REGULARITY OF POWERS OF COVER IDEALS OF BIPARTITE GRAPHS

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Abstract. Let $G = (V, E)$ be a bipartite graph over the vertex set $V = \{1, \dots, r\}$ and let $J = J(G)$ be the cover ideal of G in the polynomial ring $R = K[x_1, \dots, x_r]$. It is known that there are integers b and t_0 such that $\operatorname{reg} J^t = d(J)t + b$ is a linear function in t for all $t \geqslant t_0$. In this paper, we give effective bounds for b and t_0.

Introduction

Let $R = K[x_1, \dots, x_r]$ be a polynomial ring over a field K and let I be a homogeneous ideal of R. The Castelnuovo-Mumford regularity (or regularity) of I, denote $\operatorname{reg}(I)$, is an important invariant in commutative algebra and algebraic geometry. It is celebrated result that the $\operatorname{reg} I^t$ is a linear function in t for t large enough (see [10, 30]), i.e. there are non-negative integers b and t_0 such that

$$\operatorname{reg} I^t = dt + b \text{ for all } t \geqslant t_0.$$

While the constant d was implicitly described in [30] and made more precise in [35], the information of b and t_0 is little known, that make a great attraction for researchers. In [12], Eisenbud and Ulrich asked two following questions:

(1) What is the significance of the number b?
(2) What is a suitable bound for t_0?

In general, the stability index t_0 and the constant b are hard to compute. There have been particular attempts is identifying t_0 and b for certain classes ideals (see e.g. [3, 6, 7, 8, 11, 12, 16, 17, 18, 27, 23]).

Recently, there are a lot of work using the relationship between the combinatorial properties of graphs and the algebraic properties of ideals associated with graphs to find bounds or compute b and t_0. For instant, when $I = I(G)$ is an edge ideal of a graph G. Herzog, Hibi and Zheng [21] showed that if I has a linear resolution, so does I^t for all $t \geqslant 1$; Beyarslan, Hà and Trung [7] proved that if G is a forest then $\operatorname{reg}(I^t) = 2t + v(G) - 1$, for all $t \geqslant 1$, where $v(G)$ is the matching number of G; this formula also holds for any Cameron-Walker graph G (see [4]); Alilooee and Beyarslan and Selvaraja [2] proved that for any unicyclic graph (which is not a cycle) and any integer $t \geqslant 1$, they had $\operatorname{reg}(I^t) = 2t + \operatorname{reg}(I) - 2$.

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Moving away from edge ideals, the next class of monomial ideals to consider are cover ideals. For a finite simple graph G on the vertex set $\{1,\dots,r\}$, let $J(G)\subseteq R$ denote the cover ideal of G, which is defined by

$$J(G) := (\mathbf{x}_\tau \mid \tau \text{ is a minimal vertex cover of } G).$$

Note that cover ideal $J(G)=\bigcap_{\{i,j\}\in E(G)}(x_i,x_j)$ is the Alexander dual of the edge ideal $I(G)$. Although the connection between the algebraic properties of the edge ideals and the combinatorial properties of graphs has been studied extensively, not much is known about the connection between the properties of the cover ideals. Several results on the regularity of the ordinary powers of $J(G)$ have been recently establised only in very special cases. For example, in [18, Corollary 3.4], Hang and Trung proved that when G bipartite graphs, then there is a non-negative integer $b\leqslant |V(G)|-d(J(G))-1$ such that $\operatorname{reg} J(G)^t = d(J(G))t+b$ for all $t\geqslant r+2$; In [28], A. Kumar and R. Kumar showed that if G is a tree then $\operatorname{reg} J(G)^t = d(J(G))t$ for all $t\geqslant 1$.

As in many previous works in the literature, we are interested in two questions of Eisenbud and Ulrich for the case $I=J(G)$, the cover ideal of a bipartite graph G. Denote $d(J(G))$ to be the maximal degree of minimal monomial generators of $J(G)$. Our main result is stated as the following theorem.

Theorem 3.2. *Let G be a bipartite graph. Then there is a non-negative integer b with $0\leqslant b\leqslant e-d(J(G))$, such that*

$$\operatorname{reg} J(G)^t = d(J(G))t+b \quad \text{for all } t\geqslant \max\left\{\frac{m+1}{2}, e-b-d(J(G))+1\right\},$$

where m is the length of a longest simple path in G and $e=\max\{\operatorname{reg}(J(H)) \mid H$ is a subgraph of $G\}$.

It is worth mentioning that the following consequence of our main result improves [18, Corollary 3.4] significantly.

Corollary 3.3. *Let G be a bipartite graph with r vertices. Then, there is a non-negative integer $b\leqslant r-d(J(G))-1$ such that*

$$\operatorname{reg} J(G)^t = d(J(G))t+b, \text{ for all } t\geqslant \frac{r}{2}.$$

Our technique is based on a formula given by Takayama [34] for computing the dimension of the K-vector space $H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha}\in\mathbb{Z}^r$ in the case I is a monomial ideal, which is a generalization of Hochster's formula for the case I is squarefree [26, Theorem 4.1]. By using this formula for the case $I=J(G)^t$ where G is a bipartite graph, we are able to study the non-vanishing of $H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}}$ by searching for $\boldsymbol{\alpha}$ in a polytope in $\mathbb{R}^r$. And then we use the theory of integer programming as the key role in this paper (see e.g. [19, 24, 25] for this approach).

The paper is organized as follows. In Section 1, we recall some basic notations and terminology for simplicial complex, the relationship between simplicial complexes and cover ideals of graphs; give Takayama's formula for computing local cohomology modules. In Section 2, we consider the integer solutions of systems of linear inequalities with bipartite matrices. In the last section, we prove the main theorem, Theorem 3.2.

1. Preliminary

In this section, we recollect notation, terminology and basic results used in the paper. We follow standard texts [9, 20, 32, 33]. Throughout the paper, let K be a field, let $R = K[x_1, \ldots, x_r]$, $r \geqslant 1$ be a polynomial ring, and let $\mathfrak{m} = (x_1, \ldots, x_r)$ be the maximal homogeneous ideal of R.

