Hyperbolic algebraic varieties and holomorphic differential equations

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§0. Introduction

The goal of these notes is to explain recent results in the theory of complex varieties, mainly projective algebraic ones, through a few geometric questions pertaining to hyperbolicity in the sense of Kobayashi. A complex space X is said to be hyperbolic if analytic disks $f : \mathbb{D} \to X$ through a given point form a normal family. If X is not hyperbolic, a basic question is to analyze entire holomorphic curves $f : \mathbb{C} \to X$, and especially to understand the Zariski closure $Y \subset X$ of the union $\bigcup f(\mathbb{C})$ of all those curves. A tantalizing conjecture by Green-Griffiths and Lang says that Y is a proper algebraic subvariety of X whenever X is a projective variety of general type. It is also expected that very generic algebraic hypersurfaces X of high degree in complex projective space $\mathbb{P}^{n+1}$ are Kobayashi hyperbolic, i.e. without any entire holomorphic curves $f : \mathbb{C} \to X$. A convenient framework for this study is the category of "directed manifolds", that is, the category of pairs (X, V) where X is a complex manifold and V a holomorphic subbundle of T_X, possibly with singularities – this includes for instance the case of holomorphic foliations. If X is compact, the pair (X, V) is hyperbolic if and only if there are no nonconstant entire holomorphic curves $f : \mathbb{C} \to X$ tangent to V, as a consequence of the Brody criterion. We describe here the construction of certain jet bundles J_kX, $J_k(X, V)$, and corresponding projectivized k-jet bundles P_kV. These bundles, which were introduced in various contexts (Semple in 1954, Green-Griffiths in 1978) allow to analyze hyperbolicity in terms of certain negativity properties of the curvature. For instance, $\pi_k : P_kV \to X$ is a tower of projective bundles over X and carries a canonical line bundle $\mathcal{O}_{P_kV}(1)$; the hyperbolicity of X is then conjecturally equivalent to the existence of suitable singular hermitian metrics of negative curvature on $\mathcal{O}_{P_kV}(-1)$ for k large enough. The direct images $(\pi_k)_*\mathcal{O}_{P_kV}(m)$ can be viewed as bundles of algebraic differential operators of order k and degree m, acting on germs of curves and invariant under reparametrization.

Following an approach initiated by Green and Griffiths, one can use the Ahlfors-Schwarz lemma in the situation where the jet bundle carries a (possibly singular) metric of negative curvature, to infer that every nonconstant entire curve $f : \mathbb{C} \to V$ tangent to V must be contained in the base locus of the metric. A related result is the fundamental vanishing theorem asserting that entire curves must be solutions of the algebraic differential equations provided by global sections of jet bundles, whenever their coefficients vanish on a given ample divisor; this result was obtained in the mid 1990's as the conclusion of contributions by Bloch, Green-Griffiths, Siu-Yeung and the author. It can in its turn be used to prove various important geometric statements. One of them is the Bloch theorem, which was confirmed at the end of the 1970's by Ochiai and Kawamata, asserting that the Zariski closure of an entire curve in a complex torus is a translate of a subtorus.

Since then many developments occurred, for a large part via the technique of constructing jet differentials – either by direct calculations or by various indirect methods: Riemann-Roch calculations, vanishing theorems ... In 1997, McQuillan introduced his "diophantine approximation" method, which was soon recognized to be an important tool in the study of holomorphic foliations, in parallel with Nevanlinna theory and the construction of Ahlfors currents. Around 2000, Siu showed that generic hyperbolicity results in the direction of the Kobayashi conjecture could be investigated by combining the algebraic techniques of Clemens, Ein and Voisin with the existence of certain "vertical" meromorphic vector fields on the jet space of the universal hypersurface of high degree; these vector fields are actually used to differentiate the global sections of the jet bundles involved, so as to produce new sections with a better control on the base locus. Also, in 2007, Demailly pioneered the use of holomorphic Morse inequalities to construct jet differentials; in 2010, Diverio, Merker and Rousseau were able in that way to prove the Green-Griffiths conjecture for generic hypersurfaces of high degree in projective space – their proof also makes an essential use of Siu's differentiation technique via meromorphic vector fields, as improved by Păun and Merker in 2008. The last sections of the notes are devoted to explaining the holomorphic Morse inequality technique; as an application, one obtains a partial answer to the Green-Griffiths conjecture in a very wide context : in particular, for every projective variety of general type X, there exists a global algebraic differential operator P on X (in fact many such operators P_j) such that every entire curve $f : \mathbb{C} \to X$ must satisfy the differential equations $P_j(f; f', \ldots, f^{(k)}) = 0$. We also recover from there the result of Diverio-Merker-Rousseau on the generic Green-Griffiths conjecture (with an even better bound asymptotically as the dimension tends to infinity), as well as a recent recent of Diverio-Trapani (2010) on the hyperbolicity of generic 3-dimensional hypersurfaces in $\mathbb{P}^4$.

§1. Basic hyperbolicity concepts

§1.A. Kobayashi hyperbolicity

We first recall a few basic facts concerning the concept of hyperbolicity, according to S. Kobayashi [Kob70, Kob76]. Let X be a complex space. An *analytic disk* in X a holomorphic map from the unit disk $\Delta = D(0,1)$ to X. Given two points $p, q \in X$, consider a *chain of analytic disks* from p to q, that is a chain of points $p = p_0, p_1, \ldots, p_k = q$ of X, pairs of points $a_1, b_1, \ldots, a_k, b_k$ of Δ and holomorphic maps $f_1, \ldots, f_k : \Delta \to X$ such that

$$f_i(a_i) = p_{i-1}, \quad f_i(b_i) = p_i, \qquad i = 1, \ldots, k.$$

Denoting this chain by α, define its length $\ell(\alpha)$ by

$$\ell(\alpha) = d_P(a_1, b_1) + \cdots + d_P(a_k, b_k) \tag{1.1$'$}$$

and a pseudodistance d_X^K on X by

$$d_X^K(p, q) = \inf_\alpha \ell(\alpha). \tag{1.1$''$}$$

This is by definition the *Kobayashi pseudodistance* of X. In the terminology of Kobayashi [Kob75], a *Finsler metric* (resp. *pseudometric*) on a vector bundle E is a homogeneous positive (resp. nonnegative) positive function N on the total space E, that is,

$$N(\lambda\xi) = |\lambda|\, N(\xi) \qquad \text{for all } \lambda \in \mathbb{C} \text{ and } \xi \in E,$$

but in general N is not assumed to be subbadditive (i.e. convex) on the fibers of E. A Finsler (pseudo-)metric on E is thus nothing but a hermitian (semi-)norm on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $Y = P(E)$. The *Kobayashi-Royden infinitesimal pseudometric* on X is the Finsler pseudometric on the tangent bundle T_X defined by

$$\mathbf{k}_X(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \Delta \to X,\ f(0) = x,\ \lambda f'(0) = \xi\big\}, \qquad x \in X,\ \xi \in T_{X,x}. \tag{1.2}$$

Here, if X is not smooth at x, we take $T_{X,x} = (\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2)^*$ to be the Zariski tangent space, i.e. the tangent space of a minimal smooth ambient vector space containing the germ (X, x); all tangent vectors may not be reached by analytic disks and in those cases we put $\mathbf{k}_X(\xi) = +\infty$. When X is a smooth manifold, it follows from the work of H.L. Royden ([Roy71], [Roy74]) that d_X^K is the integrated pseudodistance associated with the pseudometric, i.e.

$$d_X^K(p, q) = \inf_\gamma \int_\gamma \mathbf{k}_X(\gamma'(t))\, dt,$$

where the infimum is taken over all piecewise smooth curves joining p to q; in the case of complex spaces, a similar formula holds, involving jets of analytic curves of arbitrary order, cf. S. Venturini [Ven96].

1.3. Definition. *A complex space X is said to be hyperbolic* (*in the sense of Kobayashi*) *if d_X^K is actually a distance, namely if $d_X^K(p, q) > 0$ for all pairs of distinct points (p, q) in X.*

When X is hyperbolic, it is interesting to investigate when the Kobayashi metric is complete: one then says that X is a *complete hyperbolic* space. However, we will be mostly concerned with compact spaces here, so completeness is irrelevant in that case.

Another important property is the *monotonicity* of the Kobayashi metric with respect to holomorphic mappings. In fact, if $\Phi : X \to Y$ is a holomorphic map, it is easy to see from the definition that

$$d_Y^K(\Phi(p), \Phi(q)) \leqslant d_X^K(p, q), \qquad \text{for all } p, q \in X. \tag{1.4}$$

The proof merely consists of taking the composition $\Phi \circ f_i$ for all clains of analytic disks connecting p and q in X. Clearly the Kobayashi pseudodistance $d_{\mathbb{C}}^K$ on $X = \mathbb{C}$ is identically zero, as one can see by looking at arbitrarily large analytic disks $\Delta \to \mathbb{C}$, $t \mapsto \lambda t$. Therefore, if there is any (non constant) *entire curve* $\Phi : \mathbb{C} \to X$, namely a non constant holomorphic map defined on the whole complex plane $\mathbb{C}$, then by monotonicity d_X^K is identically zero on the image $\Phi(\mathbb{C})$ of the curve, and therefore X cannot be hyperbolic. When X is hyperbolic, it follows that X cannot contain rational curves $C \simeq \mathbb{P}^1$, or elliptic curves $\mathbb{C}/\Lambda$, or more generally any non trivial image $\Phi : W = \mathbb{C}^p/\Lambda \to X$ of a p-dimensional complex torus (quotient of $\mathbb{C}^p$ by a lattice).

§1.B. The case of complex curves (i.e. Riemann surfaces)

The only case where hyperbolicity is easy to assess is the case of curves ($\dim_{\mathbb{C}} X = 1$). In fact, as the disk is simply connected, every holomorphic map $f : \Delta \to X$ lifts to the universal cover $\widehat{f} : \Delta \to \widehat{X}$, so that $f = \rho \circ \widehat{f}$ where $\rho : \widehat{X} \to X$ is the projection map.

Now, by the Poincaré-Koebe uniformization theorem, every simply connected Riemann surface is biholomorphic to $\mathbb{C}$, the unit disk Δ or the complex projective line $\mathbb{P}^1$. The complex projective line $\mathbb{P}^1$ has no smooth étale quotient since every automorphism of $\mathbb{P}^1$ has a fixed point; therefore the only case where $\widehat{X} \simeq \mathbb{P}^1$ is when $X \simeq \mathbb{P}^1$ already. Assume now that $\widehat{X} \simeq \mathbb{C}$. Then $\pi_1(X)$ operates by translation on $\mathbb{C}$ (all other automorphisms are affine nad have fixed points), and the discrete subgroups of $(\mathbb{C}, +)$ are isomorphic to $\mathbb{Z}^r$, $r = 0, 1, 2$. We then obtain respectively $X \simeq \mathbb{C}$, $X \simeq \mathbb{C}/2\pi i\mathbb{Z} \simeq \mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$ and $X \simeq \mathbb{C}/\Lambda$ where Λ is a lattice, i.e. X is an elliptic curve. In all those cases, any entire function $\widehat{f} : \mathbb{C} \to \mathbb{C}$ gives rise to an entire curve $f : \mathbb{C} \to X$, and the same is true when $X \simeq \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

Finally, assume that $\widehat{X} \simeq \Delta$; by what we have just seen, this must occur as soon as $X \not\simeq \mathbb{P}^1, \mathbb{C}, \mathbb{C}^*, \mathbb{C}/\Lambda$. Let us take on X the infinitesimal metric ω_P which is the quotient of the Poincaré metric on Δ. The Schwarz-Pick lemma shows that $d_\Delta^K = d_P$ coincides with the Poincaré metric on Δ, and it follows easily by the lifting argument that we have $\mathbf{k}_X = \omega_P$. In particular, d_X^K is non degenerate and is just the quotient of the Poincaré metric on Δ, i.e.

$$d_X^K(p, q) = \inf_{p' \in \rho^{-1}(p),\, q' \in \rho^{-1}(q)} d_P(p', q').$$

We can summarize this discussion as follows.

1.5. Theorem. *Up to bihomorphism, any smooth Riemann surface X belongs to one* (*and only one*) *of the following three types.*

(a) (*rational curve*) $X \simeq \mathbb{P}^1$.

(b) (*parabolic type*) $\widehat{X} \simeq \mathbb{C}$, $X \simeq \mathbb{C}$, $\mathbb{C}^*$ *or* $X \simeq \mathbb{C}/\Lambda$ (*elliptic curve*)

(c) (*hyperbolic type*) $\widehat{X} \simeq \Delta$. *All compact curves X of genus $g \geqslant 2$ enter in this category, as well as $X = \mathbb{P}^1 \smallsetminus \{a, b, c\} \simeq \mathbb{C} \smallsetminus \{0, 1\}$, or $X = \mathbb{C}/\Lambda \smallsetminus \{a\}$* (*elliptic curve minus one point*).

In some rare cases, the one-dimensional case can be used to study the case of higher dimensions. For instance, it is easy to see by looking at projections that the Kobayashi pseudodistance on a product $X \times Y$ of complex spaces is given by

$$d_{X\times Y}^K((x, y), (x', y')) = \max\big(d_X^K(x, x'), d_Y^K(y, y')\big), \tag{1.6}$$

$$\mathbf{k}_{X\times Y}(\xi, \xi') = \max\big(\mathbf{k}_X(\xi), \mathbf{k}_Y(\xi')\big), \tag{1.6$'$}$$

and from there it follows that a product of hyperbolic spaces is hyperbolic. As a consequence $(\mathbb{C} \smallsetminus \{0, 1\})^2$, which is also a complement of five lines in $\mathbb{P}^2$, is hyperbolic.

§1.C. Brody criterion for hyperbolicity

Throughout this subsection, we assume that X is a complex manifold. In this context, we have the following well-known result of Brody [Bro78]. Its main interest is to relate hyperbolicity to the non existence of entire curves.

1.7. Brody reparametrization lemma. *Let ω be a hermitian metric on X and let $f : \Delta \to X$ be a holomorphic map. For every $\varepsilon > 0$, there exists a radius $R \geqslant (1-\varepsilon)\|f'(0)\|_\omega$ and a homographic transformation ψ of the disk $D(0,R)$ onto $(1-\varepsilon)\Delta$ such that*

$$\|(f\circ\psi)'(0)\|_\omega = 1, \qquad \|(f\circ\psi)'(t)\|_\omega \leqslant \frac{1}{1-|t|^2/R^2} \quad \textit{for every } t \in D(0,R).$$

Proof. Select $t_0 \in \Delta$ such that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ reaches its maximum for $t = t_0$. The reason for this choice is that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ is the norm of the differential $f'((1-\varepsilon)t) : T_\Delta \to T_X$ with respect to the Poincaré metric $|dt|^2/(1-|t|^2)^2$ on T_Δ, which is conformally invariant under $\mathrm{Aut}(\Delta)$. One then adjusts R and ψ so that $\psi(0) = (1-\varepsilon)t_0$ and $|\psi'(0)|\,\|f'(\psi(0))\|_\omega = 1$. As $|\psi'(0)| = \frac{1-\varepsilon}{R}(1-|t_0|^2)$, the only possible choice for R is

$$R = (1-\varepsilon)(1-|t_0|^2)\|f'(\psi(0))\|_\omega \geqslant (1-\varepsilon)\|f'(0)\|_\omega.$$

The inequality for $(f\circ\psi)'$ follows from the fact that the Poincaré norm is maximum at the origin, where it is equal to 1 by the choice of R. Using the Ascoli-Arzelà theorem we obtain immediately:

1.8. Corollary (Brody). *Let (X,ω) be a compact complex hermitian manifold. Given a sequence of holomorphic mappings $f_\nu : \Delta \to X$ such that $\lim \|f_\nu'(0)\|_\omega = +\infty$, one can find a sequence of homographic transformations $\psi_\nu : D(0,R_\nu) \to (1-1/\nu)\Delta$ with $\lim R_\nu = +\infty$, such that, after passing possibly to a subsequence, $(f_\nu \circ \psi_\nu)$ converges uniformly on every compact subset of $\mathbb{C}$ towards a non constant holomorphic map $g : \mathbb{C} \to X$ with $\|g'(0)\|_\omega = 1$ and $\sup_{t\in\mathbb{C}} \|g'(t)\|_\omega \leqslant 1$.*

An entire curve $g : \mathbb{C} \to X$ such that $\sup_{\mathbb{C}} \|g'\|_\omega = M < +\infty$ is called a *Brody curve*; this concept does not depend on the choice of ω when X is compact, and one can always assume $M = 1$ by rescaling the parameter t.

1.9. Brody criterion. *Let X be a compact complex manifold. The following properties are equivalent.*

(a) *X is hyperbolic.*

(b) *X does not possess any entire curve $f : \mathbb{C} \to X$.*

(c) *X does not possess any Brody curve $g : \mathbb{C} \to X$.*

(d) *The Kobayashi infinitesimal metric $\mathbf{k}_X$ is uniformly bouded below, namely*

$$\mathbf{k}_X(\xi) \geqslant c\|\xi\|_\omega, \qquad c > 0,$$

for any hermitian metric ω on X.

Proof. (a)⇒(b) If X possesses an entire curve $f : \mathbb{C} \to X$, then by looking at arbitrary large disks $D(0,R) \subset \mathbb{C}$, it is easy to see that the Kobayashi distance of any two points in $f(\mathbb{C})$ is zero, so X is not hyperbolic.

(b)⇒(c) is trivial.

(c)⇒(d) If (d) does not hold, there exists a sequence of tangent vectors $\xi_\nu \in T_{X,x_\nu}$ with $\|\xi_\nu\|_\omega = 1$ and $\mathbf{k}_X(\xi_\nu) \to 0$. By definition, this means that there exists an analytic curve $f_\nu : \Delta \to X$ with $f(0) = x_\nu$ and $\|f'_\nu(0)\|_\omega \geqslant (1 - \frac{1}{\nu})/\mathbf{k}_X(\xi_\nu) \to +\infty$. One can then produce a Brody curve $g = \mathbb{C} \to X$ by Corollary 1.8, contradicting (c).

