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Second-Order Necessary Optimality Conditions for a Discrete Optimal Control Problem with Mixed Constraints

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Abstract. In this paper, we study second-order necessary optimality conditions for a discrete optimal control problem with nonconvex cost functions and state-control constraints. By establishing an abstract result on second-order necessary optimality conditions for a mathematical programming problem, we derive second-order necessary optimality conditions for a discrete optimal control problem.

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1 Introduction

A wide variety of the problems in discrete optimal control problem can be posed in the following form.

Determine a pair (x, u) of a path $x = (x_0, x_1, \dots, x_N) \in X_0 \times X_1 \times \cdots \times X_N$ and a control vector $u = (u_0, u_1, \dots, u_{N-1}) \in U_0 \times U_1 \times \cdots \times U_{N-1}$, which minimize the cost

$$f(x, u) = \sum_{k=0}^{N-1} h_k(x_k, u_k) + h_N(x_N), \tag{1}$$

and satisfy the state equation

$$x_{k+1} = A_k x_k + B_k u_k, \quad k = 0, 1, \dots, N-1, \tag{2}$$

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the constraints

$$\begin{cases} g_{ik}(x_k,u_k) \le 0, & i=1,2,\dots,m \text{ and } k=0,1,\dots,N-1 \\ g_{iN}(x_N) \le 0, & i=1,2,\dots,m. \end{cases} \tag{3}$$

Here:

k indexes the discrete time,

N is the horizon or number times control applied,

x_k is the state of the system which summarizes past information that is relevant to future optimization,

u_k is the control variable to be selected at time k with the knowledge of the state x_k,

$h_k : X_k \times U_k \to \mathbb{R}$ is a continuous function on $X_k \times U_k$; $h_N : X_N \to \mathbb{R}$ is a continuous function on X_N,

$A_k : X_k \to X_{k+1}$; $B_k : U_k \to X_{k+1}$; $T_k : W_k \to X_{k+1}$ are linear mappings,

X_k is a finite-dimensional space of state variables at stage k,

U_k is a finite-dimensional space of control variables at stage k,

Y_{ik} is a finite-dimensional space,

$g_{ik} : X_k \times U_k \to Y_{ik}$ is a continuous function on $X_k \times U_k$; $g_{iN} : X_N \to Y_{iN}$ is a continuous function on X_N.

This type of problems are considered and investigated in [1], [3], [7], [15–18], [20], [24] and the references therein. A classical example for problem (1)–(3) is the economic stabilization problem, see, for example, [29] and [32].

The study of optimality conditions is an important topic in variational analysis and optimization. In order to give a general idea of such optimality conditions, consider for the moment the simplest case, when optimization problem is unconstrained. Then stationary points are the first-order optimality condition. It is well known that the second-order necessary condition for stationary points to be locally optimal is that the Hessian matrix should be positive semidefinite. There have been many papers dealing with the first-order optimality condition and second-order necessary condition for mathematical programming problems; see, for example, [4–6], [11], [13], [27, 28]. By considering a set of assumptions, which involve different kinds of the critical direction and the Mangasarian-Fromovitz condition, Kawasaki [13] derived second-order optimality conditions for a mathematical programming problem. However, the results of Kawasaki cannot be applied for non-conical constraints. In [6], Cominetti extended the results of Kawasaki. He gave second-order necessary optimality conditions for optimization problem with variable and functional constraints described by sets, involving Kuhn-Tucker-Lagrange multipliers. The novelty of this result with respect to the classical positive semidefiniteness condition on the Hessian of the Lagrangian function, is that it contains an extra term which represents a kind of second-order derivative associated with the target set of the functional constraints of the problem.

Besides the study of optimality conditions in mathematical programming, the study of optimality conditions in optimal control is also of interest to many researchers. It is well known that optimal control problems with continuous variables can be transferred to discrete optimal control problems by discretization. There have been many papers dealing with the first-order optimality condition and the second-order necessary condition for discrete optimal control; see, for example, [1], [9,10], [12], [21–23], [31]. Under the convexity conditions according to control variables of cost functions, Ioffe and Tihomirov [12, Theorem 1 of §6.4] established the first-order necessary optimality conditions for discrete optimal control problems with control constraints, which are described by the sets. By applying necessary optimality conditions for a mathematical programming problem, which can be referred to [2], Marinkovíc [22] generalized their recent results obtained in [21] to derive necessary optimality conditions to the case of discrete optimal control problems with equality and inequality type of constraints on control and on endpoints. Recently, we [31] have derived second-order optimality conditions for a discrete optimal control problem with control constraints and initial conditions, which are described by the sets. However, to the best of our knowledge, we did not see second-order necessary optimality conditions for discrete optimal control problems with both the state and control constraints.

In this paper, by establishing second-order necessary optimality conditions for a mathematical programming problem, we derived the second-order necessary optimality conditions for the discrete optimal control problems in the case where objective functions are nonconvex and mixed constraints. We show that if the second-order necessary condition is not satisfied, then the admissible couple is not a solution even it satisfies first-order necessary conditions.

2 Statement of the Main Result

We now return back to problem (1)–(3). For each $x = (x_0, x_1, \ldots, x_N) \in X = X_0 \times X_1 \times \cdots \times X_N$ and $u = (u_0, u_1, \ldots, u_{N-1}) \in U = U_0 \times U_1 \times \cdots \times U_{N-1}$, we put

$$f(x,u) = \sum_{k=0}^{N-1} h_k(x_k, u_k) + h_N(x_N),$$

and

$$F(x,y) = \big(g_{10}(x_0,u_0), g_{11}(x_1,u_1), \ldots, g_{1N-1}(x_{N-1},u_{N-1}), g_{1N}(x_N), \ldots,\\ g_{m0}(x_0,u_0), g_{m1}(x_1,u_1), \ldots, g_{mN-1}(x_{N-1},u_{N-1}), g_{mN}(x_N)\big). \quad (4)$$

Let

$$D_{ik} = (-\infty, 0] \quad (i = 1, 2, \ldots, m \text{ and } k = 0, 1, \ldots, N), \quad D = \prod_{i=1}^{m} \prod_{k=0}^{N} D_{ik},$$

$$Z = X \times U, \; \tilde{X} = X_1 \times X_2 \times \cdots \times X_N,$$

and

$$Y = \prod_{i=1}^{m} \prod_{k=0}^{N} Y_{ik}.$$

Then problem (1)–(3) can be written as the following form:

$$\begin{aligned} &\text{Minimize} \;\; f(z) \\ &\text{subject to} \;\; H(z) = 0, \quad F(z) \in D, \end{aligned}$$

where

$$H(z) = Mz,$$

$M : Z \to \tilde{X}$ is defined by

$$Mz = \begin{bmatrix} -A_0 & I & 0 & 0 & \ldots & 0 & 0 & -B_0 & 0 & 0 & \ldots & 0 \\ 0 & -A_1 & I & 0 & \ldots & 0 & 0 & 0 & -B_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -A_{N-1} & I & 0 & 0 & 0 & \ldots & -B_{N-1} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \\ u_0 \\ u_1 \\ \vdots \\ u_{N-1}, \end{bmatrix}$$

and $F : Z \to Y$ is defined by (4).

From the formula of M, we have

$$M^* y^* = \begin{bmatrix} -A_0^* & 0 & 0 & \ldots & 0 \\ I & -A_1^* & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -A_{N-1}^* \\ 0 & 0 & 0 & \ldots & I \\ -B_0^* & 0 & 0 & \ldots & 0 \\ 0 & -B_1^* & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -B_{N-1}^* \end{bmatrix} \begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_N^* \end{bmatrix}, \tag{5}$$

where M^* is the adjoint operator of M.

Recall that a couple $(\overline{x}, \overline{u})$, that satisfies (2) and (3), is said to be *admissible* for problem (1)–(3). For a given admissible couple $(\overline{x}, \overline{u})$, symbols $\overline{h}_k$, $\frac{\partial \overline{h}_k}{\partial u_k}$, $\frac{\partial^2 \overline{h}_k}{\partial u_k \partial x_k}$, etc., stand, respectively, for $h_k(\overline{x}_k, \overline{u}_k), (\frac{\partial h_k}{\partial u_k})(\overline{x}_k, \overline{u}_k), (\frac{\partial^2 h_k}{\partial u_k \partial x_k})(\overline{x}_k, \overline{u}_k)$, etc. An admissible couple $(\overline{x}, \overline{u})$ is said to be a *locally optimal solution* of problem (1)–(3) if there exists $\epsilon > 0$ such that for all admissible couples (x, u), the following implication holds:

$$\|(x,u) - (\overline{x}, \overline{u})\|_Z \le \epsilon \Rightarrow f(x,u) \ge f(\overline{x}, \overline{u}).$$

We now impose assumptions for problem (1)–(3).

(A) For each $(i,k) \in I(\overline{x}, \overline{u}) = I_1(\overline{x}, \overline{u}) \cup I_2(\overline{x}, \overline{u})$ and $v_{ik} \le 0$, there exist $x_0 \in X_0, \quad u_k \in U_k$ such that

$$\begin{cases} \frac{\partial \overline{g}_{ik}}{\partial x_k} x_k + \frac{\partial \overline{g}_{ik}}{\partial u_k} u_k - v_{ik} \le 0 \text{ if } (\mathrm{i,k}) \in \mathrm{I_1(\overline{x}, \overline{u})} \\ \frac{\partial \overline{g}_{iN}}{\partial x_N} x_N - v_{iN} \le 0 \qquad\qquad \text{if } (\mathrm{i,k}) = (\mathrm{i,N}) \in \mathrm{I_2(\overline{x}, \overline{u})}, \end{cases}$$

where

$$x_{k+1} = A_k x_k + B_k u_k,$$

$$I_1(\overline{x}, \overline{u}) = \{(i,k) : i = 1, 2, \ldots, m;\ k = 0, 1, \ldots, N-1 \text{ such that } \overline{g}_{ik} = 0\}, \qquad (6)$$

and

$$I_2(\overline{x}, \overline{u}) = \{(i,N) : i = 1, 2, \ldots, m \text{ such that } \overline{g}_{iN} = 0\}. \qquad (7)$$

A pair $z = (x,u) \in X \times U$ with $x = (x_0, x_1, \ldots, x_N),\ u = (u_0, u_1, \ldots, u_{N-1})$ is said to be a *critical direction* for problem (1)–(3) at $\overline{z} = (\overline{x}, \overline{u})$ with $\overline{x} = (\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_N)$, $\overline{u} = (\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_{N-1})$ iff the following conditions hold:

$(C1)$

$$\sum_{k=0}^{N} \frac{\partial \overline{h}_k}{\partial x_k} x_k + \sum_{k=0}^{N-1} \frac{\partial \overline{h}_k}{\partial u_k} u_k = 0;$$

$(C2)$

$$x_{k+1} = A_k x_k + B_k u_k,\ k = 0, 1, \ldots, N-1;$$

$(C3)$

$$\begin{cases} \frac{\partial \overline{g}_{ik}}{\partial x_k} x_k + \frac{\partial \overline{g}_{ik}}{\partial u_k} u_k \le 0, & \forall (i,k) \in I_1(\overline{x}, \overline{u}) \\ \frac{\partial \overline{g}_{iN}}{\partial x_N} x_N \le 0, & \forall (i,N) \in I_2(\overline{x}, \overline{u}), \end{cases}$$

where $I_1(\overline{x}, \overline{u})$, $I_2(\overline{x}, \overline{u})$ are defined by (6) and (7), respectively.
We denote by $\Theta(\overline{x}, \overline{u})$ the set of all critical directions to problem (1)–(3) at $(\overline{x}, \overline{u})$. It is clear that $\Theta(\overline{x}, \overline{u})$ is a convex cone which contains $(0,0)$.

We now state our main result.

