On the complete convergence for sequences of random vectors in Hilbert spaces

Nguyen Van Huan

Abstract

Let $\{X_n, n \geqslant 1\}$ be a sequence of coordinatewise negatively associated random vectors taking values in a real separable Hilbert space with the k-th partial sum S_k, $k \geqslant 1$. We provide conditions for the convergence of $\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(\max_{1\leqslant k\leqslant n} \|S_k\| > \varepsilon n^{\alpha})$ and $\sum_{n=1}^{\infty} \frac{\log n}{n} \mathbb{P}(\max_{1\leqslant k\leqslant n} \|S_k\| > \varepsilon n^{\alpha})$ for all $\varepsilon > 0$. The converses of these results are also discussed.

1 Introduction

The concept of negative association for random variables was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [8].

Definition 1.1. A finite family $\{Y_i, 1 \leqslant i \leqslant n\}$ of random variables is said to be *negatively associated* (NA) if for any disjoint subsets A, B of $\{1, 2, ..., n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^{|A|}$, g on $\mathbb{R}^{|B|}$,

$$\mathrm{Cov}\big(f(Y_i, i \in A), g(Y_j, j \in B)\big) \leqslant 0$$

whenever the covariance exists, where $|A|$ denotes the cardinality of A. An infinite family of random variables is NA if every finite subfamily is NA.

The concept of negative association was extended to finite dimensional random vectors and to Hilbert space valued random vectors (for details see Zhang [15], Ko et al. [10]). Let H be a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle \cdot, \cdot \rangle$, let $\{e_j, j \geqslant 1\}$ be an orthonormal basis in H, let X be an H-valued random vector, and $\langle X, e_j \rangle$ will be denoted by $X^{(j)}$. In [7], the

2010 *Mathematics Subject Classification*: 60F15, 60B12.

Key words and phrases: complete convergence, Baum-Katz theorem, coordinatewise negatively associated random vectors, Hilbert space.

authors introduced the concept of coordinatewise negative association for H-valued random vectors which is more general than the concept of negative association of Ko et al. [10].

Definition 1.2 ([7]). A sequence $\{X_n, n \geqslant 1\}$ of H-valued random vectors is said to be *coordinatewise negatively associated* (CNA) if for each $j \geqslant 1$, the sequence $\{X_n^{(j)}, n \geqslant 1\}$ of random variables is NA.

Hsu and Robbins [6] introduced the concept of complete convergence and proved that the sequence of arithmetic means of independent, identically distributed (i.i.d.) random variables converges completely to the expected value of the variables, provided their variance is finite. The necessity was proved by Erdös [4, 5]. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and was later generalized and extended during a process which led to the now classical paper by Baum and Katz [3].

Theorem 1.3 ([3]). *Let r, α be real numbers ($r > 1; \alpha > 1/2; \alpha r > 1$), and let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables with zero mean. Then the following three statements are equivalent:*

(a) $\mathbb{E}|X_1|^r < \infty$.

(b) $\sum_{n=1}^{\infty} n^{\alpha r-2}\, \mathbb{P}\Big(\Big|\sum_{k=1}^{n} X_k\Big| > \varepsilon n^{\alpha}\Big) < \infty$ *for all* $\varepsilon > 0$.

(c) $\sum_{n=1}^{\infty} n^{\alpha r-2}\, \mathbb{P}\Big(\sup_{k \geqslant n} \frac{1}{k^{\alpha}}\Big|\sum_{l=1}^{k} X_l\Big| > \varepsilon\Big) < \infty$ *for all* $\varepsilon > 0$.

The Baum-Katz theorem has been extensively studied for many classes of dependent random variables. For negatively associated random variables, we refer to Shao [13], Kuczmaszewska [11] (for sequences), Baek et al. [2], Sung [14] (for triangular arrays), Ko [9], Kuczmaszewska and Lagodowski [12] (for fields), and other authors.

In [7], the authors extended the Baum-Katz theorem to sequences of H-valued CNA random vectors for the case $r > 1/\alpha$. The aim of the present paper is to study this problem for the case $r = 1/\alpha$.

