A FINITEDIMENSIONAL VERSION OF FREDHOLM REPRESENTATIONS

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Abstract. We consider pairs of maps from a discrete group Γ to the unitary group. The deficiencies of these maps from being homomorphisms may be great, but if they are close to each other then we call such pairs *balanced*. We show that balanced pairs determine elements in the K^0 group of the classifying space of the group. We also show that a Fredholm representation of Γ determines balanced pairs.

1. Introduction

It is well known that various generalizations of unitary group representations, e.g. almost representations, Fredholm representations, representations into $U(p,q)$, quasi-representations etc. can be viewed as representatives of the K-homology of the group C^*-algebra. It is interesting to know, how far we can generalize the notion of a representation (or a pair of representations) keeping the property to determine a class in K-homology. It was shown recently in [2] that K-theory elements can be represented not necessarily by pairs of projections, but by pairs satisfying weaker properties. We follow this to find a generalization for pairs of matrix-valued functions on a group called balanced pairs.

Let Γ be a finitely generated group, and let M_n denote the C^*-algebra of operators on the n-dimensional Hilbert space. Given a map $\pi : \Gamma \to M_n$, and $g,h \in \Gamma$, denote by $M_\pi(g,h) = \pi(gh) - \pi(g)\pi(h)$ the defect, i.e. the deviation of π from multiplicativity.

Definition 1. Given a finite set $F \in \Gamma$ and $\varepsilon > 0$, a pair (π^+, π^-) of maps $\Gamma \to M_n$ satisfying $\pi^\pm(g^{-1}) = \pi^\pm(g)^*$ for any $g \in F$, is (F,ε)-*admissible* if

$$\|M_{\pi^\pm}(g,h)(\pi^+(\gamma) - \pi^-(\gamma))\| < \varepsilon \tag{1}$$

for any $g,h,\gamma \in F$. A pair (π^+, π^-) satisfying $\pi^\pm(g^{-1}) = \pi^\pm(g)^*$ for any $g \in F$, is (F,ε)-*balanced* if

$$\|M_{\pi^+}(g,h) - M_{\pi^-}(g,h)\| < \varepsilon \tag{2}$$

for any $g,h \in F$, and

$$\|\pi^+(k)M_{\pi^+}(g,h) - \pi^-(k)M_{\pi^-}(g,h)\| < \varepsilon \tag{3}$$

for any $g,h,k \in F$.

A family of pairs of maps $(\pi_n^\pm)_{n\in\mathbb{N}} : \Gamma \to M_{k_n}$ is *asymptotically admissible* (resp., *asymptotically balanced*) if, for any finite $F \subset \Gamma$, the pair (π_n^+, π_n^-) is (F,ε_n)-admissible (resp., balanced) with $\varepsilon_n \to 0$ as $n \to \infty$. We also use these terms for families of maps with continuous parameter $t \in [0,\infty)$.

A similar definition appeared in [1], but there we required that the range of the maps $\pi^\pm$ lies in the unitary group of M_n. Here we don't assume $\pi^\pm(g)$ to be invertible.

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We shall show that asymptotically admissible (resp., balanced) pairs can be viewed as finitedimensional versions of Fredholm representations.

2. Making Fredholm representations finitedimensional

The definition of Fredholm representations is due to A.S.Mishchenko [3]. A Fredholm representation is a triple (π^+, F, π^-) of two representations, π^+, π^-, of Γ, on a Hilbert space H, and of a Fredholm 'intertwining' operator F on H, such that $\pi^+(g)F - F\pi^-(g)$ is compact for any $g \in \Gamma$. Well known simplifications allow to drop out either one of the two representations, or F. For example, we may change F by either an isometry or a coisometry, and then change π^- by $F\pi^-F^*$ or π^+ by $F^*\pi^+F$, which gives us a triple of the form $(\pi^+, \mathrm{Id}, \pi^-)$. So, from now on, let us consider pairs (π^+, π^-) with $\pi^+(g) - \pi^-(g)$ compact for any $g \in \Gamma$ as Fredholm representations.