(d)⇒(a). In fact (d) implies after integrating that $d_X^K(p,q) \geqslant c\, d_\omega(p,q)$ where d_ω is the geodesic distance associated with ω, so d_X^K must be non degenerate. □

Notice also that if $f : \mathbb{C} \to X$ is an entire curve such that $\|f'\|_\omega$ is unbounded, one can apply the Corollary 1.8 to $f_\nu(t) := f(t + a_\nu)$ where the sequence (a_ν) is chosen such that $\|f'_\nu(0)\|_\omega = \|f(a_\nu)\|_\omega \to +\infty$. Brody's result then produces repametrizations $\psi_\nu : D(0, R_\nu) \to D(a_\nu, 1 - 1/\nu)$ and a Brody curve $g = \lim f \circ \psi_\nu : \mathbb{C} \to X$ such that $\sup \|g'\|_\omega = 1$ and $g(\mathbb{C}) \subset \overline{f(\mathbb{C})}$. It may happen that the image $g(\mathbb{C})$ of such a limiting curve is disjoint from $f(\mathbb{C})$. In fact Winkelmann [Win07] has given a striking example, actually a projective 3-fold X obtained by blowing-up a 3-dimensional abelian variety Y, such that every Brody curve $g : \mathbb{C} \to X$ lies in the exceptional divisor $E \subset X$; however, entire curves $f : \mathbb{C} \to X$ can be dense, as one can see by taking f to be the lifting of a generic complex line embedded in the abelian variety Y. For further precise information on the localization of Brody curves, we refer the reader to the remarkable results of [Duv08].

The absence of entire holomorphic curves in a given complex manifold is often referred to as *Brody hyperbolicity*. Thus, in the compact case, Brody hyperbolicity and Kobayashi hyperbolicity coincide (but Brody hyeperbolicity is in general a strictly weaker property when X is non compact).

§1.D. Geometric applications

We give here two immediate consequences of the Brody criterion: the openness property of hyperbolicity and a hyperbolicity criterion for subvarieties of complex tori. By definition, a *holomorphic family* of compact complex manifolds is a holomorphic proper submersion $\mathcal{X} \to S$ between two complex manifolds.

1.10. Proposition. *Let $\pi : \mathcal{X} \to S$ be a holomorphic family of compact complex manifolds. Then the set of $s \in S$ such that the fiber $X_s = \pi^{-1}(s)$ is hyperbolic is open in the Euclidean topology.*

Proof. Let ω be an arbitrary hermitian metric on $\mathcal{X}$, $(X_{s_\nu})_{s_\nu \in S}$ a sequence of non hyperbolic fibers, and $s = \lim s_\nu$. By the Brody criterion, one obtains a sequence of entire maps $f_\nu : \mathbb{C} \to X_{s_\nu}$ such that $\|f'_\nu(0)\|_\omega = 1$ and $\|f'_\nu\|_\omega \leqslant 1$. Ascoli's theorem shows that there is a subsequence of f_ν converging uniformly to a limit $f : \mathbb{C} \to X_s$, with $\|f'(0)\|_\omega = 1$. Hence X_s is not hyperbolic and the collection of non hyperbolic fibers is closed in S. □

Consider now an n-dimensional complex torus W, i.e. an additive quotient $W = \mathbb{C}^n/\Lambda$, where $\Lambda \subset \mathbb{C}^n$ is a (cocompact) lattice. By taking a composition of entire curves $\mathbb{C} \to \mathbb{C}^n$ with the projection $\mathbb{C}^n \to W$ we obtain an infinite dimensional space of entire curves in W.

1.11. Theorem. *Let $X \subset W$ be a compact complex submanifold of a complex torus. Then X is hyperbolic if and only if it does not contain any translate of a subtorus.*

Proof. If X contains some translate of a subtorus, then it contains lots of entire curves and so X is not hyperbolic.

Conversely, suppose that X is not hyperbolic. Then by the Brody criterion there exists an entire curve $f : \mathbb{C} \to X$ such that $\|f'\|_\omega \leqslant \|f'(0)\|_\omega = 1$, where ω is the flat metric on W

inherited from $\mathbb{C}^n$. This means that any lifting $\widetilde{f} = (\widetilde{f}, \ldots, \widetilde{f}_\nu) : \mathbb{C} \to \mathbb{C}^n$ is such that

$$\sum_{j=1}^{n} |f_j'|^2 \leqslant 1.$$

Then, by Liouville's theorem, $\widetilde{f}'$ is constant and therefore $\widetilde{f}$ is affine. But then the closure of the image of f is a translate $a + H$ of a connected (possibly real) subgroup H of W. We conclude that X contains the analytic Zariski closure of $a + H$, namely $a + H^{\mathbb{C}}$ where $H^{\mathbb{C}} \subset W$ is the smallest closed complex subgroup of W containing H. □

§2. Directed manifolds

§2.A. Basic definitions concerning directed manifolds

Let us consider a pair (X, V) consisting of a n-dimensional complex manifold X equipped with a *linear subspace* $V \subset T_X$: assuming X connected, this is by definition an irreducible closed analytic subspace of the total space of T_X such that each fiber $V_x = V \cap T_{X,x}$ is a vector subspace of $T_{X,x}$; the rank $x \mapsto \dim_{\mathbb{C}} V_x$ is Zariski lower semicontinuous, and it may a priori jump. We will refer to such a pair as being a (complex) *directed manifold.* A morphism $\Phi : (X, V) \to (Y, W)$ in the category of (complex) directed manifolds is a holomorphic map such that $\Phi_*(V) \subset W$.

The rank $r \in \{0, 1, \ldots, n\}$ of V is by definition the dimension of V_x at a generic point. The dimension may be larger at non generic points; this happens e.g. on $X = \mathbb{C}^n$ for the rank 1 linear space V generated by the Euler vector field: $V_z = \mathbb{C} \sum_{1 \leqslant j \leqslant n} z_j \frac{\partial}{\partial z_j}$ for $z \neq 0$, and $V_0 = \mathbb{C}^n$. Our philosophy is that directed manifolds are also useful to study the "absolute case", i.e. the case $V = T_X$, because there are certain fonctorial constructions which are quite natural in the category of directed manifolds (see e.g. § 5, 6, 7). We think of directed manifolds as a kind of "relative situation", covering e.g. the case when V is the relative tangent space to a holomorphic map $X \to S$. In general, we can associate to V a sheaf $\mathcal{V} = \mathcal{O}(V) \subset \mathcal{O}(T_X)$ of holomorphic sections. These sections need not generate the fibers of V at singular points, as one sees already in the case of the Euler vector field when $n \geqslant 2$. However, $\mathcal{V}$ is a *saturated* subsheaf of $\mathcal{O}(T_X)$, i.e. $\mathcal{O}(T_X)/\mathcal{V}$ has no torsion: in fact, if the components of a section have a common divisorial component, one can always simplify this divisor and produce a new section without any such common divisorial component. Instead of defining directed manifolds by picking a linear space V, one could equivalently define them by considering saturated coherent subsheaves $\mathcal{V} \subset \mathcal{O}(T_X)$. One could also take the dual viewpoint, looking at arbitrary quotient morphisms $\Omega^1_X \to \mathcal{W} = \mathcal{V}^*$ (and recovering $\mathcal{V} = \mathcal{W}^* = \mathrm{Hom}_{\mathcal{O}}(\mathcal{W}, \mathcal{O})$, as $\mathcal{V} = \mathcal{V}^{**}$ is reflexive). We want to stress here that no assumption need be made on the Lie bracket tensor $[\ ,\ ] : \mathcal{V} \times \mathcal{V} \to \mathcal{O}(T_X)/\mathcal{V}$, i.e. we do not assume any kind of integrability for $\mathcal{V}$ or $\mathcal{W}$.

The singular set $\mathrm{Sing}(V)$ is by definition the set of points where $\mathcal{V}$ is not locally free, it can also be defined as the indeterminacy set of the (meromorphic) classifying map $\alpha : X \dashrightarrow G_r(T_X)$, $z \mapsto V_z$ to the Grasmannian of r dimensional subspaces of T_X. We thus have $V_{|X \smallsetminus \mathrm{Sing}(V)} = \alpha^* S$ where $S \to G_r(T_X)$ is the tautological subbundle of $G_r(T_X)$. The singular set $\mathrm{Sing}(V)$ is an analytic subset of X of codim $\geqslant 2$, hence V is always a holomorphic subbundle outside of codimension 2. Thanks to this remark, one can most often treat linear spaces as vector bundles (possibly modulo passing to the Zariski closure along $\mathrm{Sing}(V)$).

§2.B. Hyperbolicity properties of directed manifolds

Most of what we have done in §1 can be extended to the category of directed manifolds.

2.1. Definition. *Let (X, V) be a complex directed manifold.*

i) *The Kobayashi-Royden infinitesimal metric of (X, V) is the Finsler metric on V defined for any $x \in X$ and $\xi \in V_x$ by*

$$\mathbf{k}_{(X,V)}(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \Delta \to X,\ f(0) = x,\ \lambda f'(0) = \xi,\ f'(\Delta) \subset V\big\}.$$

Here $\Delta \subset \mathbb{C}$ is the unit disk and the map f is an arbitrary holomorphic map which is tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all $t \in \Delta$. We say that (X, V) is infinitesimally hyperbolic if $\mathbf{k}_{(X,V)}$ is positive definite on every fiber V_x and satisfies a uniform lower bound $\mathbf{k}_{(X,V)}(\xi) \geqslant \varepsilon\|\xi\|_\omega$ in terms of any smooth hermitian metric ω on X, when x describes a compact subset of X.

ii) *More generally, the Kobayashi-Eisenman infinitesimal pseudometric of (X, V) is the pseudometric defined on all decomposable p-vectors $\xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p V_x$, $1 \leqslant p \leqslant r = \operatorname{rank} V$, by*

$$\mathbf{e}^p_{(X,V)}(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \mathbb{B}_p \to X,\ f(0) = x,\ \lambda f_*(\tau_0) = \xi,\ f_*(T_{\mathbb{B}_p}) \subset V\big\}$$

where $\mathbb{B}_p$ is the unit ball in $\mathbb{C}^p$ and $\tau_0 = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_p$ is the unit p-vector of $\mathbb{C}^p$ at the origin. We say that (X, V) is infinitesimally p-measure hyperbolic if $\mathbf{e}^p_{(X,V)}$ is positive definite on every fiber $\Lambda^p V_x$ and satisfies a locally uniform lower bound in terms of any smooth metric.

If $\Phi : (X, V) \to (Y, W)$ is a morphism of directed manifolds, it is immediate to check that we have the monotonicity property

$$\mathbf{k}_{(Y,W)}(\Phi_*\xi) \leqslant \mathbf{k}_{(X,V)}(\xi), \qquad \forall \xi \in V, \tag{2.2}$$

$$\mathbf{e}^p_{(Y,W)}(\Phi_*\xi) \leqslant \mathbf{e}^p_{(X,V)}(\xi), \qquad \forall \xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p V. \tag{2.2^p}$$

The following proposition shows that virtually all reasonable definitions of the hyperbolicity property are equivalent if X is compact (in particular, the additional assumption that there is locally uniform lower bound for $\mathbf{k}_{(X,V)}$ is not needed). We merely say in that case that (X, V) is *hyperbolic.*

2.3. Proposition. *For an arbitrary directed manifold (X, V), the Kobayashi-Royden infinitesimal metric $\mathbf{k}_{(X,V)}$ is upper semicontinuous on the total space of V. If X is compact, (X, V) is infinitesimally hyperbolic if and only if there are no non constant entire curves $g : \mathbb{C} \to X$ tangent to V. In that case, $\mathbf{k}_{(X,V)}$ is a continuous (and positive definite) Finsler metric on V.*

Proof. The proof is almost identical to the standard proof for $\mathbf{k}_X$, for which we refer to Royden [Roy71, Roy74]. □

Another easy observation is that the concept of p-measure hyperbolicity gets weaker and weaker as p increases (we leave it as an exercise to the reader, this is mostly just linear algebra).

2.4. Proposition. *If (X, V) is p-measure hyperbolic, then it is $(p + 1)$-measure hyperbolic for all $p \in \{1, \dots, r - 1\}$.*

Again, an argument extremely similar to the proof of 1.10 shows that relative hyperbolicity is again an open property.

2.5. Proposition. *Let $(\mathcal{X}, \mathcal{V}) \to S$ be a holomorphic family of compact directed manifolds (by this, we mean a proper holomorphic map $\mathcal{X} \to S$ together with an analytic linear subspace $\mathcal{V} \subset T_{\mathcal{X}/S} \subset T_{\mathcal{X}}$ of the relative tangent bundle, defining a deformation $(X_s, V_s)_{s \in S}$ of the fibers). Then the set of $s \in S$ such that the fiber (X_s, V_s) is hyperbolic is open in S with respect to the Euclidean topology.*

Let us mention here an impressive result proved by Marco Brunella [Bru03, Bru05, Bru06] concerning the behavior of the Kobayashi metric on foliated varieties.

2.6. Theorem (Brunella)**.** *Let X be a compact Kähler manifold equipped with a (possibly singular) rank 1 holomorphic foliation which is not a foliation by rational curves. Then the canonical bundle $K_{\mathcal{F}} = \mathcal{F}^*$ of the foliation is pseudoeffective (i.e. the curvature of $K_{\mathcal{F}}$ is $\geqslant 0$ in the sense of currents).*

The proof is obtained by putting on $K_{\mathcal{F}}$ precisely the metric induced by the Kobayashi metric on the leaves whenever they are generically hyperbolic (i.e. covered by the unit disk). The case of parabolic leaves (covered by $\mathbb{C}$) has to be treated separately.

§3. Algebraic hyperbolicity

In the case of projective algebraic varieties, hyperbolicity is expected to be related to other properties of a more algebraic nature. Theorem 3.1 below is a first step in this direction.

3.1. Theorem. *Let (X, V) be a compact complex directed manifold and let $\sum \omega_{jk} dz_j \otimes d\overline{z}_k$ be a hermitian metric on X, with associated positive $(1,1)$-form $\omega = \frac{i}{2} \sum \omega_{jk} dz_j \wedge d\overline{z}_k$. Consider the following three properties, which may or not be satisfied by (X, V) :*

i) *(X, V) is hyperbolic.*

ii) *There exists $\varepsilon > 0$ such that every compact irreducible curve $C \subset X$ tangent to V satisfies*
$$-\chi(\overline{C}) = 2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C)$$
where $g(\overline{C})$ is the genus of the normalization $\overline{C}$ of C, $\chi(\overline{C})$ its Euler characteristic and $\deg_\omega(C) = \int_C \omega$. (This property is of course independent of ω.)

iii) *There does not exist any non constant holomorphic map $\Phi : Z \to X$ from an abelian variety Z to X such that $\Phi_*(T_Z) \subset V$.*

Then i) $\Rightarrow$ ii) $\Rightarrow$ iii)*.*

Proof. i) $\Rightarrow$ ii). If (X, V) is hyperbolic, there is a constant $\varepsilon_0 > 0$ such that $\mathbf{k}_{(X,V)}(\xi) \geqslant \varepsilon_0 \|\xi\|_\omega$ for all $\xi \in V$. Now, let $C \subset X$ be a compact irreducible curve tangent to V and let $\nu : \overline{C} \to C$ be its normalization. As (X, V) is hyperbolic, $\overline{C}$ cannot be a rational or elliptic curve, hence $\overline{C}$ admits the disk as its universal covering $\rho : \Delta \to \overline{C}$.

The Kobayashi-Royden metric $\mathbf{k}_\Delta$ is the Finsler metric $|dz|/(1 - |z|^2)$ associated with the Poincaré metric $|dz|^2/(1 - |z|^2)^2$ on Δ, and $\mathbf{k}_{\overline{C}}$ is such that $\rho^* \mathbf{k}_{\overline{C}} = \mathbf{k}_\Delta$. In other words, the metric $\mathbf{k}_{\overline{C}}$ is induced by the unique hermitian metric on $\overline{C}$ of constant Gaussian curvature -4. If $\sigma_\Delta = \frac{i}{2} dz \wedge d\overline{z}/(1 - |z|^2)^2$ and $\sigma_{\overline{C}}$ are the corresponding area measures, the Gauss-Bonnet formula (integral of the curvature $= 2\pi\,\chi(\overline{C})$) yields
$$\int_{\overline{C}} d\sigma_{\overline{C}} = -\frac{1}{4} \int_{\overline{C}} \mathrm{curv}(\mathbf{k}_{\overline{C}}) = -\frac{\pi}{2} \chi(\overline{C})$$

On the other hand, if $j : C \to X$ is the inclusion, the monotonicity property (2.2) applied to the holomorphic map $j \circ \nu : \overline{C} \to X$ shows that

$$\mathbf{k}_{\overline{C}}(t) \geqslant \mathbf{k}_{(X,V)}\big((j\circ\nu)_* t\big) \geqslant \varepsilon_0 \big\|(j\circ\nu)_* t\big\|_\omega, \qquad \forall t \in T_{\overline{C}}.$$

From this, we infer $d\sigma_{\overline{C}} \geqslant \varepsilon_0^2 (j\circ\nu)^*\omega$, thus

$$-\frac{\pi}{2}\chi(\overline{C}) = \int_{\overline{C}} d\sigma_{\overline{C}} \geqslant \varepsilon_0^2 \int_{\overline{C}} (j\circ\nu)^*\omega = \varepsilon_0^2 \int_C \omega.$$

Property ii) follows with $\varepsilon = 2\varepsilon_0^2/\pi$.