Throughout this paper, the symbol C will denote a generic positive constant which is not necessarily the same one in each appearance. The logarithms are to the base 2, for $a \in \mathbb{R}$, $\log(\max\{2; a\})$ will be denoted by $\log^+ a$.

Let $\{X, X_n, n \geqslant 1\}$ be a sequence of H-valued random vectors. We consider the following inequalities

$$C_1\, \mathbb{P}(|X^{(j)}| > t) \leqslant \frac{1}{n}\sum_{k=1}^{n} \mathbb{P}(|X_k^{(j)}| > t) \leqslant C_2\, \mathbb{P}(|X^{(j)}| > t). \tag{1.1}$$

If there exists a positive constant C_1 (C_2) such that the left-hand side (right-hand side) of (1.1) is satisfied for all $j \geqslant 1$, $n \geqslant 1$ and $t \geqslant 0$, then the sequence $\{X_n, n \geqslant 1\}$ is said to be *coordinatewise weakly lower* (*upper*) *bounded* by X. Note that (1.1) is, of course, automatic with $X = X_1$ and $C_1 = C_2 = 1$ if $\{X_n, n \geqslant 1\}$ is a sequence of identically distributed random vectors.

2 Preliminary lemmas

In this section, we give the following lemmas which will be used to prove our main results.

Lemma 2.1 ([7], Lemma 1.7). *Let $\{X_n, n \geqslant 1\}$ be a sequence of H-valued CNA random vectors with $\mathbb{E}X_n = 0$ and $\mathbb{E}\|X_n\|^2 < \infty$, $n \geqslant 1$. Then, we have*

$$\mathbb{E}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{l=1}^{k} X_l\Big\|^2\Big) \leqslant 2\sum_{k=1}^{n}\mathbb{E}\|X_k\|^2 \quad \textit{for all} \quad n \geqslant 1.$$

The proofs of the following two lemmas are quite simple and are therefore omitted.

Lemma 2.2. *Let α be a positive real number, and let X be an H-valued random vector such that $\sum\limits_{j=1}^{\infty}\mathbb{E}|X^{(j)}|^{1/\alpha} < \infty$. Then*

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{\theta\alpha}}\,\mathbb{E}\big(|X^{(j)}|^{\theta} I(|X^{(j)}| > n^{\alpha})\big) < \infty \quad \textit{if} \quad 0 \leqslant \theta < 1/\alpha;$$

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{\theta\alpha}}\,\mathbb{E}\big(|X^{(j)}|^{\theta} I(|X^{(j)}| \leqslant n^{\alpha})\big) < \infty \quad \textit{if} \quad \theta > 1/\alpha.$$

Lemma 2.3. *Let α be a positive real number, and let X be an H-valued random vector such that $\sum\limits_{j=1}^{\infty}\mathbb{E}(|X^{(j)}|^{1/\alpha}\log^{+}|X^{(j)}|) < \infty$. Then*

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{\log n}{n^{\theta\alpha}}\,\mathbb{E}\big(|X^{(j)}|^{\theta} I(|X^{(j)}| > n^{\alpha})\big) < \infty \quad \textit{if} \quad 0 \leqslant \theta < 1/\alpha; \tag{2.1}$$

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{\log n}{n^{\theta\alpha}}\,\mathbb{E}\big(|X^{(j)}|^{\theta} I(|X^{(j)}| \leqslant n^{\alpha})\big) < \infty \quad \textit{if} \quad \theta > 1/\alpha. \tag{2.2}$$

Lemma 2.4. *Let α be a positive real number, and let X be an H-valued random vector such that $\sum_{j=1}^{\infty}\mathbb{E}|X^{(j)}|^{1/\alpha} < \infty$. Then*

$$\sum_{j=1}^{\infty}\mathbb{E}\big(|X^{(j)}|^{1/\alpha} I(|X^{(j)}|^{1/\alpha} > n)\big) \to 0 \quad \textit{as } n \to \infty. \tag{2.3}$$

Proof. Set $\xi = \sum_{j=1}^{\infty} |X^{(j)}|^{1/\alpha}$. Then we have

$$\sum_{j=1}^{\infty} \mathbb{E}\big(|X^{(j)}|^{1/\alpha} I(|X^{(j)}|^{1/\alpha} > n)\big) \leqslant \mathbb{E}\big(\xi\, I(\xi > n)\big), \quad n \geqslant 1.$$