1.1. **Regularity.** Let M be a finitely generated graded R-module. For each $i = 0, \ldots, \dim M$, the *a_i-invariant* of M is defined by

$$a_i(M) := \max\{t \mid H^i_{\mathfrak{m}}(M)_t \neq 0\},$$

where $H^i_{\mathfrak{m}}(M)$ is the i-th local cohomology module of M with support in $\mathfrak{m}$, with the convention that $a_i(M) = -\infty$ if $H^i_{\mathfrak{m}}(M) = 0$. Then, the Castelnuovo-Mumford regularity (regularity for short) of M is defined by

$$\operatorname{reg}(M) = \max\{a_i(M) + i \mid 0 \leqslant i \leqslant \dim M\}.$$

This invariant can be defined via either the minimal free resolutions or the local cohomology modules.

Let M be a nonzero finitely generated graded R-module and let

$$0 \to \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \to 0$$

be the minimal free resolution of M. The *Castelnuovo–Mumford regularity* (or regularity for short) of M is defined by

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

Let us denote by $d(M)$ the maximal degree of a minimal homogeneous generator of M. The definition of the regularity implies $d(M) \leqslant \operatorname{reg}(M)$.

For any nonzero proper homogeneous ideal I of R, by looking at the minimal free resolution, it is easy to see that $\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1$, so we shall work with $\operatorname{reg}(I)$ and $\operatorname{reg}(R/I)$ interchangeably.

1.2. **Simplicial complexes and Stanley-Reisner ideals.** We recall a relationship between cover ideals of graphs and simplicial complexes. A *simplicial complex* on $V = \{1, \ldots, r\}$ is a collection of subsets of V, called faces, such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. A face of Δ not properly contained in another face of Δ is called a facet. The set of facets is denoted by $\mathcal{F}(\Delta)$.

The Stanley-Reisner ideal associated to a simplicial complex Δ is the squarefree monomial ideal

$$I_\Delta := (x_\tau \mid \tau \notin \Delta) \subseteq R.$$

Note that if I is a squarefree monomial ideal, then it is a Stanley-Reisner ideal of the simplicial complex $\Delta(I) := \{\tau \subseteq V \mid \mathbf{x}_\tau \notin I\}$. If I is a monomial ideal (maybe not squarefree) we also use $\Delta(I)$ to denote the simplicial complex corresponding to the squarefree monomial ideal $\sqrt{I}$.

Let $\mathcal{F}(\Delta)$ be the set of facets of Δ. If $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we write $\Delta = \langle F_1, \ldots, F_m \rangle$. Then, I_Δ has the primary-decomposition (see [32, Theorem 1.7]):

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} (x_i \mid i \notin F).$$

For $n \geqslant 1$, the n-th symbolic power of I_Δ is

$$I_\Delta^{(n)} = \bigcap_{F \in \mathcal{F}(\Delta)} (x_i \mid i \notin F)^n.$$

1.3. **Degree complexes.** Let I be a non-zero monomial ideal. Since R/I is an $\mathbb{N}^r-$ graded algebra, $H^i_{\mathfrak{m}}(R/I)$ is an $\mathbb{Z}^r$-graded module over R/I for every i. For each degree $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$, in order to compute $\dim_K H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}}$ we use a formula given by Takayama [34, Theorem 2.2] which is a generalization of Hochster's formula for the case I is squarefree [26, Theorem 4.1].

Set $G_{\boldsymbol{\alpha}} := \{i \mid \alpha_i < 0\}$. For a subset $F \subseteq V$, we let $R_F := R[x_i^{-1} \mid i \in F]$. Define the *degree complex* $\Delta_{\boldsymbol{\alpha}}(I)$ by

$$\Delta_\alpha(I) := \{F \subseteq V \setminus G_{\boldsymbol{\alpha}} \mid x^\alpha \notin I R_{F \cup G_{\boldsymbol{\alpha}}}\}. \tag{1}$$

Lemma 1.1. [34, Theorem 2.2] $\dim_K H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}} = \dim_K \widetilde{H}_{i-|G_\alpha|-1}(\Delta_{\boldsymbol{\alpha}}(I); K)$.

The following result of Minh and Trung is very useful for computing $\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(t)})$, which allows us to investigate $\operatorname{reg}(I_\Delta^{(t)})$ by using the theory of convex polyhedra.

Lemma 1.2. [31, Lemma 1.3] *Let Δ be a simplicial complex and $\boldsymbol{\alpha} \in \mathbb{N}^r$. Then,*

$$\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(t)})) = \left\{ F \in \mathcal{F}(\Delta) \mid \sum_{i \notin F} \alpha_i \leqslant t-1 \right\}.$$

1.4. **Graphs and their cover ideals.** Let G be a simple graph. We use the symbols $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G and $E(H)$ is the set of all edges of G with end points in $V(H)$, then H is called an *induced* subgraph of G.

Let $p\colon v_0, v_1, \ldots, v_k$ is a sequence of vertices of G. Then,

1. p is called a *path* if $\{v_{i-1}, v_i\} \in E(G)$ for $i = 1, \ldots, k$. In this case, we say that p is a path from v_0 to v_k.
2. p is called a *simple path* if it is a path and every vertex appears exactly once.
3. p is called a *cycle* if $k \geqslant 3$ and p is a path with distinct vertices except for $v_0 = v_k$.

In each case, k is called the *length* of p. A simple path is longest if it is among the simple paths of largest lengths of G.

The graph G is *bipartite* if $V(G)$ can be partitioned into two subsets X and Y so that every edge has one end in X and another end in Y; such a partition (X, Y) is called a *bipartition* of the graph. Note that G is bipartite if and only if it has no cycle of odd length (see [9, Theorem 4.7]).

A graph is *connected* if there is a path from any point to any other point in the graph. A graph that is not connected is said to be *disconnected*. A connected component of a graph G is a connected subgraph that is not part of any larger connected subgraph. The components of any graph partition its vertices into disjoint sets, and are the induced subgraphs of those sets. If a connected component of G has just one element, then this element is called an *isolated vertex* of G.

Assume that $E(G) = \{e_1, \ldots, e_n\}$. The *incidence matrix* of G is the $n \times r$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 0$ if $j \notin e_i$ and $a_{ij} = 1$ if $j \in e_i$. It is well-known that G is bipartite if and only if $A(G)$ is totally unimodular, i.e. the determinant of every its square submatrix is one of $-1, 0, 1$ (see [5, Theorem 5, page 164]).

A connected graph is a *tree* if it has no cycles. If a subgraph T of G with $V(T) = V(G)$ is a tree, then T is called a *spanning tree* of G. From [9, Theorem 4.3] and [9, Theorem 4.6], we deduce that:

Lemma 1.3. *If G is a connected graph, then $|E(G)| \geqslant |V(G)| - 1$. The equality occurs if and only if G is a tree.*

If G is a tree, then for each pair of vertices u and v of G has a unique simple path from u to v according to [9, Proposition 4.1]. The length this path is just the distance between u and v, and we denoted by $\mathrm{dist}_G(u, v)$.