ii) $\Rightarrow$ iii). First observe that ii) excludes the existence of elliptic and rational curves tangent to V. Assume that there is a non constant holomorphic map $\Phi : Z \to X$ from an abelian variety Z to X such that $\Phi_*(T_Z) \subset V$. We must have $\dim \Phi(Z) \geqslant 2$, otherwise $\Phi(Z)$ would be a curve covered by images of holomorphic maps $\mathbb{C} \to \Phi(Z)$, and so $\Phi(Z)$ would be elliptic or rational, contradiction. Select a sufficiently general curve Γ in Z (e.g., a curve obtained as an intersection of very generic divisors in a given very ample linear system $|L|$ in Z). Then all isogenies $u_m : Z \to Z$, $s \mapsto ms$ map Γ in a $1:1$ way to curves $u_m(\Gamma) \subset Z$, except maybe for finitely many double points of $u_m(\Gamma)$ (if $\dim Z = 2$). It follows that the normalization of $u_m(\Gamma)$ is isomorphic to Γ. If Γ is general enough, similar arguments show that the images

$$C_m := \Phi(u_m(\Gamma)) \subset X$$

are also generically $1:1$ images of Γ, thus $\overline{C}_m \simeq \Gamma$ and $g(\overline{C}_m) = g(\Gamma)$. We would like to show that C_m has degree $\geqslant \mathrm{Const}\, m^2$. This is indeed rather easy to check if ω is Kähler, but the general case is slightly more involved. We write

$$\int_{C_m} \omega = \int_\Gamma (\Phi\circ u_m)^*\omega = \int_Z [\Gamma] \wedge u_m^*(\Phi^*\omega),$$

where Γ denotes the current of integration over Γ. Let us replace Γ by an arbitrary translate $\Gamma + s$, $s \in Z$, and accordingly, replace C_m by $C_{m,s} = \Phi\circ u_m(\Gamma + s)$. For $s \in Z$ in a Zariski open set, $C_{m,s}$ is again a generically $1:1$ image of $\Gamma + s$. Let us take the average of the last integral identity with respect to the unitary Haar measure $d\mu$ on Z. We find

$$\int_{s\in Z} \bigg(\int_{C_{m,s}} \omega\bigg)\, d\mu(s) = \int_Z \bigg(\int_{s\in Z} [\Gamma + s]\, d\mu(s)\bigg) \wedge u_m^*(\Phi^*\omega).$$

Now, $\gamma := \int_{s\in Z} [\Gamma + s]\, d\mu(s)$ is a translation invariant positive definite form of type $(p-1, p-1)$ on Z, where $p = \dim Z$, and γ represents the same cohomology class as $[\Gamma]$, i.e. $\gamma \equiv c_1(L)^{p-1}$. Because of the invariance by translation, γ has constant coefficients and so $(u_m)_*\gamma = m^2\gamma$. Therefore we get

$$\int_{s\in Z} d\mu(s) \int_{C_{m,s}} \omega = m^2 \int_Z \gamma \wedge \Phi^*\omega.$$

In the integral, we can exclude the algebraic set of values z such that $C_{m,s}$ is not a generically $1:1$ image of $\Gamma + s$, since this set has measure zero. For each m, our integral identity implies that there exists an element $s_m \in Z$ such that $g(\overline{C}_{m,s_m}) = g(\Gamma)$ and

$$\deg_\omega(C_{m,s_m}) = \int_{C_{m,s_m}} \omega \geqslant m^2 \int_Z \gamma \wedge \Phi^*\omega.$$

As $\int_Z \gamma \wedge \Phi^*\omega > 0$, the curves C_{m,s_m} have bounded genus and their degree is growing quadratically with m, contradiction to property ii). □

3.2. Definition. *We say that a projective directed manifold* (X, V) *is "algebraically hyperbolic" if it satisfies property* 3.1 ii)*, namely, if there exists* $\varepsilon > 0$ *such that every algebraic curve* $C \subset X$ *tangent to* V *satisfies*

$$2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C).$$

A nice feature of algebraic hyperbolicity is that it satisfies an algebraic analogue of the openness property.

3.3. Proposition. *Let* $(\mathcal{X}, \mathcal{V}) \to S$ *be an algebraic family of projective algebraic directed manifolds* (*given by a projective morphism* $\mathcal{X} \to S$)*. Then the set of* $t \in S$ *such that the fiber* (X_t, V_t) *is algebraically hyperbolic is open with respect to the "countable Zariski topology" of* S (*by definition, this is the topology for which closed sets are countable unions of algebraic sets*)*.*

Proof. After replacing S by a Zariski open subset, we may assume that the total space $\mathcal{X}$ itself is quasi-projective. Let ω be the Kähler metric on $\mathcal{X}$ obtained by pulling back the Fubini-Study metric via an embedding in a projective space. If integers $d > 0$, $g \geqslant 0$ are fixed, the set $A_{d,g}$ of $t \in S$ such that X_t contains an algebraic 1-cycle $C = \sum m_j C_j$ tangent to V_t with $\deg_\omega(C) = d$ and $g(\overline{C}) = \sum m_j\, g(\overline{C}_j) \leqslant g$ is a closed algebraic subset of S (this follows from the existence of a relative cycle space of curves of given degree, and from the fact that the geometric genus is Zariski lower semicontinuous). Now, the set of non algebraically hyperbolic fibers is by definition

$$\bigcap_{k>0} \; \bigcup_{2g-2<d/k} A_{d,g}.$$

This concludes the proof (of course, one has to know that the countable Zariski topology is actually a topology, namely that the class of countable unions of algebraic sets is stable under arbitrary intersections; this can be easily checked by an induction on dimension). □

3.4. Remark. More explicit versions of the openness property have been dealt with in the literature. H. Clemens ([Cle86] and [CKL88]) has shown that on a very generic surface of degree $d \geqslant 5$ in $\mathbb{P}^3$, the curves of type (d, k) are of genus $g > kd(d-5)/2$ (recall that a very generic surface $X \subset \mathbb{P}^3$ of degree $\geqslant 4$ has Picard group generated by $\mathcal{O}_X(1)$ thanks to the Noether-Lefschetz theorem, thus any curve on the surface is a complete intersection with another hypersurface of degree k ; such a curve is said to be of type (d, k) ; genericity is taken here in the sense of the countable Zariski topology). Improving on this result of Clemens, Geng Xu [Xu94] has shown that every curve contained in a very generic surface of degree $d \geqslant 5$ satisfies the sharp bound $g \geqslant d(d-3)/2 - 2$. This actually shows that a very generic surface of degree $d \geqslant 6$ is algebraically hyperbolic. Although a very generic quintic surface has no rational or elliptic curves, it seems to be unknown whether a (very) generic quintic surface is algebraically hyperbolic in the sense of Definition 3.2.

In higher dimension, L. Ein ([Ein88], [Ein91]) proved that every subvariety of a very generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2n+1$ $(n \geqslant 2)$, is of general type. This was reproved by a simple efficient technique by C. Voisin in [Voi96].

3.5. Remark. It would be interesting to know whether algebraic hyperbolicity is open with respect to the Euclidean topology ; still more interesting would be to know whether

Kobayashi hyperbolicity is open for the countable Zariski topology (of course, both properties would follow immediately if one knew that algebraic hyperbolicity and Kobayashi hyperbolicity coincide, but they seem otherwise highly non trivial to establish). The latter openness property has raised an important amount of work around the following more particular question: is a (very) generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d large enough (say $d \geqslant 2n+1$) Kobayashi hyperbolic ? Again, "very generic" is to be taken here in the sense of the countable Zariski topology. Brody-Green [BrGr77] and Nadel [Nad89] produced examples of hyperbolic surfaces in $\mathbb{P}^3$ for all degrees $d \geqslant 50$, and Masuda-Noguchi [MaNo93] gave examples of such hypersurfaces in $\mathbb{P}^n$ for arbitrary $n \geqslant 2$, of degree $d \geqslant d_0(n)$ large enough. The question of studying the hyperbolicity of complements $\mathbb{P}^n \smallsetminus D$ of generic divisors is in principle closely related to this; in fact if $D = \{P(z_0, \ldots, z_n) = 0\}$ is a smooth generic divisor of degree d, one may look at the hypersurface

$$X = \big\{z_{n+1}^d = P(z_0, \ldots, z_n)\big\} \subset \mathbb{P}^{n+1}$$

which is a cyclic $d\!:\!1$ covering of $\mathbb{P}^n$. Since any holomorphic map $f : \mathbb{C} \to \mathbb{P}^n \smallsetminus D$ can be lifted to X, it is clear that the hyperbolicity of X would imply the hyperbolicity of $\mathbb{P}^n \smallsetminus D$. The hyperbolicity of complements of divisors in $\mathbb{P}^n$ has been investigated by many authors. $\square$

In the "absolute case" $V = T_X$, it seems reasonable to expect that properties 3.1 i), ii) are equivalent, i.e. that Kobayashi and algebraic hyperbolicity coincide. However, it was observed by Serge Cantat [Can00] that property 3.1 (iii) is not sufficient to imply the hyperbolicity of X, at least when X is a general complex surface: a general (non algebraic) K3 surface is known to have no elliptic curves and does not admit either any surjective map from an abelian variety; however such a surface is not Kobayashi hyperbolic. We are uncertain about the sufficiency of 3.1 (iii) when X is assumed to be projective.

§4. The Ahlfors-Schwarz lemma for metrics of negative curvature

One of the most basic ideas is that hyperbolicity should somehow be related with suitable negativity properties of the curvature. For instance, it is a standard fact already observed in Kobayashi [Kob70] that the negativity of T_X (or the ampleness of T^*_X) implies the hyperbolicity of X. There are many ways of improving or generalizing this result. We present here a few simple examples of such generalizations.

§4.A. Exploiting curvature via potential theory

If (V, h) is a holomorphic vector bundle equipped with a smooth hermitian metric, we denote by $\nabla_h = \nabla'_h + \nabla''_h$ the associated Chern connection and by $\Theta_{V,h} = \frac{i}{2\pi}\nabla_h^2$ its Chern curvature tensor.

4.1. Proposition. *Let (X, V) be a compact directed manifold. Assume that V is non singular and that V^* is ample. Then (X, V) is hyperbolic.*

Proof (from an original idea of [Kob75]). Recall that a vector bundle E is said to be ample if $S^m E$ has enough global sections $\sigma_1, \ldots, \sigma_N$ so as to generate 1-jets of sections at any point, when m is large. One obtains a Finsler metric N on E^* by putting

$$N(\xi) = \Big(\sum_{1 \leqslant j \leqslant N} |\sigma_j(x) \cdot \xi^m|^2\Big)^{1/2m}, \qquad \xi \in E^*_x,$$

and N is then a strictly plurisubharmonic function on the total space of E^* minus the zero section (in other words, the line bundle $\mathcal{O}_{P(E^*)}(1)$ has a metric of positive curvature). By the ampleness assumption on V^*, we thus have a Finsler metric N on V which is strictly plurisubharmonic outside the zero section. By the Brody lemma, if (X,V) is not hyperbolic, there is a non constant entire curve $g:\mathbb{C}\to X$ tangent to V such that $\sup_{\mathbb{C}}\|g'\|_\omega \leqslant 1$ for some given hermitian metric ω on X. Then $N(g')$ is a bounded subharmonic function on $\mathbb{C}$ which is strictly subharmonic on $\{g'\neq 0\}$. This is a contradiction, for any bounded subharmonic function on $\mathbb{C}$ must be constant. □

§4.B. Ahlfors-Schwarz lemma

Proposition 4.1 can be generalized a little bit further by means of the Ahlfors-Schwarz lemma (see e.g. [Lang87]; we refer to [Dem85] for the generalized version presented here; the proof is merely an application of the maximum principle plus a regularization argument).

4.2. Ahlfors-Schwarz lemma. *Let $\gamma(t)=\gamma_0(t)\,i\,dt\wedge d\overline{t}$ be a hermitian metric on Δ_R where $\log\gamma_0$ is a subharmonic function such that $i\,\partial\overline{\partial}\log\gamma_0(t)\geqslant A\,\gamma(t)$ in the sense of currents, for some positive constant A. Then γ can be compared with the Poincaré metric of Δ_R as follows:*

$$\gamma(t)\leqslant\frac{2}{A}\frac{R^{-2}|dt|^2}{(1-|t|^2/R^2)^2}.$$

More generally, let $\gamma=i\sum\gamma_{jk}dt_j\wedge d\overline{t}_k$ be an almost everywhere positive hermitian form on the ball $B(0,R)\subset\mathbb{C}^p$, such that $-\operatorname{Ricci}(\gamma):=i\,\partial\overline{\partial}\log\det\gamma\geqslant A\gamma$ in the sense of currents, for some constant $A>0$ (this means in particular that $\det\gamma=\det(\gamma_{jk})$ is such that $\log\det\gamma$ is plurisubharmonic). Then

$$\det(\gamma)\leqslant\Big(\frac{p+1}{AR^2}\Big)^p\frac{1}{(1-|t|^2/R^2)^{p+1}}.$$

4.C. Applications of the Ahlfors-Schwarz lemma to hyperbolicity

Let (X,V) be a *compact* directed manifold. We assume throughout this subsection that V is *non singular*.

4.3. Proposition. *Assume V^* is "very big" in the following sense: there exists an ample line bundle L and a sufficiently large integer m such that the global sections in $H^0(X,S^mV^*\otimes L^{-1})$ generate all fibers over $X\smallsetminus Y$, for some analytic subset $Y\subsetneq X$. Then all entire curves $f:\mathbb{C}\to X$ tangent to V satisfy $f(\mathbb{C})\subset Y$ [under our assumptions, X is a projective algebraic manifold and Y is an algebraic subvariety, thus it is legitimate to say that the entire curves are "algebraically degenerate"].*

Proof. Let $\sigma_1,\ldots,\sigma_N\in H^0(X,S^mV^*\otimes L^{-1})$ be a basis of sections generating $S^mV^*\otimes L^{-1}$ over $X\smallsetminus Y$. If $f:\mathbb{C}\to X$ is tangent to V, we define a semipositive hermitian form $\gamma(t)=\gamma_0(t)\,|dt|^2$ on $\mathbb{C}$ by putting

$$\gamma_0(t)=\sum\|\sigma_j(f(t))\cdot f'(t)^m\|_{L^{-1}}^{2/m}$$

where $\|\ \|_L$ denotes a hermitian metric with positive curvature on L. If $f(\mathbb{C})\not\subset Y$, the form γ is not identically 0 and we then find

$$i\,\partial\overline{\partial}\log\gamma_0\geqslant\frac{2\pi}{m}f^*\Theta_L$$

where Θ_L is the curvature form. The positivity assumption combined with an obvious homogeneity argument yield

$$\frac{2\pi}{m} f^*\Theta_L \geqslant \varepsilon \|f'(t)\|^2_\omega \, |dt|^2 \geqslant \varepsilon' \, \gamma(t)$$

for any given hermitian metric ω on X. Now, for any t_0 with $\gamma_0(t_0) > 0$, the Ahlfors-Schwarz lemma shows that f can only exist on a disk $D(t_0, R)$ such that $\gamma_0(t_0) \leqslant \frac{2}{\varepsilon'} R^{-2}$, contradiction. □

There are similar results for p-measure hyperbolicity, e.g.

4.4. Proposition. *Assume that $\Lambda^p V^*$ is ample. Then (X, V) is infinitesimally p-measure hyperbolic. More generally, assume that $\Lambda^p V^*$ is very big with base locus contained in $Y \subsetneq X$ (see 3.3). Then $\mathbf{e}^p$ is non degenerate over $X \smallsetminus Y$.*

Proof. By the ampleness assumption, there is a smooth Finsler metric N on $\Lambda^p V$ which is strictly plurisubharmonic outside the zero section. We select also a hermitian metric ω on X. For any holomorphic map $f : \mathbb{B}_p \to X$ we define a semipositive hermitian metric $\widetilde{\gamma}$ on $\mathbb{B}_p$ by putting $\widetilde{\gamma} = f^*\omega$. Since ω need not have any good curvature estimate, we introduce the function $\delta(t) = N_{f(t)}(\Lambda^p f'(t) \cdot \tau_0)$, where $\tau_0 = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_p$, and select a metric $\gamma = \lambda \widetilde{\gamma}$ conformal to $\widetilde{\gamma}$ such that $\det \gamma = \delta$. Then λ^p is equal to the ratio $N/\Lambda^p \omega$ on the element $\Lambda^p f'(t) \cdot \tau_0 \in \Lambda^p V_{f(t)}$. Since X is compact, it is clear that the conformal factor λ is bounded by an absolute constant independent of f. From the curvature assumption we then get

$$i\,\partial\overline{\partial} \log \det \gamma = i\,\partial\overline{\partial} \log \delta \geqslant (f, \Lambda^p f')^*(i\,\partial\overline{\partial} \log N) \geqslant \varepsilon f^*\omega \geqslant \varepsilon' \, \gamma.$$

By the Ahlfors-Schwarz lemma we infer that $\det \gamma(0) \leqslant C$ for some constant C, i.e., $N_{f(0)}(\Lambda^p f'(0) \cdot \tau_0) \leqslant C'$. This means that the Kobayashi-Eisenman pseudometric $\mathbf{e}^p_{(X,V)}$ is positive definite everywhere and uniformly bounded from below. In the case $\Lambda^p V^*$ is very big with base locus Y, we use essentially the same arguments, but we then only have N being positive definite on $X \smallsetminus Y$. □

4.5. Corollary ([Gri71], KobO71]). *If X is a projective variety of general type, the Kobayashi-Eisenmann volume form $\mathbf{e}^n$, $n = \dim X$, can degenerate only along a proper algebraic set $Y \subsetneq X$.*

§4.C. Main conjectures concerning hyperbolicity

One of the earliest conjectures in hyperbolicity theory is the following statement due to Kobayashi ([Kob70], [Kob76]).

4.6. Conjecture (Kobayashi).

(a) *A (very) generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ large enough is hyperbolic.*

(b) *The complement $\mathbb{P}^n \smallsetminus H$ of a (very) generic hypersurface $H \subset \mathbb{P}^n$ of degree $d \geqslant d'_n$ large enough is hyperbolic.*

In its original form, Kobayashi conjecture did not give the lower bounds d_n and d'_n. Zaidenberg proposed the bounds $d_n = 2n + 1$ (for $n \geqslant 2$) and $d'_n = 2n + 1$ (for $n \geqslant 1$), based on the results of Clemens, Xu, Ein and Voisin already mentioned, and the following observation (cf. [Zai87], [Zai93]).