Let $L_n \in \mathbb{B}(H)$ be an increasing sequence of subspaces, such that $\dim L_n = n$ and $\cup_{n\in\mathbb{N}} L_n$ is dense in H, and let P_n denote the projection onto L_n. For any operator A, set $A_n = P_n A|_{L_n}$. Similarly, we write π_n for the map given by $g \mapsto (\pi(g))_n$.

Theorem 2. *For any Fredholm representation (π^+, π^-), the sequence (π_n^+, π_n^-) is asymptotically admissible and asymptotically balanced.*

Proof. One obviously has $\pi_n^{\pm}(g^{-1}) = \pi_n^{\pm}(g)^*$ for any $g \in \Gamma$. Let us check (1). Since $\pi^+(\gamma) - \pi^-(\gamma)$ is compact for any $\gamma \in \Gamma$, so for any $\varepsilon > 0$ and for any finite $F \subset \Gamma$, there is K such that for any $n > k > K$ one has

$$\|\pi_n^+(\gamma) - \pi_n^-(\gamma) - P_k(\pi_n^+(\gamma) - \pi_n^-(\gamma))P_k\| < \varepsilon$$

for any $\gamma \in F$. Fix $k > K$, then it suffices to show that for sufficiently large n, $\|P_k(\pi_n^{\pm}(gh) - \pi_n^{\pm}(g)\pi_n^{\pm}(h))\|$ and $\|(\pi_n^{\pm}(gh) - \pi_n^{\pm}(g)\pi_n^{\pm}(h))P_k\|$ can be made smaller than ε. As the two terms are similar, let us estimate the first one.

We have

$$\|P_k(\pi_n^{\pm}(gh) - \pi_n^{\pm}(g)\pi_n^{\pm}(h))\| = \|P_k P_n \pi^{\pm}(g)(1 - P_n)\pi^{\pm}(h)P_n\|$$
$$= \|P_k \pi^{\pm}(g)(1 - P_n)\pi^{\pm}(h)P_n\| \leq \|P_k \pi^{\pm}(g)(1 - P_n)\|.$$

Then for a finite number of elements $g \in F \subset \Gamma$ and for the fixed k it is always possible to find N such that for any $n > N$ one has $\|P_k \pi^{\pm}(g)(1 - P_n)\| < \varepsilon$.

Now let us check (2). We use the notation from the previous paragraph, namely the projections P_k and P_n. Decompose the Hilbert space H as $H = P_k H \oplus (P_n - P_k)H \oplus (1 - P_n)H$, and write operators on H as 3×3 matrices with respect to this decomposition.

Let $\pi^{\pm}(g) = A^{\pm} = (a_{ij}^{\pm})_{i,j=1}^3$, $\pi^{\pm}(h) = B^{\pm} = (b_{ij}^{\pm})_{i,j=1}^3$, $\pi^{\pm}(gh) = C^{\pm} = (c_{ij}^{\pm})_{i,j=1}^3$. We know that

$$A^{\pm}B^{\pm} = C^{\pm}, \tag{4}$$

and that $\|x_{ij}^+ - x_{ij}^-\| < \varepsilon$, where $x = a, b, c$, for all $i, j = 1, 2, 3$ except the case $i = j = 1$. Using (4), we obtain that

$$M_{\pi_n^{\pm}}(g,h) = \begin{pmatrix} c_{11}^{\pm} & c_{12}^{\pm} \\ c_{21}^{\pm} & c_{22}^{\pm} \end{pmatrix} - \begin{pmatrix} a_{11}^{\pm} & a_{12}^{\pm} \\ a_{21}^{\pm} & a_{22}^{\pm} \end{pmatrix} \begin{pmatrix} b_{11}^{\pm} & b_{12}^{\pm} \\ b_{21}^{\pm} & b_{22}^{\pm} \end{pmatrix} = \begin{pmatrix} a_{13}^{\pm}b_{31}^{\pm} & a_{13}^{\pm}b_{32}^{\pm} \\ a_{23}^{\pm}b_{21}^{\pm} & a_{23}^{\pm}b_{32}^{\pm} \end{pmatrix},$$

hence $\|M_{\pi_n^+}(g,h) - M_{\pi_n^-}(g,h)\| \leq 4\max_{(i,j),(k,l)\neq(1,1)} \|a_{ij}^+ b_{kl}^+ - a_{ij}^- b_{kl}^-\| < 8\varepsilon$.