In the sequence we need the following fact on bipartite graphs.

Lemma 1.4. *Let G be a bipartite graph with at least one edge. Assume that for each edge $\{i, j\}$ of G we have a real number a_{ij}. Then, the linear system:*

$$\begin{cases} x_i + x_j = a_{ij}, \\ \{i, j\} \in E(G) \end{cases}$$

has no unique solution.

Proof. It suffices to show that the corresponding homogeneous system

$$\begin{cases} x_i + x_j = 0, \\ \{i, j\} \in E(G) \end{cases}$$

has a non-trivial solution. In order to prove this assertion, let (A, B) be a bipartition of G. Then, for $i = 1, \ldots, r$, put

$$y_i = \begin{cases} 1 & \text{if } i \in A, \\ -1 & \text{if } i \in B. \end{cases}$$

It is obvious that $(y_1, \dots, y_r)$ is a non-trivial solution of the homogeneous system, and hence the lemma follows. $\square$

A *vertex cover* of G is a subset of $V(G)$ which meets every edge of G; a vertex cover is *minimal* if none of its proper subsets is itself a cover. For a subset $\tau = \{i_1, \dots, i_t\}$ of V, set $\mathbf{x}_\tau := x_{i_1} \cdots x_{i_t}$. Then, the *cover ideal* of G is defined by

$$J(G) := (\mathbf{x}_\tau \mid \tau \text{ is a minimal vertex cover of } G),$$

Note that $J(G)$ can be written as

$$J(G) = \bigcap_{\{u,v\} \in E(G)} (x_u, x_v), \tag{2}$$

and $J(G)$ is the Stanley-Reisner ideal corresponding with the simplicial complex

$$\Delta(J(G)) = \langle V \setminus e \mid e \in E \rangle . \tag{3}$$

When G is bipartite graph, the cover ideal $J(G)$ is normally torsion-free, i.e. $J(G)^{(t)} = J(G)^t$ for all $t \geqslant 1$ by [15, Corollary 2.6]. Therefore, Lemma 1.2 can be written as follows.

Lemma 1.5. *Let $G = (V, E)$ be a bipartite graph with vertex set $V = \{1, \dots, r\}$ and the edge set E. For every $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$ and $t \geqslant 1$, we have*

$$\Delta_{\boldsymbol{\alpha}}(J(G)^t) = \langle V \setminus \{u, v\} \mid \{u, v\} \in E \text{ and } \alpha_u + \alpha_v \leqslant t - 1 \rangle .$$

2. Integer polytopes

For a vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$, we set $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_r$ and for a nonempty bounded closed subset S of $\mathbb{R}^r$ we set

$$\delta(S) := \max\{|\boldsymbol{\alpha}| \mid \boldsymbol{\alpha} \in S\}.$$

Let $G = (V, E)$ be a bipartite graph on the vertex set $V = \{1, \dots, r\}$, and edge set E. Assume that

$$H^i_{\mathfrak{m}}(R/J(G)^t)_{\boldsymbol{\beta}} \neq 0$$

for some $i \geqslant 0$, $t \geqslant 1$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_r) \in \mathbb{N}^r$.

By Lemma 1.1 we have

$$\dim_K \widetilde{H}_{i-1}(\Delta_{\boldsymbol{\beta}}(J(G)^t); K) = \dim_K H^i_{\mathfrak{m}}(R/J(G)^t)_{\boldsymbol{\beta}} \neq 0. \tag{4}$$

In particular, $\Delta_{\boldsymbol{\beta}}(J(G)^t)$ is not acyclic.

Let $E = \{e_1, \dots, e_n\}$ where $n \geqslant 1$. Then, by Equation (3)

$$\Delta(J(G)) = \langle V \setminus e_1, \dots, V \setminus e_n \rangle .$$

Since $\Delta_{\boldsymbol{\beta}}(J(G)^t)$ is not acyclic, by Lemma 1.5 we may assume that

$$\Delta_{\boldsymbol{\beta}}(J(G)^t) = \langle V \setminus e_1, \dots, V \setminus e_k \rangle$$

where $1 \leqslant k \leqslant n$.

For each integer $t \geqslant 1$, let $\mathcal{P}_t$ be the set of solutions in $\mathbb{R}^r$ of the following system:

$$
\begin{cases} x_u + x_v \leqslant t-1 & \text{for } \{u,v\} \in E_1,\\ x_u + x_v \geqslant t & \text{for } \{u,v\} \in E_2,\\ x_1 \geqslant 0, \ldots, x_r \geqslant 0, \end{cases} \tag{5}
$$

where $E_1 = \{e_1, \ldots, e_k\},\ E_2 = \{e_{k+1}, \ldots, e_n\},\ 1 \leqslant k \leqslant n$. Then, $\boldsymbol{\beta} \in \mathcal{P}_t$. Moreover, by Lemma 1.5 one has

$$\Delta_{\boldsymbol{\alpha}}(J(G)^s) = \langle V \setminus e_1, \ldots, V \setminus e_k \rangle = \Delta_{\boldsymbol{\beta}}(J(G)^t) \ \text{ whenever } \boldsymbol{\alpha} \in \mathcal{P}_s \cap \mathbb{N}^r\,.$$

In order to study the set $\mathcal{P}_t$, we consider $\mathcal{C}_t$ to be the set of solutions in $\mathbb{R}^r$ of the following system:

$$
\begin{cases} x_u + x_v \leqslant t & \text{for } \{u,v\} \in E_1,\\ x_u + x_v \geqslant t & \text{for } \{u,v\} \in E_2,\\ x_1 \geqslant 0, \ldots, x_r \geqslant 0. \end{cases} \tag{6}
$$

Note that if G is a bipartite graph, then G is a unimodular hypergraph by [5, Theorem 5, page 164]. So, we have both $\mathcal{P}_t$ and $\mathcal{C}_t$ are integer convex polyhedra by [33, Theorem 19.1], i.e. all their vertices have integral coordinates. Especially, we have:

Lemma 2.1. *$\mathcal{C}_1$ is a polytope with $\dim \mathcal{C}_1 = r$. Moreover, if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$ is a vertex of $\mathcal{C}_1$, then $\boldsymbol{\alpha} \in \{0,1\}^r$.*

Proof. By [19, Lemma 2.1], we imply that $\mathcal{C}_1$ is a polytope with $\dim \mathcal{C}_1 = r$.