Theorem 2.1 *Suppose that $(\overline{x}, \overline{u})$ is a locally optimal solution of problem* (1)–(3). *For each $i = 1, 2, \ldots, m$ and $k = 0, 1, \ldots, N-1$, assume that the functions $h_k : X_k \times U_k \to \mathbb{R}$, $g_{ik} : X_k \times U_k \to Y_{ik}$ are twice differentiable at $(\overline{x}_k, \overline{u}_k)$, and the functions $h_N : X_N \to \mathbb{R}$, $g_{iN} : X_N \to Y_{iN}$ are twice differentiable at $\overline{x}_N$ and assumption (A) is satisfied. Then, for each $(x, u) \in \Theta(\overline{x}, \overline{u})$, there exist $w^* = (x^*_{10}, w^*_{11}, \ldots, w^*_{1N}, \ldots, w^*_{m0}, w^*_{m1}, \ldots, w^*_{mN}) \in Y$ and $y^* = (y^*_1, y^*_2, \ldots, y^*_N) \in \tilde{X}$ such that the following conditions are fulfilled:*

(a) *Adjoint equation:*

$$\begin{cases} \frac{\partial \overline{h}_0}{\partial x_0} + \sum_{i=1}^m \frac{\partial \overline{g}_{i0}}{\partial x_0} w^*_{i0} - A^*_0 y^*_1 = 0 \\ \frac{\partial \overline{h}_k}{\partial x_k} + \sum_{i=1}^m \frac{\partial \overline{g}_{ik}}{\partial x_k} w^*_{ik} + y^*_k - A^*_k y^*_{k+1} = 0, \ k = 1, 2, \ldots, N-1 \\ \frac{\partial \overline{h}_N}{\partial x_N} + \sum_{i=1}^m \frac{\partial \overline{g}_{iN}}{\partial x_N} w^*_{iN} + y^*_N = 0 \\ \frac{\partial \overline{h}_k}{\partial u_k} + \sum_{i=1}^m \frac{\partial \overline{g}_{ik}}{\partial u_k} w^*_{ik} - B^*_k y^*_{k+1} = 0, \qquad k = 0, 1, \ldots, N-1; \end{cases}$$

(b) *Non-negative second-order condition:*

$$\sum_{k=0}^{N-1} \Big(\frac{\partial^2 \overline{h}_k}{\partial x_k^2} x_k + \frac{\partial^2 \overline{h}_k}{\partial x_k \partial u_k} u_k \Big) x_k + \frac{\partial^2 \overline{h}_N}{\partial x_N^2} x_N^2 + \sum_{k=0}^{N-1} \Big(\frac{\partial^2 \overline{h}_k}{\partial u_k \partial x_k} x_k + \frac{\partial^2 \overline{h}_k}{\partial u_k^2} u_k \Big) u_k$$
$$+ \sum_{i=1}^{m} \sum_{k=0}^{N-1} \Big[\frac{\partial^2 \overline{g}_{ik}}{\partial x_k^2} x_k^2 + \big(\frac{\partial^2 \overline{g}_{ik}}{\partial x_k \partial u_k} + \frac{\partial^2 \overline{g}_{ik}}{\partial u_k \partial x_k} \big) x_k u_k + \frac{\partial^2 \overline{g}_{ik}}{\partial u_k^2} u_k^2 \Big] w^*_{ik}$$
$$+ \sum_{i=1}^{m} \frac{\partial^2 \overline{g}_{iN}}{\partial x_N^2} x_N^2 w^*_{iN} \geq 0;$$

(c) *Complementarity condition:*

$$w^*_{ik} \geq 0 \quad (i = 1, 2, \ldots, m; \quad k = 0, 1, \ldots, N),$$

and

$$\langle w^*_{ik}, \overline{g}_{ik} \rangle = 0 \quad (i = 1, 2, \ldots, m; \quad k = 0, 1, \ldots, N).$$

In order to prove Theorem 2.1, we first reduce the problem to a programming problem and then establish an abstract result on second-order necessary optimality conditions for a mathematical programming problem. This procedure is presented in Section 4. The complete proof for Theorem 2.1 will be provided in Section 5.

3 Basic Definitions and Preliminaries

In this section, we recall some notions and facts from variational analysis and generalized differentiation which will be used in the sequel. These notations and facts can be found in [6], [8], [14], [19], [25, 26], and [30].

Let E_1 and E_2 be finite-dimensional Euclidean spaces and $F : E_1 \rightrightarrows E_2$ be a multifunction. The effective domain, denoted by $\mathrm{dom}F$, and the graph of F, denoted by $\mathrm{gph}F$, are defined as

$$\mathrm{dom}F := \{z \in E_1 : F(z) \neq \emptyset\},$$

and

$$\mathrm{gph}F := \{(z, v) \in E_1 \times E_2 : v \in F(z)\}.$$

Let E be a finite-dimensional Euclidean space, D be a nonempty closed convex subset of E and $\overline{z} \in D$. We define

$$D(\overline{z}) = \mathrm{cone}(D - \overline{z}) = \{\lambda(d - \overline{z}) : d \in D, \lambda > 0\}.$$

The set

$$T(D; \overline{z}) = \liminf_{t \to 0^+} \frac{D - \overline{z}}{t} = \big\{h \in E : \forall t_n \to 0^+, \exists h_n \to h, \overline{z} + t_n h_n \in D\big\}$$

is called the *tangent cone* to D at $\overline{z}$. It is well known that

$$T(D; \overline{z}) = \mathrm{cl}\big(\mathrm{D}(\overline{\mathrm{z}})\big) = \mathrm{cl}\big(\mathrm{cone}(\mathrm{D} - \overline{\mathrm{z}})\big).$$

The *second-order tangent cone* to D at $\overline{z}$ in the direction $v \in E$ is defined by

$$\begin{aligned} T^2(D; \overline{z}, v) &= \liminf_{t \to 0^+} \frac{D - \overline{z} - tv}{\frac{t^2}{2}} \\ &= \left\{w : \forall t_n \to 0^+, \exists w_n \to w, \overline{z} + t_n v + \frac{t_n^2}{2} w_n \in D\right\}. \end{aligned}$$

When $v \in D(\overline{z}) = \mathrm{cone}(D - \overline{z})$, then there exists $\lambda > 0$ such that $v = \lambda(z - \overline{z})$ for some $z \in D$. By the convexity of D, for any $t_n \to 0^+$, we have

$$t_n v = t_n \lambda z + (1 - t_n \lambda)\overline{z} - \overline{z} \in D - \overline{z}.$$

This implies that $\overline{z} + t_n v \in D$, and so, $0 \in T^2(D; \overline{z}, v)$. By [6, Proposition 3.1], we have

$$T^2(D; \overline{z}, v) = T\big(T(D; \overline{z}); v\big).$$

The set

$$N(D; \overline{z}) = \{z^* \in E : \langle z^*, z\rangle \leq 0, \ \forall z \in T(D; \overline{z})\}$$

is called *normal cone* to D at $\overline{z}$. It is known that

$$N(D; \overline{z}) = \{z^* \in E : \langle z^*, z - \overline{z}\rangle \leq 0, \ \forall z \in D\}.$$

4 The Optimal Control Problem as a Programming Problem

In this section, we suppose that Z and Y are finite-dimensional spaces. Assume moreover that $f : Z \to \mathbb{R}$, $F : Z \to Y$ are functions and the sets $A \subset Z$ and $D \subset Y$ are closed convex. Let us consider the programming problem

$$(P) \qquad\qquad \text{Minimize}\{f(z) : z \in A \text{ and } F(z) \in D\}.$$

Let Q be a subset of Z. The *usual support function* $\sigma(\cdot, Q) : Z \to \mathbb{R}$ of the set Q is defined by

$$\sigma(z^*, Q) := \sup_{z \in Q} \langle z^*, z\rangle.$$

The following theorem is a shaper version of Commineti, which gives second-order necessary optimality conditions for mathematical programming problem (P).

Theorem 4.1 *Suppose $\overline{z}$ is a local minimum for (P) at which the following regularity condition is satisfied:*

$$\nabla F(\overline{z})\big(A(\overline{z})\big) - D\big(F(\overline{z})\big) = Y.$$

Assume that the functions f and F are continuous on A and twice differentiable at $\overline{z}$. For each $z \in Z$, the following conditions hold:
(C'1) $\langle \nabla f(\overline{z}), z\rangle = 0,$
(C'2) $z \in T(A; \overline{z}), \quad \nabla F(\overline{z})z \in T\big(D; F(\overline{z})\big).$
Then, there exists $w^* \in N\big(D; F(\overline{z})\big)$ such that the Lagrangian function $L = f + w^* \circ F$ satisfies the following properties:

(i) *(Euler-Lagrange inclusion)*

$$-\nabla L(\overline{z}) \in N(A; \overline{z});$$

(ii) *(Legendre inequality)*

$$\langle \nabla L(\overline{z}), v\rangle + \langle \nabla^2 L(\overline{z})z, z\rangle \geq \sigma\Big(w^*, T^2\big(D; F(\overline{z}), \nabla F(\overline{z})z\big)\Big),$$

for every $v \in T^2(A; \overline{z}, z)$;
(iii) $\langle \nabla L(\overline{z}), z\rangle = 0.$

When D is in fact a cone, then we also have
(iv) *(Complementarity condition)*

$$L(\overline{z}) = f(\overline{z}); \quad w^* \in N(D; 0).$$

Proof. Our proof is based on the scheme of the proof in [6, Theorem 4.2]. Fixing any $z \in Z$ which satisfies the conditions $(C'1)$ and $(C'2)$, we consider two cases:
Case 1. $T^2(A; \overline{z}, z) = \emptyset$ or $T^2\big(D; F(\overline{z}), \nabla F(\overline{z})z\big) = \emptyset$. In this case, the Legendre inequality is automatically fulfilled because

$$T^2(A; \overline{z}, z) = \emptyset \ \text{ or } \ \sigma\Big(w^*, T^2\big(D; F(\overline{z}), \nabla F(\overline{z})z\big)\Big) = -\infty.$$

To obtain the assertions (i) and (iii), we shall separate the sets B and $T\big(A \cap F^{-1}(D); \overline{z}\big)$. Here,

$$B = \{v \in Z : \langle \nabla f(\overline{z})v < 0\}.$$

From Robinson's condition, we obtain

$$Y = \nabla F(\overline{z})T(A; \overline{z}) - T\big(D; F(\overline{z})\big). \tag{8}$$

So, we can find $w \in T(A; \overline{z})$ such that

$$\nabla F(\overline{z})w \in T\big(D; F(\overline{z})\big).$$

By [6, Theorem 3.1], $w \in T\big(A \cap F^{-1}(D); \overline{z}\big)$. Now, if $\nabla f(\overline{z}) = 0$, we may just take $w^* = 0$, so let us assume the contrary, in which case $B \neq \emptyset$. We note that

$$B \cap T\big(A \cap F^{-1}(D); \overline{z}\big) = \emptyset.$$

Indeed, if $w \in T\big(A \cap F^{-1}(D); \overline{z}\big)$ we may choose $w_t \to w$ so that for $t > 0$ small enough we have

$$\overline{z} + tw_t \in A \cap F^{-1}(D)$$

and

$$f(\overline{z}) \leq f(\overline{z} + tw_t) = f(\overline{z}) + t\langle \nabla f(\overline{z}), w_t \rangle + o(t).$$

So

$$\langle \nabla f(\overline{z}), w \rangle \geq 0,$$

which is equivalent to $w \notin B$. Thus, sets B and $T\big(A \cap F^{-1}(D); \overline{z}\big)$ being nonvoid, convex, open and closed respectively. The strict separation theorem implies that there exist a nonzero functional $z^* \in Z$ and a real $r \in \mathbb{R}$ such that

$$\langle z^*, v \rangle < r \leq \langle z^*, z \rangle, \ \forall v \in B, \ z \in T\big(A \cap F^{-1}(D); \overline{z}\big),$$

or equivalently

$$\sigma(z^*, B) + \sigma\Big(- z^*, T\big(A \cap F^{-1}(D); \overline{z}\big)\Big) \leq 0. \tag{9}$$

