta, \quad t \ge 1, \quad n \in \mathbb{N}.$$

Clearly, this sequence converges pointwise to $f(t) = \frac{1}{t^s}$. For $n \ge 3$, from the inequality $(x+y)^\beta \le C_\beta(x^\beta + y^\beta)$, for some $C_\beta > 0$, we derive:

$$f_n(t) = \frac{1}{t^s}\left(1 + \frac{\ln t}{\ln n}\right)^\beta < \frac{1}{t^s}\big(1 + \ln t\big)^\beta \le C_\beta \frac{1}{t^s}\big(1 + (\ln t)^\beta\big) := g(t).$$

Since $g(t)$ is integrable on $[1, +\infty)$, the desired result follows from Lebesgue's dominated convergence theorem. □

The asymptotic constants of best n-term approximation widths of embeddings of the classes $F_{\omega,p}(\mathbb{T}^d)$ in $F_q(\mathbb{T}^d)$ are given in the following theorem.

Theorem 3.4. *Let $s > 0$, $\beta \geq 0$ and let ω be a given weight. Assume that there exists $C > 0$ such that*

$$\lim_{n\to\infty} \frac{\lambda_n}{n^{-s}(\ln n)^\beta} = \lim_{n\to\infty} \frac{a_n\big(id : F_{\omega,2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^\beta} = C\,. \tag{3.3}$$

(i) *If $0 < p \leq q \leq \infty$ we have*

$$\lim_{n\to\infty} \frac{\sigma_n\big(id : F_{\omega,p}(\mathbb{T}^d) \to F_q(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s-\frac{1}{p}+\frac{1}{q}}(\ln n)^\beta} = \frac{(s+\frac{1}{p}-\frac{1}{q})^{s+\frac{1}{p}-\frac{1}{q}}}{(s+\frac{1}{p})^s} \frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}} C\,.$$

If $q = \infty$ and/or $p = \infty$, the asymptotic constant is understood as the limit of the right-hand side when $q \to \infty$ and/or $p \to \infty$.

(ii) *If $0 < q < p < \infty$ and $s > \frac{1}{q} - \frac{1}{p}$ we have*

$$\lim_{n\to\infty} \frac{\sigma_n\big(id : F_{\omega,p}(\mathbb{T}^d) \to F_q(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s-\frac{1}{p}+\frac{1}{q}}(\ln n)^\beta} = \left(\frac{s}{s+\frac{1}{p}}\right)^s \left(\frac{\frac{1}{q}}{s+\frac{1}{p}-\frac{1}{q}}\right)^{\frac{1}{q}-\frac{1}{p}} C.$$

Proof. We prove the case $0 < p, q < \infty$. The cases $p = \infty$ and/or $q = \infty$ are carried out similarly with slight modification. In this proof, for simplicity we denote

$$\sigma_n := \sigma_n\big(id : F_{\omega,p}(\mathbb{T}^d) \to F_q(\mathbb{T}^d), \mathcal{T}^d\big).$$

Step 1. We need some preparations. Assumption (3.3) indicates that for any $\varepsilon > 0$ there exists $n_1 := n_1(\varepsilon) \in \mathbb{N}$ such that for $k > n_1$ we have

$$\left|\frac{\lambda_k}{k^{-s}(\ln k)^\beta} - C\right| \leq \varepsilon \quad \Longleftrightarrow \quad C - \varepsilon \leq \frac{\lambda_k}{k^{-s}(\ln k)^\beta} \leq \varepsilon + C. \tag{3.4}$$

Since the function $\psi(t) = t^{ps}(\ln t)^{-p\beta}$ is increasing when $t \geq n_2 := e^{\beta/s}$, for $m > n_0 := \max\{n_1, n_2\}$ we have

$$\sum_{k=1}^m \lambda_k^{-p} = \sum_{k=1}^{n_0} \lambda_k^{-p} + \sum_{k=n_0+1}^{m} \lambda_k^{-p} \leq \sum_{k=1}^{n_0} \lambda_k^{-p} + \frac{1}{(C-\varepsilon)^p} \sum_{k=n_0+1}^{m} k^{ps}(\ln k)^{-p\beta}\,.$$