Let $\boldsymbol{\alpha}$ is a vertex of $\mathcal{C}_1$, by [33, Formula 23 in Page 104], $\boldsymbol{\alpha}$ is the unique solution of a system of linear equations of the form

$$
\begin{cases} x_u + x_v = 1 & \text{for } \{u,v\} \in S_1,\\ x_j = 0, & \text{for } j \in S_2, \end{cases} \tag{7}
$$

where $S_1 \subseteq E_1 \cup E_2,\ S_2 \subseteq \{1, \ldots, r\}$ and $|S_1| + |S_2| = r$.

Note that the incidence matrix $A(G)$ of G is totally unimodular as G is bipartite. It follows that the matrix of the System (7) is also totally unimodular. By [33, Theorem 2.17], we have $\boldsymbol{\alpha}$ is a $\{0,1\}-$vector, as required. $\square$

Remark 2.2. Since $\mathcal{C}_t = tC_1$, $\mathcal{C}_t$ is also a polytope. Observe that $\mathcal{P}_t \subseteq \mathcal{C}_t$, so $\mathcal{P}_t$ is a polytope as well.

Since $\mathcal{C}_1$ is a polytope of dimension r, there is a vertex $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_r)$ of $\mathcal{C}_1$ such that

$$\delta(\mathcal{C}_1) = |\boldsymbol{\gamma}| = \gamma_1 + \cdots + \gamma_r.$$

Let $a := |\boldsymbol{\gamma}| = \delta(\mathcal{C}_1)$, we have $a \geqslant 1$. Note that $t\boldsymbol{\gamma}$ is also a vertex of $\mathcal{C}_t$ and $\delta(\mathcal{C}_t) = at$. Since $\mathcal{P}_t \subseteq \mathcal{C}_t$, we have $\delta(P_t) \leqslant at$, so we can write

$$\delta(P_t) = at - b_t \text{ for some integer } b_t \geqslant 0.$$

Lemma 2.3. *If $\mathcal{P}_t \neq \emptyset$, then $\mathcal{P}_{t+1} \neq \emptyset$ and $b_t \geqslant b_{t+1}$.*

Proof. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathcal{P}_t$ such that $\delta(\mathcal{P}_t) = |\boldsymbol{\alpha}|$. Since $\boldsymbol{\alpha}$ is a solution of the System (5), and $\boldsymbol{\gamma}$ is a solution of the System (6) with $t = 1$, by Lemma 2.1 we have $\boldsymbol{\gamma} \in \{0,1\}^r$. Let $\boldsymbol{\alpha} + \boldsymbol{\gamma} = \boldsymbol{\theta} = (\theta_1, \dots, \theta_r)$, we deduce that

$$\begin{cases} \theta_u + \theta_v = (\alpha_u + \alpha_v) + (\gamma_u + \gamma_v) \leqslant t - 1 + 1 = t & \text{for } \{u,v\} \in E_1, \\ \theta_u + \theta_v = (\alpha_u + \alpha_v) + (\gamma_u + \gamma_v) \geqslant t + 1 & \text{for } \{u,v\} \in E_2. \end{cases}$$

In other words, $\boldsymbol{\theta} \in \mathcal{P}_{t+1}$. Therefore, $\mathcal{P}_{t+1} \neq \emptyset$ and $\delta(P_{t+1}) \geqslant |\alpha| + |\gamma|$. Since $\delta(\mathcal{P}_{t+1}) = a(t+1) - b_{t+1}$ and $|\boldsymbol{\alpha}| + |\boldsymbol{\gamma}| = a(t+1) - b_t$, we have $b_t \geqslant b_{t+1}$. $\square$

Let m be the length of a longest simple path in G. In the following lemma, we show that $\delta(\mathcal{P}_t)$ is a linear function in t for all $t \geqslant \dfrac{m+1}{2}$.

Lemma 2.4. *There exists a non-negative integer b such that*

$$\delta(\mathcal{P}_t) = \delta(C_1)t - b \textit{ for all } t \geqslant \frac{m+1}{2}.$$

Proof. Let $a = \delta(\mathcal{C}_1)$. For $t \geqslant 1$ with $P_t \neq \emptyset$, represent $\delta(P_t) = at - b_t$ where b_t is an integer. By Lemma 2.3 we have $b_t \geqslant b_{t+1} \geqslant \cdots \geqslant 0$. It follows that there is $t_0 \geqslant 1$ such that $b_t = b_{t_0}$ for $t \geqslant t_0$. Let $b := b_{t_0}$. Then,

$$\delta(\mathcal{P}_t) = at - b, \text{ for all } t \geqslant t_0.$$

By Lemma 2.3 again, we deduce that

$$\delta(\mathcal{P}_t) \leqslant at - b, \tag{8}$$

whenever $\mathcal{P}_t \neq \emptyset$.

Let s be an integer such that $s \geqslant \max\{2r^2 + b, t_0\}$. Then, we have

$$\delta(\mathcal{P}_s) = as - b.$$

Since $\mathcal{P}_s$ is a polytope, then $\delta(\mathcal{P}_s) = |\boldsymbol{\alpha}|$ for some vertex $\boldsymbol{\alpha}$ of $\mathcal{P}_s$. Note that the polytope $\mathcal{P}_s$ is defined by the following system

$$\begin{cases} x_u + x_v \leqslant s - 1 & \text{for } \{u,v\} \in E_1 \\ x_u + x_v \geqslant s & \text{for } \{u,v\} \in E_2, \\ x_1 \geqslant 0, \dots, x_r \geqslant 0. \end{cases}$$

By [33, Formula 23 in Page 104], $\boldsymbol{\alpha}$ is the unique solution of a system of linear equations of the form

$$\begin{cases} x_u + x_v = s - 1 & \text{for } \{u,v\} \in S_1, \\ x_u + x_v = s & \text{for } \{u,v\} \in S_2, \\ x_t = 0, & \text{for } t \in S_3, \end{cases} \tag{9}$$

where $S_1 \subseteq E_1,\ S_2 \subseteq E_2,\ S_3 \subseteq [r]$ such that $|S_1| + |S_2| + |S_3| = r$.

Let H be the subgraph of G with $V(H) = V(G)$ and $E(H) = S_1 \cup S_2$. Let

$$H_1, \ldots, H_p$$

be connected components of H.

We next prove following claims:

Claim 1: H_i is a tree and $|V(H_i) \cap S_3| = 1$ for each $i = 1, \ldots, p$.