Since $\mathbb{E}\xi < \infty$, it follows that (2.3) holds. $\square$

Lemma 2.5 ([16], Lemma A.6)**.** *Suppose that the events $A_1, A_2 ..., A_n$ satisfy*

$$\mathrm{Var}\Big(\sum_{k=1}^{n} I(A_k)\Big) \leqslant \theta \sum_{k=1}^{n} \mathbb{P}(A_k) \quad \textit{for some} \quad \theta > 0.$$

Then

$$\Big(1 - \mathbb{P}\Big(\bigcup_{k=1}^{n} A_k\Big)\Big)^2 \sum_{k=1}^{n} \mathbb{P}(A_k) \leqslant \theta\, \mathbb{P}\Big(\bigcup_{k=1}^{n} A_k\Big).$$

3 Main results

With the preliminaries accounted for, the main results may now be established. In the following theorem, we state Theorem 2.1 in [7] for the case $r = 1/\alpha$. Note that we cannot prove this result by using the method in the proof of Theorem 2.1 in [7].

Theorem 3.1. *Let α be a real number $(1/2 < \alpha < 1)$, and let $\{X_n, n \geqslant 1\}$ be a sequence of H-valued CNA random vectors with zero means. Suppose that $\{X_n, n \geqslant 1\}$ is coordinatewise weakly upper bounded by a random vector X. If*

$$\sum_{j=1}^{\infty} \mathbb{E}|X^{(j)}|^{1/\alpha} < \infty, \tag{3.1}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\Big(\max_{1 \leqslant k \leqslant n} \Big\| \sum_{l=1}^{k} X_l \Big\| > \varepsilon n^{\alpha}\Big) < \infty \quad \textit{for all } \varepsilon > 0. \tag{3.2}$$

Proof. For $n, k \geqslant 1$, set

$$\begin{aligned} Y_{nk}^{(j)} &= X_k^{(j)} I(|X_k^{(j)}| \leqslant n^{\alpha}) + n^{\alpha} I(X_k^{(j)} > n^{\alpha}) - n^{\alpha} I(X_k^{(j)} < -n^{\alpha}); \\ Z_{nk}^{(j)} &= X_k^{(j)} - Y_{nk}^{(j)} \ (j \geqslant 1); \quad Y_{nk} = \sum_{j=1}^{\infty} Y_{nk}^{(j)} e_j; \quad Z_{nk} = \sum_{j=1}^{\infty} Z_{nk}^{(j)} e_j. \end{aligned}$$

Then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\Big(\max_{1 \leqslant k \leqslant n} \Big\| \sum_{l=1}^{k} X_l \Big\| > \varepsilon n^{\alpha}\Big)$$

$$
\begin{aligned}
&\leqslant \sum_{n=1}^{\infty} \frac{1}{n}\,\mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \Big\| \sum_{l=1}^{k} (Y_{nl} - \mathbb{E}Y_{nl}) \Big\| > \varepsilon n^{\alpha}/2 \Big) \\
&+ \sum_{n=1}^{\infty} \frac{1}{n}\,\mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \Big\| \sum_{l=1}^{k} (Z_{nl} - \mathbb{E}Z_{nl}) \Big\| > \varepsilon n^{\alpha}/2 \Big) \\
&= I_1 + I_2.
\end{aligned}
$$

Noting that for each $n \geqslant 1$, $\{Y_{nk}, k \geqslant 1\}$ is CNA. By the Markov inequality, Lemmas 2.1 and 2.2, we have