4.7. Theorem (Zaidenberg). *The complement of a general hypersurface of degree $2n$ in $\mathbb{P}^n$ is not hyperbolic.*

The converse of Corollary 4.5 is also expected to be true, namely, the generic non degeneracy of $\mathbf{e}^n$ should imply that X is of general type, but this is only known for surfaces (see [GrGr80] and [MoMu82]):

4.8. Conjecture (Green-Griffiths [GrGr80]). *A projective algebraic variety X is measure hyperbolic* (*i.e.* $\mathbf{e}^n$ *degenerates only along a proper algebraic subvariety*) *if and only if X is of general type.*

An essential step in the proof of the necessity of having general type subvarieties would be to show that manifolds of Kodaira dimension 0 (say, Calabi-Yau manifolds and holomorphic symplectic manifolds, all of which have $c_1(X) = 0$) are not measure hyperbolic, e.g. by exhibiting enough families of curves $C_{s,\ell}$ covering X such that $(2g(\overline{C}_{s,\ell})-2)/\deg(C_{s,\ell}) \to 0$. Another (even stronger) conjecture which we will investigate at the end of these notes is

4.9. Conjecture (Green-Griffiths [GrGr80]). *If X is a variety of general type, there exists a proper algebraic set $Y \subsetneq X$ such that every entire holomorphic curve $f : \mathbb{C} \to X$ is contained in Y.*

One of the early important result in the direction of Conjecture 4.9 is the proof of the Bloch theorem, as proposed by Bloch [Blo26a] and Ochiai [Och77]. The Bloch theorem is the special case of 4.9 when the irregularity of X satisfies $q = h^0(X, \Omega^1_X) > \dim X$. Various solutions have then been obtained in fundamental papers of Noguchi [Nog77, 81, 84], Kawamata [Kaw80] and Green-Griffiths [GrGr80], by means of different techniques. See section § 10 for a proof based on jet bundle techniques. A much more recent result is the striking statement due to Diverio, Merker and Rousseau [DMR10], confirming 4.9 when $X \subset \mathbb{P}^{n+1}$ is a generic non singular hypersurface of sufficiently large degree $d \geqslant 2^{n^5}$ (cf. §16). Conjecture 4.9 was also considered by S. Lang [Lang86, Lang87] in view of arithmetic counterparts of the above geometric statements.

4.10. Conjecture (Lang). *A projective algebraic variety X is hyperbolic if and only if all its algebraic subvarieties* (*including X itself*) *are of general type.*

4.11. Conjecture (Lang). *Let X be a projective variety defined over a number field K.*

(a) *If X is hyperbolic, then the set of K-rational points is finite.*

(a′) *Conversely, if the set of K'-rational points is finite for every finite extension $K' \supset K$, then X is hyperbolic.*

(b) *If X is of general type, then the set of K-rational points is not Zariski dense.*

(b′) *Conversely, if the set of K'-rational points is not Zariski dense for any extension $K' \supset K$, then X is of general type.*

In fact, in 4.11 (b), if $Y \subsetneq X$ is the "Green-Griffiths locus" of X, it is expected that $X \smallsetminus Y$ contains only finitely many rational K-points. Even when dealing only with the geometric statements, there are several interesting connections between these conjectures.

4.12. Proposition. *Conjecture* 4.9 *implies the "if" part of conjecture* 4.8, *and Conjecture* 4.8 *implies the "only if" part of Conjecture* 4.8, *hence* (4.8 *and* 4.9) $\Rightarrow$ (4.10).

Proof. In fact if Conjecture 4.9 holds and every subariety Y of X is of general type, then it is easy to infer that every entire curve $f : \mathbb{C} \to X$ has to be constant by induction on $\dim X$, because in fact f maps $\mathbb{C}$ to a certain subvariety $Y \subsetneq X$. Therefore X is hyperbolic.

Conversely, if Conjecture 4.8 holds and X has a certain subvariety Y which is not of general type, then Y is not measure hyperbolic. However Proposition 2.4 shows that hyperbolicity implies measure hyperbolicity. Therefore Y is not hyperbolic and so X itself is not hyperbolic either. □

4.13. Proposition. *Assume that the Green-Griffiths conjecture* 4.9 *holds. Then the Kobayashi conjecture* 4.6 (a) *holds with* $d_n = 2n + 1$.

Proof. We know by Ein [Ein88, Ein91] and Voisin [Voi96] that a very generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2n + 1$, $n \geqslant 2$, has all its subvarieties that are of general type. We have seen that the Green-Griffiths conjecture 4.9 implies the hyperbolicity of X in this circumstance. □

§5. Projectivization of a directed manifold

§5.A. The 1-jet fonctor

The basic idea is to introduce a fonctorial process which produces a new complex directed manifold $(\widetilde{X}, \widetilde{V})$ from a given one (X, V). The new structure $(\widetilde{X}, \widetilde{V})$ plays the role of a space of 1-jets over X. We let

$$\widetilde{X} = P(V), \qquad \widetilde{V} \subset T_{\widetilde{X}}$$

be the projectivized bundle of lines of V, together with a subbundle $\widetilde{V}$ of $T_{\widetilde{X}}$ defined as follows: for every point $(x, [v]) \in \widetilde{X}$ associated with a vector $v \in V_x \smallsetminus \{0\}$,

$$\widetilde{V}_{(x,[v])} = \big\{\xi \in T_{\widetilde{X},\,(x,[v])}\,;\ \pi_*\xi \in \mathbb{C}v\big\}, \qquad \mathbb{C}v \subset V_x \subset T_{X,x}, \tag{5.1}$$

where $\pi : \widetilde{X} = P(V) \to X$ is the natural projection and $\pi_* : T_{\widetilde{X}} \to \pi^* T_X$ is its differential. On $\widetilde{X} = P(V)$ we have a tautological line bundle $\mathcal{O}_{\widetilde{X}}(-1) \subset \pi^* V$ such that $\mathcal{O}_{\widetilde{X}}(-1)_{(x,[v])} = \mathbb{C}v$. The bundle $\widetilde{V}$ is characterized by the two exact sequences

$$0 \longrightarrow T_{\widetilde{X}/X} \longrightarrow \widetilde{V} \xrightarrow{\ \pi_*\ } \mathcal{O}_{\widetilde{X}}(-1) \longrightarrow 0, \tag{5.2}$$

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \pi^* V \otimes \mathcal{O}_{\widetilde{X}}(1) \longrightarrow T_{\widetilde{X}/X} \longrightarrow 0, \tag{5.2$'$}$$

where $T_{\widetilde{X}/X}$ denotes the relative tangent bundle of the fibration $\pi : \widetilde{X} \to X$. The first sequence is a direct consequence of the definition of $\widetilde{V}$, whereas the second is a relative version of the Euler exact sequence describing the tangent bundle of the fibers $P(V_x)$. From these exact sequences we infer

$$\dim \widetilde{X} = n + r - 1, \qquad \operatorname{rank} \widetilde{V} = \operatorname{rank} V = r, \tag{5.3}$$

and by taking determinants we find $\det(T_{\widetilde{X}/X}) = \pi^* \det V \otimes \mathcal{O}_{\widetilde{X}}(r)$, thus

$$\det \widetilde{V} = \pi^* \det V \otimes \mathcal{O}_{\widetilde{X}}(r - 1). \tag{5.4}$$

By definition, $\pi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a morphism of complex directed manifolds. Clearly, our construction is fonctorial, i.e., for every morphism of directed manifolds $\Phi : (X, V) \to (Y, W)$, there is a commutative diagram

$$\begin{tikzcd} (\widetilde{X}, \widetilde{V}) \arrow[r, "\pi"] \arrow[d, dashed, "\widetilde{\Phi}"'] & (X, V) \arrow[d, "\Phi"] \\ (\widetilde{Y}, \widetilde{W}) \arrow[r, "\pi"] & (Y, W) \end{tikzcd} \tag{5.5}$$

where the left vertical arrow is the meromorphic map $P(V) \dashrightarrow P(W)$ induced by the differential $\Phi_* : V \to \Phi^* W$ ($\widetilde{\Phi}$ is actually holomorphic if $\Phi_* : V \to \Phi^* W$ is injective).

§5.B. Lifting of curves to the 1-jet bundle

Suppose that we are given a holomorphic curve $f : \Delta_R \to X$ parametrized by the disk Δ_R of centre 0 and radius R in the complex plane, and that f is a tangent curve of the directed manifold, i.e., $f'(t) \in V_{f(t)}$ for every $t \in \Delta_R$. If f is non constant, there is a well defined and unique tangent line $[f'(t)]$ for every t, even at stationary points, and the map

$$\widetilde{f} : \Delta_R \to \widetilde{X}, \qquad t \mapsto \widetilde{f}(t) := (f(t), [f'(t)]) \tag{5.6}$$

is holomorphic (at a stationary point t_0, we just write $f'(t) = (t - t_0)^s u(t)$ with $s \in \mathbb{N}^*$ and $u(t_0) \neq 0$, and we define the tangent line at t_0 to be $[u(t_0)]$, hence $\widetilde{f}(t) = (f(t), [u(t)])$ near t_0 ; even for $t = t_0$, we still denote $[f'(t_0)] = [u(t_0)]$ for simplicity of notation). By definition $f'(t) \in \mathcal{O}_{\widetilde{X}}(-1)_{\widetilde{f}(t)} = \mathbb{C}\, u(t)$, hence the derivative f' defines a section

$$f' : T_{\Delta_R} \to \widetilde{f}^* \mathcal{O}_{\widetilde{X}}(-1). \tag{5.7}$$

Moreover $\pi \circ \widetilde{f} = f$, therefore

$$\pi_* \widetilde{f}'(t) = f'(t) \in \mathbb{C}u(t) \Longrightarrow \widetilde{f}'(t) \in \widetilde{V}_{(f(t),u(t))} = \widetilde{V}_{\widetilde{f}(t)}$$

and we see that $\widetilde{f}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$. We say that $\widetilde{f}$ is the *canonical lifting* of f to $\widetilde{X}$. Conversely, if $g : \Delta_R \to \widetilde{X}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$, then by definition of $\widetilde{V}$ we see that $f = \pi \circ g$ is a tangent trajectory of (X, V) and that $g = \widetilde{f}$ (unless g is contained in a vertical fiber $P(V_x)$, in which case f is constant).

For any point $x_0 \in X$, there are local coordinates $(z_1, \ldots, z_n)$ on a neighborhood Ω of x_0 such that the fibers $(V_z)_{z \in \Omega}$ can be defined by linear equations

$$V_z = \Big\{ \xi = \sum_{1 \leqslant j \leqslant n} \xi_j \frac{\partial}{\partial z_j} \,;\, \xi_j = \sum_{1 \leqslant k \leqslant r} a_{jk}(z) \xi_k \text{ for } j = r+1, \ldots, n \Big\}, \tag{5.8}$$

where (a_{jk}) is a holomorphic $(n-r) \times r$ matrix. It follows that a vector $\xi \in V_z$ is completely determined by its first r components $(\xi_1, \ldots, \xi_r)$, and the affine chart $\xi_j \neq 0$ of $P(V)_{\restriction \Omega}$ can be described by the coordinate system

$$\Big(z_1, \ldots, z_n; \frac{\xi_1}{\xi_j}, \ldots, \frac{\xi_{j-1}}{\xi_j}, \frac{\xi_{j+1}}{\xi_j}, \ldots, \frac{\xi_r}{\xi_j} \Big). \tag{5.9}$$

Let $f \simeq (f_1, \ldots, f_n)$ be the components of f in the coordinates $(z_1, \ldots, z_n)$ (we suppose here R so small that $f(\Delta_R) \subset \Omega$). It should be observed that f is uniquely determined by its initial value x and by the first r components $(f_1, \ldots, f_r)$. Indeed, as $f'(t) \in V_{f(t)}$, we can recover the other components by integrating the system of ordinary differential equations

$$f_j'(t) = \sum_{1 \leqslant k \leqslant r} a_{jk}(f(t)) f_k'(t), \qquad j > r, \tag{5.10}$$

on a neighborhood of 0, with initial data $f(0) = x$. We denote by $m = m(f, t_0)$ the *multiplicity* of f at any point $t_0 \in \Delta_R$, that is, $m(f, t_0)$ is the smallest integer $m \in \mathbb{N}^*$ such that $f_j^{(m)}(t_0) \neq 0$ for some j. By (5.10), we can always suppose $j \in \{1, \ldots, r\}$, for example $f_r^{(m)}(t_0) \neq 0$. Then $f'(t) = (t - t_0)^{m-1} u(t)$ with $u_r(t_0) \neq 0$, and the lifting $\widetilde{f}$ is described in the coordinates of the affine chart $\xi_r \neq 0$ of $P(V)_{\restriction \Omega}$ by

$$\widetilde{f} \simeq \Big(f_1, \ldots, f_n; \frac{f_1'}{f_r'}, \ldots, \frac{f_{r-1}'}{f_r'} \Big). \tag{5.11}$$

§5.C. Curvature properties of the 1-jet bundle

We end this section with a few curvature computations. Assume that V is equipped with a smooth hermitian metric h. Denote by $\nabla_h = \nabla'_h + \nabla''_h$ the associated Chern connection and by $\Theta_{V,h} = \frac{i}{2\pi}\nabla_h^2$ its Chern curvature tensor. For every point $x_0 \in X$, there exists a "normalized" holomorphic frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ on a neighborhood of x_0, such that

$$\langle e_\lambda, e_\mu\rangle_h = \delta_{\lambda\mu} - \sum_{1\leqslant j,k\leqslant n} c_{jk\lambda\mu} z_j \overline{z}_k + O(|z|^3), \tag{5.12}$$

with respect to any holomorphic coordinate system $(z_1, \ldots, z_n)$ centered at x_0. A computation of $d'\langle e_\lambda, e_\mu\rangle_h = \langle\nabla'_h e_\lambda, e_\mu\rangle_h$ and $\nabla_h^2 e_\lambda = d''\nabla'_h e_\lambda$ then gives

$$\nabla'_h e_\lambda = -\sum_{j,k,\mu} c_{jk\lambda\mu}\overline{z}_k\, dz_j \otimes e_\mu + O(|z|^2),$$

$$\Theta_{V,h}(x_0) = \frac{i}{2\pi}\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e_\lambda^* \otimes e_\mu. \tag{5.13}$$

The above curvature tensor can also be viewed as a hermitian form on $T_X \otimes V$. In fact, one associates with $\Theta_{V,h}$ the hermitian form $\langle\Theta_{V,h}\rangle$ on $T_X \otimes V$ defined for all $(\zeta, v) \in T_X \times_X V$ by

$$\langle\Theta_{V,h}\rangle(\zeta\otimes v) = \sum_{1\leqslant j,k\leqslant n,\, 1\leqslant\lambda,\mu\leqslant r} c_{jk\lambda\mu}\zeta_j\overline{\zeta}_k v_\lambda \overline{v}_\mu. \tag{5.14}$$

Let h_1 be the hermitian metric on the tautological line bundle $\mathcal{O}_{P(V)}(-1) \subset \pi^*V$ induced by the metric h of V. We compute the curvature $(1,1)$-form $\Theta_{h_1}(\mathcal{O}_{P(V)}(-1))$ at an arbitrary point $(x_0, [v_0]) \in P(V)$, in terms of $\Theta_{V,h}$. For simplicity, we suppose that the frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ has been chosen in such a way that $[e_r(x_0)] = [v_0] \in P(V)$ and $|v_0|_h = 1$. We get holomorphic local coordinates $(z_1, \ldots, z_n\,;\, \xi_1, \ldots, \xi_{r-1})$ on a neighborhood of $(x_0, [v_0])$ in $P(V)$ by assigning

$$(z_1, \ldots, z_n\,;\, \xi_1, \ldots, \xi_{r-1}) \longmapsto (z, [\xi_1 e_1(z) + \cdots + \xi_{r-1}e_{r-1}(z) + e_r(z)]) \in P(V).$$

Then the function

$$\eta(z,\xi) = \xi_1 e_1(z) + \cdots + \xi_{r-1}e_{r-1}(z) + e_r(z)$$

defines a holomorphic section of $\mathcal{O}_{P(V)}(-1)$ in a neighborhood of $(x_0, [v_0])$. By using the expansion (5.12) for h, we find

$$|\eta|^2_{h_1} = |\eta|^2_h = 1 + |\xi|^2 - \sum_{1\leqslant j,k\leqslant n} c_{jkrr} z_j\overline{z}_k + O((|z|+|\xi|)^3),$$

$$\Theta_{h_1}(\mathcal{O}_{P(V)}(-1))_{(x_0,[v_0])} = -\frac{i}{2\pi}\partial\overline{\partial}\log|\eta|^2_{h_1}$$

$$= \frac{i}{2\pi}\Big(\sum_{1\leqslant j,k\leqslant n} c_{jkrr} dz_j\wedge d\overline{z}_k - \sum_{1\leqslant\lambda\leqslant r-1} d\xi_\lambda \wedge d\overline{\xi}_\lambda\Big). \tag{5.15}$$

§6. Jets of curves and Semple jet bundles

Let X be a complex n-dimensional manifold. Following ideas of Green-Griffiths [GrGr80], we let $J_k \to X$ be the bundle of k-jets of germs of parametrized curves in X, that is, the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \leqslant j \leqslant k$, when computed in some local coordinate system of X near x. The projection map $J_k \to X$ is simply $f \mapsto f(0)$. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on an open set $\Omega \subset X$, the elements f of any fiber $J_{k,x}$, $x \in \Omega$, can be seen as $\mathbb{C}^n$-valued maps

$$f = (f_1, \ldots, f_n) : (\mathbb{C}, 0) \to \Omega \subset \mathbb{C}^n,$$

and they are completetely determined by their Taylor expansion of order k at $t = 0$

$$f(t) = x + t\, f'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1}).$$

In these coordinates, the fiber $J_{k,x}$ can thus be identified with the set of k-tuples of vectors $(\xi_1, \ldots, \xi_k) = (f'(0), \ldots, f^{(k)}(0)) \in (\mathbb{C}^n)^k$. It follows that J_k is a holomorphic fiber bundle with typical fiber $(\mathbb{C}^n)^k$ over X (however, J_k is not a vector bundle for $k \geqslant 2$, because of the nonlinearity of coordinate changes; see formula (7.2) in § 7).

According to the philosophy developed throughout this paper, we describe the concept of jet bundle in the general situation of complex directed manifolds. If X is equipped with a holomorphic subbundle $V \subset T_X$, we associate to V a k-jet bundle $J_k V$ as follows.