Indeed, since the System (9) has unique solution, it implies that the system

$$\begin{cases} x_u + x_v = s - 1 & \text{for } \{u, v\} \in S_1 \cap V(H_i), \\ x_u + x_v = s & \text{for } \{u, v\} \in S_2 \cap V(H_i), \\ x_t = 0, & \text{for } t \in S_3 \cap V(H_i), \end{cases} \tag{10}$$

also has unique solution. In particular, the number of equations equals the number of variables, i.e.

$$|V(H_i)| = |E(H_i)| + |S_3 \cap V(H_i)|. \tag{11}$$

Note that $S_3 \cap V(H_i) \neq \emptyset$ by Lemma 1.4. Together this fact with the Equality (11) and Lemma 1.3, we imply that

$$|E(H_i)| = |V(H_i)| - 1, \ |S_3 \cap V(H_i)| = 1, \text{ and } H_i \text{ is a tree},$$

as claimed.

From Claim 1, for $i = 1, \ldots, p$, denote the unique vertex in $V(H_i) \cap S_3$ by u_i. Since H_i is a tree, for every vertex v of H_i, there is a unique simple path in H_i from v to u_i, and we assume that this path is of the form

$$u_i = v_0, v_1, \ldots, v_n = v,$$

where $n = \mathrm{dist}_{H_i}(v, u_i)$ is the distance between v and u_i.

From the system (10) we have

$$\alpha_{v_{j-1}} + \alpha_{v_j} = s - \epsilon_j, \quad \text{for } j = 1, \ldots, n,$$

where

$$\epsilon_j = \begin{cases} 1 & \text{if } \{v_{j-1}, v_j\} \in E(H_i) \cap S_1 \\ 0 & \text{if } \{v_{j-1}, v_j\} \in E(H_i) \cap S_2. \end{cases}$$

Let $a_v = \sum_{k=0}^{n} (-1)^{k+1} \epsilon_k$ where we make a convention that $\epsilon_0 = 0$.

Claim 2. For every vertex v of H_i, we have $0 \leqslant a_v \leqslant \lceil \mathrm{dist}(v, u_i)/2 \rceil$ and

$$\alpha_v = \begin{cases} a_v & \text{if } \mathrm{dist}(v, u_i) \text{ is even}, \\ s - a_v & \text{if } \mathrm{dist}(v, u_i) \text{ is odd}. \end{cases}$$

Indeed, for each $m = 0, \ldots, n$, put

$$b_m = \sum_{k=0}^{m} (-1)^{k+1} \epsilon_k.$$

Then, $a_v = b_n$. In order to prove the claim, it suffices to show that

$$0 \leqslant b_{2l} \leqslant l, \text{ and } 0 \leqslant b_{2l+1} \leqslant l+1,$$

and

$$\alpha_{v_{2l}} = b_{2l}, \quad \text{and } \alpha_{v_{2l+1}} = s - b_{2l+1},$$

whenever the indices do not exceed n.

We proceed by induction on l. If $l = 0$, we have $b_0 = 0$ and $\alpha_{v_0} = \alpha_{u_i} = 0$ since $u_i \in S_3$. Note that $b_1 = \epsilon_1 \in \{0, 1\}$, so that $0 \leqslant b_1 \leqslant 1$. On the other hand, since

$$\alpha_{v_1} + \alpha_{v_0} = s - \epsilon_1$$

one has $\alpha_{v_1} = s - \epsilon_1 = s - b_1$, and the case $l = 0$ holds.

Assume that $l \geqslant 1$. By the induction hypothesis, $0 \leqslant b_{2l-1} \leqslant l$ and $\alpha_{2l-1} = s - b_{2l-1}$. From the equation $\alpha_{2l-1} + \alpha_{2l} = s - \epsilon_{2l}$, we have

$$\alpha_{2l} = s - \epsilon_{2l} - (s - b_{2l-1}) = b_{2l-1} - \epsilon_{2l} = b_{2l}.$$

Since $b_{2l-1} \leqslant l$ by the induction hypothesis, we get $b_{2l} \leqslant l$. On the other hand, since $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in P_s$, we have $\alpha_{2l} \geqslant 0$, and so $b_{2l} \geqslant 0$.

From the equation $\alpha_{2l} + \alpha_{2l+1} = s - \epsilon_{2l+1}$, we have

$$\alpha_{2l+1} = s - \epsilon_{2l+1} - b_{2l} = s - (\epsilon_{2l+1} + b_{2l}) = s - b_{2l+1}.$$

Note that $0 \leqslant b_{2l} \leqslant l$, so $0 \leqslant b_{2l+1} \leqslant l+1$, and the claim follows.

For each $t \geqslant (m+1)/2$, we consider the integer point $\boldsymbol{\beta}(t) = (\beta_1(t), \dots, \beta_r(t)) \in \mathbb{Z}^r$ where

$$\beta_v(t) = \begin{cases} a_v & \text{if } v \in H_i \text{ and } \operatorname{dist}(v, u_i) \text{ is even,} \\ t - a_v & \text{if } v \in H_i \text{ and } \operatorname{dist}(v, u_i) \text{ is odd.} \end{cases}$$

Then, $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}$ by Claim 2.

Claim 3: $\boldsymbol{\beta}(t) \in P_t$ for all $t \geqslant (m+1)/2$.

Firstly, we show that $\boldsymbol{\beta}(t) \in \mathbb{N}^r$. By Claim 2, it suffices to show that $\beta_v(t) \geqslant 0$ if $v \in V(H_i)$ and $\operatorname{dist}(v, u_i)$ is odd for some $i = 1, \dots, p$. In this case, $\beta_v(t) = t - a_v$. By Claim 2 again, $a_v \leqslant \lceil \operatorname{dist}_{H_i}(v, u_i) \rceil \leqslant (m+1)/2$, and thus $\beta_v(t) \geqslant 0$.