Estimating the summation by an integral and afterwards changing variable $y = \frac{t}{m+1}$ we find

$$\sum_{k=n_0+1}^{m} k^{ps}(\ln k)^{-p\beta} \leq \int_{n_0+1}^{m+1} t^{ps}(\ln t)^{-p\beta}\mathrm{d}t = \frac{(m+1)^{ps+1}}{(\ln(m+1))^{p\beta}} \int_{\frac{n_0+1}{m+1}}^{1} y^{ps}\left(\frac{\ln(m+1)}{\ln(y(m+1))}\right)^{p\beta} \mathrm{d}y\,.$$

By Lemma 3.3 (i) we can choose $n_3 > n_0$ such that for $m \geq n_3$ we have

$$\int_{\frac{n_0+1}{m+1}}^{1} y^{ps}\left(\frac{\ln(m+1)}{\ln(ym+y)}\right)^{p\beta} \mathrm{d}y \leq \frac{1+\varepsilon}{ps+1} \qquad \text{and} \qquad \frac{(\ln(m+1))^{p\beta}}{(m+1)^{ps+1}} \sum_{k=1}^{n_0} \lambda_k^{-p} \leq \varepsilon$$

which leads to

$$\sum_{k=1}^m \lambda_k^{-p} \leq \frac{(m+1)^{ps+1}}{(\ln(m+1))^{p\beta}} \left(\varepsilon + \frac{1+\varepsilon}{(C-\varepsilon)^p(ps+1)}\right) \tag{3.5}$$

for $m \geq n_3$. Similarly we have

$$\sum_{k=n_0+1}^{m} k^{ps}(\ln k)^{-p\beta} \geq \int_{n_0}^{m} \frac{t^{ps}}{(\ln t)^{p\beta}}\mathrm{d}t \geq \frac{m^{ps+1}}{(\ln m)^{p\beta}} \int_{\frac{n_0}{m}}^{1} y^{ps}\left(\frac{\ln m}{\ln(ym)}\right)^{p\beta} \mathrm{d}y \geq \frac{m^{ps+1}}{(\ln m)^{p\beta}} \frac{1-\varepsilon}{ps+1}\,,$$

which implies

$$\sum_{k=1}^{m} \lambda_k^{-p} \geq \sum_{k=n_0+1}^{m} \lambda_k^{-p} \geq \frac{1}{(C+\varepsilon)^p} \sum_{k=n_0+1}^{m} k^{ps} (\ln k)^{-p\beta} \geq \frac{m^{ps+1}}{(\ln m)^{p\beta}} \cdot \frac{1-\varepsilon}{(C+\varepsilon)^p (ps+1)}. \tag{3.6}$$

Step 2. Proof of the case $0 < p \leq q < \infty$. From (3.6) we get

$$\frac{m-n}{\left(\sum_{k=1}^{m} \lambda_k^{-p}\right)^{\frac{q}{p}}} \leq \frac{(m-n)(\ln m)^{q\beta}}{m^{qs+\frac{q}{p}}} \left(\frac{(C+\varepsilon)^p (ps+1)}{1-\varepsilon}\right)^{\frac{q}{p}}.$$

Considering the function $g(t) := \frac{t-n}{t^{qs+\frac{q}{p}}} (\ln t)^{q\beta},\ t \in [n, \infty)$, we have

$$g'(t) = \left(\frac{-t\left(qs+\frac{q}{p}-1\right) + n\left(qs+\frac{q}{p}\right)}{t^{qs+\frac{q}{p}+1}}\right)(\ln t)^{q\beta} + \left(\frac{t-n}{t^{qs+\frac{q}{p}+1}}\right) q\beta (\ln t)^{q\beta-1}.$$

We put

$$f(t) := \left[-t\left(qs+\frac{q}{p}-1\right) + n\left(qs+\frac{q}{p}\right)\right] \ln t + \left(t-n\right) q\beta, \qquad t \in [n, \infty).$$