Finally, to obtain (3), we have to combine (1) and (2).

$\square$

One can replace the discrete parameter by a continuous one. This follows from the following Lemma.

Lemma 3. *For $t \in [n, n+1]$ set $\pi_t^{\pm}(g) = t\pi_n^{\pm}(g) + (1-t)\pi_{n+1}^{\pm}(g)$. Then the family of pairs (π_t^+, π_t^-) is asymptotically admissible and asymptotically balanced.*

Proof. Direct calculation similar to that above. □

Let X be a compact Hausdorff space, $\pi_1(X) = \Gamma$, $\{U_i\}_{i\in I}$ a finite covering, and let φ_i, $i \in I$, be continuous functions on X such that $0 \le \varphi_i(x) \le 1$, $i \in I$, $x \in X$, $\operatorname{supp} \varphi_i \subset U_i$ and $\sum_{i\in I} \varphi_i^2(x) = 1$ for any $x \in X$. Let $\gamma = \{\gamma_{ij}\}_{i,j\in I}$ be a Γ-valued cocycle, i.e. $\gamma_{ji} = \gamma_{ij}^{-1}$ for any $i, j \in I$, and $\gamma_{ij} \in \Gamma$ and $\gamma_{ij}\gamma_{jk} = \gamma_{ik}$ whenever $U_i \cap U_j \cap U_k$ is not empty.

Then

$$p(x) = (p_{ij}(x))_{i,j\in I}, \text{ where } p_{ij}(x) = \varphi_i(x)\varphi_j(x)\gamma_{ij}, \tag{5}$$

is known to be idempotent for each $x \in X$.

For a map $\pi : \Gamma \to M_n$, put

$$A_\pi(x) = \pi(p(x)) = (\varphi_i(x)\varphi_j(x)\pi(\gamma_{ij}))_{i,j\in I}. \tag{6}$$

When π is a (unitary) group representation then A_π is a (selfadjoint) projection.

For shortness' sake set $A_{\pi^+} = a$, $A_{\pi^-} = b$. Let $\delta = |I| \cdot \varepsilon$.

If (π^+, π^-) is (F, ε)-admissible then the pair (a, b) satisfies the following conditions:

$$a^* = a; \quad b^* = b; \tag{7}$$

$$\|(a^2 - a)(a - b)\| < \delta; \quad \|(b^2 - b)(a - b)\| < \delta. \tag{8}$$

If (π^+, π^-) is (F, ε)-balanced then the pair (a, b) satisfies (7) and

$$\|f(a) - f(b)\| < \delta \tag{9}$$

for $f(t) = t(1-t)$ and $f(t) = t^2(1-t)$.

3. Relation to K-theory

Consider the following two sets of relations on selfadjoints a and b:

$$0 \le a, b \le 1; \quad (a^2 - a)(a - b) = (b^2 - b)(a - b) = 0; \tag{10}$$

and

$$p(a) = p(b) \text{ for } p(t) = t(1-t) \text{ and } p(t) = t^2(1-t). \tag{11}$$

In [2] it was shown that the K_0 group of a C^*-algebra A is the set of homotopy classes of selfadjoint pairs (a, b) of matrices over A satisfying either (10) or (11).

As we may replace exact projections by almost projections (i.e. selfadjoints A with $\|A^2 - A\| < \frac{1}{4}$), so the relations (10) and (11) can be replced by their approximate versions: (8) plus $0 \le a, b \le 1$, and (9), respectively, for sufficiently small δ. It was shown in [2] in particular, that the element of the K_0 group corresponding to a pair (a, b) satisfying (10) is given by the formal difference $[P] - [Q]$, where

$$P = P(a, b) = \begin{pmatrix} 1-b & g(a) \\ g(a) & a \end{pmatrix}$$

is a projection ($g(t) = \sqrt{t - t^2}$) and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. If a and b satisfy (8) and $0 \le a, b \le 1$ then P is only an almost projection, but $[P] - [Q]$ still determines an element in K_0.

Note that if a, b are genuine projections then $P = P(a,b) = \begin{pmatrix} 1-b & 0 \\ 0 & a \end{pmatrix}$, hence $[P]-[Q] = [1-b]+[a]-[1] = [a]-[b]$.