So, we have

$$\sigma(z^*, B) < +\infty. \tag{10}$$

We will prove that $z^* = \lambda\nabla f(\overline{z})$ for some positive λ. Indeed, suppose that $z^* \notin \{\lambda\nabla f(\overline{z}) : \lambda > 0\}$. It follows from the strict separation theorem that there exists $z_1 \neq 0$ such that

$$\langle \lambda\nabla f(\overline{z}), z_1\rangle \leq 0 < \langle z^*, z_1\rangle,\ \forall\lambda \geq 0.$$

Hence, $\nabla f(\overline{z})z_1 \leq 0$. Let $z_2 \in B$ then

$$\langle \nabla f(\overline{z}), z_2 + \alpha z_1\rangle \leq \langle \nabla f(\overline{z}), z_2\rangle < 0,\ \forall\alpha > 0.$$

Therefore, $z_2 + \alpha z_1 \in B$ for all $\alpha > 0$. One the other hand, $\langle z^*, z_2 + \alpha z_1\rangle \to +\infty$ as $\alpha \to +\infty$, this implies that $\sigma(z^*, B) = +\infty$, which contradicts (10). By eventually dividing by this λ we may assume that $z^* = \nabla f(\overline{z})$ and then a direct calculation gives us

$$\sigma(z^*, B) = 0. \tag{11}$$

Concerning the second term in (9), we notice that [6, Theorem 3.1] implies that

$$T\big(A \cap F^{-1}(D); \overline{z}\big) = P \cap L^{-1}(Q),$$

where

$$\begin{aligned} P &= T(A; \overline{z}),\\ Q &= T\big(D, F(\overline{z})\big),\\ L &= \nabla F(\overline{z}). \end{aligned}$$

Moreover, (8) gives us $0 \in \operatorname{core}[L(P) - Q]$, so that we may use [6, Lemma 3] in order to find $w^* \in Y^*$ such that

$$\sigma\Big(-z^*, T\big(A \cap F^{-1}(D); \overline{z}\big)\Big) = \sigma\big(-\nabla f(\overline{z}) - w^* \circ \nabla F(\overline{z}), T(A; \overline{z})\big) + \sigma\Big(w^*, T\big(D; F(\overline{z})\big)\Big).$$

Defining $L = f + w^* \circ F$ and combining (9), (11) we have

$$\langle \nabla L(\overline{z}), w\rangle \geq \sigma\Big(w^*, T\big(D; F(\overline{z})\big)\Big),\ \forall w \in T(A; \overline{z}). \tag{12}$$

Choosing $w = 0 \in T(A; \overline{z})$, we get

$$\langle w^*, z\rangle \leq 0,\ \forall z \in T\big(D; F(\overline{z})\big).$$

So, $w^* \in N\big(D; F(\overline{z})\big)$. Since $0 \in T\big(D; F(\overline{z})\big)$ and (12), we get

$$\langle -\nabla L(\overline{z}), w\rangle \leq 0, \forall w \in T(A; \overline{z}).$$

Hence $-\nabla L(\overline{z}) \in N(A; \overline{z})$, this is the Euler-Lagrange inclusion. From $z \in T(A; \overline{z})$ and $-\nabla L(\overline{z}) \in N(A; \overline{z})$, we have

$$\langle \nabla L(\overline{z}), z\rangle \geq 0. \tag{13}$$

Besides,

$$\langle \nabla L(\overline{z}), z\rangle = \langle \nabla f(\overline{z}), z\rangle + \langle w^* \circ \nabla F(\overline{z}), z\rangle = \langle w^*, \nabla F(\overline{z})z\rangle.$$

Since $\nabla F(\overline{z})z \in T\big(D; F(\overline{z})\big)$ and $w^* \in N\big(D; F(\overline{z})\big)$, we get $\langle w^*, \nabla F(\overline{z})z\rangle \leq 0$. Hence,

$$\langle \nabla L(\overline{z}), z\rangle \leq 0. \tag{14}$$

Combining (13) and (14), we obtain $\langle \nabla L(\overline{z}), z\rangle = 0$, this is the assertions (iii).
Case 2. $T^2(A; \overline{z}, z) \neq \emptyset$ and $T^2\big(D; F(\overline{z}), \nabla F(\overline{z})z\big) \neq \emptyset$. This case was proved by Cominetti in [6, Theorem 4.2]. $\square$

5 Proof of the Main Result

We now return to problem (1)–(3). Let

$$A := \{z \in Z : H(z) = 0\} \tag{15}$$

and define a mapping $F : Z \to Y$ by (4). We now rewrite problem (1)–(3) in the form

$$\begin{cases} \text{Minimize} \ \ f(z) \\ \text{subject to} \ \ z \in A \cap F^{-1}(D). \end{cases}$$

Note that A is a nonempty closed convex set and D is a nonempty closed convex cone. The next step is to apply Theorem 4.1 to the problem. In order to use this theorem, we have to check all conditions of Theorem 4.1.

Let F, H, M and A be the same as defined above. The first, we have the following result.

Lemma 5.1 *Suppose that* $I(\overline{z}) = I(\overline{x}, \overline{u}) = I_1(\overline{x}, \overline{u}) \cup I_2(\overline{x}, \overline{u})$, *where* $I_1(\overline{x}, \overline{u})$ *and* $I_2(\overline{x}, \overline{u})$ *are defined by* (6) *and* (7), *respectively. Then*

$$\begin{aligned} \mathrm{cl}\big(\mathrm{cone(D - F(\overline{z}))}\big) = \mathrm{cone}(D - F(\overline{z})) = \{&(v_{10}, v_{11}, \ldots, v_{1N}, \ldots, v_{m0}, v_{m1}, \\ &\ldots, v_{mN}) \in Y : v_{ik} \leq 0, \forall (i,k) \in I(\overline{z})\} := E. \end{aligned} \tag{16}$$

Proof. Take any

$$\begin{aligned} y = (y_{10}, y_{11}, \ldots, y_{1N}, \ldots, y_{m0}, y_{m1}, \ldots, y_{mN}) \in{} & \mathrm{cone}(D - F(\overline{z})) \\ &= \prod_{i=1}^{m} \prod_{k=0}^{N} \mathrm{cone}(D_{ik} - \overline{g}_{ik}) \end{aligned}$$

and $(i,k) \in I(\overline{z})$, we have $\overline{g}_{ik} = 0$. So, $y_{ik} \in \text{cone}(D_{ik}) = D_{ik}$. This implies that $y_{ik} \leq 0$. Hence $y \in E$. Conversely, take any

$$v = (v_{10}, v_{11}, \ldots, v_{1N}, \ldots, v_{m0}, v_{m1}, \ldots, v_{mN}) \in E.$$

If $\overline{g}_{ik} = 0$ then by the definition of E, $v_{ik} \leq 0$. Hence,

$$v_{ik} = v_{ik} - \overline{g}_{ik} \in D_{ik} = \text{cone}(D_{ik}) = \text{cone}(D_{ik} - \overline{g}_{ik}).$$

If $\overline{g}_{ik} < 0$ then there exist a constant $\lambda > 0$ such that $\frac{1}{\lambda}v_{ik} + \overline{g}_{ik} \leq 0$. So,

$$\frac{1}{\lambda}v_{ik} + \overline{g}_{ik} \in D_{ik}.$$

Hence,

$$v_{ik} = \lambda[\frac{1}{\lambda}v_{ik} + \overline{g}_{ik} - \overline{g}_{ik}] \in \text{cone}(D_{ik} - \overline{g}_{ik}).$$

This implies that

$$\begin{aligned} v = (v_{10}, v_{11}, \ldots, v_{1N}, \ldots, v_{m0}, v_{m1}, \ldots, v_{mN}) \in \prod_{i=1}^{m}\prod_{k=0}^{N} \text{cone}(D_{ik} - \overline{g}_{ik}) \\ = \text{cone}(D - F(\overline{z})). \end{aligned}$$

Thus,

$$\text{cone}(D - F(\overline{z})) = E.$$

It is easy to see that the set E is closed. So,

$$\mathrm{cl}\big(\mathrm{cone(D - F(\overline{z}))}\big) = \mathrm{E}.$$

Hence,

$$\mathrm{cl}\big(\mathrm{cone(D - F(\overline{z}))}\big) = \mathrm{cone(D - F(\overline{z}))} = \mathrm{E},$$

the proof of the lemma is complete. $\square$

We now have the following result on the regularity condition for mathematical programming problem (P).

Lemma 5.2 *Suppose that assumption (A) is satisfied. Then, the regularity condition is fulfilled, that is*

$$\nabla F(\overline{z})\big(A(\overline{z})\big) - D\big(F(\overline{z})\big) = Y.$$

Proof. We first claim that

$$N(A;(x^1,u^1)) = \{M^*y^* : y^* \in \tilde{X}\}, \forall (x^1,u^1) = z^1 \in A,$$

where M^* is defined by (5). Indeed, we see that H is a continuous linear mapping and it's adjoint mapping is

$$\begin{aligned} H^* : \tilde{X} &\to Z \\ y^* &\mapsto H^*(y^*) = M^*y^*. \end{aligned}$$

Since A is a vector space, we have

$$N(A;z^1) = (\mathrm{ker}H)^\perp, \quad T(A;z^1) = A, \quad A(\overline{z}) = \mathrm{cone}(A - \overline{z}) = A.$$

Hence, the proof will be completed if we show that

$$\nabla F(\overline{z})\big(A\big) - D\big(F(\overline{z})\big) = Y.$$

Since

$$\begin{aligned} F(x,y) = \big(&g_{10}(x_0,u_0), g_{11}(x_1,u_1), \ldots, g_{1N-1}(x_{N-1},u_{N-1}), g_{1N}(x_N), \ldots, \\ &g_{m0}(x_0,u_0), g_{m1}(x_1,u_1), \ldots, g_{mN-1}(x_{N-1},u_{N-1}), g_{mN}(x_N)\big), \end{aligned}$$

we have
$\nabla F(\overline{z})z =$

$$\begin{bmatrix} \frac{\partial \overline{g}_{10}}{\partial x_0} & 0 & 0 & \ldots & 0 & 0 & \frac{\partial \overline{g}_{10}}{\partial u_0} & 0 & 0 & \ldots & 0 \\ 0 & \frac{\partial \overline{g}_{11}}{\partial x_1} & 0 & \ldots & 0 & 0 & 0 & \frac{\partial \overline{g}_{11}}{\partial u_1} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{\partial \overline{g}_{1N-1}}{\partial x_{N-1}} & 0 & 0 & 0 & 0 & \ldots & \frac{\partial \overline{g}_{1N-1}}{\partial u_{N-1}} \\ 0 & 0 & 0 & \ldots & 0 & \frac{\partial \overline{g}_{1N}}{\partial x_N} & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \overline{g}_{m0}}{\partial x_0} & 0 & 0 & \ldots & 0 & 0 & \frac{\partial \overline{g}_{m0}}{\partial u_0} & 0 & 0 & \ldots & 0 \\ 0 & \frac{\partial \overline{g}_{m1}}{\partial x_1} & 0 & \ldots & 0 & 0 & 0 & \frac{\partial \overline{g}_{m1}}{\partial u_1} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{\partial \overline{g}_{mN-1}}{\partial x_{N-1}} & 0 & 0 & 0 & 0 & \ldots & \frac{\partial \overline{g}_{mN-1}}{\partial u_{N-1}} \\ 0 & 0 & 0 & \ldots & 0 & \frac{\partial \overline{g}_{mN}}{\partial x_N} & 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \\ u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$\begin{aligned} = \Big(&\frac{\partial \overline{g}_{10}}{\partial x_0}x_0 + \frac{\partial \overline{g}_{10}}{\partial u_0}u_0, \frac{\partial \overline{g}_{11}}{\partial x_1}x_1 + \frac{\partial \overline{g}_{11}}{\partial u_1}u_1, \ldots, \frac{\partial \overline{g}_{1N-1}}{\partial x_{N-1}}x_{N-1} + \\ &\frac{\partial \overline{g}_{1N-1}}{\partial u_{N-1}}u_{N-1}, \frac{\partial \overline{g}_{1N}}{\partial x_N}x_N, \ldots, \frac{\partial \overline{g}_{m0}}{\partial x_0}x_0 + \frac{\partial \overline{g}_{m0}}{\partial u_0}u_0, \frac{\partial \overline{g}_{m1}}{\partial x_1}x_1 + \frac{\partial \overline{g}_{m1}}{\partial u_1}u_1, \ldots, \\ &\frac{\partial \overline{g}_{mN-1}}{\partial x_{N-1}}x_{N-1} + \frac{\partial \overline{g}_{mN-1}}{\partial u_{N-1}}u_{N-1}, \frac{\partial \overline{g}_{mN}}{\partial x_N}x_N\Big). \end{aligned}$$