6.1. Definition. *Let (X, V) be a complex directed manifold. We define $J_k V \to X$ to be the bundle of k-jets of curves $f : (\mathbb{C}, 0) \to X$ which are tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all t in a neighborhood of 0, together with the projection map $f \mapsto f(0)$ onto X.*

It is easy to check that $J_k V$ is actually a subbundle of J_k. In fact, by using (5.8) and (5.10), we see that the fibers $J_k V_x$ are parametrized by

$$\big((f_1'(0), \ldots, f_r'(0)); (f_1''(0), \ldots, f_r''(0)); \ldots; (f_1^{(k)}(0), \ldots, f_r^{(k)}(0))\big) \in (\mathbb{C}^r)^k$$

for all $x \in \Omega$, hence $J_k V$ is a locally trivial $(\mathbb{C}^r)^k$-subbundle of J_k. Alternatively, we can pick a local holomorphic connection ∇ on V, defined on some open set $\Omega \subset X$, and compute inductively the successive derivatives

$$\nabla f = f', \qquad \nabla^j f = \nabla_{f'}(\nabla^{j-1} f)$$

with respect to ∇ along the cure $t \mapsto f(t)$. Then

$$(\xi_1, \xi_2, \ldots, \xi_k) = (\nabla f(0), \nabla^2 f(0), \ldots, \nabla^k f(0)) \in V_x^{\oplus k}$$

provides a "trivialization" $J^k V_{|\Omega} \simeq V_{|\Omega}^{\oplus k}$. This identification depends of course on the choice of ∇ and cannot be defined globally in general (unless we are in the rare situation where V has a global holomorphic connection). □

We now describe a convenient process for constructing "projectivized jet bundles", which will later appear as natural quotients of our jet bundles $J_k V$ (or rather, as suitable desingularized compactifications of the quotients). Such spaces have already been considered since a long time, at least in the special case $X = \mathbb{P}^2$, $V = T_{\mathbb{P}^2}$ (see Gherardelli [Ghe41], Semple [Sem54]), and they have been mostly used as a tool for establishing enumerative formulas dealing with the order of contact of plane curves (see [Coll88], [CoKe94]); the article [ASS92] is also concerned with such generalizations of jet bundles, as well as [LaTh96] by Laksov and Thorup.

We define inductively the *projectivized k-jet bundle* $P_kV = X_k$ (or *Semple k-jet bundle*) and the associated subbundle $V_k \subset T_{X_k}$ by

$$(X_0, V_0) = (X, V), \qquad (X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1}). \tag{6.2}$$

In other words, $(P_kV, V_k) = (X_k, V_k)$ is obtained from (X, V) by iterating k-times the lifting construction $(X, V) \mapsto (\widetilde{X}, \widetilde{V})$ described in § 5. By (5.2–5.7), we find

$$\dim P_kV = n + k(r-1), \qquad \operatorname{rank} V_k = r, \tag{6.3}$$

together with exact sequences

$$0 \longrightarrow T_{P_kV/P_{k-1}V} \longrightarrow V_k \xrightarrow{(\pi_k)_*} \mathcal{O}_{P_kV}(-1) \longrightarrow 0, \tag{6.4}$$

$$0 \longrightarrow \mathcal{O}_{P_kV} \longrightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{P_kV}(1) \longrightarrow T_{P_kV/P_{k-1}V} \longrightarrow 0. \tag{6.4'}$$

where π_k is the natural projection $\pi_k : P_kV \to P_{k-1}V$ and $(\pi_k)_*$ its differential. Formula (5.4) yields

$$\det V_k = \pi_k^* \det V_{k-1} \otimes \mathcal{O}_{P_kV}(r-1). \tag{6.5}$$

Every non constant tangent trajectory $f : \Delta_R \to X$ of (X, V) lifts to a well defined and unique tangent trajectory $f_{[k]} : \Delta_R \to P_kV$ of (P_kV, V_k). Moreover, the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]} : T_{\Delta_R} \to f_{[k]}^* \mathcal{O}_{P_kV}(-1). \tag{6.6}$$

In coordinates, one can compute $f_{[k]}$ in terms of its components in the various affine charts (5.9) occurring at each step: we get inductively

$$f_{[k]} = (F_1, \ldots, F_N), \qquad f_{[k+1]} = \Big(F_1, \ldots, F_N, \frac{F'_{s_1}}{F'_{s_r}}, \ldots, \frac{F'_{s_{r-1}}}{F'_{s_r}}\Big) \tag{6.7}$$

where $N = n + k(r-1)$ and $\{s_1, \ldots, s_r\} \subset \{1, \ldots, N\}$. If $k \geqslant 1$, $\{s_1, \ldots, s_r\}$ contains the last $r-1$ indices of $\{1, \ldots, N\}$ corresponding to the "vertical" components of the projection $P_kV \to P_{k-1}V$, and in general, s_r is an index such that $m(F_{s_r}, 0) = m(f_{[k]}, 0)$, that is, F_{s_r} has the smallest vanishing order among all components F_s (s_r may be vertical or not, and the choice of $\{s_1, \ldots, s_r\}$ need not be unique).

By definition, there is a canonical injection $\mathcal{O}_{P_kV}(-1) \hookrightarrow \pi_k^* V_{k-1}$, and a composition with the projection $(\pi_{k-1})_*$ (analogue for order $k-1$ of the arrow $(\pi_k)_*$ in sequence (6.4)) yields for all $k \geqslant 2$ a canonical line bundle morphism

$$\mathcal{O}_{P_kV}(-1) \hookrightarrow \pi_k^* V_{k-1} \xrightarrow{(\pi_k)^*(\pi_{k-1})_*} \pi_k^* \mathcal{O}_{P_{k-1}V}(-1), \tag{6.8}$$

which admits precisely $D_k = P(T_{P_{k-1}V/P_{k-2}V}) \subset P(V_{k-1}) = P_kV$ as its zero divisor (clearly, D_k is a hyperplane subbundle of P_kV). Hence we find

$$\mathcal{O}_{P_kV}(1) = \pi_k^* \mathcal{O}_{P_{k-1}V}(1) \otimes \mathcal{O}(D_k). \tag{6.9}$$

Now, we consider the composition of projections

$$\pi_{j,k} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : P_kV \longrightarrow P_jV. \tag{6.10}$$

Then $\pi_{0,k} : P_kV \to X = P_0V$ is a locally trivial holomorphic fiber bundle over X, and the fibers $P_kV_x = \pi_{0,k}^{-1}(x)$ are k-stage towers of $\mathbb{P}^{r-1}$-bundles. Since we have (in both directions) morphisms $(\mathbb{C}^r, T_{\mathbb{C}^r}) \leftrightarrow (X, V)$ of directed manifolds which are bijective on the level of bundle morphisms, the fibers are all isomorphic to a "universal" nonsingular projective algebraic variety of dimension $k(r-1)$ which we will denote by $\mathbb{R}_{r,k}$; it is not hard to see that $\mathbb{R}_{r,k}$ is rational (as will indeed follow from the proof of Theorem 6.8 below). The following Proposition will help us to understand a little bit more about the geometric structure of P_kV. As usual, we define the *multiplicity* $m(f, t_0)$ of a curve $f : \Delta_R \to X$ at a point $t \in \Delta_R$ to be the smallest integer $s \in \mathbb{N}^*$ such that $f^{(s)}(t_0) \neq 0$, i.e., the largest s such that $\delta(f(t), f(t_0)) = O(|t - t_0|^s)$ for any hermitian or riemannian geodesic distance δ on X. As $f_{[k-1]} = \pi_k \circ f_{[k]}$, it is clear that the sequence $m(f_{[k]}, t)$ is non increasing with k.

6.11. Proposition. *Let $f : (\mathbb{C}, 0) \to X$ be a non constant germ of curve tangent to V. Then for all $j \geqslant 2$ we have $m(f_{[j-2]}, 0) \geqslant m(f_{[j-1]}, 0)$ and the inequality is strict if and only if $f_{[j]}(0) \in D_j$. Conversely, if $w \in P_kV$ is an arbitrary element and $m_0 \geqslant m_1 \geqslant \cdots \geqslant m_{k-1} \geqslant 1$ is a sequence of integers with the property that*

$$\forall j \in \{2, \ldots, k\}, \qquad m_{j-2} > m_{j-1} \quad \textit{if and only if } \pi_{j,k}(w) \in D_j,$$

there exists a germ of curve $f : (\mathbb{C}, 0) \to X$ tangent to V such that $f_{[k]}(0) = w$ and $m(f_{[j]}, 0) = m_j$ for all $j \in \{0, \ldots, k-1\}$.

Proof. i) Suppose first that f is given and put $m_j = m(f_{[j]}, 0)$. By definition, we have $f_{[j]} = (f_{[j-1]}, [u_{j-1}])$ where $f'_{[j-1]}(t) = t^{m_{j-1}-1}u_{j-1}(t) \in V_{j-1}$, $u_{j-1}(0) \neq 0$. By composing with the differential of the projection $\pi_{j-1} : P_{j-1}V \to P_{j-2}V$, we find $f'_{[j-2]}(t) = t^{m_{j-1}-1}(\pi_{j-1})_* u_{j-1}(t)$. Therefore

$$m_{j-2} = m_{j-1} + \mathrm{ord}_{t=0}(\pi_{j-1})_* u_{j-1}(t),$$

and so $m_{j-2} > m_{j-1}$ if and only if $(\pi_{j-1})_* u_{j-1}(0) = 0$, that is, if and only if $u_{j-1}(0) \in T_{P_{j-1}V/P_{j-2}V}$, or equivalently $f_{[j]}(0) = (f_{[j-1]}(0), [u_{j-1}(0)]) \in D_j$.

ii) Suppose now that $w \in P_kV$ and $m_0, \ldots, m_{k-1}$ are given. We denote by $w_{j+1} = (w_j, [\eta_j])$, $w_j \in P_jV$, $\eta_j \in V_j$, the projection of w to $P_{j+1}V$. Fix coordinates $(z_1, \ldots, z_n)$ on X centered at w_0 such that the r-th component $\eta_{0,r}$ of η_0 is non zero. We prove the existence of the germ f by induction on k, in the form of a Taylor expansion

$$f(t) = a_0 + t\, a_1 + \cdots + t^{d_k} a_{d_k} + O(t^{d_k+1}), \qquad d_k = m_0 + m_1 + \cdots + m_{k-1}.$$

If $k = 1$ and $w = (w_0, [\eta_0]) \in P_1V_x$, we simply take $f(t) = w_0 + t^{m_0}\eta_0 + O(t^{m_0+1})$. In general, the induction hypothesis applied to $P_kV = P_{k-1}(V_1)$ over $X_1 = P_1V$ yields a curve $g : (\mathbb{C}, 0) \to X_1$ such that $g_{[k-1]} = w$ and $m(g_{[j]}, 0) = m_{j+1}$ for $0 \leqslant j \leqslant k-2$. If $w_2 \notin D_2$, then $[g'_{[1]}(0)] = [\eta_1]$ is not vertical, thus $f = \pi_1 \circ g$ satisfies $m(f, 0) = m(g, 0) = m_1 = m_0$ and we are done.

If $w_2 \in D_2$, we express $g = (G_1, \ldots, G_n; G_{n+1}, \ldots, G_{n+r-1})$ as a Taylor expansion of order $m_1 + \cdots + m_{k-1}$ in the coordinates (5.9) of the affine chart $\xi_r \neq 0$. As $\eta_1 = \lim_{t \to 0} g'(t)/t^{m_1-1}$ is vertical, we must have $m(G_s, 0) > m_1$ for $1 \leqslant j \leqslant n$. It follows from (6.7) that $G_1, \ldots, G_n$ are never involved in the calculation of the liftings $g_{[j]}$. We can therefore replace g by $f \simeq (f_1, \ldots, f_n)$ where $f_r(t) = t^{m_0}$ and $f_1, \ldots, f_{r-1}$ are obtained by integrating the equations $f_j'(t)/f_r'(t) = G_{n+j}(t)$, i.e., $f_j'(t) = m_0 t^{m_0-1} G_{n+j}(t)$, while $f_{r+1}, \ldots, f_n$ are obtained by integrating (5.10). We then get the desired Taylor expansion of order d_k for f. □

Since we can always take $m_{k-1} = 1$ without restriction, we get in particular:

6.12. Corollary. *Let $w \in P_k V$ be an arbitrary element. Then there is a germ of curve $f : (\mathbb{C}, 0) \to X$ such that $f_{[k]}(0) = w$ and $f'_{[k-1]}(0) \neq 0$ (thus the liftings $f_{[k-1]}$ and $f_{[k]}$ are regular germs of curve). Moreover, if $w_0 \in P_k V$ and w is taken in a sufficiently small neighborhood of w_0, then the germ $f = f_w$ can be taken to depend holomorphically on w.*

Proof. Only the holomorphic dependence of f_w with respect to w has to be guaranteed. If f_{w_0} is a solution for $w = w_0$, we observe that $(f_{w_0})'_{[k]}$ is a non vanishing section of V_k along the regular curve defined by $(f_{w_0})_{[k]}$ in $P_k V$. We can thus find a non vanishing section ξ of V_k on a neighborhood of w_0 in $P_k V$ such that $\xi = (f_{w_0})'_{[k]}$ along that curve. We define $t \mapsto F_w(t)$ to be the trajectory of ξ with initial point w, and we put $f_w = \pi_{0,k} \circ F_w$. Then f_w is the required family of germs. □

Now, we can take $f : (\mathbb{C}, 0) \to X$ to be regular at the origin (by this, we mean $f'(0) \neq 0$) if and only if $m_0 = m_1 = \cdots = m_{k-1} = 1$, which is possible by Proposition 6.11 if and only if $w \in P_k V$ is such that $\pi_{j,k}(w) \notin D_j$ for all $j \in \{2, \ldots, k\}$. For this reason, we define

$$\begin{aligned} P_k V^{\mathrm{reg}} &= \bigcap_{2 \leqslant j \leqslant k} \pi_{j,k}^{-1}(P_j V \smallsetminus D_j), \\ P_k V^{\mathrm{sing}} &= \bigcup_{2 \leqslant j \leqslant k} \pi_{j,k}^{-1}(D_j) = P_k V \smallsetminus P_k V^{\mathrm{reg}}, \end{aligned} \tag{6.13}$$

in other words, $P_k V^{\mathrm{reg}}$ is the set of values $f_{[k]}(0)$ reached by all regular germs of curves f. One should take care however that there are singular germs which reach the same points $f_{[k]}(0) \in P_k V^{\mathrm{reg}}$, e.g., any s-sheeted covering $t \mapsto f(t^s)$. On the other hand, if $w \in P_k V^{\mathrm{sing}}$, we can reach w by a germ f with $m_0 = m(f, 0)$ as large as we want.

6.14. Corollary. *Let $w \in P_k V^{\mathrm{sing}}$ be given, and let $m_0 \in \mathbb{N}$ be an arbitrary integer larger than the number of components D_j such that $\pi_{j,k}(w) \in D_j$. Then there is a germ of curve $f : (\mathbb{C}, 0) \to X$ with multiplicity $m(f, 0) = m_0$ at the origin, such that $f_{[k]}(0) = w$ and $f'_{[k-1]}(0) \neq 0$.*

§7. Jet differentials

§7.A. Green-Griffiths jet differentials

We first introduce the concept of jet differentials in the sense of Green-Griffiths [GrGr80]. The goal is to provide an intrinsic geometric description of holomorphic differential equations that a germ of curve $f : (\mathbb{C}, 0) \to X$ may satisfy. In the sequel, we fix a directed manifold (X, V) and suppose implicitly that all germs of curves f are tangent to V.

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \qquad a_1 \in \mathbb{C}^*,\ a_j \in \mathbb{C},\ j \geqslant 2,$$

in which the composition law is taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ is a k-dimensional nilpotent complex Lie group, which admits a natural fiberwise right action on J_kV. The action consists of reparametrizing k-jets of maps $f : (\mathbb{C}, 0) \to X$ by a biholomorphic change of parameter $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, that is, $(f, \varphi) \mapsto f \circ \varphi$. There is an exact sequence of groups

$$1 \to \mathbb{G}'_k \to \mathbb{G}_k \to \mathbb{C}^* \to 1$$

where $\mathbb{G}_k \to \mathbb{C}^*$ is the obvious morphism $\varphi \mapsto \varphi'(0)$, and $\mathbb{G}'_k = [\mathbb{G}_k, \mathbb{G}_k]$ is the group of k-jets of biholomorphisms tangent to the identity. Moreover, the subgroup $\mathbb{H} \simeq \mathbb{C}^*$ of homotheties $\varphi(t) = \lambda t$ is a (non normal) subgroup of $\mathbb{G}_k$, and we have a semidirect decomposition $\mathbb{G}_k = \mathbb{G}'_k \ltimes \mathbb{H}$. The corresponding action on k-jets is described in coordinates by

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

Following [GrGr80], we introduce the vector bundle $E^{\mathrm{GG}}_{k,m}V^* \to X$ whose fibers are complex valued polynomials $Q(f', f'', \ldots, f^{(k)})$ on the fibers of J_kV, of weighted degree m with respect to the $\mathbb{C}^*$ action defined by H, that is, such that

$$Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)}) \tag{7.1}$$

for all $\lambda \in \mathbb{C}^*$ and $(f', f'', \ldots, f^{(k)}) \in J_kV$. Here we view $(f', f'', \ldots, f^{(k)})$ as indeterminates with components

$$\big((f'_1, \ldots, f'_r); (f''_1, \ldots, f''_r); \ldots; (f^{(k)}_1, \ldots, f^{(k)}_r)\big) \in (\mathbb{C}^r)^k.$$

Notice that the concept of polynomial on the fibers of J_kV makes sense, for all coordinate changes $z \mapsto w = \Psi(z)$ on X induce polynomial transition automorphisms on the fibers of J_kV, given by a formula

$$(\Psi \circ f)^{(j)} = \Psi'(f) \cdot f^{(j)} + \sum_{s=2}^{s=j} \sum_{j_1+j_2+\cdots+j_s=j} c_{j_1\ldots j_s} \Psi^{(s)}(f) \cdot (f^{(j_1)}, \ldots, f^{(j_s)}) \tag{7.2}$$

with suitable integer constants $c_{j_1\ldots j_s}$ (this is easily checked by induction on s). In the "absolute case" $V = T_X$, we simply write $E^{\mathrm{GG}}_{k,m}T^*_X = E^{\mathrm{GG}}_{k,m}$. If $V \subset W \subset T_X$ are holomorphic subbundles, there are natural inclusions

$$J_kV \subset J_kW \subset J_k, \qquad P_kV \subset P_kW \subset P_k.$$

The restriction morphisms induce surjective arrows

$$E^{\mathrm{GG}}_{k,m} \to E^{\mathrm{GG}}_{k,m}W^* \to E^{\mathrm{GG}}_{k,m}V^*,$$

in particular $E^{\mathrm{GG}}_{k,m}V^*$ can be seen as a quotient of $E^{\mathrm{GG}}_{k,m}$. (The notation V^* is used here to make the contravariance property implicit from the notation). Another useful consequence of these inclusions is that one can extend the definition of J_kV and P_kV to the case where V is an arbitrary linear space, simply by taking the closure of $J_kV_{X\smallsetminus\mathrm{Sing}(V)}$ and $P_kV_{X\smallsetminus\mathrm{Sing}(V)}$ in the smooth bundles J_k and P_k, respectively.