No explicit formula was given in [2] for pairs satisfying (11). Since $A_{\pi^\pm}$ do not need to satisfy $0 \le A_{\pi^\pm} \le 1$, there are two ways to proceed. We may apply the cutting function h, $h(t) = \begin{cases} 0, & t<0; \\ t, & 0 \le t \le 1; \\ 1, & t>1 \end{cases}$ to make $h(A_{\pi^\pm})$ satisfy it. This approach was used in [1], but it does not give explicit formulas. In this paper we present an explicit formula for an almost projection P when a, b satisfy the relations (9).

Note that P is unitarily equivalent to

$$P' = P'(a,b) = \begin{pmatrix} 1+(1-a)^{1/2}(a-b)(1-a)^{1/2} & (1-a)^{1/2}(b-a)a^{1/2} \\ a^{1/2}(b-a)(1-a)^{1/2} & a^{1/2}(a-b)a^{1/2} \end{pmatrix}$$

via the unitary $U = \begin{pmatrix} (1-a)^{1/2} & -a^{1/2} \\ a^{1/2} & (1-a)^{1/2} \end{pmatrix}$, $P' = U^*PU$.

Set

$$P'' = P''(a,b) = \begin{pmatrix} 1+(1-a)(a-b)(1-a) & (1-a)(b-a)a \\ a(b-a)(1-a) & a(a-b)a \end{pmatrix}.$$

Lemma 4. *Let a, b be selfadjoints satisfying $0 \le a, b \le 1$ and $p(a) = p(b)$ for $p(t) = t(1-t)$ and $p(t) = t^2(1-t)$. Then $P'(a,b) = P''(a,b)$.*

Proof. Suppose that $a-a^2 = b-b^2$ and $a^2-a^3 = b^2-b^3$. Then

$$(a^2-a)(a-b) = a^3-a^2-(a^2-a)b = a^3-a^2-(b^2-b)b = 0.$$

Similarly, $(b^2-b)(a-b) = 0$.

There is (see [2]) a universal C^*-algebra D generated by two selfadjoint positive contractions a, b subject to the relations $(a^2-a)(a-b) = (b^2-b)(a-b) = 0$. It was shown in [2] that $D \subset C([-1,1]; M_2)$ is a subalgebra of matrix-valued functions

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in C([-1,1]; M_2)$$

such that

$$f_{11}(-1) = 0; \quad f_{12}(t) = f_{21}(t) = f_{22}(t) = 0 \text{ for } t \in [-1,0]; \quad f_{12}(1) = f_{21}(1) = 0.$$

The generators $a, b \in D$ are given by the formulas

$$a(t) = \begin{cases} \begin{pmatrix} \cos^2 \frac{\pi}{2}t & 0 \\ 0 & 0 \end{pmatrix} & \text{for } t \in [-1,0]; \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } t \in [0,1], \end{cases}$$

$$b(t) = \begin{cases} \begin{pmatrix} \cos^2 \frac{\pi}{2}t & 0 \\ 0 & 0 \end{pmatrix} & \text{for } t \in [-1,0]; \\ \begin{pmatrix} \cos^2 \frac{\pi}{2}t & \cos\frac{\pi}{2}t \sin\frac{\pi}{2}t \\ \cos\frac{\pi}{2}t \sin\frac{\pi}{2}t & \sin^2 \frac{\pi}{2}t \end{pmatrix} & \text{for } t \in [0,1]. \end{cases}$$

Notice that $a(t)-b(t) = 0$ for $t \le 0$, and that $a(t)^{1/2} = a(t)$, $(1-a(t))^{1/2} = 1-a(t)$ for $t \ge 0$, therefore, P' equals P''.

$\square$

Lemma 5. *Let a, b be two selfadjoints satisfying $p(a) = p(b)$, where $p(t)$ is either $t(1-t)$ or $t^2(1-t)$ (but not necessarily $0 \le a, b \le 1$). Then P'' is a projection.*

Proof. Direct calculation. □

Remark 6. It follows from Lemma 5 that if the relations $p(a) = p(b)$ are true only up to some small value then P'' is an almost projection.

The advantage of P'' compared with P' is that P'' is polynomial in a and b, hence easier to use in calculations.