By Lemma 5.1,

$$D(F(\overline{z})) = \text{cone}(D - F(\overline{z})) = E.$$

Therefore, we need to prove that

$$\nabla F(\overline{z})(A) - E = Y.$$

Take any

$$v = (v_{10}, v_{11}, \dots, v_{1N}, \dots, v_{m0}, v_{m1}, \dots, v_{mN}) \in Y,$$

the proof will be completed if we show that

$$v \in \nabla F(\overline{z})(A) - E.$$

For each $(i, k) \in \{1, 2, \dots, m\} \times \{0, 1, \dots, N\}$, we get $\overline{g}_{ik} \leq 0$. If $\overline{g}_{ik} < 0$ and $z \in A$, we choose

$$y_{ik} = \begin{cases} \frac{\partial \overline{g}_{ik}}{\partial x_k} x_k + \frac{\partial \overline{g}_{ik}}{\partial u_k} u_k - v_{ik} & \text{if } k < N \\ \frac{\partial \overline{g}_{ik}}{\partial x_k} x_k - v_{ik} & \text{if } k = N. \end{cases}$$

It is easy to see that

$$\begin{cases} \frac{\partial \overline{g}_{ik}}{\partial x_k} x_k + \frac{\partial \overline{g}_{ik}}{\partial u_k} u_k - y_{ik} = v_{ik} & \text{if } k < N \\ \frac{\partial \overline{g}_{ik}}{\partial x_k} x_k - y_{ik} = v_{ik} & \text{if } k = N. \end{cases}$$

If $\overline{g}_{ik} = 0$, that is $(i, k) \in I(\overline{z})$. We now represent

$$v_{ik} = v_{ik}^1 - v_{ik}^2,$$

where $v_{ik}^1, v_{ik}^2 \leq 0$. By assumption (A), there exist $x_0 \in X_0,\ u_k \in U_k$ such that

$$\begin{cases} \frac{\partial \overline{g}_{ik}}{\partial x_k} x_k + \frac{\partial \overline{g}_{ik}}{\partial u_k} u_k - v_{ik}^1 \leq 0 \text{ if } (\mathrm{i,k}) \in \mathrm{I}_1(\overline{\mathrm{z}}) \\ \frac{\partial \overline{g}_{iN}}{\partial x_N} x_N - v_{iN}^1 \leq 0 \qquad \text{if } (\mathrm{i,k}) = (\mathrm{i,N}) \in \mathrm{I}_2(\overline{\mathrm{z}}), \end{cases}$$

where

$$x_{k+1} = A_k x_k + B_k u_k.$$

Define

$$\begin{cases} y_{ik} = \frac{\partial \overline{g}_{ik}}{\partial x_k} x_k + \frac{\partial \overline{g}_{ik}}{\partial u_k} u_k - v_{ik} = \frac{\partial \overline{g}_{ik}}{\partial x_k} x_k + \frac{\partial \overline{g}_{ik}}{\partial u_k} u_k - v_{ik}^1 + v_{ik}^2 \text{ if } (\mathrm{i,k}) \in \mathrm{I}_1(\overline{\mathrm{z}}) \\ y_{iN} = \frac{\partial \overline{g}_{iN}}{\partial x_N} x_N - v_{iN} = \frac{\partial \overline{g}_{iN}}{\partial x_N} x_N - v_{iN}^1 + v_{iN}^2 \qquad \text{if } (\mathrm{i,k}) = (\mathrm{i,N}) \in \mathrm{I}_2(\overline{\mathrm{z}}). \end{cases}$$

We see that

$$y_{ik} \leq 0, \quad \forall (i, k) \in I(\overline{z}),$$

$$z = (x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}) \in A,$$

and

$$\begin{cases} \frac{\partial \overline{g}_{ik}}{\partial x_k} x_k + \frac{\partial \overline{g}_{ik}}{\partial u_k} u_k - y_{ik} = v_{ik}, & \forall (i,k) \in I_1(\overline{z}) \\ \frac{\partial \overline{g}_{iN}}{\partial x_N} x_N - y_{iN} = v_{iN}, & \forall (i,N) \in I_2(\overline{z}). \end{cases}$$

We note that

$$\begin{aligned}\Big(\frac{\partial \overline{g}_{10}}{\partial x_0} x_0 + \frac{\partial \overline{g}_{10}}{\partial u_0} u_0, \frac{\partial \overline{g}_{11}}{\partial x_1} x_1 + \frac{\partial \overline{g}_{11}}{\partial u_1} u_1, \ldots, \frac{\partial \overline{g}_{1N-1}}{\partial x_{N-1}} x_{N-1} + \\ \frac{\partial \overline{g}_{1N-1}}{\partial u_{N-1}} u_{N-1}, \frac{\partial \overline{g}_{1N}}{\partial x_N} x_N, \ldots, \frac{\partial \overline{g}_{m0}}{\partial x_0} x_0 + \frac{\partial \overline{g}_{m0}}{\partial u_0} u_0, \frac{\partial \overline{g}_{m1}}{\partial x_1} x_1 + \frac{\partial \overline{g}_{m1}}{\partial u_1} u_1, \ldots, \\ \frac{\partial \overline{g}_{mN-1}}{\partial x_{N-1}} x_{N-1} + \frac{\partial \overline{g}_{mN-1}}{\partial u_{N-1}} u_{N-1}, \frac{\partial \overline{g}_{mN}}{\partial x_N} x_N\Big) = \nabla F(\overline{z})(z),\end{aligned}$$

and

$$y = (y_{10}, y_{11}, \ldots, y_{1N}, \ldots, y_{m0}, y_{m1}, \ldots, y_{mN}) \in E.$$

Hence, the proof of the lemma is complete. □

Proof of the Main Result. From Lemma 5.2, we see that all conditions of Theorem 4.1 are fulfilled. Since

$$f(z) = f(x,u) = \sum_{k=0}^{N-1} h_k(x_k, u_k) + h_N(x_N),$$

we have

$$\begin{aligned}\nabla f(\overline{z}) = \nabla f(\overline{x}, \overline{u}) &= \sum_{k=0}^{N} \big(\nabla_x h_k(\overline{x}_k, \overline{u}_k), \nabla_u h_k(\overline{x}_k, \overline{u}_k)\big) \\ &= \Big(\frac{\partial h_0}{\partial x_0}(\overline{x}_0, \overline{u}_0), \frac{\partial h_1}{\partial x_1}(\overline{x}_1, \overline{u}_1), \ldots, \frac{\partial h_{N-1}}{\partial x_{N-1}}(\overline{x}_{N-1}, \overline{u}_{N-1}), \frac{\partial h_N}{\partial x_N}(\overline{x}_N), \\ &\qquad \frac{\partial h_0}{\partial u_0}(\overline{x}_0, \overline{u}_0), \frac{\partial h_1}{\partial u_1}(\overline{x}_1, \overline{u}_1), \ldots, \frac{\partial h_{N-1}}{\partial u_{N-1}}(\overline{x}_{N-1}, \overline{u}_{N-1})\Big).\end{aligned}$$

So, for each $z = (x,u) = (x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}) \in Z$, we get

$$\langle \nabla f(\overline{z}), z \rangle = \sum_{k=0}^{N} \frac{\partial \overline{h}_k}{\partial x_k} x_k + \sum_{k=0}^{N-1} \frac{\partial \overline{h}_k}{\partial u_k} u_k.$$

Take any $z = (x,u) \in \Theta(\overline{x}, \overline{u}) = \Theta(\overline{z})$. By condition $(C1)$, we obtain

$$\langle \nabla f(\overline{z}), z \rangle = 0;$$

this is, the condition (C_1') of Theorem 4.1. From assumption $(C2)$, we get

$$z \in A = T(A; \overline{z}). \tag{17}$$

By Lemma 5.1, we have

$$D(F(\overline{z})) = \text{cone}(D - F(\overline{z})) = \text{cl}\big(\text{cone}(\text{D} - \text{F}(\overline{\text{z}}))\big) = \text{E},$$

where E is defined by (16). Since condition $(C3)$, we have

$$\begin{aligned}\nabla F(\overline{z})z = \Big(&\frac{\partial \overline{g}_{10}}{\partial x_0}x_0 + \frac{\partial \overline{g}_{10}}{\partial u_0}u_0, \frac{\partial \overline{g}_{11}}{\partial x_1}x_1 + \frac{\partial \overline{g}_{11}}{\partial u_1}u_1, \ldots, \frac{\partial \overline{g}_{1N-1}}{\partial x_{N-1}}x_{N-1} + \\ &\frac{\partial \overline{g}_{1N-1}}{\partial u_{N-1}}u_{N-1}, \frac{\partial \overline{g}_{1N}}{\partial x_N}x_N, \ldots, \frac{\partial \overline{g}_{m0}}{\partial x_0}x_0 + \frac{\partial \overline{g}_{m0}}{\partial u_0}u_0, \frac{\partial \overline{g}_{m1}}{\partial x_1}x_1 + \frac{\partial \overline{g}_{m1}}{\partial u_1}u_1, \ldots, \\ &\frac{\partial \overline{g}_{mN-1}}{\partial x_{N-1}}x_{N-1} + \frac{\partial \overline{g}_{mN-1}}{\partial u_{N-1}}u_{N-1}, \frac{\partial \overline{g}_{mN}}{\partial x_N}x_N\Big) \in E = D(F(\overline{z})).\end{aligned}$$

Hence,

$$\nabla F(\overline{z})z \in T\big(D; F(\overline{z})\big) \tag{18}$$

and

$$0 \in T^2\big(D; F(\overline{z}), \nabla F(\overline{z})z\big).$$

Combining (17) and (18), the condition $(C'2)$ of Theorem 4.1 is fulfilled. Thus, each

$$z = (x, u) \in \Theta(\overline{x}, \overline{u})$$

satisfies all the conditions of Theorem 4.1. According to Theorem 4.1, there exists

$$w^* = (w_{10}^*, w_{11}^*, \ldots, w_{1N}^*, \ldots, w_{m0}^*, w_{m1}^*, \ldots, w_{mN}^*) \in Y$$

such that the Lagrangian function $L = f + w^* \circ F$ satisfies the following properties:

$(a1)$ (Euler-Lagrange inclusion)

$$-\nabla L(\overline{z}) \in N(A; \overline{z});$$

$(a2)$ (Legendre inequality)

$$\langle \nabla L(\overline{z}), v\rangle + \langle \nabla^2 L(\overline{z})z, z\rangle \geq \sigma\big(w^*, T^2\big(D; F(\overline{z}), \nabla F(\overline{z})z\big)\big)$$

for every $v \in T^2(A; \overline{z}, z)$;

$(a3)$ (Complementarity condition)

$$L(\overline{z}) = f(\overline{z}); \quad w^* \in N(D; 0).$$

The complementarity condition is equivalent to

$$w^* = (w_{10}^*, w_{11}^*, \dots, w_{1N}^*, \dots, w_{m0}^*, w_{m1}^*, \dots, w_{mN}^*) \in N(D;0),$$

and

$$w^* \circ F(\overline{z}) = 0.$$

Since

$$N(D;0) = \prod_{i=1}^{m}\prod_{k=0}^{N} N(D_{ik};0),$$

we obtain

$$w_{ik}^* \in N(D_{ik};0) \ \ (i=1,2,\dots,m; \quad k=0,1,\dots,N),$$

and

$$w^* \circ F(\overline{z}) = \sum_{i=1}^{m}\sum_{k=0}^{N}\langle w_{ik}^*, \overline{g}_{ik}\rangle = 0. \tag{19}$$