If $Q \in E_{k,m}^{\rm GG}V^*$ is decomposed into multihomogeneous components of multidegree $(\ell_1, \ell_2, \ldots, \ell_k)$ in $f', f'', \ldots, f^{(k)}$ (the decomposition is of course coordinate dependent), these multidegrees must satisfy the relation

$$\ell_1 + 2\ell_2 + \cdots + k\ell_k = m.$$

The bundle $E_{k,m}^{\rm GG}V^*$ will be called the *bundle of jet differentials of order k and weighted degree m*. It is clear from (7.2) that a coordinate change $f \mapsto \Psi\circ f$ transforms every monomial $(f^{(\bullet)})^\ell = (f')^{\ell_1}(f'')^{\ell_2}\cdots(f^{(k)})^{\ell_k}$ of partial weighted degree $|\ell|_s := \ell_1 + 2\ell_2 + \cdots + s\ell_s$, $1 \leqslant s \leqslant k$, into a polynomial $((\Psi\circ f)^{(\bullet)})^\ell$ in $(f', f'', \ldots, f^{(k)})$ which has the same partial weighted degree of order s if $\ell_{s+1} = \cdots = \ell_k = 0$, and a larger or equal partial degree of order s otherwise. Hence, for each $s = 1, \ldots, k$, we get a well defined (i.e., coordinate invariant) decreasing filtration $F_s^\bullet$ on $E_{k,m}^{\rm GG}V^*$ as follows:

$$F_s^p(E_{k,m}^{\rm GG}V^*) = \left\{ \begin{matrix} Q(f', f'', \ldots, f^{(k)}) \in E_{k,m}^{\rm GG}V^* \text{ involving} \\ \text{only monomials } (f^{(\bullet)})^\ell \text{ with } |\ell|_s \geqslant p \end{matrix} \right\}, \qquad \forall p \in \mathbb{N}. \tag{7.3}$$

The graded terms $\mathrm{Gr}_{k-1}^p(E_{k,m}^{\rm GG}V^*)$ associated with the filtration $F_{k-1}^p(E_{k,m}^{\rm GG}V^*)$ are precisely the homogeneous polynomials $Q(f', \ldots, f^{(k)})$ whose monomials $(f^\bullet)^\ell$ all have partial weighted degree $|\ell|_{k-1} = p$ (hence their degree ℓ_k in $f^{(k)}$ is such that $m - p = k\ell_k$, and $\mathrm{Gr}_{k-1}^p(E_{k,m}^{\rm GG}V^*) = 0$ unless $k|m-p$). The transition automorphisms of the graded bundle are induced by coordinate changes $f \mapsto \Psi\circ f$, and they are described by substituting the arguments of $Q(f', \ldots, f^{(k)})$ according to formula (7.2), namely $f^{(j)} \mapsto (\Psi\circ f)^{(j)}$ for $j < k$, and $f^{(k)} \mapsto \Psi'(f)\circ f^{(k)}$ for $j = k$ (when $j = k$, the other terms fall in the next stage F_{k-1}^{p+1} of the filtration). Therefore $f^{(k)}$ behaves as an element of $V \subset T_X$ under coordinate changes. We thus find

$$G_{k-1}^{m-k\ell_k}(E_{k,m}^{\rm GG}V^*) = E_{k-1,m-k\ell_k}^{\rm GG}V^* \otimes S^{\ell_k}V^*. \tag{7.4}$$

Combining all filtrations $F_s^\bullet$ together, we find inductively a filtration $F^\bullet$ on $E_{k,m}^{\rm GG}V^*$ such that the graded terms are

$$\mathrm{Gr}^\ell(E_{k,m}^{\rm GG}V^*) = S^{\ell_1}V^* \otimes S^{\ell_2}V^* \otimes \cdots \otimes S^{\ell_k}V^*, \qquad \ell \in \mathbb{N}^k, \quad |\ell|_k = m. \tag{7.5}$$

The bundles $E_{k,m}^{\rm GG}V^*$ have other interesting properties. In fact,

$$E_{k,\bullet}^{\rm GG}V^* := \bigoplus_{m\geqslant 0} E_{k,m}^{\rm GG}V^*$$

is in a natural way a bundle of graded algebras (the product is obtained simply by taking the product of polynomials). There are natural inclusions $E_{k,\bullet}^{\rm GG}V^* \subset E_{k+1,\bullet}^{\rm GG}V^*$ of algebras, hence $E_{\infty,\bullet}^{\rm GG}V^* = \bigcup_{k\geqslant 0} E_{k,\bullet}^{\rm GG}V^*$ is also an algebra. Moreover, the sheaf of holomorphic sections $\mathcal{O}(E_{\infty,\bullet}^{\rm GG}V^*)$ admits a canonical derivation $\nabla^{\rm GG}$ given by a collection of $\mathbb{C}$-linear maps

$$\nabla^{\rm GG} : \mathcal{O}(E_{k,m}^{\rm GG}V^*) \to \mathcal{O}(E_{k+1,m+1}^{\rm GG}V^*),$$

constructed in the following way. A holomorphic section of $E_{k,m}^{\rm GG}V^*$ on a coordinate open set $\Omega \subset X$ can be seen as a differential operator on the space of germs $f : (\mathbb{C}, 0) \to \Omega$ of the form

$$Q(f) = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} a_{\alpha_1\ldots\alpha_k}(f)\,(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k} \tag{7.6}$$

in which the coefficients $a_{\alpha_1\ldots\alpha_k}$ are holomorphic functions on Ω. Then ∇Q is given by the formal derivative $(\nabla Q)(f)(t) = d(Q(f))/dt$ with respect to the 1-dimensional parameter t in $f(t)$. For example, in dimension 2, if $Q \in H^0(\Omega, \mathcal{O}(E^{\mathrm{GG}}_{2,4}))$ is the section of weighted degree 4

$$Q(f) = a(f_1, f_2)\, f_1'^3 f_2' + b(f_1, f_2)\, f_1''^2,$$

we find that $\nabla Q \in H^0(\Omega, \mathcal{O}(E^{\mathrm{GG}}_{3,5}))$ is given by

$$\begin{aligned}(\nabla Q)(f) = {}& \frac{\partial a}{\partial z_1}(f_1, f_2)\, f_1'^4 f_2' + \frac{\partial a}{\partial z_2}(f_1, f_2)\, f_1'^3 f_2'^2 + \frac{\partial b}{\partial z_1}(f_1, f_2)\, f_1' f_1''^2 \\ &+ \frac{\partial b}{\partial z_2}(f_1, f_2)\, f_2' f_1''^2 + a(f_1, f_2)\,\big(3 f_1'^2 f_1'' f_2' + f_1'^3 f_2''\big) + b(f_1, f_2)\; 2 f_1'' f_1'''.\end{aligned}$$

Associated with the graded algebra bundle $E^{\mathrm{GG}}_{k,\bullet}V^*$, we have an analytic fiber bundle

$$X^{\mathrm{GG}}_k := \mathrm{Proj}(E^{\mathrm{GG}}_{k,\bullet}V^*) = (J_kV \smallsetminus \{0\})/\mathbb{C}^* \tag{7.7}$$

over X, which has weighted projective spaces $\mathbb{P}(1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ as fibers (these weighted projective spaces are singular for $k > 1$, but they only have quotient singularities, see [Dol81] ; here $J_kV \smallsetminus \{0\}$ is the set of non constant jets of order k ; we refer e.g. to Hartshorne's book [Har77] for a definition of the Proj fonctor). As such, it possesses a canonical sheaf $\mathcal{O}_{X^{\mathrm{GG}}_k}(1)$ such that $\mathcal{O}_{X^{\mathrm{GG}}_k}(m)$ is invertible when m is a multiple of $\mathrm{lcm}(1, 2, \ldots, k)$. Under the natural projection $\pi_k : X^{\mathrm{GG}}_k \to X$, the direct image $(\pi_k)_*\mathcal{O}_{X^{\mathrm{GG}}_k}(m)$ coincides with polynomials

$$P(z\,;\xi_1, \ldots, \xi_k) = \sum_{\alpha_\ell \in \mathbb{N}^r,\, 1 \leqslant \ell \leqslant k} a_{\alpha_1\ldots\alpha_k}(z)\, \xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k} \tag{7.8}$$

of weighted degree $|\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k| = m$ on J^kV with holomorphic coefficients; in other words, we obtain precisely the sheaf of sections of the bundle $E^{\mathrm{GG}}_{k,m}V^*$ of jet differentials of order k and degree m.

7.9. Proposition. *By construction, if $\pi_k : X^{\mathrm{GG}}_k$ is the natural projection, we have the direct image formula*

$$(\pi_k)_*\mathcal{O}_{X^{\mathrm{GG}}_k}(m) = \mathcal{O}(E^{\mathrm{GG}}_{k,m}V^*)$$

for all k and m.

§7.B. Invariant jet differentials

In the geometric context, we are not really interested in the bundles $(J_kV \smallsetminus \{0\})/\mathbb{C}^*$ themselves, but rather on their quotients $(J_kV \smallsetminus \{0\})/\mathbb{G}_k$ (would such nice complex space quotients exist!). We will see that the Semple bundle P_kV constructed in § 6 plays the role of such a quotient. First we introduce a canonical bundle subalgebra of $E^{\mathrm{GG}}_{k,\bullet}V^*$.

7.10. Definition. *We introduce a subbundle $E_{k,m}V^* \subset E^{\mathrm{GG}}_{k,m}V^*$, called the bundle of invariant jet differentials of order k and degree m, defined as follows: $E_{k,m}V^*$ is the set of polynomial differential operators $Q(f', f'', \ldots, f^{(k)})$ which are invariant under arbitrary changes of parametrization, i.e., for every $\varphi \in \mathbb{G}_k$*

$$Q\big((f\circ\varphi)', (f\circ\varphi)'', \ldots, (f\circ\varphi)^{(k)}\big) = \varphi'(0)^m Q(f', f'', \ldots, f^{(k)}).$$

Alternatively, $E_{k,m}V^* = (E^{\rm GG}_{k,m}V^*)^{\mathbb{G}'_k}$ is the set of invariants of $E^{\rm GG}_{k,m}V^*$ under the action of $\mathbb{G}'_k$. Clearly, $E_{\infty,\bullet}V^* = \bigcup_{k\geqslant 0}\bigoplus_{m\geqslant 0} E_{k,m}V^*$ is a subalgebra of $E^{\rm GG}_{k,m}V^*$ (observe however that this algebra is not invariant under the derivation $\nabla^{\rm GG}$, since e.g. $f''_j = \nabla^{\rm GG} f_j$ is not an invariant polynomial). In addition to this, there are natural induced filtrations $F^p_s(E_{k,m}V^*) = E_{k,m}V^* \cap F^p_s(E^{\rm GG}_{k,m}V^*)$ (all locally trivial over X). These induced filtrations will play an important role later on.

7.11. Theorem. *Suppose that V has rank $r \geqslant 2$. Let $\pi_{0,k} : P_kV \longrightarrow X$ be the Semple jet bundles constructed in section 6, and let $J_kV^{\rm reg}$ be the bundle of regular k-jets of maps $f : (\mathbb{C},0) \to X$, that is, jets f such that $f'(0) \neq 0$.*

i) *The quotient $J_kV^{\rm reg}/\mathbb{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding $J_kV^{\rm reg}/\mathbb{G}_k \hookrightarrow P_kV$ over X, which identifies $J_kV^{\rm reg}/\mathbb{G}_k$ with $P_kV^{\rm reg}$ (thus P_kV is a relative compactification of $J_kV^{\rm reg}/\mathbb{G}_k$ over X).*

ii) *The direct image sheaf*
$$(\pi_{0,k})_*\mathcal{O}_{P_kV}(m) \simeq \mathcal{O}(E_{k,m}V^*)$$
can be identified with the sheaf of holomorphic sections of $E_{k,m}V^*$.

iii) *For every $m > 0$, the relative base locus of the linear system $|\mathcal{O}_{P_kV}(m)|$ is equal to the set $P_kV^{\rm sing}$ of singular k-jets. Moreover, $\mathcal{O}_{P_kV}(1)$ is relatively big over X.*

Proof. i) For $f \in J_kV^{\rm reg}$, the lifting $\widetilde{f}$ is obtained by taking the derivative $(f, [f'])$ without any cancellation of zeroes in f', hence we get a uniquely defined $(k-1)$-jet $\widetilde{f} : (\mathbb{C},0) \to \widetilde{X}$. Inductively, we get a well defined $(k-j)$-jet $f_{[j]}$ in P_jV, and the value $f_{[k]}(0)$ is independent of the choice of the representative f for the k-jet. As the lifting process commutes with reparametrization, i.e., $(f\circ\varphi)^\sim = \widetilde{f}\circ\varphi$ and more generally $(f\circ\varphi)_{[k]} = f_{[k]}\circ\varphi$, we conclude that there is a well defined set-theoretic map
$$J_kV^{\rm reg}/\mathbb{G}_k \to P_kV^{\rm reg}, \qquad f \bmod \mathbb{G}_k \mapsto f_{[k]}(0).$$

This map is better understood in coordinates as follows. Fix coordinates $(z_1,\ldots,z_n)$ near a point $x_0 \in X$, such that $V_{x_0} = \mathrm{Vect}(\partial/\partial z_1,\ldots,\partial/\partial z_r)$. Let $f = (f_1,\ldots,f_n)$ be a regular k-jet tangent to V. Then there exists $i \in \{1,2,\ldots,r\}$ such that $f'_i(0) \neq 0$, and there is a unique reparametrization $t = \varphi(\tau)$ such that $f\circ\varphi = g = (g_1,g_2,\ldots,g_n)$ with $g_i(\tau) = \tau$ (we just express the curve as a graph over the z_i-axis, by means of a change of parameter $\tau = f_i(t)$, i.e. $t = \varphi(\tau) = f_i^{-1}(\tau)$). Suppose $i = r$ for the simplicity of notation. The space P_kV is a k-stage tower of $\mathbb{P}^{r-1}$-bundles. In the corresponding inhomogeneous coordinates on these $\mathbb{P}^{r-1}$'s, the point $f_{[k]}(0)$ is given by the collection of derivatives
$$\big((g'_1(0),\ldots,g'_{r-1}(0));(g''_1(0),\ldots,g''_{r-1}(0));\ldots;(g^{(k)}_1(0),\ldots,g^{(k)}_{r-1}(0))\big).$$

[Recall that the other components $(g_{r+1},\ldots,g_n)$ can be recovered from $(g_1,\ldots,g_r)$ by integrating the differential system (5.10)]. Thus the map $J_kV^{\rm reg}/\mathbb{G}_k \to P_kV$ is a bijection onto $P_kV^{\rm reg}$, and the fibers of these isomorphic bundles can be seen as unions of r affine charts $\simeq (\mathbb{C}^{r-1})^k$, associated with each choice of the axis z_i used to describe the curve as a graph. The change of parameter formula $\frac{d}{d\tau} = \frac{1}{f'_r(t)}\frac{d}{dt}$ expresses all derivatives $g^{(j)}_i(\tau) = d^jg_i/d\tau^j$ in terms of the derivatives $f^{(j)}_i(t) = d^jf_i/dt^j$
$$(g'_1,\ldots,g'_{r-1}) = \Big(\frac{f'_1}{f'_r},\ldots,\frac{f'_{r-1}}{f'_r}\Big);$$

$$(7.12)\qquad (g_1'',\ldots,g_{r-1}'')=\Big(\frac{f_1''f_r'-f_r''f_1'}{f_r'^3},\ldots,\frac{f_{r-1}''f_r'-f_r''f_{r-1}'}{f_r'^3}\Big);\ \ldots\ ;$$
$$(g_1^{(k)},\ldots,g_{r-1}^{(k)})=\Big(\frac{f_1^{(k)}f_r'-f_r^{(k)}f_1'}{f_r'^{k+1}},\ldots,\frac{f_{r-1}^{(k)}f_r'-f_r^{(k)}f_{r-1}'}{f_r'^{k+1}}\Big)+(\text{order}<k).$$

Also, it is easy to check that $f_r'^{2k-1}g_i^{(k)}$ is an invariant polynomial in f', f'', $\ldots$, $f^{(k)}$ of total degree $2k-1$, i.e., a section of $E_{k,2k-1}$.

ii) Since the bundles P_kV and $E_{k,m}V^*$ are both locally trivial over X, it is sufficient to identify sections σ of $\mathcal{O}_{P_kV}(m)$ over a fiber $P_kV_x=\pi_{0,k}^{-1}(x)$ with the fiber $E_{k,m}V_x^*$, at any point $x\in X$. Let $f\in J_kV_x^{\rm reg}$ be a regular k-jet at x. By (6.6), the derivative $f_{[k-1]}'(0)$ defines an element of the fiber of $\mathcal{O}_{P_kV}(-1)$ at $f_{[k]}(0)\in P_kV$. Hence we get a well defined complex valued operator

$$(7.13)\qquad Q(f',f'',\ldots,f^{(k)})=\sigma(f_{[k]}(0))\cdot(f_{[k-1]}'(0))^m.$$

Clearly, Q is holomorphic on $J_kV_x^{\rm reg}$ (by the holomorphicity of σ), and the $\mathbb{G}_k$-invariance condition of Def. 7.10 is satisfied since $f_{[k]}(0)$ does not depend on reparametrization and $(f\circ\varphi)_{[k-1]}'(0)=f_{[k-1]}'(0)\varphi'(0)$. Now, $J_kV_x^{\rm reg}$ is the complement of a linear subspace of codimension n in J_kV_x, hence Q extends holomorphically to all of $J_kV_x\simeq(\mathbb{C}^r)^k$ by Riemann's extension theorem (here we use the hypothesis $r\geqslant 2$; if $r=1$, the situation is anyway not interesting since $P_kV=X$ for all k). Thus Q admits an everywhere convergent power series

$$Q(f',f'',\ldots,f^{(k)})=\sum_{\alpha_1,\alpha_2,\ldots,\alpha_k\in\mathbb{N}^r}a_{\alpha_1\ldots\alpha_k}\,(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}.$$

The $\mathbb{G}_k$-invariance (7.10) implies in particular that Q must be multihomogeneous in the sense of (7.1), and thus Q must be a polynomial. We conclude that $Q\in E_{k,m}V_x^*$, as desired.