Then $g'(t) = 0$ is equivalent to $f(t) = 0$. We have

$$f'(t) = -\left(qs+\frac{q}{p}-1\right)(\ln t+1) + \frac{n(qs+\frac{q}{p})}{t} + q\beta < -\left(qs+\frac{q}{p}-1\right)\ln t + 1 + q\beta.$$

This implies $f'(t) < 0$ if $t > e^{(1+q\beta)/(qs+q/p-1)}$. Observe that

$$f\left(n + \frac{n}{qs+\frac{q}{p}-1}\right) > 0, \qquad f\left(n + \frac{2n}{qs+\frac{q}{p}-1}\right) < 0 \quad \text{and}$$

for $n \geq n_4$ depending only on p, q, s and β. Consequently the equation $f(t) = 0$ (or $g'(t) = 0$) has a unique solution belonging to the interval $I_n := \left[n + \frac{n}{qs+\frac{q}{p}-1},\ n + \frac{2n}{qs+\frac{q}{p}-1}\right]$. From this we deduce

$$\sigma_n^q = \sup_{m \geq n} \left(\frac{m-n}{\left(\sum_{k=1}^{m} \lambda_k^{-p}\right)^{\frac{q}{p}}}\right) \leq \sup_{t \in I_n} \left(\frac{(t-n)(\ln t)^{q\beta}}{t^{qs+\frac{q}{p}}}\right) \left(\frac{(C+\varepsilon)^p (ps+1)}{1-\varepsilon}\right)^{\frac{q}{p}}$$

which leads to

$$\begin{aligned}
\frac{\sigma_n^q}{n^{1-qs-\frac{q}{p}}(\ln n)^{q\beta}} &\leq \sup_{t \in I_n} \left(\frac{(t-n)(\ln t)^{q\beta}}{n(tn^{-1})^{qs+\frac{q}{p}}(\ln n)^{q\beta}}\right) \left(\frac{(C+\varepsilon)^p (ps+1)}{1-\varepsilon}\right)^{\frac{q}{p}} \\
&\leq \sup_{t \in \mathbb{R},\, t \geq n} \left(\frac{t-n}{n(tn^{-1})^{qs+\frac{q}{p}}}\right) \left(\frac{(C+\varepsilon)^p (ps+1)}{(1-\varepsilon)^2}\right)^{\frac{q}{p}}
\end{aligned}$$

if n is large enough. It is easy to see that the function $h(t) := \frac{t-n}{t^{qs+q/p}}$, $t \in [n, \infty)$, attains its maximum at $t_0 = \left(1 + \frac{1}{qs+q/p-1}\right)n$. Hence, we find

$$\left(\frac{\sigma_n}{n^{-s-\frac{1}{p}+\frac{1}{q}}(\ln n)^{\beta}}\right)^q \leq \frac{1}{(qs+\frac{q}{p}-1)\left(1+\frac{1}{qs+q/p-1}\right)^{qs+q/p}} \left(\frac{(C+\varepsilon)^p (ps+1)}{(1-\varepsilon)^2}\right)^{\frac{q}{p}} \tag{3.7}$$

if n is large enough. Taking the limits $n \to \infty$ and afterwards $\varepsilon \downarrow 0$ in (3.7) we obtain the upper bound. In view of Theorem 2.1 (i), by choosing $m \sim \left(1 + \frac{1}{qs+q/p-1}\right)n$ we also obtain the lower bound

in this case.
Step 3. Proof of the case $0 < q < p < \infty$. Firstly, we estimate n_* in (2.4). From (2.5) we have

$$(n_* - n)\lambda_{n_*}^{-p} \le \sum_{k=1}^{n_*} \lambda_k^{-p}.$$

In view of (3.4) and (3.5) we get for $n \ge n_3$

$$(n_* - n)(C+\varepsilon)^{-p} \frac{n_*^{ps}}{(\ln n_*)^{p\beta}} \le \frac{(n_*+1)^{ps+1}}{(\ln(n_*+1))^{p\beta}} \left(\varepsilon + \frac{1+\varepsilon}{(C-\varepsilon)^p (ps+1)} \right)$$

which implies

$$\frac{n_* - n}{n_*} \le \left(1 + \frac{1}{n_*}\right)^{ps+1} \left(\varepsilon (C+\varepsilon)^p + \frac{(1+\varepsilon)(C+\varepsilon)^p}{(C-\varepsilon)^p (ps+1)} \right).$$