$$
\begin{aligned}
I_1 &\leqslant C \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha+1}}\,\mathbb{E}\Big(\max_{1\leqslant k\leqslant n} \Big\| \sum_{l=1}^{k} (Y_{nl} - \mathbb{E}Y_{nl}) \Big\| \Big)^2 \\
&\leqslant C \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha+1}} \sum_{k=1}^{n} \mathbb{E}\|Y_{nk} - \mathbb{E}Y_{nk}\|^2 \\
&\leqslant C \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha+1}} \sum_{k=1}^{n} \mathbb{E}(Y_{nk}^{(j)})^2 \\
&= C \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha+1}} \sum_{k=1}^{n} n^{2\alpha}\,\mathbb{P}(|X_k^{(j)}| > n^{\alpha}) \\
&+ C \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha+1}} \sum_{k=1}^{n} \mathbb{E}\big((X_k^{(j)})^2 I(|X_k^{(j)}| \leqslant n^{\alpha}) \big) \\
&\leqslant C \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \mathbb{P}(|X^{(j)}| > n^{\alpha}) \\
&+ C \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}}\,\mathbb{E}\big((X^{(j)})^2 I(|X^{(j)}| \leqslant n^{\alpha}) \big) \\
&+ C \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \mathbb{P}(|X^{(j)}| > n^{\alpha}) < \infty.
\end{aligned}
$$

Thus, it suffices to show that $I_2 < \infty$. Indeed, using the Markov inequality and Lemma 2.2 again, we get

$$
\begin{aligned}
I_2 &\leqslant C \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}}\,\mathbb{E}\Big(\max_{1\leqslant k\leqslant n} \Big\| \sum_{l=1}^{k} (Z_{nl} - \mathbb{E}Z_{nl}) \Big\| \Big) \\
&\leqslant C \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}}\,\mathbb{E}\Big(\sum_{k=1}^{n} \|Z_{nk} - \mathbb{E}Z_{nk}\| \Big) \\
&\leqslant C \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \sum_{k=1}^{n} \mathbb{E}\|Z_{nk}\| \\
&\leqslant C \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \sum_{k=1}^{n} \mathbb{E}|X_k^{(j)} I(|X_k^{(j)}| > n^{\alpha}) - n^{\alpha} I(X_k^{(j)} > n^{\alpha}) + n^{\alpha} I(X_k^{(j)} < -n^{\alpha})|
\end{aligned}
$$

$$\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}\mathbb{E}|X_k^{(j)}I(|X_k^{(j)}|>n^{\alpha})|+C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbb{P}(|X_k^{(j)}|>n^{\alpha})$$

$$\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}\mathbb{E}|X^{(j)}I(|X^{(j)}|>n^{\alpha})|+C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\mathbb{P}(|X^{(j)}|>n^{\alpha})<\infty.$$

The proof is completed. □

Theorem 3.1 shows that the condition (3.1) implies (3.2). However, the reverse is not true. We will now show, via an example, that under the assumptions of Theorem 3.1, (3.2) does not imply (3.1). Note that the sufficient conditions for (3.1) were provided in [7, Theorem 2.6].

Example 3.2. We consider the space ℓ_2 consisting of square summable real sequences $x=\{x_k,k\geqslant 1\}$ with norm $\|x\|=\big(\sum_{k=1}^{\infty}x_k^2\big)^{1/2}$. Let α be a real number $(1/2<\alpha<1)$, and let $\{X,X_n,n\geqslant 1\}$ be a sequence of ℓ_2-valued i.i.d. random vectors with $\mathbb{P}\big(X^{(j)}=\pm j^{-\alpha}\big)=1/2$ for all $j\geqslant 1$. It is well known that the space ℓ_2 is of type 2. Then for every $\varepsilon>0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{l=1}^{k}X_l\Big\|>\varepsilon n^{\alpha}\Big)\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n^{2\alpha+1}}\mathbb{E}\Big(\max_{1\leqslant k\leqslant n}\Big\|\sum_{l=1}^{k}X_l\Big\|\Big)^2\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n^{2\alpha+1}}\sum_{k=1}^{n}\mathbb{E}\|X_k\|^2\\
&=C\sum_{n=1}^{\infty}\frac{1}{n^{2\alpha}}\Big(\sum_{j=1}^{\infty}\frac{1}{j^{2\alpha}}\Big)<\infty,
\end{aligned}$$

so that (3.2) holds. But

$$\sum_{j=1}^{\infty}\mathbb{E}|X^{(j)}|^{1/\alpha}=\sum_{j=1}^{\infty}\frac{1}{j}=\infty,$$

and therefore (3.1) fails.

In the following theorem, we provide a variant of Theorem 3.1.