From

$$w_{ik}^* \in N(D_{ik};0),$$

we have

$$\langle w_{ik}^*, w\rangle \le 0, \ \forall w \le 0 \ \ (i=1,2,\dots,m; \ k=0,1,\dots,N).$$

This implies that

$$w_{ik}^* \ge 0 \ \ (i=1,2,\dots,m; \quad k=0,1,\dots,N), \tag{20}$$

and

$$\langle w_{ik}^*, \overline{g}_{ik}\rangle \le 0 \ \ (i=1,2,\dots,m; \ k=0,1,\dots,N). \tag{21}$$

Combining (19) and (21), we get

$$\langle w_{ik}^*, \overline{g}_{ik}\rangle = 0 \ \ (i=1,2,\dots,m; \ k=0,1,\dots,N). \tag{22}$$

Since (20) and (22), we obtain the complementarity condition of Theorem 2.1. We have

$$N(A;\overline{z}) = \{M^*y^* : y^* \in \tilde{X}\}.$$

Since the Euler-Lagrange inclusion, there exist $y^* = (y_1^*, y_2^*, \dots, y_N^*) \in \tilde{X}$ such that

$$\nabla L(\overline{z}) + M^*y^* = 0.$$

This is equivalent to

$$\nabla f(\overline{z}) + w^* \circ \nabla F(\overline{z}) + M^*y^* = 0. \tag{23}$$

We get

$$\nabla f(\overline{z}) = \nabla f(\overline{x}, \overline{u}) = \Big(\frac{\partial \overline{h}_0}{\partial x_0}, \frac{\partial \overline{h}_1}{\partial x_1}, \ldots, \frac{\partial \overline{h}_N}{\partial x_N}, \frac{\partial \overline{h}_0}{\partial u_0}, \frac{\partial \overline{h}_1}{\partial u_1}, \ldots, \frac{\partial \overline{h}_{N-1}}{\partial u_{N-1}} \Big),$$

$$w^* \circ \nabla F(\overline{z}) = \Big(\sum_{i=1}^m \frac{\partial \overline{g}_{i0}}{\partial x_0} w_{i0}^*, \sum_{i=1}^m \frac{\partial \overline{g}_{i1}}{\partial x_1} w_{i1}^*, \ldots, \sum_{i=1}^m \frac{\partial \overline{g}_{iN-1}}{\partial x_{N-1}} w_{iN-1}^*, \sum_{i=1}^m \frac{\partial \overline{g}_{iN}}{\partial x_N} w_{iN}^*,$$
$$\sum_{i=1}^m \frac{\partial \overline{g}_{i0}}{\partial u_0} w_{i0}^*, \sum_{i=1}^m \frac{\partial \overline{g}_{i1}}{\partial u_1} w_{i1}^*, \sum_{i=1}^m \frac{\partial \overline{g}_{iN-1}}{\partial u_{N-1}} w_{iN-1}^* \Big),$$

and

$$M^* y^* = \big(-A_0^* y_1^*, y_1^* - A_1^* y_2^*, \, y_2^* - A_2^* y_3^*, \ldots, y_{N-1}^* - A_{N-1}^* y_N^*,$$
$$y_N^*, -B_0^* y_1^*, -B_1^* y_2^*, \ldots, -B_{N-1}^* y_N^* \big).$$

So,

$$(23) \Leftrightarrow \begin{cases} \frac{\partial \overline{h}_0}{\partial x_0} + \sum_{i=1}^m \frac{\partial \overline{g}_{i0}}{\partial x_0} w_{i0}^* - A_0^* y_1^* = 0 \\ \frac{\partial \overline{h}_k}{\partial x_k} + \sum_{i=1}^m \frac{\partial \overline{g}_{ik}}{\partial x_k} w_{ik}^* + y_k^* - A_k^* y_{k+1}^* = 0, \ k = 1, 2, \ldots, N-1 \\ \frac{\partial \overline{h}_N}{\partial x_N} + \sum_{i=1}^m \frac{\partial \overline{g}_{iN}}{\partial x_N} w_{iN}^* + y_N^* = 0 \\ \frac{\partial \overline{h}_k}{\partial u_k} + \sum_{i=1}^m \frac{\partial \overline{g}_{ik}}{\partial u_k} w_{ik}^* - B_k^* y_{k+1}^* = 0, \qquad k = 0, 1, \ldots, N-1; \end{cases}$$

this is the adjoint equation of Theorem 2.1. From

$$0 \in T^2\big(D; F(\overline{z}), \nabla F(\overline{z}) z\big),$$

we get

$$\sigma\big(z^*, T^2\big(D; F(\overline{z}), \nabla F(\overline{z}) z\big)\big) = \sup_{z \in T^2(D; F(\overline{z}), \nabla F(\overline{z}) z)} \langle z^*, z \rangle \geq \langle z^*, 0 \rangle = 0.$$

Since $z \in A(\overline{z}) = A = T(A; \overline{z})$, we have

$$T^2(A; \overline{z}, z) = T\big(T(A; \overline{z}); z\big) = T(A; z) = A.$$

So, for $w = z \in A = T^2(A; \overline{z}, z)$, the Legendre inequality implies that

$$\langle \nabla L(\overline{z}), z \rangle + \langle \nabla^2 L(\overline{z}) z, z \rangle \geq 0. \tag{24}$$

We have

$$\langle \nabla L(\overline{z}), z \rangle = \langle \nabla f(\overline{z}), z \rangle + \langle w^* \circ F(\overline{z}), z \rangle.$$

Since condition (C_1') and (19), we obtain

$$\langle \nabla L(\overline{z}), z\rangle = 0. \tag{25}$$

From (24) and (25), we get $\langle \nabla^2 L(\overline{z})z, z\rangle \geq 0$. This is equivalent to

$$\langle \nabla^2 f(\overline{z})z, z\rangle + \langle w^* \circ \nabla^2 F(\overline{z})z, z\rangle \geq 0. \tag{26}$$

We have

$$\nabla^2 f(\overline{z})z =$$

$$\begin{bmatrix}
\frac{\partial^2\overline{h}_0}{\partial x_0^2} & 0 & 0 & \dots & 0 & 0 & \frac{\partial^2\overline{h}_0}{\partial x_0\partial u_0} & 0 & 0 & \dots & 0\\
0 & \frac{\partial^2\overline{h}_1}{\partial x_1^2} & 0 & \dots & 0 & 0 & 0 & \frac{\partial^2\overline{h}_1}{\partial x_1\partial u_1} & 0 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \dots & \frac{\partial^2\overline{h}_{N-1}}{\partial x_{N-1}^2} & 0 & 0 & 0 & 0 & \dots & \frac{\partial^2\overline{h}_{N-1}}{\partial x_{N-1}\partial u_{N-1}}\\
0 & 0 & 0 & \dots & 0 & \frac{\partial^2\overline{h}_N}{\partial x_N^2} & 0 & 0 & 0 & \dots & 0\\
\frac{\partial^2\overline{h}_0}{\partial u_0\partial x_0} & 0 & 0 & \dots & 0 & 0 & \frac{\partial^2\overline{h}_0}{\partial u_0^2} & 0 & 0 & \dots & 0\\
0 & \frac{\partial^2\overline{h}_1}{\partial u_1\partial x_1} & 0 & \dots & 0 & 0 & 0 & \frac{\partial^2\overline{h}_1}{\partial u_1^2} & 0 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \dots & \frac{\partial^2\overline{h}_{N-1}}{\partial u_{N-1}\partial x_{N-1}} & 0 & 0 & 0 & 0 & \dots & \frac{\partial^2\overline{h}_{N-1}}{\partial u_{N-1}^2}
\end{bmatrix}
\begin{bmatrix} x_0\\ x_1\\ \vdots\\ x_N\\ u_0\\ u_1\\ \vdots\\ u_{N-1}\end{bmatrix}$$

$$\begin{aligned}
= \Big(\frac{\partial^2\overline{h}_0}{\partial x_0^2}x_0 + \frac{\partial^2\overline{h}_0}{\partial x_0\partial u_0}u_0, \frac{\partial^2\overline{h}_1}{\partial x_1^2}x_1 + \frac{\partial^2\overline{h}_1}{\partial x_1\partial u_1}u_1, \dots, \frac{\partial^2\overline{h}_{N-1}}{\partial x_{N-1}^2}x_{N-1}&\\
+ \frac{\partial^2\overline{h}_{N-1}}{\partial x_{N-1}\partial u_{N-1}}u_{N-1}, \frac{\partial^2\overline{h}_N}{\partial x_N^2}x_N, \frac{\partial^2\overline{h}_0}{\partial u_0\partial x_0}x_0 + \frac{\partial^2\overline{h}_0}{\partial u_0^2}u_0, \frac{\partial^2\overline{h}_1}{\partial u_1\partial x_1}x_1&\\
+ \frac{\partial^2\overline{h}_1}{\partial u_1^2}u_1, \dots, \frac{\partial^2\overline{h}_{N-1}}{\partial u_{N-1}\partial x_{N-1}}x_{N-1} + \frac{\partial^2\overline{h}_{N-1}}{\partial u_{N-1}^2}u_{N-1}\Big)&.
\end{aligned}$$

So,

$$\begin{aligned}
\langle \nabla^2 f(\overline{z})z, z\rangle = \sum_{k=0}^{N-1}\Big(\frac{\partial^2\overline{h}_k}{\partial x_k^2}x_k + \frac{\partial^2\overline{h}_k}{\partial x_k\partial u_k}u_k\Big)x_k + \frac{\partial^2\overline{h}_N}{\partial x_N^2}x_N^2&\\
+ \sum_{k=0}^{N-1}\Big(\frac{\partial^2\overline{h}_k}{\partial u_k\partial x_k}x_k + \frac{\partial^2\overline{h}_k}{\partial u_k^2}u_k\Big)u_k&.
\end{aligned}$$

Morever,

$$\nabla^2 F(\overline{z})zz = \Big(\frac{\partial^2 \overline{g}_{10}}{\partial x_0^2}x_0^2 + \big(\frac{\partial^2 \overline{g}_{10}}{\partial x_0 \partial u_0} + \frac{\partial^2 \overline{g}_{10}}{\partial u_0 \partial x_0}\big)x_0 u_0 + \frac{\partial^2 \overline{g}_{10}}{\partial u_0^2}u_0^2, \ldots, \frac{\partial^2 \overline{g}_{1N-1}}{\partial x_{N-1}^2}x_{N-1}^2$$
$$+ \big(\frac{\partial^2 \overline{g}_{1N-1}}{\partial x_{N-1} \partial u_{N-1}} + \frac{\partial^2 \overline{g}_{1N-1}}{\partial u_{N-1} \partial x_{N-1}}\big)x_{N-1}u_{N-1} + \frac{\partial^2 \overline{g}_{1N-1}}{\partial u_{N-1}^2}u_{N-1}^2, \frac{\partial^2 \overline{g}_{1N}}{\partial x_N^2}x_N^2,$$
$$\ldots, \frac{\partial^2 \overline{g}_{m0}}{\partial x_0^2}x_0^2 + \big(\frac{\partial^2 \overline{g}_{m0}}{\partial x_0 \partial u_0} + \frac{\partial^2 \overline{g}_{m0}}{\partial u_0 \partial x_0}\big)x_0 u_0 + \frac{\partial^2 \overline{g}_{m0}}{\partial u_0^2}u_0^2, \ldots, \frac{\partial^2 \overline{g}_{mN-1}}{\partial x_{N-1}^2}x_{N-1}^2$$
$$+ \big(\frac{\partial^2 \overline{g}_{mN-1}}{\partial x_{N-1} \partial u_{N-1}} + \frac{\partial^2 \overline{g}_{mN-1}}{\partial u_{N-1} \partial x_{N-1}}\big)x_{N-1}u_{N-1} + \frac{\partial^2 \overline{g}_{mN-1}}{\partial u_{N-1}^2}u_{N-1}^2, \frac{\partial^2 \overline{g}_{mN}}{\partial x_N^2}x_N^2\Big).$$