Conversely, Corollary 6.12 implies that there is a holomorphic family of germs $f_w:(\mathbb{C},0)\to X$ such that $(f_w)_{[k]}(0)=w$ and $(f_w)_{[k-1]}'(0)\neq 0$, for all w in a neighborhood of any given point $w_0\in P_kV_x$. Then every $Q\in E_{k,m}V_x^*$ yields a holomorphic section σ of $\mathcal{O}_{P_kV}(m)$ over the fiber P_kV_x by putting

$$(7.14)\qquad \sigma(w)=Q(f_w',f_w'',\ldots,f_w^{(k)})(0)\,\big((f_w)_{[k-1]}'(0)\big)^{-m}.$$

iii) By what we saw in i-ii), every section σ of $\mathcal{O}_{P_kV}(m)$ over the fiber P_kV_x is given by a polynomial $Q\in E_{k,m}V_x^*$, and this polynomial can be expressed on the Zariski open chart $f_r'\neq 0$ of $P_kV_x^{\rm reg}$ as

$$(7.15)\qquad Q(f',f'',\ldots,f^{(k)})=f_r'^m\widehat{Q}(g',g'',\ldots,g^{(k)}),$$

where $\widehat{Q}$ is a polynomial and g is the reparametrization of f such that $g_r(\tau)=\tau$. In fact $\widehat{Q}$ is obtained from Q by substituting $f_r'=1$ and $f_r^{(j)}=0$ for $j\geqslant 2$, and conversely Q can be recovered easily from $\widehat{Q}$ by using the substitutions (7.12).

In this context, the jet differentials $f\mapsto f_1',\ldots,f\mapsto f_r'$ can be viewed as sections of $\mathcal{O}_{P_kV}(1)$ on a neighborhood of the fiber P_kV_x. Since these sections vanish exactly on $P_kV^{\rm sing}$, the relative base locus of $\mathcal{O}_{P_kV}(m)$ is contained in $P_kV^{\rm sing}$ for every $m>0$. We see that $\mathcal{O}_{P_kV}(1)$ is big by considering the sections of $\mathcal{O}_{P_kV}(2k-1)$ associated with the polynomials

$Q(f',\ldots,f^{(k)}) = f_r'^{2k-1} g_i^{(j)}$, $1 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant k$; indeed, these sections separate all points in the open chart $f_r' \neq 0$ of $P_kV_x^{\mathrm{reg}}$.

Now, we check that every section σ of $\mathcal{O}_{P_kV}(m)$ over P_kV_x must vanish on $P_kV_x^{\mathrm{sing}}$. Pick an arbitrary element $w \in P_kV^{\mathrm{sing}}$ and a germ of curve $f : (\mathbb{C},0) \to X$ such that $f_{[k]}(0) = w$, $f'_{[k-1]}(0) \neq 0$ and $s = m(f,0) \gg 0$ (such an f exists by Corollary 6.14). There are local coordinates $(z_1,\ldots,z_n)$ on X such that $f(t) = (f_1(t),\ldots,f_n(t))$ where $f_r(t) = t^s$. Let Q, $\widehat{Q}$ be the polynomials associated with σ in these coordinates and let $(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$ be a monomial occurring in Q, with $\alpha_j \in \mathbb{N}^r$, $|\alpha_j| = \ell_j$, $\ell_1 + 2\ell_2 + \cdots + k\ell_k = m$. Putting $\tau = t^s$, the curve $t \mapsto f(t)$ becomes a Puiseux expansion $\tau \mapsto g(\tau) = (g_1(\tau),\ldots,g_{r-1}(\tau),\tau)$ in which g_i is a power series in $\tau^{1/s}$, starting with exponents of τ at least equal to 1. The derivative $g^{(j)}(\tau)$ may involve negative powers of τ, but the exponent is always $\geqslant 1 + \frac{1}{s} - j$. Hence the Puiseux expansion of $\widehat{Q}(g',g'',\ldots,g^{(k)})$ can only involve powers of τ of exponent $\geqslant -\max_\ell((1-\frac{1}{s})\ell_2 + \cdots + (k-1-\frac{1}{s})\ell_k)$. Finally $f_r'(t) = st^{s-1} = s\tau^{1-1/s}$, thus the lowest exponent of τ in $Q(f',\ldots,f^{(k)})$ is at least equal to

$$\begin{aligned}\Big(1-\frac{1}{s}\Big)m - \max_\ell\Big(\Big(1-\frac{1}{s}\Big)\ell_2 + \cdots + \Big(k-1-\frac{1}{s}\Big)\ell_k\Big)&\\ \geqslant \min_\ell \Big(1-\frac{1}{s}\Big)\ell_1 + \Big(1-\frac{1}{s}\Big)\ell_2 + \cdots + \Big(1-\frac{k-1}{s}\Big)\ell_k&\end{aligned}$$

where the minimum is taken over all monomials $(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$, $|\alpha_j| = \ell_j$, occurring in Q. Choosing $s \geqslant k$, we already find that the minimal exponent is positive, hence $Q(f',\ldots,f^{(k)})(0) = 0$ and $\sigma(w) = 0$ by (7.14). □

Theorem (7.11 iii) shows that $\mathcal{O}_{P_kV}(1)$ is never relatively ample over X for $k \geqslant 2$. In order to overcome this difficulty, we define for every $\boldsymbol{a} = (a_1,\ldots,a_k) \in \mathbb{Z}^k$ a line bundle $\mathcal{O}_{P_kV}(\boldsymbol{a})$ on P_kV such that

(7.16) $$\mathcal{O}_{P_kV}(\boldsymbol{a}) = \pi_{1,k}^*\mathcal{O}_{P_1V}(a_1) \otimes \pi_{2,k}^*\mathcal{O}_{P_2V}(a_2) \otimes \cdots \otimes \mathcal{O}_{P_kV}(a_k).$$

By (6.9), we have $\pi_{j,k}^*\mathcal{O}_{P_jV}(1) = \mathcal{O}_{P_kV}(1) \otimes \mathcal{O}_{P_kV}(-\pi_{j+1,k}^*D_{j+1} - \cdots - D_k)$, thus by putting $D_j^* = \pi_{j+1,k}^*D_{j+1}$ for $1 \leqslant j \leqslant k-1$ and $D_k^* = 0$, we find an identity

(7.17) $$\begin{aligned}&\mathcal{O}_{P_kV}(\boldsymbol{a}) = \mathcal{O}_{P_kV}(b_k) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^*), \qquad \text{where}\\ &\boldsymbol{b} = (b_1,\ldots,b_k) \in \mathbb{Z}^k, \quad b_j = a_1 + \cdots + a_j,\\ &\boldsymbol{b}\cdot D^* = \sum_{1\leqslant j\leqslant k-1} b_j\,\pi_{j+1,k}^*D_{j+1}.\end{aligned}$$

In particular, if $\boldsymbol{b} \in \mathbb{N}^k$, i.e., $a_1 + \cdots + a_j \geqslant 0$, we get a morphism

(7.18) $$\mathcal{O}_{P_kV}(\boldsymbol{a}) = \mathcal{O}_{P_kV}(b_k) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^*) \to \mathcal{O}_{P_kV}(b_k).$$

7.19. Remark. As in Green-Griffiths [GrGr80], Riemann's extension theorem shows that for every meromorphic map $\Phi : X \dashrightarrow Y$ there are well-defined pullback morphisms

$$\Phi^* : H^0(Y, E_{k,m}^{\mathrm{GG}}) \to H^0(X, E_{k,m}^{\mathrm{GG}}), \qquad \Phi^* : H^0(Y, E_{k,m}) \to H^0(X, E_{k,m}).$$

In particular the dimensions $h^0(X, E_{k,m}^{\mathrm{GG}})$ and $h^0(X, E_{k,m}^{\mathrm{GG}})$ are bimeromorphic invariants of X. The same is true for spaces of sections of any subbundle of $E_{k,m}^{\mathrm{GG}}$ or $E_{k,m}$ constructed by means of the canonical filtrations $F_s^\bullet$.

7.20. Remark. As $\mathbb{G}_k$ is a non reductive group, it is not a priori clear that the graded ring $\mathcal{A}_{n,k,r} = \bigoplus_{m\in\mathbb{Z}} E_{k,m}V^\star$ is finitely generated (pointwise). This can be checked by hand ([Dem07a], [Dem07b]) for $n = 2$ and $k \geqslant 4$. Rousseau [Rou06b] also checked the case $n = 3$, $k = 3$, and then Merker [Mer08] proved the finiteness for $n = 2$, $k = 5$. Recently, Bérczi and Kirwan [BeKi10] found a nice geometric argument proving the finiteness in full generality.

§8. k-jet metrics with negative curvature

The goal of this section is to show that hyperbolicity is closely related to the existence of k-jet metrics with suitable negativity properties of the curvature. The connection between these properties is in fact a simple consequence of the Ahlfors-Schwarz lemma. Such ideas have been already developed long ago by Grauert-Reckziegel [GRec65], Kobayashi [Kob75] for 1-jet metrics (i.e., Finsler metrics on T_X) and by Cowen-Griffiths [CoGr76], Green-Griffiths [GrGr80] and Grauert [Gra89] for higher order jet metrics.

§8.A. Definition of k-jet metrics

Even in the standard case $V = T_X$, the definition given below differs from that of [GrGr80], in which the k-jet metrics are not supposed to be $\mathbb{G}'_k$-invariant. We prefer to deal here with $\mathbb{G}'_k$-invariant objects, because they reflect better the intrinsic geometry. Grauert [Gra89] actually deals with $\mathbb{G}'_k$-invariant metrics, but he apparently does not take care of the way the quotient space $J_k^{\mathrm{reg}}V/\mathbb{G}_k$ can be compactified; also, his metrics are always induced by the Poincaré metric, and it is not at all clear whether these metrics have the expected curvature properties (see 8.14 below). In the present situation, it is important to allow also hermitian metrics possessing some singularities ("singular hermitian metrics" in the sense of [Dem90]).

8.1. Definition. *Let $L \to X$ be a holomorphic line bundle over a complex manifold X. We say that h is a singular metric on L if for any trivialization $L_{\restriction U} \simeq U \times \mathbb{C}$ of L, the metric is given by $|\xi|_h^2 = |\xi|^2 e^{-\varphi}$ for some real valued weight function $\varphi \in L^1_{\mathrm{loc}}(U)$. The curvature current of L is then defined to be the closed $(1,1)$-current $\Theta_{L,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi$, computed in the sense of distributions. We say that h admits a closed subset $\Sigma \subset X$ as its degeneration set if φ is locally bounded on $X \smallsetminus \Sigma$ and is unbounded on a neighborhood of any point of Σ.*

An especially useful situation is the case when the curvature of h is positive definite. By this, we mean that there exists a smooth positive definite hermitian metric ω and a continuous positive function ε on X such that $\Theta_{L,h} \geqslant \varepsilon\omega$ in the sense of currents, and we write in this case $\Theta_{L,h} \gg 0$. We need the following basic fact (quite standard when X is projective algebraic; however we want to avoid any algebraicity assumption here, so as to be able to cover the case of general complex tori in § 10).

8.2. Proposition. *Let L be a holomorphic line bundle on a compact complex manifold X.*

i) *L admits a singular hermitian metric h with positive definite curvature current $\Theta_{L,h} \gg 0$ if and only if L is big.*

Now, define B_m to be the base locus of the linear system $|H^0(X, L^{\otimes m})|$ and let

$$\Phi_m : X \smallsetminus B_m \to \mathbb{P}^N$$

be the corresponding meromorphic map. Let Σ_m be the closed analytic set equal to the union of B_m and of the set of points $x \in X \smallsetminus B_m$ such that the fiber $\Phi_m^{-1}(\Phi_m(x))$ is positive dimensional.

ii) *If $\Sigma_m \neq X$ and G is any line bundle, the base locus of $L^{\otimes k} \otimes G^{-1}$ is contained in Σ_m for k large. As a consequence, L admits a singular hermitian metric h with degeneration set Σ_m and with $\Theta_{L,h}$ positive definite on X.*

iii) *Conversely, if L admits a hermitian metric h with degeneration set Σ and positive definite curvature current $\Theta_{L,h}$, there exists an integer $m > 0$ such that the base locus B_m is contained in Σ and $\Phi_m : X \smallsetminus \Sigma \to \mathbb{P}_m$ is an embedding.*

Proof. i) is proved e.g. in [Dem90, 92], and ii) and iii) are well-known results in the basic theory of linear systems. □

We now come to the main definitions. By (6.6), every regular k-jet $f \in J_kV$ gives rise to an element $f'_{[k-1]}(0) \in \mathcal{O}_{P_kV}(-1)$. Thus, measuring the "norm of k-jets" is the same as taking a hermitian metric on $\mathcal{O}_{P_kV}(-1)$.

8.3. Definition. *A smooth,* (*resp. continuous, resp. singular*) *k-jet metric on a complex directed manifold (X,V) is a hermitian metric h_k on the line bundle $\mathcal{O}_{P_kV}(-1)$ over P_kV* (*i.e. a Finsler metric on the vector bundle V_{k-1} over $P_{k-1}V$*)*, such that the weight functions φ representing the metric are smooth* (*resp. continuous, $L^1_{\rm loc}$*)*. We let $\Sigma_{h_k} \subset P_kV$ be the singularity set of the metric, i.e., the closed subset of points in a neighborhood of which the weight φ is not locally bounded.*

We will always assume here that the weight function φ is quasi psh. Recall that a function φ is said to be quasi psh if φ is locally the sum of a plurisubharmonic function and of a smooth function (so that in particular $\varphi \in L^1_{\rm loc}$). Then the curvature current

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) = \frac{i}{2\pi}\partial\overline{\partial}\varphi.$$

is well defined as a current and is locally bounded from below by a negative $(1,1)$-form with constant coefficients.

8.4. Definition. *Let h_k be a k-jet metric on (X,V). We say that h_k has negative jet curvature* (*resp. negative total jet curvature*) *if $\Theta_{h_k}(\mathcal{O}_{P_kV}(-1))$ is negative definite along the subbundle $V_k \subset T_{P_kV}$* (*resp. on all of T_{P_kV}*)*, i.e., if there is $\varepsilon > 0$ and a smooth hermitian metric ω_k on T_{P_kV} such that*

$$\langle \Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_k}, \qquad \forall\xi \in V_k \subset T_{P_kV} \quad (\textit{resp.}\quad \forall \xi \in T_{P_kV}).$$

(*If the metric h_k is not smooth, we suppose that its weights φ are quasi psh, and the curvature inequality is taken in the sense of distributions.*)

It is important to observe that for $k \geqslant 2$ there cannot exist any smooth hermitian metric h_k on $\mathcal{O}_{P_kV}(1)$ with positive definite curvature along $T_{X_k/X}$, since $\mathcal{O}_{P_kV}(1)$ is not relatively ample over X. However, it is relatively big, and Prop. 8.2 i) shows that $\mathcal{O}_{P_kV}(-1)$ admits a singular hermitian metric with negative total jet curvature (whatever the singularities of the metric are) if and only if $\mathcal{O}_{P_kV}(1)$ is big over P_kV. It is therefore crucial to allow singularities in the metrics in Def. 8.4.

§8.B. Special case of 1-jet metrics

A 1-jet metric h_1 on $\mathcal{O}_{P_1V}(-1)$ is the same as a Finsler metric $N = \sqrt{h_1}$ on $V \subset T_X$. Assume until the end of this paragraph that h_1 is smooth. By the well known Kodaira embedding theorem, the existence of a smooth metric h_1 such that $\Theta_{h_1^{-1}}(\mathcal{O}_{P_1V}(1))$ is positive on all of T_{P_1V} is equivalent to $\mathcal{O}_{P_1V}(1)$ being ample, that is, V^* ample.

8.5 Remark. In the absolute case $V = T_X$, there are only few examples of varieties X such that T_X^* is ample, mainly quotients of the ball $\mathbb{B}_n \subset \mathbb{C}^n$ by a discrete cocompact group of automorphisms.

The 1-jet negativity condition considered in Definition 8.4 is much weaker. For example, if the hermitian metric h_1 comes from a (smooth) hermitian metric h on V, then formula (5.16) implies that h_1 has negative total jet curvature (i.e. $\Theta_{h_1^{-1}}(\mathcal{O}_{P_1V}(1))$ is positive) if and only if $\langle\Theta_{V,h}\rangle(\zeta\otimes v) < 0$ for all $\zeta \in T_X \smallsetminus \{0\}$, $v \in V \smallsetminus \{0\}$, that is, if (V, h) is *negative in the sense of Griffiths*. On the other hand, $V_1 \subset T_{P_1V}$ consists by definition of tangent vectors $\tau \in T_{P_1V,(x,[v])}$ whose horizontal projection ${}^H\tau$ is proportional to v, thus $\Theta_{h_1}(\mathcal{O}_{P_1V}(-1))$ is negative definite on V_1 if and only if $\Theta_{V,h}$ satisfies the much weaker condition that the *holomorphic sectional curvature* $\langle\Theta_{V,h}\rangle(v\otimes v)$ is negative on every complex line. □

§8.C. Vanishing theorem for invariant jet differentials

We now come back to the general situation of jets of arbitrary order k. Our first observation is the fact that the k-jet negativity property of the curvature becomes actually weaker and weaker as k increases.