Therefore, for any $\epsilon > 0$, exist $N_1 > 0$ such that for $n > N_1$ we have

$$\frac{n_* - n}{n_*} \le \frac{1}{ps+1} + \epsilon \qquad \text{or} \qquad n_* \le \frac{n}{\frac{ps}{ps+1} - \epsilon}. \tag{3.8}$$

Using (3.4) and (3.6) the condition $(m-n)\lambda_m^{-p} \le \sum_{k=1}^m \lambda_k^{-p}$ is satisfied if

$$\frac{m^{ps+1}}{(\ln m)^{p\beta}} \cdot \frac{1-\varepsilon}{(C+\varepsilon)^p (ps+1)} \ge \frac{(m-n)(C-\varepsilon)^{-p} m^{ps}}{(\ln m)^{p\beta}}$$

which is equivalent to

$$\frac{m-n}{m} \le \frac{(1-\varepsilon)(C-\varepsilon)^p}{(C+\varepsilon)^p (ps+1)}.$$

Hence, for any $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that for $n > N_2$ we have

$$\frac{(1-\varepsilon)(C-\varepsilon)^p}{(C+\varepsilon)^p (ps+1)} \ge \frac{1}{ps+1} - \epsilon.$$

Therefore, the condition $(m-n)\lambda_m^{-p} \le \sum_{k=1}^m \lambda_k^{-p}$ is satisfied if

$$\frac{m-n}{m} \le \frac{1}{ps+1} - \epsilon \qquad \text{or} \qquad m \le \frac{n}{\frac{ps}{ps+1} + \epsilon}.$$

This leads to $n_* \ge \frac{n}{\frac{ps}{ps+1}+\epsilon}$. From this and (3.8) we deduce

$$n_* \sim \left(1 + \frac{1}{ps}\right) n, \quad n \to +\infty\,.$$

Denoting $\alpha = \frac{pq}{p-q}$, from (2.4) we have

$$\sigma_n^\alpha = \left(\frac{(n_* - n)^{1/q}}{\left(\sum_{k=1}^{n_*} \lambda_k^{-p}\right)^{1/p}} \right)^\alpha + \sum_{k=n_*+1}^{\infty} \lambda_k^\alpha. \tag{3.9}$$

Using (3.5) and (3.6) again we get

$$\sum_{k=1}^{n_*} \lambda_k^{-p} \sim \frac{1}{C^p (ps+1)} \frac{n_*^{ps+1}}{(\ln n_*)^{p\beta}} \sim \frac{1}{C^p(ps+1)} \frac{(1+\frac{1}{ps})^{ps+1} n^{ps+1}}{(\ln n)^{p\beta}}, \quad n \to +\infty.$$

Therefore, the first term in (3.9) can be estimated:

$$\left(\frac{(n_*-n)^{\frac{1}{q}}}{\left(\sum_{k=1}^{n_*}\lambda_k^{-p}\right)^{\frac{1}{p}}}\right)^{\alpha} \overset{n\to+\infty}{\sim} \left(\frac{C(ps+1)^{\frac{1}{p}}(\ln n)^{\beta}}{(1+\frac{1}{ps})^{s+\frac{1}{p}}n^{s+\frac{1}{p}}}\left(\frac{n}{ps}\right)^{\frac{1}{q}}\right)^{\alpha} = \frac{C^{\alpha}}{ps}\left(\frac{ps}{1+ps}\right)^{s\alpha}\frac{(\ln n)^{\alpha\beta}}{n^{\alpha(s+\frac{1}{p}-\frac{1}{q})}}. \tag{3.10}$$