Secondly, we prove that $\beta_u(t) + \beta_v(t) \leqslant t - 1$ for $\{u, v\} \in E_1$. We may assume that $u \in V(H_i)$ and $v \in V(H_j)$. We now consider three possible cases:

Case 1: $\operatorname{dist}(u, u_i)$ and $\operatorname{dist}(v, u_j)$ are even. If $i = j$, there are two even paths from u and v to u_i, respectively. Since $\{u, v\}$ is an edge of G, we deduce that G contains an odd cycle, which contradicts the fact that G is bipartite. Thus, $i \neq j$. In this case $\beta_u(t) = a_u$ and $\beta_v(t) = a_v$, so that

$$\beta_u(t) + \beta_v(t) = a_u + a_v \leqslant \frac{\operatorname{dist}_{H_i}(u, u_i)}{2} + \frac{\operatorname{dist}_{H_j}(v, u_j)}{2} = \frac{\operatorname{dist}_{H_i}(u, u_i) + \operatorname{dist}_{H_j}(v, u_j)}{2}.$$

If we have a simple path, say p_1 in H_i from u_i to u, and a simple path, say p_2, in H_j from v to u_j, then we have a simple path

$$p_1, u, v, p_2$$

from u_i to u_j in G. This implies that $\operatorname{dist}_{H_i}(u,u_i)+\operatorname{dist}_{H_j}(v,u_j)\leqslant m-1$. Together with the inequality above, it gives

$$\beta_u(t)+\beta_v(t)\leqslant \frac{m-1}{2}\leqslant t-1.$$

Case 2: $\operatorname{dist}(u,u_i)$ is even and $\operatorname{dist}(v,u_j)$ is odd. In this case, by Claim 2, one has

$$\alpha_u+\alpha_v=s+a_u-a_v\leqslant s-1,$$

hence $a_u-a_v\leqslant -1$. It follows that $\beta_u(t)+\beta_v(t)=t+a_u-a_v\leqslant t-1$.

Case 3: $\operatorname{dist}(u,u_i)$ is odd and $\operatorname{dist}(v,u_j)$ is even. In this case the proof is similar to the previous case.

Case 4: $\operatorname{dist}(u,u_i)$ and $\operatorname{dist}(v,u_j)$ are odd. In this case, by Claim 2 we have

$$\alpha_u+\alpha_v=2s-a_u-a_v\leqslant s-1.$$

But this is not true, since $s\geqslant 2r$ and $a_u\leqslant r-1$ and $a_v\leqslant r-1$.

Therefore, we have proven that $\beta_u(t)+\beta_v(t)\leqslant t-1$ for $\{u,v\}\in E_1$.

Similarly, we can verify $\beta_u(t)+\beta_v(t)\geqslant t$ for $\{u,v\}\in E_2$.

In summary, we have $\boldsymbol{\beta}(t)\in\mathcal{P}_t$ for $t\geqslant (m+1)/2$, and the claim follows.

Claim 4: $|\boldsymbol{\beta}(t)|=at-b$.

Indeed, by Claim 2 we have

$$|\boldsymbol{\alpha}|=gs+h$$

where g is the number vertex v such that $v\in H_i$ with $\operatorname{dist}_{H_i}(v,u_i)$ is odd for some i, and h is the sum of all a_v for which $v\in H_i$ with $\operatorname{dist}_{H_i}(v,u_i)$ is even for some i minus the sum of all a_u for which $u\in H_j$ with $\operatorname{dist}_{H_j}(v,u_j)$ is odd for some j. It is obvious that $0\leqslant g\leqslant r$ and $|h|\leqslant r^2$. On the other hand, $|\boldsymbol{\alpha}|=as-b$, and hence $as-b=gs+h$, or equivalently

$$(a-g)s=h+b.$$

Note that, $s\geqslant 2r^2+b>|h|+b$. Together with the equality above, it forces $a=g$ and $h=-b$. By the definition of $\boldsymbol{\beta}(t)$, we obtain $|\boldsymbol{\beta}(t)|=gt+h=at-b$, as claimed.

We now turn to prove the lemma. For any $t\geqslant (m+1)/2$, by Claim 3, we have $\boldsymbol{\beta}(t)\in\mathcal{P}_t$, and so $\delta(P_t)\geqslant|\boldsymbol{\beta}(t)|$. Together with Claim 4, it yields $\delta(P_t)\geqslant at-b$.

On the other hand, because $\mathcal{P}_t\neq\emptyset$, we have $\delta(\mathcal{P}_t)\leqslant at-b$ by (8). Hence, $\delta(\mathcal{P}_t)=at-b$ for all $t\geqslant (m+1)/2$, and the proof is complete. □

3. Regularity of powers of cover ideals

Let G be a bipartite graph. In this section we will give a bound for t_0 such that $\operatorname{reg} J(G)^t$ is a linear function in t for $t\geqslant t_0$. We start with the following lemma.

Lemma 3.1. *For every $s\geqslant (m+1)/2$, there are non-negative integers a and b such that*

(1) $\operatorname{reg} J^s=a(s-1)+b$.
(2) $b\leqslant\max\{\operatorname{reg} J(H)\mid H$ *is a subgraph of* $G\}$.

(3) $\operatorname{reg} J^t \geqslant a(t-1)+b$ *for every* $t \geqslant (m+1)/2$.

Proof. If $E(G)$ has just one edge, then we have $\operatorname{reg} J(G)^s = s$ for all $s \geqslant 1$. The lemma holds true in this case, so we assume that $E(G)$ has at least two edges.

For simplicity, we set $J = J(G)$. For any $s \geqslant (m+1)/2$, assume that

$$\operatorname{reg}(R/J^s) = a_i(R/J^s) + i, \text{ for some } 0 \leqslant i \leqslant \dim(R/J),$$

and

$$a_i(R/J^s) = |\boldsymbol{\alpha}| \text{ where } \boldsymbol{\alpha} \in \mathbb{Z}^r \text{ such that } H^i_{\mathfrak{m}}(R/J^s)_{\boldsymbol{\alpha}} \neq 0.$$

By Lemma 1.1, we have

$$\dim_K \widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}}(J^s); K) = \dim_K H^i_{\mathfrak{m}}(R/J^s)_{\boldsymbol{\alpha}} \neq 0. \tag{12}$$

In particular, $\Delta_{\boldsymbol{\alpha}}(J^s)$ is not acyclic.

If $G_{\boldsymbol{\alpha}} = [r]$, then $\Delta_{\boldsymbol{\alpha}}(J^s)$ is either $\{\emptyset\}$ or a void complex. Because it is not acyclic, $\Delta_{\boldsymbol{\alpha}}(J^s) = \{\emptyset\}$. By Formula (1) we deduce that J^s is an $\mathfrak{m}$-primary ideal of R. This imply that G consists of exactly one edge, a contradiction.