Set $h(t) = \begin{cases} 1, & \text{for } t > 1; \\ t, & \text{for } 0 \le t \le 1; \\ 0, & \text{for } t < 0. \end{cases}$ Then one has $0 \le h(a), h(b) \le 1$.

Lemma 7. *Let a, b be selfadjoints satisfying $p(a) = p(b)$ for $p(t) = t(1-t)$ and $p(t) = t^2(1-t)$. Then $h(a)$ and $h(b)$ also satisfy these relations, and the projections $P''(h(a), h(b))$ and $P''(a, b)$ are homotopic.*

Proof. The first claim follows from the continuous functional calculus:

$$p(h(a)) = h(p(a)) = h(p(b)) = p(h(b)).$$

Let $h_0(t) = t$, $h_1(t) = h(t)$ and let h_s, $s \in [0, 1]$, be a (linear) homotopy connecting h_0 with h_1. Then $P''_s = P''(h_s(a), h_s(b))$ provides the required homotopy.

□

Remark 8. If the relations in Lemma 7 are satisfied only up to some small value then the homotopy constructed above lies in the set of almost projections.

Thus, if a pair (π^+, π^-) is (F, ε)-balanced then

$$p(\pi^+, \pi^-) = [P''(A_{\pi^+}, A_{\pi^-})] - [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)]$$

determines a class in $K^0(X)$ when F is large and ε is small.

Remark 9. One can write P'' as

$$P''(a, b) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1-a \\ -a \end{pmatrix} (a - b) \begin{pmatrix} 1-a, & -a \end{pmatrix}. \tag{12}$$

4. Example

Let X be a manifold with $\pi_1(X) = \Gamma$, $\widetilde{X}$ its universal covering, $x \in \widetilde{X}$. Set $Y = \Gamma x \cap B_R$, where $B_R \subset \widetilde{X}$ is the ball of radius R centered at x. Then the space $l^2(Y)$ of functions on Y is a finitedimensional subspace of $l^2(\Gamma)$. Let p denote the projection in $l^2(\Gamma)$ onto $l^2(Y)$, and let V be an n-dimensional complex vector space. Set $H = l^2(Y) \otimes V$.

Define $\pi(g)$ on $l^2(Y) \otimes V$ by

$$\pi(g) = p\lambda(g)|_{l^2(Y)} \otimes \mathrm{id}_V,$$

where λ denotes the left regular representation. Let $B_\pm$ be selfadjoint $\mathrm{End}(V)$-valued functions on $\widetilde{X}$, and let $M_{B_\pm}$ denote the operator of multiplication by the function $B_\pm$ on $l^2(\Gamma) \otimes V$. Set

$$\pi^\pm(g) = \pi(g) M_{B_\pm}|_{l^2(Y)} \otimes \mathrm{id}_V\,.$$

Let us check when the pair (π^+,π^-) is (F,ε)-balanced. Let $y\in Y$, $\delta_y\in l^2(Y)$ the corresponding delta-function. Then

$$(\pi^\pm(gh)-\pi^\pm(g)\pi^\pm(h))\delta_y=\left\{\begin{array}{cl}(1-B_\pm(hy))B_\pm(y)\delta_{ghy} & \text{if } hy\in Y, ghy\in Y;\\ B_\pm(y)\delta_{ghy} & \text{if } hy\notin Y, ghy\in Y;\\ 0 & \text{otherwise.}\end{array}\right.$$

Set

$$m_1=\sup\{\|B_+(y)-B_-(y)\|: y\in Y, ghy\in Y, hy\notin Y\};$$
$$m_2=\sup\{\|(1-B_+(hy))B_+(y)-(1-B_-(hy))B_-(y)\|: y\in Y, ghy\in Y, hy\in Y\}.$$

Then

$$\|M_{\pi^+}(g,h)-M_{\pi^-}(g,h)\|=\|(\pi^+(gh)-\pi^+(g)\pi^+(h))-(\pi^-(gh)-\pi^-(g)\pi^-(h))\|=\max(m_1,m_2).$$