Now, we estimate the second term in (3.9). Observe that $\varphi(t)=t^{-s\alpha}(\ln t)^{\alpha\beta}$ is a decreasing function when $t\geq t_0$ for some $t_0>0$. Hence, when n is large enough, in view of (3.4) we can bound

$$\begin{aligned}\sum_{k=n_*+1}^{\infty}\lambda_k^{\alpha} &\leq (C+\varepsilon)^{\alpha}\sum_{k=n_*+1}^{\infty}\frac{(\ln k)^{\alpha\beta}}{k^{s\alpha}} \leq (C+\varepsilon)^{\alpha}\int_{n_*}^{+\infty}\frac{(\ln t)^{\alpha\beta}}{t^{s\alpha}}\mathrm{d}t \\ &= (C+\varepsilon)^{\alpha}\frac{(\ln n_*)^{\alpha\beta}}{n_*^{s\alpha-1}}\int_1^{+\infty}\frac{1}{t^{s\alpha}}\left(\frac{\ln n_* t}{\ln n_*}\right)^{\alpha\beta}\mathrm{d}t. \end{aligned}\tag{3.11}$$

Similarly, we also have the estimate

$$\begin{aligned}\sum_{k=n_*+1}^{\infty}\lambda_k^{\alpha} &\geq (C-\varepsilon)^{\alpha}\sum_{k=n_*+1}^{\infty}\frac{(\ln k)^{\alpha\beta}}{k^{s\alpha}} \geq (C-\varepsilon)^{\alpha}\int_{n_*+1}^{+\infty}\frac{(\ln t)^{\alpha\beta}}{t^{s\alpha}}\mathrm{d}t \\ &= (C-\varepsilon)^{\alpha}\frac{(\ln(n_*+1))^{\alpha\beta}}{(n_*+1)^{s\alpha-1}}\int_1^{+\infty}\frac{1}{t^{s\alpha}}\left(\frac{\ln(n_*+1)t}{\ln(n_*+1)}\right)^{\alpha\beta}\mathrm{d}t. \end{aligned}\tag{3.12}$$

Note that the condition $s>\frac{1}{q}-\frac{1}{p}$ implies $s\alpha>1$. Using Lemma 3.3 (ii), from (3.11) and (3.12) we get

$$\begin{aligned}\sum_{k=n_*+1}^{\infty}\lambda_k^{\alpha} &\overset{n\to+\infty}{\sim} \frac{1}{s\alpha-1}C^{\alpha}\left(1+\frac{1}{ps}\right)^{1-s\alpha}\frac{(\ln n)^{\alpha\beta}}{n^{s\alpha-1}} \\ &= \frac{p-q}{spq-p+q}\left(1+\frac{1}{ps}\right)C^{\alpha}\left(\frac{ps}{ps+1}\right)^{\alpha s}\frac{(\ln n)^{\alpha\beta}}{n^{\alpha(s+\frac{1}{p}-\frac{1}{q})}}. \end{aligned}$$

From this and (3.10) we finally obtain

$$\begin{aligned}\sigma_n^{\alpha} &\overset{n\to+\infty}{\sim} \left[\frac{1}{ps}+\frac{p-q}{spq-p+q}\left(1+\frac{1}{ps}\right)\right]C^{\alpha}\left(\frac{ps}{ps+1}\right)^{s\alpha}\frac{(\ln n)^{\alpha\beta}}{n^{\alpha(s+\frac{1}{p}-\frac{1}{q})}} \\ &= \frac{p}{spq-p+q}C^{\alpha}\left(\frac{ps}{ps+1}\right)^{s\alpha}\frac{(\ln n)^{\alpha\beta}}{n^{\alpha(s+\frac{1}{p}-\frac{1}{q})}} \end{aligned}$$

which proves the second statement. □

4 Best n-term approximation of function spaces with mixed smoothness

In this section we shall apply the result in Section 3 to the family of weights

$$\omega_{s,r}(k) := \prod_{i=1}^{d}\left(1+|k_i|^r\right)^{s/r}, \qquad 0<r<\infty,$$