Theorem 3.3. *Let α be a real number $(1/2<\alpha\leqslant 1)$, and let $\{X_n,n\geqslant 1\}$ be a sequence of H-valued CNA random vectors with zero means. Suppose that $\{X_n,n\geqslant 1\}$ is coordinatewise weakly upper bounded by a random vector X. If*

$$\sum_{j=1}^{\infty}\mathbb{E}\big(|X^{(j)}|^{1/\alpha}\log^{+}|X^{(j)}|\big)<\infty, \tag{3.3}$$

then

$$\sum_{n=1}^{\infty} \frac{\log n}{n} \mathbb{P}\Big(\max_{1 \leqslant k \leqslant n} \Big\| \sum_{l=1}^{k} X_l \Big\| > \varepsilon n^{\alpha} \Big) < \infty \quad \textit{for all } \varepsilon > 0. \tag{3.4}$$

Proof. For $n, k \geqslant 1$, set

$$Y_{nk}^{(j)} = X_k^{(j)} I(|X_k^{(j)}| \leqslant n^{\alpha}) + n^{\alpha} I(X_k^{(j)} > n^{\alpha}) - n^{\alpha} I(X_k^{(j)} < -n^{\alpha}) \quad (j \geqslant 1);$$

$$Y_{nk} = \sum_{j=1}^{\infty} Y_{nk}^{(j)} e_j.$$

Then for every $\varepsilon > 0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\log n}{n} \mathbb{P}\Big(\max_{1 \leqslant k \leqslant n} \Big\| \sum_{l=1}^{k} X_l \Big\| > \varepsilon n^{\alpha} \Big) \\
&\leqslant \sum_{n=1}^{\infty} \frac{\log n}{n} \sum_{j=1}^{\infty} \sum_{k=1}^{n} \mathbb{P}(|X_k^{(j)}| > n^{\alpha}) \\
&+ \sum_{n=1}^{\infty} \frac{\log n}{n} \mathbb{P}\Big(\max_{1 \leqslant k \leqslant n} \Big\| \sum_{l=1}^{k} Y_{nl} \Big\| > \varepsilon n^{\alpha} \Big) \\
&\leqslant C \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \log n \, \mathbb{P}(|X^{(j)}| > n^{\alpha}) \\
&+ \sum_{n=1}^{\infty} \frac{\log n}{n} \mathbb{P}\Big(\max_{1 \leqslant k \leqslant n} \Big\| \sum_{l=1}^{k} (Y_{nl} - \mathbb{E} Y_{nl}) \Big\| > \varepsilon n^{\alpha}/2 \Big) \\
&+ \sum_{n=1}^{\infty} \frac{\log n}{n} \mathbb{P}\Big(\frac{1}{n^{\alpha}} \max_{1 \leqslant k \leqslant n} \Big\| \sum_{l=1}^{k} \mathbb{E} Y_{nl} \Big\| > \varepsilon/2 \Big) \\
&= C + J_1 + J_2 \qquad \text{(by Lemma 2.3).}
\end{aligned}$$

For J_1, by the Markov inequality, Lemma 2.1, Lemma 2.3 and the similar arguments used in proving Theorem 3.1, we obtain

$$\begin{aligned}
J_1 &\leqslant C \sum_{n=1}^{\infty} \frac{\log n}{n^{2\alpha+1}} \mathbb{E}\Big(\max_{1 \leqslant k \leqslant n} \Big\| \sum_{l=1}^{k} (Y_{nl} - \mathbb{E} Y_{nl}) \Big\| \Big)^2 \\
&\leqslant C \sum_{n=1}^{\infty} \frac{\log n}{n^{2\alpha+1}} \sum_{k=1}^{n} \mathbb{E} \| Y_{nk} - \mathbb{E} Y_{nk} \|^2 \\
&\leqslant C \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{\log n}{n^{2\alpha+1}} \sum_{k=1}^{n} \mathbb{E} (Y_{nk}^{(j)})^2 \\
&\leqslant C \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \log n \, \mathbb{P}(|X^{(j)}| > n^{\alpha})
\end{aligned}$$

$$+ C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{\log n}{n^{2\alpha}}\,\mathbb{E}\big((X^{(j)})^2 I(|X^{(j)}|\leqslant n^{\alpha})\big)<\infty.$$