So,

$$\langle w^* \circ \nabla^2 F(\overline{z})z, z\rangle = \sum_{i=1}^{m}\sum_{k=0}^{N-1}\Big[\frac{\partial^2 \overline{g}_{ik}}{\partial x_k^2}x_k^2 + \big(\frac{\partial^2 \overline{g}_{ik}}{\partial x_k \partial u_k} + \frac{\partial^2 \overline{g}_{ik}}{\partial u_k \partial x_k}\big)x_k u_k + \frac{\partial^2 \overline{g}_{ik}}{\partial u_k^2}u_k^2\Big]w_{ik}^*$$
$$+ \sum_{i=1}^{m}\frac{\partial^2 \overline{g}_{iN}}{\partial x_N^2}x_N^2 w_{iN}^*.$$

By (26), we obtain

$$\sum_{k=0}^{N-1}\Big(\frac{\partial^2 \overline{h}_k}{\partial x_k^2}x_k + \frac{\partial^2 \overline{h}_k}{\partial x_k \partial u_k}u_k\Big)x_k + \frac{\partial^2 \overline{h}_N}{\partial x_N^2}x_N^2 + \sum_{k=0}^{N-1}\Big(\frac{\partial^2 \overline{h}_k}{\partial u_k \partial x_k}x_k + \frac{\partial^2 \overline{h}_k}{\partial u_k^2}u_k\Big)u_k$$
$$+ \sum_{i=1}^{m}\sum_{k=0}^{N-1}\Big[\frac{\partial^2 \overline{g}_{ik}}{\partial x_k^2}x_k^2 + \big(\frac{\partial^2 \overline{g}_{ik}}{\partial x_k \partial u_k} + \frac{\partial^2 \overline{g}_{ik}}{\partial u_k \partial x_k}\big)x_k u_k + \frac{\partial^2 \overline{g}_{ik}}{\partial u_k^2}u_k^2\Big]w_{ik}^*$$
$$+ \sum_{i=1}^{m}\frac{\partial^2 \overline{g}_{iN}}{\partial x_N^2}x_N^2 w_{iN}^* \geq 0;$$

which is non-negative second-order condition of Theorem 2.1. The proof of Theorem 2.1 is complete. □

6 Some Examples

To illustrate Theorem 2.1, we provide the following examples.

Example 6.1 Let $N = 2$, $X_0 = X_1 = X_2 = \mathbb{R}$, $U_0 = U_1 = \mathbb{R}$. We consider the problem of finding $u = (u_0, u_1) \in \mathbb{R}^2$ and $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ such that

$$\begin{cases} f(x,u) = \sum_{k=0}^{1}(x_k + u_k)^2 + \frac{1}{1+x_2^2} \to \inf, \\ x_{k+1} = x_k + u_k, \quad k = 0, 1, \\ x_0 - u_0 - 1 \le 0, \\ u_1 \le 0, \\ x_2 \le 0. \end{cases}$$

Suppose that $(\overline{x}, \overline{u})$ is a locally optimal solution of the problem. Then,

$$\overline{x} = (\alpha, 0, 0, 0); \quad \overline{u} = (-\alpha, 0, 0) \quad (\alpha \le \frac{1}{2}).$$

Indeed, it is easy to check that the functions

$$h_k = (x_k + u_k)^2 \ (k = 0, 1), \quad h_2 = \frac{1}{1 + x_2^2}$$

are second-order differentiable. We have

$$g_{10} = x_0 - u_0 - 1; \quad \frac{\partial g_{10}}{\partial x_0} = 1; \quad \frac{\partial g_{10}}{\partial u_0} = -1,$$

$$g_{11} = u_1; \quad \frac{\partial g_{11}}{\partial x_1} = 0; \quad \frac{\partial g_{11}}{\partial u_1} = 1,$$

and

$$g_{12} = x_2; \quad \frac{\partial g_{12}}{\partial x_2} = 1.$$

For each $(1, k) \in I(\overline{x}, \overline{u})$ and $v_{1k} \le 0$. We consider the following cases occur:
$(*)$ $I(\overline{x}, \overline{u}) = \emptyset$. It is easy to see that ssumption (A) is satisfied.
$(*)$ $I(\overline{x}, \overline{u}) = \{(1, 0)\}$. We choose

$$u_0 \in \mathbb{R}, \quad x_0 = u_0 + v_{10}.$$

Then,

$$\frac{\partial \overline{g}_{10}}{\partial x_0} x_0 + \frac{\partial \overline{g}_{10}}{\partial u_0} u_0 - v_{10} = x_0 - u_0 - v_{10} = 0.$$

Hence, assumption (A) is also satisfied.
$(*)$ $I(\overline{x}, \overline{u}) = \{(1, 1)\}$. We choose $x_0, u_0 \in \mathbb{R}$ such that $x_0 - u_0 - 1 \le 0$ and $u_1 = v_{11}$.
So,

$$x_1 = x_0 + u_0.$$

Then,

$$\frac{\partial \overline{g}_{11}}{\partial x_1}x_1 + \frac{\partial \overline{g}_{11}}{\partial u_1}u_1 - v_{11} = u_1 - v_{11} = 0.$$

Hence, assumption (A) is also satisfied.
$(*)$ $I(\overline{x}, \overline{u}) = \{(1,2)\}$. We choose

$$x_0 = u_0 = 0, \quad u_1 = v_{12}.$$

So,

$$x_1 = x_0 + u_0 = 0, \quad x_2 = x_1 + u_1 = v_{12}.$$

Then,

$$\frac{\partial \overline{g}_{12}}{\partial x_2}x_2 - v_{12} = x_2 - v_{12} = 0.$$

Hence, assumption (A) is also satisfied.
$(*)$ $I(\overline{x}, \overline{u}) = \{(1,0);(1,1)\}$. We choose

$$u_0 \in \mathbb{R}, \quad x_0 = u_0 + v_{10}, \quad u_1 = v_{11}.$$

So,

$$x_1 = x_0 + u_0.$$

Then,

$$\begin{cases} \frac{\partial \overline{g}_{10}}{\partial x_0}x_0 + \frac{\partial \overline{g}_{10}}{\partial u_0}u_0 - v_{10} = x_0 - u_0 - v_{10} = 0 \\ \frac{\partial \overline{g}_{11}}{\partial x_1}x_1 + \frac{\partial \overline{g}_{11}}{\partial u_1}u_1 - v_{11} = u_1 - v_{11} = 0. \end{cases}$$

Hence, assumption (A) is also satisfied.
$(*)$ $I(\overline{x}, \overline{u}) = \{(1,0);(1,2)\}$. We choose

$$u_0 = 0, \quad x_0 = v_{10}, \quad u_1 = v_{12}.$$

So,

$$x_1 = x_0 + u_0 = v_{10}, \quad x_2 = x_1 + u_1 = v_{10} + v_{12}.$$

Then,

$$\begin{cases} \frac{\partial \overline{g}_{10}}{\partial x_0}x_0 + \frac{\partial \overline{g}_{10}}{\partial u_0}u_0 - v_{10} = x_0 - u_0 - v_{10} = 0 \\ \frac{\partial \overline{g}_{12}}{\partial x_2}x_2 - v_{12} = x_2 - v_{12} = v_{10} \le 0. \end{cases}$$

Hence, assumption (A) is also satisfied.
$(*)$ $I(\overline{x}, \overline{u}) = \{(1,1);(1,2)\}$. We choose

$$u_0 = x_0 = 0, \quad u_1 = v_{11} + v_{12}.$$

So,

$$x_1 = x_0 + u_0 = 0, \quad x_2 = x_1 + u_1 = v_{11} + v_{12}.$$

Then,

$$\begin{cases} \frac{\partial \overline{g}_{11}}{\partial x_1} x_1 + \frac{\partial \overline{g}_{11}}{\partial u_1} u_1 - v_{11} = u_1 - v_{11} = v_{12} \leq 0 \\ \frac{\partial \overline{g}_{12}}{\partial x_2} x_2 - v_{12} = x_2 - v_{12} = v_{11} \leq 0. \end{cases}$$

Hence, assumption (A) is also satisfied.

$(*)$ $I(\overline{x}, \overline{u}) = \{(1,0); (1,1); (1,2)\}$. We choose

$$x_0 = v_{10} + v_{11} + v_{12}; \quad u_0 = v_{11} + v_{12}; \quad u_1 = v_{11}.$$

So,

$$x_1 = x_0 + u_0 = v_{10} + 2v_{11} + 2v_{12}; \quad x_2 = x_1 + u_1 = v_{10} + 3v_{11} + 2v_{12}.$$

Then,

$$\begin{cases} \frac{\partial \overline{g}_{10}}{\partial x_0} x_0 + \frac{\partial \overline{g}_{10}}{\partial u_0} u_0 - v_{10} = x_0 - u_0 - v_{10} = 0 \\ \frac{\partial \overline{g}_{11}}{\partial x_1} x_1 + \frac{\partial \overline{g}_{11}}{\partial u_1} u_1 - v_{11} = u_1 - v_{11} = 0 \\ \frac{\partial \overline{g}_{12}}{\partial x_2} x_2 - v_{12} = x_2 - v_{12} = v_{10} + 3v_{11} + v_{12} \leq 0. \end{cases}$$

Hence, assumption (A) of Theorem 2.1 is also satisfied. We have

$$A_0 = A_1 = B_0 = B_1 = 1,$$

$$A_0^* = A_1^* = B_0^* = B_1^* = 1,$$

and

$$\begin{aligned}
&\frac{\partial \overline{h}_k}{\partial x_k} = \frac{\partial \overline{h}_k}{\partial u_k} = 2(\overline{x}_k + \overline{u}_k), && k = 0, 1, \\
&\frac{\partial \overline{h}_2}{\partial x_2} = \frac{-2\overline{x}_2}{(1 + \overline{x}_2^2)^2}, \\
&\frac{\partial^2 \overline{h}_k}{\partial x_k^2} = \frac{\partial^2 \overline{h}_k}{\partial x_k \partial u_k} = \frac{\partial^2 \overline{h}_k}{\partial u_k \partial x_k} = \frac{\partial^2 \overline{h}_k}{\partial u_k^2} = 2, \quad k = 0, 1, \\
&\frac{\partial^2 \overline{h}_2}{\partial x_2^2} = \frac{6\overline{x}_2^4 + 4\overline{x}_2^2 - 2}{(1 + \overline{x}_2^2)^4}, \\
&\frac{\partial^2 \overline{g}_{1k}}{\partial x_k^2} = \frac{\partial^2 \overline{g}_{1k}}{\partial x_k \partial u_k} = \frac{\partial^2 \overline{g}_{1k}}{\partial u_k \partial x_k} = \frac{\partial^2 \overline{g}_{1k}}{\partial u_k^2} = 0, \quad k = 0, 1, \\
&\frac{\partial^2 \overline{g}_{12}}{\partial x_2^2} = 0.
\end{aligned}$$

By Theorem 2.1, for each $(x, u) \in \Theta(\overline{x}, \overline{u})$, there exist $w^* = (w_{10}^*, w_{11}^*, w_{12}^*) \in \mathbb{R}^3$ and $y^* = (y_1^*, y_2^*) \in \mathbb{R}^2$ such that the following conditions are fulfilled:

(a^*) Adjoint equation:

$$\begin{aligned}
&2(\overline{x}_0+\overline{u}_0)+w_{10}^*-y_1^*=0, \quad (27)\\
&2(\overline{x}_1+\overline{u}_1)+y_1^*-y_2^*=0,\\
&\frac{-2\overline{x}_2}{(1+\overline{x}_2^2)^2}+w_{12}^*+y_2^*=0,\\
&2(\overline{x}_0+\overline{u}_0)-w_{10}^*-y_1^*=0, \quad (28)\\
&2(\overline{x}_1+\overline{u}_1)+w_{11}^*-y_2^*=0;
\end{aligned}$$

(b^*) Non-negative second-order condition:

$$\sum_{k=0}^{1}2(x_k+u_k)x_k+\frac{6\overline{x}_2^4+4\overline{x}_2^2-2}{(1+\overline{x}_2^2)^4}x_2^2+\sum_{k=0}^{1}2(x_k+u_k)u_k\geq 0,$$

which is equivalent to

$$2\sum_{k=0}^{1}(x_k+u_k)^2+\frac{6\overline{x}_2^4+4\overline{x}_2^2-2}{(1+\overline{x}_2^2)^4}x_2^2\geq 0; \quad (29)$$

(c^*) Complementarity condition:

$$w_{1k}^*\geq 0 \quad (k=0,1,2),$$

and

$$\langle w_{1k}^*,\overline{g}_{1k}\rangle=0 \quad (k=0,1,2).$$

Since (27) and (28), we have $w_{10}^*=0$. From the complementarity condition, we get

$$w_{11}^*,w_{12}^*\geq 0,$$

and

$$\begin{cases} w_{11}^*\overline{u}_1=0\\ w_{12}^*\overline{x}_2=0.\end{cases}$$

We now consider the following four cases:

Case 1, $w_{11}^*=w_{12}^*=0$. Substituting $w_{10}^*=0$ and $w_{11}^*=w_{12}^*=0$ into the adjoint equation, we get

$$2(\overline{x}_0+\overline{u}_0)-y_1^*=0, \quad (30)$$

$$2(\overline{x}_1+\overline{u}_1)+y_1^*-y_2^*=0, \quad (31)$$

$$\frac{-2\overline{x}_2}{(1+\overline{x}_2^2)^2}+y_2^*=0, \quad (32)$$

$$2(\overline{x}_1+\overline{u}_1)-y_2^*=0. \quad (33)$$

From (31) and (33), we obtain $y_1^* = 0$. Since $\overline{x}_1 = \overline{x}_0 + \overline{u}_0$, $y_1^* = 0$ and (30), we have

$$\overline{x}_0 + \overline{u}_0 = 0, \quad \overline{x}_1 = 0.$$

So $\overline{x}_2 = \overline{x}_1 + \overline{u}_1 = \overline{u}_1$. From $\overline{x}_1 = 0$, $\overline{u}_1 = \overline{x}_2$ and equations (32), (33), we get

$$\frac{2\overline{x}_2}{(1 + \overline{x}_2^2)^2} = 2\overline{x}_2.$$

This is equivelent to $\overline{x}_2 = 0$. Hence, $\overline{u}_1 = \overline{x}_2 = 0$. Substituting $\overline{x}_2 = 0$ into (29), we get

$$(x_0 + u_0)^2 + (x_1 + u_1)^2 - x_2^2 \geq 0. \tag{34}$$

Since $(x, u) \in \Theta(\overline{x}, \overline{u})$, we have $x_2 = x_1 + u_1$. Hence, (34) is fulfilled. Thus, if $(\overline{x}, \overline{u})$ is a locally optimal solution of the problem then

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0),$$

with

$$\overline{x}_0 - \overline{u}_0 - 1 = 2\alpha - 1 \leq 0 \Leftrightarrow \alpha \leq \frac{1}{2}.$$

Case 2, $w_{11}^* = 0$ and $\overline{x}_2 = 0$. Substituting $w_{10}^* = 0$, $w_{11}^* = 0$ and $\overline{x}_2 = 0$ into the adjoint equation, we have

$$\begin{aligned}
&2(\overline{x}_0 + \overline{u}_0) - y_1^* = 0, \\
&2(\overline{x}_1 + \overline{u}_1) + y_1^* - y_2^* = 0, \\
&w_{12}^* + y_2^* = 0, && (35) \\
&2(\overline{x}_1 + \overline{u}_1) - y_2^* = 0. && (36)
\end{aligned}$$

Since $\overline{x}_1 + \overline{u}_1 = \overline{x}_2 = 0$ and (36), we have $y_2^* = 0$. Substituting $y_2^* = 0$ into (35), we get $w_{12}^* = 0$. By using similar Case 1, we can also prove that if $(\overline{x}, \overline{u})$ is a locally optimal solution of the problem then

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0),$$

with $\alpha \leq \frac{1}{2}$.

Case 3, $w_{12}^* = 0$ and $\overline{u}_1 = 0$. Substituting $w_{10}^* = 0$, $w_{12}^* = 0$ and $\overline{u}_1 = 0$ into the adjoint equation, we have

$$\begin{aligned}
&2(\overline{x}_0 + \overline{u}_0) - y_1^* = 0, && (37) \\
&2\overline{x}_1 + y_1^* - y_2^* = 0, && (38) \\
&\frac{-2\overline{x}_2}{(1 + \overline{x}_2^2)^2} + y_2^* = 0, && (39) \\
&2\overline{x}_1 + w_{11}^* - y_2^* = 0.
\end{aligned}$$

Since $\overline{x}_1 = \overline{x}_0 + \overline{u}_0$ and (37), we have $y_1^* = 2\overline{x}_1$. Substituting $y_1^* = 2\overline{x}_1$ into (38), we get $y_2^* = 4\overline{x}_1$. From $\overline{x}_2 = \overline{x}_1 + \overline{u}_1 = \overline{x}_1$, $y_2^* = 4\overline{x}_1$ and (39), we have

$$\frac{2\overline{x}_2}{(1+\overline{x}_2^2)^2} = 4\overline{x}_2.$$

This is equivalent to $\overline{x}_2 = 0$. So $\overline{x}_1 = \overline{x}_2 = 0$, $\overline{x}_0 + \overline{u}_0 = \overline{x}_1 = 0$. In the Case 1, we checked that

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0) \quad (\alpha \leq \frac{1}{2})$$

satisfies the non-negative second-order condition.

Case 4, $\overline{u}_1 = 0$ and $\overline{x}_2 = 0$. Since $\overline{x}_1 = \overline{x}_1 + \overline{u}_1 = \overline{x}_2 = 0$, we have $\overline{x}_0 + \overline{u}_0 = \overline{x}_1 = 0$. As in Case 1, we can also check that

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0) \quad (\alpha \leq \frac{1}{2})$$

satisfies the non-negative second-order condition.

The following example show that if the second-order necessary condition is not satisfied then the admissible couple is not solution even it satisfies first-necessary conditions.

Example 6.2 Let $N = 2$, $X_0 = X_1 = X_2 = \mathbb{R}$, $U_0 = U_1 = \mathbb{R}$. We consider the problem of finding $u = (u_0, u_1) \in \mathbb{R}^2$ and $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ such that

$$\begin{cases} f(x,u) = \frac{1}{4}\sum_{k=0}^{1}(x_k + u_k)^4 + \frac{2}{1+x_2^2} \to \inf, \\ x_{k+1} = x_k + u_k, \quad k = 0, 1, \\ x_0 - u_0 - 1 \leq 0, \\ u_1 \leq 0, \\ x_2 \leq 0. \end{cases}$$

Suppose that $(\overline{x}, \overline{u})$ is a locally optimal solution of the problem. Then, by first-order optimality conditions, we obtain

$$\overline{x} = (\alpha, 0, 0); \; \overline{u} = (-\alpha, 0) \quad (\alpha \leq \frac{1}{2}),$$

or

$$\overline{x} = (\alpha, 0, 1); \; \overline{u} = (-\alpha, 1) \quad (\alpha \leq \frac{1}{2}),$$

or

$$\overline{x} = (\alpha, 0, -1); \; \overline{u} = (-\alpha, -1) \quad (\alpha \leq \frac{1}{2}),$$

or

$$\overline{x} = (\alpha, \sqrt{a}, \sqrt{a}); \quad \overline{u} = (\sqrt{a} - \alpha, 0) \ \ (\alpha \le \frac{1+\sqrt{a}}{2}),$$

where $a \in (\frac{1}{2}, 1) \subset [0, \infty)$ is the unique solution of the following equation

$$X^3 + 2X^2 + X - 2 = 0.$$

If we let

$$\overline{x}^1 = (\alpha, 0, 0); \ \overline{u}^1 = (-\alpha, 0) \ \ (\alpha \le \frac{1}{2}),$$

then $(\overline{x}^1, \overline{u}^1)$ does not satisfy the second-order optimality conditions for any $\alpha \le \frac{1}{2}$. Hence, $(\overline{x}^1, \overline{u}^1)$ is not a locally optimal solution of the problem. Thus, if $(\overline{x}; \overline{u})$ is a locally optimal solution of the problem, then

$$\overline{x} = (\alpha, 0, 1); \quad \overline{u} = (-\alpha, 1) \ \ (\alpha \le \frac{1}{2}),$$

or

$$\overline{x} = (\alpha, 0, -1); \quad \overline{u} = (-\alpha, -1) \ \ (\alpha \le \frac{1}{2}),$$

or

$$\overline{x} = (\alpha, \sqrt{a}, \sqrt{a}); \quad \overline{u} = (\sqrt{a} - \alpha, 0) \ \ (\alpha \le \frac{1+\sqrt{a}}{2}),$$

where $a \in (\frac{1}{2}, 1) \subset [0, \infty)$ is the unique solution of the following equation

$$X^3 + 2X^2 + X - 2 = 0.$$

Indeed, it is easy to check that the functions

$$h_k = \frac{1}{4}(x_k + u_k)^4 \ (k = 0, 1), \quad h_2 = \frac{2}{1 + x_2^2}$$

are second-order differentiable. We have

$$g_{10} = x_0 - u_0 - 1; \quad \frac{\partial g_{10}}{\partial x_0} = 1; \quad \frac{\partial g_{10}}{\partial u_0} = -1,$$

$$g_{11} = u_1; \quad \frac{\partial g_{11}}{\partial x_1} = 0; \quad \frac{\partial g_{11}}{\partial u_1} = 1,$$

and

$$g_{12} = x_2; \quad \frac{\partial g_{12}}{\partial x_2} = 1.$$

In Example 6.1, we checked that condition (A) of the Theorem 2.1 is satisfied. Hence, the assumptions of Theorem 2.1 are fulfilled. We have

$$A_0 = A_1 = B_0 = B_1 = 1,$$

$$A_0^* = A_1^* = B_0^* = B_1^* = 1,$$

and

$$\begin{aligned}
&\frac{\partial \overline{h}_k}{\partial x_k} = \frac{\partial \overline{h}_k}{\partial u_k} = (\overline{x}_k + \overline{u}_k)^3, && k = 0, 1,\\
&\frac{\partial \overline{h}_2}{\partial x_2} = \frac{-4\overline{x}_2}{(1+\overline{x}_2^2)^2},\\
&\frac{\partial^2 \overline{h}_k}{\partial x_k^2} = \frac{\partial^2 \overline{h}_k}{\partial x_k \partial u_k} = \frac{\partial^2 \overline{h}_k}{\partial u_k \partial x_k} = \frac{\partial^2 \overline{h}_k}{\partial u_k^2} = 3(\overline{x}_k + \overline{u}_k)^2, && k = 0, 1,\\
&\frac{\partial^2 \overline{h}_2}{\partial x_2^2} = \frac{12\overline{x}_2^4 + 8\overline{x}_2^2 - 4}{(1+\overline{x}_2^2)^4},\\
&\frac{\partial^2 \overline{g}_{1k}}{\partial x_k^2} = \frac{\partial^2 \overline{g}_{1k}}{\partial x_k \partial u_k} = \frac{\partial^2 \overline{g}_{1k}}{\partial u_k \partial x_k} = \frac{\partial^2 \overline{g}_{1k}}{\partial u_k^2} = 0, && k = 0, 1,\\
&\frac{\partial^2 \overline{g}_{12}}{\partial x_2^2} = 0.
\end{aligned}$$