$$\omega_{s,r}(k) := \prod_{i=1}^{d}\max(1,|k_i|)^s, \qquad r=\infty,$$

$k \in \mathbb{Z}^d$, where the parameter s satisfies $0 < s < \infty$. We shall use the notation $H^{s,r}_{\text{mix}}(\mathbb{T}^d) := F_{\omega_{s,r},2}(\mathbb{T}^d)$ and $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) := F_{\omega_{s,r},1}(\mathbb{T}^d)$, respectively. The classes $H^{s,r}_{\text{mix}}(\mathbb{T}^d)$ are called periodic Sobolev spaces with mixed smoothness and well-known in approximation theory, see, e.g., [34, 35, 15]. The classes $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d)$ are the weighted Wiener algebras. These spaces have been studied extensively recently in [24, 6, 25, 33]. In both spaces $H^{s,r}_{\text{mix}}(\mathbb{T}^d)$ and $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d)$, for different r, we obtain the same sets of functions. A change of the parameter r leads to a change of the quasinorm only.

Let $m \in \mathbb{N}$. We define the space $H^m_{\text{mix}}(\mathbb{T}^d)$ to be the collection of all functions $f \in L_2(\mathbb{T}^d)$ such that all distributional derivatives $D^\alpha f$ with $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\max_{j=1,\ldots,d} \alpha_j \le m$ belong to $L_2(\mathbb{T}^d)$. The space $H^m_{\text{mix}}(\mathbb{T}^d)$ is equipped with the norm

$$\|\, f \,\|_{H^m_{\text{mix}}(\mathbb{T}^d)} := \Bigg(\sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_d)\in\mathbb{N}_0^d \\ \alpha_j \le m,\, j=1,\ldots,d}} \|\, D^\alpha f \|^2_{L_2(\mathbb{T}^d)} \Bigg)^{1/2} .$$

Then $H^m_{\text{mix}}(\mathbb{T}^d) = H^{m,r}_{\text{mix}}(\mathbb{T}^d)$ for all r in the sense of equivalent quasinorms. If $m = 1$, then we have $\|\cdot\|_{H^{1,2}_{\text{mix}}(\mathbb{T}^d)} = \|\cdot\|_{H^1_{\text{mix}}(\mathbb{T}^d)}$. If $m \ge 2$, then the norm $\|\cdot\|_{H^m_{\text{mix}}(\mathbb{T}^d)}$ itself does not belong to the family of norms $\|\cdot\|_{H^{m,r}_{\text{mix}}(\mathbb{T}^d)}$, $0 < r \le \infty$. But the choice $r = 2m$ leads to the following standard norm

$$\|\, f \,\|_{H^{m,2m}_{\text{mix}}(\mathbb{T}^d)} = \Bigg(\sum_{\alpha\in\{0,m\}^d} \|\, D^\alpha f \,\|^2_{L_2(\mathbb{T}^d)} \Bigg)^{1/2} ,$$

see [31].

Let $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ denote the non-increasing rearrangement of the sequence $(1/\omega_{s,r}(k))_{k\in\mathbb{Z}^d}$. That leads to $\lambda_n = a_n\big(id : H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)$. We recall a result obtained in [31].

Proposition 4.1. *Let $0 < s < \infty$ and $0 < r \le \infty$. Then it holds*

$$\lim_{n\to\infty} \frac{\lambda_n}{n^{-s}(\ln n)^{s(d-1)}} = \lim_{n\to\infty} \frac{a_n\big(id : H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{s(d-1)}} = \left(\frac{2^d}{(d-1)!}\right)^s .$$

From this and Theorem 3.4 we get the following.