In order to prove $J_2<\infty$, we will show that

$$J_{2n}:=\frac{1}{n^{\alpha}}\max_{1\leqslant k\leqslant n}\Big\|\sum_{l=1}^{k}\mathbb{E}Y_{nl}\Big\|\to 0\quad\text{as } n\to\infty.$$

Indeed,

$$\begin{aligned}
J_{2n}&\leqslant\frac{1}{n^{\alpha}}\max_{1\leqslant k\leqslant n}\sum_{j=1}^{\infty}\Big|\sum_{l=1}^{k}\mathbb{E}\big(X_l^{(j)}I(|X_l^{(j)}|\leqslant n^{\alpha})+n^{\alpha}I(X_l^{(j)}>n^{\alpha})-n^{\alpha}I(X_l^{(j)}<-n^{\alpha})\big)\Big|\\
&\leqslant\frac{1}{n^{\alpha}}\max_{1\leqslant k\leqslant n}\sum_{j=1}^{\infty}\Big|\sum_{l=1}^{k}\mathbb{E}\big(X_l^{(j)}I(|X_l^{(j)}|\leqslant n^{\alpha})\big)\Big|+\frac{1}{n^{\alpha}}\sum_{j=1}^{\infty}\sum_{k=1}^{n}n^{\alpha}\,\mathbb{P}\big(|X_k^{(j)}|>n^{\alpha}\big)\\
&\leqslant\frac{1}{n^{\alpha}}\sum_{j=1}^{\infty}\sum_{k=1}^{n}\mathbb{E}\big(|X_k^{(j)}|I(|X_k^{(j)}|>n^{\alpha})\big)+C\sum_{j=1}^{\infty}n\,\mathbb{P}\big(|X^{(j)}|>n^{\alpha}\big)\\
&\leqslant\frac{C}{n^{\alpha-1}}\sum_{j=1}^{\infty}\mathbb{E}\big(|X^{(j)}|I(|X^{(j)}|>n^{\alpha})\big)+C\sum_{j=1}^{\infty}n\,\mathbb{P}\big(|X^{(j)}|>n^{\alpha}\big)\\
&\leqslant C\sum_{j=1}^{\infty}\mathbb{E}\big(|X^{(j)}|^{1/\alpha}\,I(|X^{(j)}|^{1/\alpha}>n)\big)\to 0\quad\text{as } n\to\infty\qquad\text{(by Lemma 2.4).}
\end{aligned}$$

Combining the above arguments, this completes the proof of Theorem 3.3. □

Remark 3.4. Let α be a real number ($1/2<\alpha\leqslant 1$). We consider the sequence $\{X,X_n,n\geqslant 1\}$ in Example 3.2. By using the same arguments as in Example 3.2, we can show that (3.4) holds while (3.3) fails. Therefore, under the assumptions of Theorem 3.3, (3.4) does not imply (3.3).

The following theorem provides sufficient conditions for (3.3) to hold.

Theorem 3.5. *Let α be a positive real number, and let $\{X_n,n\geqslant 1\}$ be a sequence of H-valued CNA random vectors with zero means. Suppose that $\{X_n,n\geqslant 1\}$ is coordinatewise weakly lower bounded by a random vector X with*

$$\sum_{j=1}^{\infty}\mathbb{E}\big(|X^{(j)}|^{1/\alpha}\log^{+}|X^{(j)}|\,I(|X^{(j)}|^{1/\alpha}\leqslant 2)\big)<\infty. \tag{3.5}$$

If

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{\log n}{n}\,\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{l=1}^{k}X_l^{(j)}\Big|>\varepsilon n^{\alpha}\Big)<\infty\quad\textit{for all }\varepsilon>0, \tag{3.6}$$

then (3.3) holds.