Therefore, we may assume that $G_{\boldsymbol{\alpha}} = \{n+1, \ldots, r\}$ for some $1 \leqslant n \leqslant r$. Note that, if we consider the point

$$\boldsymbol{\beta} = (\alpha_1, \ldots, \alpha_n, -1, \ldots, -1) \in \mathbb{Z}^r.$$

Then, $\Delta_{\boldsymbol{\alpha}}(J^s) = \Delta_{\boldsymbol{\beta}}(J^s)$ by Formula (1). Together with Lemma 1.1 and Formula (12) we get

$$H^i_{\mathfrak{m}}(R/J^s)_{\boldsymbol{\beta}} \neq 0.$$

In particular, $a_i(R/J^s) \geqslant |\boldsymbol{\beta}|$. Obviously, $\alpha_j \leqslant \beta_j$ for $j = n+1, \ldots, r$, so that $\alpha_j = -1$ for $j = n+1, \ldots, r$. Thus, $a_i(R/J^s) = |\boldsymbol{\alpha}'| - |G_{\boldsymbol{\alpha}}|$ where $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and thus

$$\operatorname{reg}(R/J^s) = |\boldsymbol{\alpha}'| + i - |G_{\boldsymbol{\alpha}}|. \tag{13}$$

Let $S = K[x_1, \ldots, x_n]$ and let G' be the graph on the vertex set $V' = \{1, \ldots, n\}$ with the edge set $E' = \{e \in E \mid e \subseteq V'\}$. By Formula (1) we obtain:

$$JR_{G_{\boldsymbol{\alpha}}} \cap S = J(G') =: J'. \tag{14}$$

By using Formulae (1) and (14) we get

$$\Delta_{\boldsymbol{\alpha}'}(J'^k) = \Delta_{\boldsymbol{\alpha}}(J^k) \text{ for any } k \geqslant 1,$$

and thus $\Delta_{\boldsymbol{\alpha}'}(J'^k)$ is not acyclic.

Then, by Equation (3) we have

$$\Delta(J') = \langle V' \setminus e \mid e \in E' \rangle.$$

By Lemma 1.5 we can partition $E' = E_1 \cup E_2$ such that

$$\Delta_{\boldsymbol{\alpha}'}(J'^s) = \langle V' \setminus e \mid e \in E_1 \rangle.$$

For each integer $t \geqslant 1$, let $\mathcal{P}_t$ be the set of solutions in $\mathbb{R}^n$ of the following system:

$$\begin{cases} x_u + x_v \leqslant t-1 & \text{for } \{u,v\} \in E_1,\\ x_u + x_v \geqslant t & \text{for } \{u,v\} \in E_2,\\ x_1 \geqslant 0, \ldots, x_n \geqslant 0. \end{cases}$$

Then $\boldsymbol{\alpha}' \in \mathcal{P}_s$.

Recall the the associated polytope $\mathcal{C}_t$ of $\mathcal{P}_t$ is defined by

$$\begin{cases} x_u + x_v \leqslant t & \text{for } \{u,v\} \in E_1,\\ x_u + x_v \geqslant t & \text{for } \{u,v\} \in E_2,\\ x_1 \geqslant 0, \ldots, x_n \geqslant 0. \end{cases}$$

Let $a = \delta(\mathcal{C}_1)$. We now prove that

$$\delta(\mathcal{P}_t) \leqslant a(t-1), \text{ whenever } \mathcal{P}_t \neq \emptyset.$$

Indeed, let $\boldsymbol{\beta}' = (\beta_1, \ldots, \beta_n)$ be a vertex of $\mathcal{P}_t$ such that $\delta(\mathcal{P}_t) = |\boldsymbol{\beta}'|$. Let

$$\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) = \frac{1}{t-1}\boldsymbol{\beta}' \in \mathbb{R}^n\,.$$

By Lemma 1.2 again, we obtain

$$\begin{cases} \sigma_u + \sigma_v = \frac{1}{t-1}(\beta_u + \beta_v) \leqslant 1 & \text{for } \{u,v\} \in E_1,\\ \sigma_u + \sigma_v = \frac{1}{t-1}(\beta_u + \beta_v) \geqslant \frac{t}{t-1} > 1 & \text{for } \{u,v\} \in E_2. \end{cases}$$

Clearly, all coordinates of $\boldsymbol{\sigma}$ are non-negative, so that $\boldsymbol{\sigma} \in C_1$. In particular, $|\boldsymbol{\sigma}| \leqslant \delta(C_1) = a$, hence

$$\delta(\mathcal{P}_t) = |\boldsymbol{\beta}'| = |\boldsymbol{\sigma}|(t-1) \leqslant a(t-1),$$

and the assertion follows.

Together with Lemma 2.4, it implies that there is a non-negative integer f such that for all $t \geqslant (m+1)/2$ we have

$$\delta(\mathcal{P}_t) = a(t-1) - f. \tag{15}$$

Next we observe that for an integer point $\boldsymbol{\beta}' = (\beta_1, \ldots, \beta_n) \in \mathcal{P}_t \cap \mathbb{N}^n$, let

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n, -1, \ldots, -1) \in \mathbb{Z}^r\,.$$

Then,

$$\Delta_{\boldsymbol{\beta}}(J^t) = \Delta_{\boldsymbol{\beta}'}(J'^t) = \Delta_{\boldsymbol{\alpha}'}(J'^s) = \Delta_{\boldsymbol{\alpha}}(J^s).$$

Together with the Equation (12) and Lemma 1.1, it yields

$$\dim_K H^i_{\mathfrak{m}}(R/J^t)_{\boldsymbol{\beta}} = \dim_K H^i_{\mathfrak{m}}(R/J^s)_{\boldsymbol{\alpha}} \neq 0.$$

In particular, $a_i(R/J^t) \geqslant |\boldsymbol{\beta}| = |\boldsymbol{\beta}'| - |G_{\boldsymbol{\alpha}}|$. It follows that for all $t \geqslant (m+1)/2$ we have

$$\operatorname{reg}(R/J^t) \geqslant \delta(\mathcal{P}_t) + i - |G_{\boldsymbol{\alpha}}| = a(t-1) + i - |G_{\boldsymbol{\alpha}}| - f. \tag{16}$$

Since $\boldsymbol{\alpha}' \in \mathcal{P}_s$, from (13), (15) and (16) we get

$$\operatorname{reg}(R/J^s) = \delta(\mathcal{P}_s) + i - |G_{\boldsymbol{\alpha}}| = a(s-1) + i - |G_{\boldsymbol{\alpha}}| - f. \tag{17}$$

Note that for any non-zero homogeneous ideal I of R, we have $\operatorname{reg}(I) = \operatorname{reg}(R/I)+1$. Thus, from (16) and (17), it remains to prove that $b \leqslant e$ where $b = i - |G_{\boldsymbol{\alpha}}| - f + 1$ and $e = \max\{\operatorname{reg}(J(H)) \mid H$ is a subgraph of $G\}$. By the same argument as the last part in the proof of Theorem 2.3 [22] we obtain $i - |G_{\boldsymbol{\alpha}}| + 1 \leqslant e$. It follows that $b \leqslant e$, and the proof is complete. ☐

We now in position to prove the main result of this paper.