Theorem 4.2. *Let $0 < s < \infty$ and $0 < r \le \infty$. Then it holds*

$$\lim_{n\to\infty} \frac{\sigma_n\big(id : H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s}(\ln n)^{s(d-1)}} = \frac{s^s}{(s+\frac{1}{2})^s}\left(\frac{2^d}{(d-1)!}\right)^s$$

and if $s > 1/2$

$$\lim_{n\to\infty} \frac{\sigma_n\big(id : H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s+\frac{1}{2}}(\ln n)^{s(d-1)}} = \left(\frac{s}{s+\frac{1}{2}}\right)^s \left(\frac{1}{s-\frac{1}{2}}\right)^{\frac{1}{2}} \left(\frac{2^d}{(d-1)!}\right)^s .$$

The asymptotic constants for embeddings of best n-term approximation widths of embedding of weighted Wiener classes $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d)$ are also obtained from Theorem 3.4.

Theorem 4.3. *Let $0 < s < \infty$ and $0 < r \le \infty$. Then it holds*

$$\lim_{n\to\infty} \frac{\sigma_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s-\frac{1}{2}}(\ln n)^{s(d-1)}} = \frac{(s+\frac{1}{2})^{s+\frac{1}{2}}}{\sqrt{2}(s+1)^s}\left(\frac{2^d}{(d-1)!}\right)^s$$

and

$$\lim_{n\to\infty} \frac{\sigma_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s}(\ln n)^{s(d-1)}} = \frac{s^s}{(s+1)^s}\left(\frac{2^d}{(d-1)!}\right)^s .$$

Remark 4.4. Let us compare the asymptotic decay of a_n and σ_n. The equivalence

$$a_n\big(id: H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \asymp \sigma_n\big(id: H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d), \mathcal{T}^d\big)$$

has been known with a long history, see, e.g., [15, Chapters 4 and 7] for comments. From Theorem 4.2 and [33] we also have

$$a_n\big(id: H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d), \mathcal{T}^d\big) \asymp \sigma_n\big(id: H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d), \mathcal{T}^d\big).$$

However, by Theorem 4.3 and [33] we find

$$a_n\big(id: \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \asymp n^{\frac{1}{2}}\sigma_n\big(id: \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d), \mathcal{T}^d\big).$$

This indicates that approximating functions in the class $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d)$ by n-term improves the convergence rate $1/2$ compared to linear method.

We are also interested in asymptotic constants of best n-term approximation of embeddings of function spaces with mixed smoothness into $H^1(\mathbb{T}^d)$. Here $H^1(\mathbb{T}^d)$ is equipped with the norm

$$\|\, f\,\|_{H^1(\mathbb{T}^d)} := \bigg(\sum_{k\in\mathbb{Z}^d}\Big(1+\sum_{j=1}^d |k_j|^2\Big)|\hat{f}(k)|^2\bigg)^{1/2} = \bigg(\|\, f\,\|^2_{L_2(\mathbb{T}^d)} + \sum_{j=1}^d \Big\|\, \frac{\partial f}{\partial x_j}\,\Big\|^2_{L_2(\mathbb{T}^d)}\bigg)^{1/2}.$$

I.e., $H^1(\mathbb{T}^d)$ is the standard isotropic periodic Sobolev space with smoothness 1. We define a weight $\tilde{\omega}$ by

$$\tilde{\omega}(k) := \frac{\prod_{j=1}^d (1+|k_j|^2)^{s/2}}{\big(1+\sum_{j=1}^d |k_j|^2\big)^{1/2}}, \qquad k=(k_1,\ldots,k_d)\in\mathbb{Z}^d\,. \tag{4.1}$$

Rearranging non-increasingly the sequence $(1/\tilde{\omega}(k))_{k\in\mathbb{Z}^d}$ with the outcome denoted by $(\tilde{\lambda}_n)_{n\in\mathbb{N}}$, we obtain $\tilde{\lambda}_n = a_n\big(id: H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)$. The asymptotic constant of $a_n\big(id: H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)$ was obtained recently in [33].