Proof. By (3.5), we have

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\mathbb{E}\big(|X^{(j)}|^{1/\alpha}\log^{+}|X^{(j)}|\big)\\
&= C+\sum_{j=1}^{\infty}\mathbb{E}\big(|X^{(j)}|^{1/\alpha}\log^{+}|X^{(j)}|\,I(|X^{(j)}|^{1/\alpha}>2)\big)\\
&\leqslant C+\sum_{j=1}^{\infty}\sum_{k=2}^{\infty}(k+1)\log^{+}(k+1)^{\alpha}\,\mathbb{P}(k<|X^{(j)}|^{1/\alpha}\leqslant k+1)\\
&\leqslant C+C\sum_{j=1}^{\infty}\sum_{k=2}^{\infty}k\log k\,\mathbb{P}(k<|X^{(j)}|^{1/\alpha}\leqslant k+1)\\
&\leqslant C+C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Big(\sum_{n=1}^{k}\log n\Big)\,\mathbb{P}(k<|X^{(j)}|^{1/\alpha}\leqslant k+1)\\
&= C+C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\log n\,\mathbb{P}(|X^{(j)}|>n^{\alpha}).
\end{aligned}
$$

It suffices to show that

$$
\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\log n\,\mathbb{P}(|X^{(j)}|>n^{\alpha})<\infty. \tag{3.7}
$$

Noting that for all $n,j\geqslant 1$, $\{I(X_k^{(j)}>n^{\alpha}),k\geqslant 1\}$ and $\{I(X_k^{(j)}<-n^{\alpha}),k\geqslant 1\}$ are NA. Then by Lemma 2.1,

$$
\begin{aligned}
\mathrm{Var}\Big(\sum_{k=1}^{n}I(|X_k^{(j)}|>n^{\alpha})\Big)&\leqslant 2\mathrm{Var}\Big(\sum_{k=1}^{n}I(X_k^{(j)}>n^{\alpha})\Big)+2\mathrm{Var}\Big(\sum_{k=1}^{n}I(X_k^{(j)}<-n^{\alpha})\Big)\\
&\leqslant 4\sum_{k=1}^{n}\mathrm{Var}\big(I(X_k^{(j)}>n^{\alpha})\big)+4\sum_{k=1}^{n}\mathrm{Var}\big(I(X_k^{(j)}<-n^{\alpha})\big)\\
&\leqslant 4\sum_{k=1}^{n}\mathbb{P}(|X_k^{(j)}|>n^{\alpha}),
\end{aligned}
$$

and so Lemma 2.5 ensures that

$$
\big(1-\mathbb{P}(\max_{1\leqslant k\leqslant n}|X_k^{(j)}|>n^{\alpha})\big)^2\sum_{k=1}^{n}\mathbb{P}(|X_k^{(j)}|>n^{\alpha})\leqslant 4\,\mathbb{P}(\max_{1\leqslant k\leqslant n}|X_k^{(j)}|>n^{\alpha}). \tag{3.8}
$$

On the other hand, from (3.6) we get

$$
\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{\log n}{n}\,\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}|X_k^{(j)}|>\varepsilon n^{\alpha}\Big)<\infty\quad\text{for all }\varepsilon>0. \tag{3.9}
$$

Then for every $\varepsilon > 0$,

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} n\,\mathbb{P}\Big(\max_{1\leqslant k\leqslant 2^n}|X_k^{(j)}| > \varepsilon\, 2^{n\alpha}\Big)\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m=2^n}^{2^{n+1}-1}\frac{\log m}{m}\,\mathbb{P}\Big(\max_{1\leqslant k\leqslant 2^n}|X_k^{(j)}| > \varepsilon\, 2^{n\alpha}\Big)\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\frac{\log m}{m}\,\mathbb{P}\Big(\max_{1\leqslant k\leqslant m}|X_k^{(j)}| > (\varepsilon/2^{\alpha})\, m^{\alpha}\Big) < \infty.
\end{aligned}
$$

This implies

$$
\sum_{j=1}^{\infty}\mathbb{P}\big(\max_{1\leqslant k\leqslant n}|X_k^{(j)}| > n^{\alpha}\big) \to 0 \quad \text{as } n \to \infty.
$$

Therefore, by (3.8), there exists a positive integer n_0, which does not depend on j, such that

$$
\sum_{k=1}^{n}\mathbb{P}(|X_k^{(j)}| > n^{\alpha}) \leqslant C\,\mathbb{P}(\max_{1\leqslant k\leqslant n}|X_k^{(j)}| > n^{\alpha}) \quad \text{for all } n > n_0, j \geqslant 1. \tag{3.10}
$$