Theorem 3.2. *Let G be a bipartite graph. Then there is a non-negative integer b with $0 \leqslant b \leqslant e - d(J(G))$, such that*

$$\operatorname{reg} J(G)^t = d(J(G))t + b \quad \textit{for all } t \geqslant \max\left\{\frac{m+1}{2}, e - b - d(J(G)) + 1\right\},$$

where m is the length of a longest simple path in G and $e = \max\{\operatorname{reg}(J(H)) \mid H$ is a subgraph of $G\}$.

Proof. For simplicity, let $J = J(G)$ and $d = d(J)$. We will prove the theorem in the following equivalent form: there is a non-negative integer b with $d \leqslant b \leqslant e$, such that

$$\operatorname{reg} J(G)^t = d(t-1) + b \quad \text{for all } t \geqslant \max\left\{\frac{m+1}{2}, e - b + 1\right\}.$$

We now prove this assertion. It is well-known that there are integers $b \geqslant d$ and $s_0 \geqslant 1$ such that

$$\operatorname{reg}(J^s) = d(s-1) + b \text{ for all } s \geqslant s_0. \tag{18}$$

Since J is torsion-free, by [22, Theorem 2.3] we get $b \leqslant e$, and note also that $e \leqslant r - 1$.

We now claim that

$$\operatorname{reg}(J^s) \geqslant d(s-1) + b \text{ for all } s \geqslant (m+1)/2. \tag{19}$$

Indeed, let $t \geqslant \max\{s_0, r+1\}$. By Lemma 3.1 again, there are two non-negative integers d_1 and b_1 with $b_1 \leqslant e$ such that

$$\operatorname{reg}(J^t) = d_1(t-1) + b_1,$$

and

$$\operatorname{reg}(J^s) \geqslant d_1(s-1) + b_1 \text{ for all } s \geqslant (m+1)/2.$$

Thus, it suffices to show that $d = d_1$ and $b = b_1$. From the equalities

$$\operatorname{reg}(J^t) = d(t-1) + b = d_1(t-1) + b_1,$$

we have $(d - d_1)(t-1) = (b_1 - b)$. Since $|b_1 - b| \leqslant \max\{b, b_1\} \leqslant e$ and $t - 1 \geqslant r > e$, it follows that $d = d_1$ and hence $b = b_1$, as claimed.

Now, for any $t \geqslant \max\{(m+1)/2, e-b+1\}$. We will prove that $\operatorname{reg}(J^t) = d(t-1)+b$. Indeed, by Lemma 3.1 again, there are integers $\delta \geqslant 0$ and $c \leqslant e$ such that

$$\operatorname{reg}(J^t) = \delta(t-1) + c. \tag{20}$$

and

$$\operatorname{reg}(J^s) \geqslant \delta(s-1) + c \text{ for all } s \geqslant (m+1)/2. \tag{21}$$

From Equation (18) and Inequality (21) we deduce that $\delta \leqslant d$. We now consider two cases:

Case 1: $\delta = d$. From Equation (18) and Inequality (21) we get $b \geqslant c$. On the other hand, from Inequality (19) and Equation (20), we get $b \leqslant c$. Therefore, $b = c$. By Equation (20) we obtain $\operatorname{reg}(J^t) = d(t-1) + b$.

Case 2: $\delta < d$. From Inequalities (19) and Equation (20), we have

$$\operatorname{reg}(J^t) = \delta(t-1) + c \geqslant d(t-1) + b,$$

and so $(d-\delta)(t-1) \leqslant c - b$. It follows that $t - 1 \leqslant c - b$. By the definition of t, we deduce that $t = c - b + 1$ and hence $d - \delta = 1$. In this case we have

$$\operatorname{reg} J^t = \delta(t-1) + c = d(t-1) + b,$$

and the proof of the theorem is complete. $\square$

The following corollary improves [18, Corollary 3.4] significantly.

Corollary 3.3. *Let G be a bipartite graph with r vertices. Then, there is a non-negative integer $b \leqslant r - d(J(G)) - 1$ such that*

$$\operatorname{reg} J(G)^t = d(J(G))t + b, \text{ for all } t \geqslant \frac{r}{2}.$$

Proof. By Theorem 3.2, there is a non-negative integer b with $0 \leqslant b \leqslant e - d(J(G))$, such that

$$\operatorname{reg} J(G)^t = d(J(G))t + b \quad \text{for all } t \geqslant \max\left\{\frac{m+1}{2}, e - b - d(J(G)) + 1\right\},$$

where m is the length of a longest simple path in G and $e = \max\{\operatorname{reg}(J(H)) \mid H \text{ is a subgraph of } G\}$.

For any graph H with $V(H) \subseteq \{1, \ldots, r\}$ and $E(H) \neq \emptyset$, we have

$$\operatorname{reg}(J(H)) = \operatorname{reg}(R/J(H)) + 1 \leqslant \dim(R/J(H)) + 1 = r - 1.$$

Thus, $e \leqslant r - 1$, and thus $b \leqslant r - 1 - d(J(G))$.

It remains to show that

$$\max\left\{\frac{m+1}{2}, e - b - d(J(G)) + 1\right\} \leqslant \frac{r}{2}.$$

In order to prove this, let (X, Y) be a bipartition of G. Then both X and Y are minimal vertex covers of G. Because $d(J(G))$ is the maximal cardinal of minimal vertex covers of G, we have

$$d(J(G)) \geqslant \max\{|X|, |Y|\} \geqslant r/2.$$

Therefore, $e-b-d(J(G))+1 \leqslant r-1-d(J(G))+1 \leqslant r-r/2=r/2$.

Finally, observe that $m \leqslant r-1$, thus $(m+1)/2 \leqslant r/2$, and therefore

$$\max\left\{\frac{m+1}{2}, e-b-d(J(G))+1\right\} \leqslant \frac{r}{2},$$

and the corollary follows. □

Remark 3.4. The bound of b in Theorem 3.2 is sharp (see [13]).

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