Proposition 4.5. *Let $d\in\mathbb{N}$, $s>1$ and*

$$S := \sum_{k=1}^{+\infty} \frac{1}{(k^2+1)^{\frac{s}{2(s-1)}}}. \tag{4.2}$$

Then we have

$$\lim_{n\to+\infty}\frac{\tilde{\lambda}_n}{n^{1-s}} = \lim_{n\to+\infty}\frac{a_n\big(id: H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)}{n^{1-s}} = (2d)^{s-1}(2S+1)^{(s-1)(d-1)}.$$

From this and Theorem 3.4 we obtain the following.

Theorem 4.6. *Let $d\in\mathbb{N}$, $s>1$ and S be given in (4.2). Then it holds*

$$\lim_{n\to\infty}\frac{\sigma_n\big(id: H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s+1}} = \bigg(\frac{s-1}{s-\frac{1}{2}}\bigg)^{s-1}(2d)^{s-1}(2S+1)^{(s-1)(d-1)}$$

and

$$\lim_{n\to\infty}\frac{\sigma_n\big(id: \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d), \mathcal{T}^d\big)}{n^{-s+\frac{1}{2}}} = \frac{(s-\frac{1}{2})^{s-\frac{1}{2}}}{\sqrt{2}\, s^{s-1}}(2d)^{s-1}(2S+1)^{(s-1)(d-1)}.$$

Proof. Let $\tilde{\omega}$ be given in (4.1). We will show that

$$\sigma_n\big(id: H^{s,2}_{\rm mix}(\mathbb{T}^d)\to H^1(\mathbb{T}^d),\mathcal{T}^d\big) = \sigma_n\big(id: F_{\tilde{\omega},2}(\mathbb{T}^d)\to F_2(\mathbb{T}^d),\mathcal{T}^d\big) \tag{4.3}$$

and

$$\sigma_n\big(id: \mathcal{A}^{s,2}_{\rm mix}(\mathbb{T}^d)\to H^1(\mathbb{T}^d),\mathcal{T}^d\big) = \sigma_n\big(id: F_{\tilde{\omega},1}(\mathbb{T}^d)\to F_2(\mathbb{T}^d),\mathcal{T}^d\big), \tag{4.4}$$

by using standard lifting arguments. We consider the diagram

$$\begin{tikzcd}
H^{s,2}_{\rm mix}(\mathbb{T}^d) \arrow[r, "id"] \arrow[d, "A"] & H^1(\mathbb{T}^d) \\
F_{\tilde{\omega},2}(\mathbb{T}^d) \arrow[r, "id"] & F_2(\mathbb{T}^d) \arrow[u, "B"']
\end{tikzcd}$$

where the linear operators A and B are defined for $f\in H^{s,2}_{\rm mix}(\mathbb{T}^d)$ and $g\in F_2(\mathbb{T}^d)$ respectively by

$$\widehat{Af}(k) := \Big(1+\sum_{j=1}^d |k_j|^2\Big)^{1/2}\hat{f}(k),\quad \widehat{Bg}(k) := \Big(1+\sum_{j=1}^d |k_j|^2\Big)^{-1/2}\hat{g}(k),\quad k=(k_1,\ldots,k_d)\in\mathbb{Z}^d.$$

It is obvious that $\|A\|=\|B\|=1$. Now by the property (2.1), we obtain

$$\sigma_n\big(id: H^{s,2}_{\rm mix}(\mathbb{T}^d)\to H^1(\mathbb{T}^d),\mathcal{T}^d\big) \le \sigma_n\big(id: F_{\tilde{\omega},2}(\mathbb{T}^d)\to F_2(\mathbb{T}^d),\mathcal{T}^d\big).$$

The reverse inequality follows from the modified diagram

$$\begin{tikzcd}
H^{s,2}_{\rm mix}(\mathbb{T}^d) \arrow[r, "id"] & H^1(\mathbb{T}^d) \arrow[d, "B^{-1}"] \\
F_{\tilde{\omega},2}(\mathbb{T}^d) \arrow[r, "id"] \arrow[u, "A^{-1}"'] & F_2(\mathbb{T}^d)\,.
\end{tikzcd}$$

Hence (4.3) is proved. Proof of (4.4) is carried out similarly. Now the assertion follows from Proposition 4.5 and Theorem 3.4. □

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