Combining (1.1), (3.9) and (3.10), we have

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\log n\,\mathbb{P}\big(|X^{(j)}| > n^{\alpha}\big)\\
&= \sum_{j=1}^{\infty}\sum_{n=1}^{n_0}\log n\,\mathbb{P}\big(|X^{(j)}| > n^{\alpha}\big) + \sum_{j=1}^{\infty}\sum_{n=n_0+1}^{\infty}\log n\,\mathbb{P}\big(|X^{(j)}| > n^{\alpha}\big)\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{n_0}\frac{\log n}{n}\sum_{k=1}^{n}\mathbb{P}\big(|X_k^{(j)}| > n^{\alpha}\big)\\
&+ C\sum_{j=1}^{\infty}\sum_{n=n_0+1}^{\infty}\frac{\log n}{n}\,\mathbb{P}\big(\max_{1\leqslant k\leqslant n}|X_k^{(j)}| > n^{\alpha}\big)\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{\log n}{n}\,\mathbb{P}\big(\max_{1\leqslant k\leqslant n}|X_k^{(j)}| > n^{\alpha}\big) < \infty,
\end{aligned}
$$

and so (3.7) holds. This ends the proof of Theorem 3.5. □

Remark 3.6. It is interesting that the above theorem does not require the coordinatewise weakly upper bounded condition on the random vectors $\{X_n, n \geqslant 1\}$. Therefore, we cannot prove Theorem 3.5 by using the same arguments as in the proof of Theorem 2.6 in [7].

Remark 3.7. If H is finite dimensional, then in Theorem 3.5, the condition (3.5) can be removed. Now we will consider this condition in the case where H is infinite dimensional. Let α be a real number ($\alpha > 1/2$), and let $\{X, X_n, n \geqslant 1\}$ be as in Example 3.2. Then for every $\varepsilon > 0$, we have

$$\begin{aligned}
&\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{\log n}{n}\,\mathbb{P}\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{l=1}^{k}X_l^{(j)}\Big| > \varepsilon n^{\alpha}\Big)\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{\log n}{n^{2\alpha+1}}\,\mathbb{E}\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{l=1}^{k}X_l^{(j)}\Big|\Big)^2\\
&\leqslant C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{\log n}{n^{2\alpha+1}}\,\sum_{k=1}^{n}\mathbb{E}|X_k^{(j)}|^2 < \infty,
\end{aligned}$$

so that (3.6) holds. We also see that

$$\sum_{j=1}^{\infty}\mathbb{E}\big(|X^{(j)}|^{1/\alpha}\log^{+}|X^{(j)}|\,I(|X^{(j)}|^{1/\alpha}\leqslant 2)\big) = \sum_{j=1}^{\infty}\frac{1}{j} = \infty,$$

and the conclusion (3.3) fails.

Thus, in Theorem 3.5, we cannot remove the condition (3.5) or even replace it by the weaker condition $\mathbb{E}\big(|X^{(j)}|^{1/\alpha}\log^{+}|X^{(j)}|\,I(|X^{(j)}|^{1/\alpha}\leqslant 2)\big) \to 0$ as $j \to \infty$.

Remark 3.8. Let α, β be real numbers ($1/2 < \alpha < \beta$), and let $\{X, X_n, n \geqslant 1\}$ be a sequence of ℓ_2-valued i.i.d. random vectors with $\mathbb{P}\big(X^{(j)} = \pm j^{-\beta}\big) = 1/2$ for all $j \geqslant 1$. It is easy to verify that the conditions (3.5) and (3.6) are satisfied. Thus, the conclusion (3.3) follows immediately from Theorem 3.5.

Acknowledgement. This work was done during a stay of the author at the Vietnam Institute for Advanced Study in Mathematics (VIASM).

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Nguyen Van Huan,
Department of Mathematics and Applications, Saigon University,
273 An Duong Vuong, Ward 3, District 5, Ho Chi Minh City, Vietnam
E-mail: vanhuandhdt@yahoo.com