A similar estimate involving also $\gamma\in\Gamma$ can be written for $\|\pi^+(\gamma)M_{\pi^+}(g,h)-\pi^-(\gamma)M_{\pi^-}(g,h)\|$. Now suppose that the $\operatorname{End}(V)$-valued functions $B_\pm$ have small variation, i.e. satisfy the estimate

$$\|B_\pm(hy)-B_\pm(y)\|<\delta \ \text{ for any } y\in Y \text{ and any } h\in F\subset\Gamma. \tag{13}$$

Lemma 10. *Assume that $B_+(y)=B_-(y)$ for all y with $d(y,x)\geq R$, and that (13) holds. There exists a constant C such that if the pair (π^+,π^-) is (F,ε)-balanced then $\|p(B_+)-p(B_-)\|<\varepsilon+C\delta$ for $p=t(1-t)$ and $p(t)=t^2(1-t)$, and, conversely, if $\|p(B_+)-p(B_-)\|<\varepsilon$ then the pair (π^+,π^-) is $(F,\varepsilon+C\delta)$-balanced.*

Proof. Up to $C\delta$, we may not distinguish between $B_\pm(hy)$ and $B_\pm(y)$. Then the claim becomes obvious.

□

Remark that the pairs (B_+,B_-) with the above properties can be considered as elements of $K_0(C_0(\widetilde{X}))=K^0_c(\widetilde{X})$.

Starting from a class $z\in K_0(C_0(\widetilde{X}))$, take a pair (B_+,B_-) that represents z, then construct (π^+_R,π^-_R), where R is the radius of the ball that determines Y, as above. The pair (π^+_R,π^-_R) is asymptotically balanced as $R\to\infty$. Then, using $\pi^\pm_R$, define $A_{\pm,R}$ as in (6), where the cocycle γ determines the Mishchenko line bundle. Finally take $P''(A_{+,R},A_{-,R})$, which determines a class in $K^0(X)$.

Lemma 11. *The construction described above defines a map*

$$z\mapsto[P''(A_{+,R},A_{-,R})]-[\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)],\quad K^0_c(\widetilde{X})\to K^0(X),$$

which coinsides with the direct image map.

Proof. Let B_- be a constant projection on $\widetilde{X}$, and let B_+ be a projection-valued function on $\widetilde{X}$, which is equal to B_- at infinity. Let $\pi^-(g)=\lambda(g)\otimes\mathrm{id}_V$ be the left regular representation on $l^2(\Gamma)\otimes V$, and let $\pi^+(g)=\pi^-(g)M_{B_+}$. Then $\pi^+(g)-\pi^-(g)$ is compact for any $g\in\Gamma$, so (π^+,π^-) is a Fredholm representation. Define $A_\pm$ by $A_\pm(x)=\pi^\pm(p(x))$, where p is defined in (5), $x\in X$. Then $A_\pm$ are projection-valued functions on X, and $[P''(A_+,A_-)]-[\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)]$ is unitarily equivalent to $[A_+]-[A_-]$.

It is shown in [4] that $i_!([B_+]-[B_-])=[A_+]-[A_-]$, where $i_!:K^0_c(\widetilde{X})\to K^0(X)$ is the direct image map. Our proof would follow if we show that $P''(A_{+,R},A_{-,R})$ is convergent to $P''(A_+,A_-)$ in norm, as $R\to\infty$ ($*$-strong convergence is obvious).

Denote by L_R the orthogonal complement to $l^2(Y)\otimes V$ in $l^2(\Gamma)\otimes V$. As the word length metric on Γ is quasi-equivalent to the metric on $\widetilde{X}$, so there is a constant $0<C<1$

such that $A_\pm\xi, A_{\pm,R}\xi \in L_{CR}$ when $\xi \in L_R$, for sufficiently great R. Therefore, for the restriction onto L_R we have, using (12), the following estimate:

$$\|(P''(A_{+,R}, A_{-,R}) - \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right))\,|_{L_R}\| \le \|(A_{+,R} - A_{-,R})|_{L_{CR}}\| \le \sup_{x\notin B_{CR}} |B_+(x) - B_-(x)| \to 0 \text{ as } R \to \infty,$$

and, similarly,

$$\lim_{R\to\infty} \|(P''(A_{+,R}, A_{-,R}) - \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right))\,|_{L_R}\| = 0.$$

As all the operators involved are selfadjoint, so this, together with the $*$-strong convergence, implies the norm convergence.

$\square$

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