By Theorem 2.1, for each $(x, u) \in \Theta(\overline{x}, \overline{u})$, there exist $w^* = (w_{10}^*, w_{11}^*, w_{12}^*) \in \mathbb{R}^3$ and $y^* = (y_1^*, y_2^*) \in \mathbb{R}^2$ such that the following conditions are fulfilled:

(a_1^*) Adjoint equation:

$$\begin{aligned}
&(\overline{x}_0 + \overline{u}_0)^3 + w_{10}^* - y_1^* = 0, && (40)\\
&(\overline{x}_1 + \overline{u}_1)^3 + y_1^* - y_2^* = 0,\\
&\frac{-4\overline{x}_2}{(1+\overline{x}_2^2)^2} + w_{12}^* + y_2^* = 0,\\
&(\overline{x}_0 + \overline{u}_0)^3 - w_{10}^* - y_1^* = 0, && (41)\\
&(\overline{x}_1 + \overline{u}_1)^3 + w_{11}^* - y_2^* = 0;
\end{aligned}$$

(b_1^*) Non-negative second-order condition:

$$\sum_{k=0}^{1} 3(\overline{x}_k + \overline{u}_k)^2 (x_k + u_k) x_k + \frac{12\overline{x}_2^4 + 8\overline{x}_2^2 - 4}{(1+\overline{x}_2^2)^4} x_2^2 + \sum_{k=0}^{1} 3(\overline{x}_k + \overline{u}_k)^2 (x_k + u_k) u_k \geq 0,$$

which is equivalent to

$$\sum_{k=0}^{1} 3(\overline{x}_k + \overline{u}_k)^2 (x_k + u_k)^2 + \frac{12\overline{x}_2^4 + 8\overline{x}_2^2 - 4}{(1+\overline{x}_2^2)^4} x_2^2 \geq 0; \tag{42}$$

(c_1^*) Complementarity condition:

$$w_{1k}^* \geq 0 \quad (k = 0, 1, 2),$$

and

$$\langle w_{1k}^*, \overline{g}_{1k}\rangle = 0 \quad (k = 0, 1, 2).$$

Since (40) and (41), we have $w_{10}^* = 0$. From the complementarity condition, we get

$$w_{11}^*, w_{12}^* \geq 0, \tag{43}$$

and

$$\begin{cases} w_{11}^* \overline{u}_1 = 0 \\ w_{12}^* \overline{x}_2 = 0. \end{cases}$$

We now consider the following four cases:

Case 1, $w_{11}^* = w_{12}^* = 0$. Substituting $w_{10}^* = 0$ and $w_{11}^* = w_{12}^* = 0$ into the adjoint equation, we get

$$(\overline{x}_0 + \overline{u}_0)^3 - y_1^* = 0, \tag{44}$$
$$(\overline{x}_1 + \overline{u}_1)^3 + y_1^* - y_2^* = 0, \tag{45}$$
$$\frac{-4\overline{x}_2}{(1+\overline{x}_2^2)^2} + y_2^* = 0, \tag{46}$$
$$(\overline{x}_1 + \overline{u}_1)^3 - y_2^* = 0. \tag{47}$$

From (45) and (47), we obtain $y_1^* = 0$. Since $\overline{x}_1 = \overline{x}_0 + \overline{u}_0$, $y_1^* = 0$ and (44), we have

$$\overline{x}_0 + \overline{u}_0 = 0, \quad \overline{x}_1 = 0.$$

So $\overline{x}_2 = \overline{x}_1 + \overline{u}_1 = \overline{u}_1$. From $\overline{x}_1 = 0$, $\overline{u}_1 = \overline{x}_2$ and equations (46), (47), we get

$$\frac{4\overline{x}_2}{(1+\overline{x}_2^2)^2} = \overline{x}_2^3.$$

This implies that

$$\overline{u}_1 = \overline{x}_2 = 0,$$

or

$$\overline{u}_1 = \overline{x}_2 = 1$$

or

$$\overline{u}_1 = \overline{x}_2 = -1.$$

Thus, if $(\overline{x}, \overline{u})$ is a locally optimal solution of the problem, then by first-order optimality conditions, we obtain

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0) \quad (\alpha \leq \frac{1}{2}),$$

or

$$\overline{x} = (\alpha, 0, 1); \quad \overline{u} = (-\alpha, 1) \quad (\alpha \leq \frac{1}{2}),$$

or

$$\overline{x} = (\alpha, 0, -1); \quad \overline{u} = (-\alpha, -1) \quad (\alpha \leq \frac{1}{2}).$$

+) Substituting

$$\overline{x}_0 = \alpha, \ \overline{u}_0 = -\alpha, \ \overline{x}_1 = \overline{u}_1 = 0, \ \overline{x}_2 = 0$$

into (42), we obtain

$$-4x_2^2 \geq 0. \tag{48}$$

But, (48) is not fulfilled if $x = (-1, -1, -3)$, $u = (0, -2)$, $(x, u) \in \Theta(\overline{x}, \overline{u})$. Hence,

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0) \quad (\alpha \leq \frac{1}{2})$$

is not a locally optimal solution of the problem.
+) Substituting

$$\overline{x} = (\alpha, 0, 1); \quad \overline{u} = (-\alpha, 1) \quad (\alpha \leq \frac{1}{2}),$$

or

$$\overline{x} = (\alpha, 0, -1); \quad \overline{u} = (-\alpha, -1) \quad (\alpha \leq \frac{1}{2})$$

into (42), we obtain

$$3(x_1 + u_1)^2 + x_2^2 \geq 0;$$

this is always fulfilled. Thus, if $(\overline{x}; \overline{u})$ is a locally optimal solution of the problem then

$$\overline{x} = (\alpha, 0, 1); \quad \overline{u} = (-\alpha, 1) \quad (\alpha \leq \frac{1}{2}),$$

or

$$\overline{x} = (\alpha, 0, -1); \quad \overline{u} = (-\alpha, -1) \quad (\alpha \leq \frac{1}{2}).$$

Case 2, $w_{11}^* = 0$ and $\overline{x}_2 = 0$. Substituting $w_{10}^* = 0$, $w_{11}^* = 0$ and $\overline{x}_2 = 0$ into the adjoint equation, we have

$$(\overline{x}_0 + \overline{u}_0)^3 - y_1^* = 0, \tag{49}$$

$$(\overline{x}_1 + \overline{u}_1)^3 + y_1^* - y_2^* = 0, \tag{50}$$

$$w_{12}^* + y_2^* = 0,$$

$$(\overline{x}_1 + \overline{u}_1)^3 - y_2^* = 0. \tag{51}$$

From (50) and (51), we get $y_1^* = 0$. Since (49) and $y_1^* = 0$, we have $2(\overline{x}_0 + \overline{u}_0) = 0$. So $\overline{x}_1 = \overline{x}_0 + \overline{u}_0 = 0$. Hence, $\overline{u}_1 = \overline{x}_1 + \overline{u}_1 = \overline{x}_2 = 0$. Thus, if $(\overline{x}, \overline{u})$ is a locally optimal solution of the problem, then by first-order optimality conditions, we obtain

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0) \quad (\alpha \leq \frac{1}{2}).$$

In the Case 1, we showed that

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0) \quad (\alpha \leq \frac{1}{2})$$

does not satisfy the second-order optimality conditions.

Case 3, $\overline{u}_1 = 0$ and $\overline{x}_2 = 0$. Since $\overline{x}_1 = \overline{x}_1 + \overline{u}_1 = \overline{x}_2 = 0$, we have $\overline{x}_0 + \overline{u}_0 = \overline{x}_1 = 0$. As in Case 1, we can also check that

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0) \quad (\alpha \leq \frac{1}{2})$$

is not a locally optimal solution of the problem.

Case 4, $w_{12}^* = 0$ and $\overline{u}_1 = 0$. Substituting $w_{10}^* = 0$, $w_{12}^* = 0$ and $\overline{u}_1 = 0$ into the adjoint equation, we have

$$(\overline{x}_0 + \overline{u}_0)^3 - y_1^* = 0, \tag{52}$$

$$\overline{x}_1^3 + y_1^* - y_2^* = 0, \tag{53}$$

$$\frac{-4\overline{x}_2}{(1 + \overline{x}_2^2)^2} + y_2^* = 0, \tag{54}$$

$$\overline{x}_1^3 + w_{11}^* - y_2^* = 0. \tag{55}$$

Since $\overline{x}_1 = \overline{x}_0 + \overline{u}_0$ and (52), we have $y_1^* = \overline{x}_1^3$. Substituting $y_1^* = \overline{x}_1^3$ into (53), we get $y_2^* = 2\overline{x}_1^3$. From $\overline{x}_2 = \overline{x}_1 + \overline{u}_1 = \overline{x}_1$, $y_2^* = 2\overline{x}_1^3$ and (54), we have

$$\frac{4\overline{x}_2}{(1 + \overline{x}_2^2)^2} = 2\overline{x}_2^3.$$

This implies that

$$\overline{x}_1 = \overline{x}_2 = 0,$$

or

$$\overline{x}_1 = \overline{x}_2 = \sqrt{a},$$

or

$$\overline{x}_1 = \overline{x}_2 = -\sqrt{a},$$

where $a \in (\frac{1}{2}, 1) \subset [0, \infty)$ is the unique solution of the following equation

$$X^3 + 2X^2 + X - 2 = 0.$$

If we let $\overline{x}_1 = -\sqrt{a}$, $y_2^* = 2\overline{x}_1^3$ then by (55), we get

$$w_{11}^* = \overline{x}_1^3 = -a\sqrt{a} < 0;$$

this is not satisfied (43). Thus, if $(\overline{x}, \overline{u})$ is a locally optimal solution of the problem, then by first-order optimality conditions, we obtain

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0) \quad (\alpha \leq \frac{1}{2}),$$

or

$$\overline{x} = (\alpha, \sqrt{a}, \sqrt{a}); \quad \overline{u} = (\sqrt{a} - \alpha, 0) \quad (\alpha \leq \frac{1+\sqrt{a}}{2}).$$

As in Case 1, we can also check that

$$\overline{x} = (\alpha, 0, 0); \quad \overline{u} = (-\alpha, 0) \quad (\alpha \leq \frac{1}{2})$$

is not a locally optimal solution of the problem. Substituting

$$\overline{x} = (\alpha, \sqrt{a}, \sqrt{a}); \quad \overline{u} = (\sqrt{a} - \alpha, 0) \quad (\alpha \leq \frac{1+\sqrt{a}}{2})$$

into (42), we obtain

$$\sum_{k=0}^{1} 3a(x_k + u_k)^2 + \frac{12a^2 + 8a - 4}{(1+a)^4} x_2^3 \geq 0. \tag{56}$$

Since $a \in (\frac{1}{2}, 1)$, we have $8a - 4 > 0$. So, (56) is always fulfilled. Thus, if $(\overline{x}; \overline{u})$ is a locally optimal solution of the problem then

$$\overline{x} = (\alpha, \sqrt{a}, \sqrt{a}); \quad \overline{u} = (\sqrt{a} - \alpha, 0) \quad (\alpha \leq \frac{1+\sqrt{a}}{2}).$$

7 Perspectives

In this paper, we derived the second-order necessary optimality conditions for the discrete optimal control problems in the case where objective functions are nonconvex and mixed constraints.

There are many open problems related to this research topic. Some problems are stated directly in this paper. In particular, Theorem 2.1 obtained the first-order and the second-order necessary optimality conditions for discrete optimal control problem (1)–(3) in the case where dynamics (2) are linear. It is noted that if dynamics are linear then set A defined by (15), is convex. So, we can apply Theorem 4.1. However, the situation will be more complicated if dynamics are nonlinear. The existence of similar results as in Theorems 2.1 and 4.1 is an open question. Moreover, sufficient optimality conditions for problem $(1) - (3)$ and the above mentioned problem are still open.

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