8.6. Lemma. *Let (X, V) be a compact complex directed manifold. If (X, V) has a $(k-1)$-jet metric h_{k-1} with negative jet curvature, then there is a k-jet metric h_k with negative jet curvature such that $\Sigma_{h_k} \subset \pi_k^{-1}(\Sigma_{h_{k-1}}) \cup D_k$. (The same holds true for negative total jet curvature).*

Proof. Let ω_{k-1}, ω_k be given smooth hermitian metrics on $T_{P_{k-1}V}$ and T_{P_kV}. The hypothesis implies

$$\langle\Theta_{h_{k-1}^{-1}}(\mathcal{O}_{P_{k-1}V}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_{k-1}}, \qquad \forall\xi \in V_{k-1}$$

for some constant $\varepsilon > 0$. On the other hand, as $\mathcal{O}_{P_kV}(D_k)$ is relatively ample over $P_{k-1}V$ (D_k is a hyperplane section bundle), there exists a smooth metric $\widetilde{h}$ on $\mathcal{O}_{P_kV}(D_k)$ such that

$$\langle\Theta_{\widetilde{h}}(\mathcal{O}_{P_kV}(D_k))\rangle(\xi) \geqslant \delta|\xi|^2_{\omega_k} - C|(\pi_k)_*\xi|^2_{\omega_{k-1}}, \qquad \forall\xi \in T_{P_kV}$$

for some constants $\delta, C > 0$. Combining both inequalities (the second one being applied to $\xi \in V_k$ and the first one to $(\pi_k)_*\xi \in V_{k-1}$), we get

$$\begin{aligned}\langle\Theta_{(\pi_k^*h_{k-1})^{-p}\widetilde{h}}(\pi_k^*\mathcal{O}_{P_{k-1}V}(p)\otimes\mathcal{O}_{P_kV}(D_k))\rangle(\xi) &\geqslant \\ \geqslant \delta|\xi|^2_{\omega_k} + (p\varepsilon - C)|(\pi_k)_*\xi|^2_{\omega_{k-1}}&, \qquad \forall\xi \in V_k.\end{aligned}$$

Hence, for p large enough, $(\pi_k^*h_{k-1})^{-p}\widetilde{h}$ has positive definite curvature along V_k. Now, by (6.9), there is a sheaf injection

$$\mathcal{O}_{P_kV}(-p) = \pi_k^*\mathcal{O}_{P_{k-1}V}(-p)\otimes\mathcal{O}_{P_kV}(-pD_k) \hookrightarrow \big(\pi_k^*\mathcal{O}_{P_{k-1}V}(p)\otimes\mathcal{O}_{P_kV}(D_k)\big)^{-1}$$

obtained by twisting with $\mathcal{O}_{P_kV}((p-1)D_k)$. Therefore $h_k := ((\pi_k^*h_{k-1})^{-p}\widetilde{h})^{-1/p} = (\pi_k^*h_{k-1})\widetilde{h}^{-1/p}$ induces a singular metric on $\mathcal{O}_{P_kV}(1)$ in which an additional degeneration divisor $p^{-1}(p-1)D_k$ appears. Hence we get $\Sigma_{h_k} = \pi_k^{-1}\Sigma_{h_{k-1}} \cup D_k$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) = \frac{1}{p}\Theta_{(\pi_k^*h_{k-1})^{-p}\widetilde{h}} + \frac{p-1}{p}[D_k]$$

is positive definite along V_k. The same proof works in the case of negative total jet curvature. □

One of the main motivations for the introduction of k-jets metrics is the following list of algebraic sufficient conditions.

8.7. Algebraic sufficient conditions. We suppose here that X is projective algebraic, and we make one of the additional assumptions i), ii) or iii) below.

i) Assume that there exist integers $k, m > 0$ and $\boldsymbol{b} \in \mathbb{N}^k$ such that the line bundle $\mathcal{O}_{P_kV}(m) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^*)$ is ample over P_kV. Set $A = \mathcal{O}_{P_kV}(m) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^*)$. Then there is a smooth hermitian metric h_A on A with positive definite curvature on P_kV. By means of the morphism $\mu : \mathcal{O}_{P_kV}(-m) \to A^{-1}$, we get an induced metric $h_k = (\mu^* h_A^{-1})^{1/m}$ on $\mathcal{O}_{P_kV}(-1)$ which is degenerate on the support of the zero divisor $\mathrm{div}(\mu) = \boldsymbol{b}\cdot D^*$. Hence $\Sigma_{h_k} = \mathrm{Supp}(\boldsymbol{b}\cdot D^*) \subset P_kV^{\mathrm{sing}}$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) = \frac{1}{m}\Theta_{h_A}(A) + \frac{1}{m}[\boldsymbol{b}\cdot D^*] \geqslant \frac{1}{m}\Theta_{h_A}(A) > 0.$$

In particular h_k has negative total jet curvature.

ii) Assume more generally that there exist integers $k, m > 0$ and an ample line bundle L on X such that $H^0(P_kV, \mathcal{O}_{P_kV}(m) \otimes \pi_{0,k}^* L^{-1})$ has non zero sections $\sigma_1, \ldots, \sigma_N$. Let $Z \subset P_kV$ be the base locus of these sections; necessarily $Z \supset P_kV^{\mathrm{sing}}$ by 7.11 iii). By taking a smooth metric h_L with positive curvature on L, we get a singular metric h'_k on $\mathcal{O}_{P_kV}(-1)$ such that

$$h'_k(\xi) = \Big(\sum_{1\leqslant j\leqslant N} |\sigma_j(w)\cdot \xi^m|^2_{h_L^{-1}}\Big)^{1/m}, \qquad w \in P_kV, \quad \xi \in \mathcal{O}_{P_kV}(-1)_w.$$

Then $\Sigma_{h'_k} = Z$, and by computing $\frac{i}{2\pi}\partial\overline{\partial}\log h'_k(\xi)$ we obtain

$$\Theta_{h_k'^{-1}}(\mathcal{O}_{P_kV}(1)) \geqslant \frac{1}{m}\pi_{0,k}^*\Theta_L.$$

By (7.18) and 7.19 iii), there exists $\boldsymbol{b} \in \mathbb{Q}_+^k$ such that $\mathcal{O}_{P_kV}(1) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^*)$ is relatively ample over X. Hence $A = \mathcal{O}_{P_kV}(1) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^*) \otimes \pi_{0,k}^* L^{\otimes p}$ is ample on X for $p \gg 0$. The arguments used in i) show that there is a k-jet metric h''_k on $\mathcal{O}_{P_kV}(-1)$ with $\Sigma_{h''_k} = \mathrm{Supp}(\boldsymbol{b}\cdot D^*) = P_kV^{\mathrm{sing}}$ and

$$\Theta_{h_k''^{-1}}(\mathcal{O}_{P_kV}(1)) = \Theta_A + [\boldsymbol{b}\cdot D^*] - p\,\pi_{0,k}^*\Theta_L,$$

where Θ_A is positive definite on P_kV. The metric $h_k = (h_k'^{mp} h''_k)^{1/(mp+1)}$ then satisfies $\Sigma_{h_k} = \Sigma_{h'_k} = Z$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) \geqslant \frac{1}{mp+1}\Theta_A > 0.$$

iii) If $E_{k,m}V^*$ is ample, there is an ample line bundle L and a sufficiently high symmetric power such that $S^p(E_{k,m}V^*) \otimes L^{-1}$ is generated by sections. These sections can be viewed as sections of $\mathcal{O}_{P_kV}(mp) \otimes \pi_{0,k}^* L^{-1}$ over P_kV, and their base locus is exactly $Z = P_kV^{\mathrm{sing}}$ by 7.11 iii). Hence the k-jet metric h_k constructed in ii) has negative total jet curvature and satisfies $\Sigma_{h_k} = P_kV^{\mathrm{sing}}$. □

An important fact, first observed by [GRe65] for 1-jet metrics and by [GrGr80] in the higher order case, is that k-jet negativity implies hyperbolicity. In particular, the existence of enough global jet differentials implies hyperbolicity.

8.8. Theorem. *Let (X,V) be a compact complex directed manifold. If (X,V) has a k-jet metric h_k with negative jet curvature, then every entire curve $f:\mathbb{C}\to X$ tangent to V is such that $f_{[k]}(\mathbb{C})\subset\Sigma_{h_k}$. In particular, if $\Sigma_{h_k}\subset P_kV^{\rm sing}$, then (X,V) is hyperbolic.*

Proof. The main idea is to use the Ahlfors-Schwarz lemma, following the approach of [GrGr80]. However we will give here all necessary details because our setting is slightly different. Assume that there is a k-jet metric h_k as in the hypotheses of Theorem 8.8. Let ω_k be a smooth hermitian metric on T_{P_kV}. By hypothesis, there exists $\varepsilon>0$ such that

$$\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi)\geqslant\varepsilon|\xi|^2_{\omega_k}\qquad\forall\xi\in V_k.$$

Moreover, by (6.4), $(\pi_k)_*$ maps V_k continuously to $\mathcal{O}_{P_kV}(-1)$ and the weight e^φ of h_k is locally bounded from above. Hence there is a constant $C>0$ such that

$$|(\pi_k)_*\xi|^2_{h_k}\leqslant C|\xi|^2_{\omega_k},\qquad\forall\xi\in V_k.$$

Combining these inequalities, we find

$$\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi)\geqslant\frac{\varepsilon}{C}|(\pi_k)_*\xi|^2_{h_k},\qquad\forall\xi\in V_k.$$

Now, let $f:\Delta_R\to X$ be a non constant holomorphic map tangent to V on the disk Δ_R. We use the line bundle morphism (6.6)

$$F=f'_{[k-1]}:T_{\Delta_R}\to f^*_{[k]}\mathcal{O}_{P_kV}(-1)$$

to obtain a pullback metric

$$\gamma=\gamma_0(t)\,dt\otimes d\overline{t}=F^*h_k\qquad\text{on }T_{\Delta_R}.$$

If $f_{[k]}(\Delta_R)\subset\Sigma_{h_k}$ then $\gamma\equiv 0$. Otherwise, $F(t)$ has isolated zeroes at all singular points of $f_{[k-1]}$ and so $\gamma(t)$ vanishes only at these points and at points of the degeneration set $(f_{[k]})^{-1}(\Sigma_{h_k})$ which is a polar set in Δ_R. At other points, the Gaussian curvature of γ satisfies

$$\frac{i\,\partial\overline{\partial}\log\gamma_0(t)}{\gamma(t)}=\frac{-2\pi\,(f_{[k]})^*\Theta_{h_k}(\mathcal{O}_{P_kV}(-1))}{F^*h_k}=\frac{\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(f'_{[k]}(t))}{|f'_{[k-1]}(t)|^2_{h_k}}\geqslant\frac{\varepsilon}{C},$$

since $f'_{[k-1]}(t)=(\pi_k)_*f'_{[k]}(t)$. The Ahlfors-Schwarz lemma 4.2 implies that γ can be compared with the Poincaré metric as follows:

$$\gamma(t)\leqslant\frac{2C}{\varepsilon}\frac{R^{-2}|dt|^2}{(1-|t|^2/R^2)^2}\quad\Longrightarrow\quad|f'_{[k-1]}(t)|^2_{h_k}\leqslant\frac{2C}{\varepsilon}\frac{R^{-2}}{(1-|t|^2/R^2)^2}.$$

If $f:\mathbb{C}\to X$ is an entire curve tangent to V such that $f_{[k]}(\mathbb{C})\not\subset\Sigma_{h_k}$, the above estimate implies as $R\to+\infty$ that $f_{[k-1]}$ must be a constant, hence also f. Now, if $\Sigma_{h_k}\subset P_kV^{\rm sing}$, the inclusion $f_{[k]}(\mathbb{C})\subset\Sigma_{h_k}$ implies $f'(t)=0$ at every point, hence f is a constant and (X,V) is hyperbolic. □

Combining Theorem 8.8 with 8.7 ii) and iii), we get the following consequences.

8.9. Corollary. *Assume that there exist integers $k, m > 0$ and an ample line bundle L on X such that $H^0(P_kV, \mathcal{O}_{P_kV}(m) \otimes \pi_{0,k}^* L^{-1}) \simeq H^0(X, E_{k,m}(V^*) \otimes L^{-1})$ has non zero sections $\sigma_1, \ldots, \sigma_N$. Let $Z \subset P_kV$ be the base locus of these sections. Then every entire curve $f : \mathbb{C} \to X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k$-invariant polynomial differential operator P with values in L^{-1}, every entire curve f must satisfy the algebraic differential equation $P(f) = 0$.* □

8.10. Corollary. *Let (X, V) be a compact complex directed manifold. If $E_{k,m}V^*$ is ample for some positive integers k, m, then (X, V) is hyperbolic.* □

8.11. Remark. Green and Griffiths [GrGr80] stated that Corollary 8.9 is even true with sections $\sigma_j \in H^0(X, E_{k,m}^{\mathrm{GG}}(V^*) \otimes L^{-1})$, in the special case $V = T_X$ they consider. We refer to [SiYe97] by Siu and Yeung for a detailed proof of this fact, based on a use of the well-known logarithmic derivative lemma in Nevanlinna theory (the original proof given in [GrGr80] does not seem to be complete, as it relies on an unsettled pointwise version of the Ahlfors-Schwarz lemma for general jet differentials); other proofs seem to have been circulating in the literature in the last years. We give here a very short proof for the case when f is supposed to have a bounded derivative (thanks to the Brody criterion, this is enough if one is merely interested in proving hyperbolicity, thus Corollary 8.10 will be valid with $E_{k,m}^{\mathrm{GG}}V^*$ in place of $E_{k,m}V^*$). In fact, if f' is bounded, one can apply the Cauchy inequalities to all components f_j of f with respect to a finite collection of coordinate patches covering X. As f' is bounded, we can do this on sufficiently small discs $D(t, \delta) \subset \mathbb{C}$ of constant radius $\delta > 0$. Therefore all derivatives f', f'', $\ldots$ $f^{(k)}$ are bounded. From this we conclude that $\sigma_j(f)$ is a bounded section of f^*L^{-1}. Its norm $|\sigma_j(f)|_{L^{-1}}$ (with respect to any positively curved metric $|\ \ |_L$ on L) is a bounded subharmonic function, which is moreover strictly subharmonic at all points where $f' \neq 0$ and $\sigma_j(f) \neq 0$. This is a contradiction unless f is constant or $\sigma_j(f) \equiv 0$. □

The above results justify the following definition and problems.

8.12. Definition. *We say that X, resp. (X, V), has non degenerate negative k-jet curvature if there exists a k-jet metric h_k on $\mathcal{O}_{P_kV}(-1)$ with negative jet curvature such that $\Sigma_{h_k} \subset P_kV^{\mathrm{sing}}$.*

8.13. Conjecture. *Let (X, V) be a compact directed manifold. Then (X, V) is hyperbolic if and only if (X, V) has nondegenerate negative k-jet curvature for k large enough.*

This is probably a hard problem. In fact, we will see in the next section that the smallest admissible integer k must depend on the geometry of X and need not be uniformly bounded as soon as $\dim X \geqslant 2$ (even in the absolute case $V = T_X$). On the other hand, if (X, V) is hyperbolic, we get for each integer $k \geqslant 1$ a generalized Kobayashi-Royden metric $\mathbf{k}_{(P_{k-1}V, V_{k-1})}$ on V_{k-1} (see Definitions 1.2 and 2.1), which can be also viewed as a k-jet metric h_k on $\mathcal{O}_{P_kV}(-1)$; we will call it the *Grauert k-jet metric* of (X, V), although it formally differs from the jet metric considered in [Gra89] (see also [DGr91]). By looking at the projection $\pi_k : (P_kV, V_k) \to (P_{k-1}V, V_{k-1})$, we see that the sequence h_k is monotonic, namely $\pi_k^* h_k \leqslant h_{k+1}$ for every k. If (X, V) is hyperbolic, then h_1 is nondegenerate and therefore by monotonicity $\Sigma_{h_k} \subset P_kV^{\mathrm{sing}}$ for $k \geqslant 1$. Conversely, if the Grauert metric satisfies $\Sigma_{h_k} \subset P_kV^{\mathrm{sing}}$, it is easy to see that (X, V) is hyperbolic. The following problem is thus especially meaningful.

8.14. Problem. *Estimate the k-jet curvature $\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))$ of the Grauert metric h_k on (P_kV, V_k) as k tends to $+\infty$.*

§8.D. Vanishing theorem for non invariant k-jet differentials

We prove here a more general vanishing theorem which strengthens Theorem 8.8 and Corollary 8.9. In this form, the result is due to Siu and Yeung ([SiYe96a, SiYe97], [Siu97], cf. also [Dem97] for a more detailed account (in French)).

8.15. Fundamental vanishing theorem. *Let (X, V) be a directed projective variety and $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ an entire curve tangent to V. Then for every global section $P \in H^0(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}(-A))$ where A is an ample divisor of X, one has $P(f\,; f', f'', \ldots, f^{(k)}) = 0$.*

Proof. After raising P to a power P^s and replacing $\mathcal{O}(-A)$ with $\mathcal{O}(-sA)$, one can always assume that A is very ample divisor. We interpret $E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}(-A)$ as the bundle of complex valued differential operators whose coefficients $a_\alpha(z)$ vanish along A.

Let us first give the proof of (8.15) in the special case where f is a brody curve, i.e. $\sup_{t\in\mathbb{C}} \|f'(t)\|_\omega < +\infty$ with respect to a given Hermitian metric ω on X. Fix a finite open covering of X by coordinate balls $B(p_j, R_j)$ such that the balls $B_j(p_j, R_j/4)$ still cover X. As f' is bounded, there exists $\delta > 0$ such that for $f(t_0) \in B(p_j, R_j/4)$ we have $f(t) \in B(p_j, R_j/2)$ whenever $|t - t_0| < \delta$, uniformly for every $t_0 \in \mathbb{C}$. The Cauchy inequalities applied to the components of f in each of the balls imply that the derivatives $f^{(j)}(t)$ are bounded on $\mathbb{C}$, and therefore, since the coefficients $a_\alpha(z)$ of P are also uniformly bounded on each of the balls $B(p_j, R_j/2)$ we conclude that $g := P(f\,; f', f'', \ldots, f^{(k)})$ is a bounded holomorphic function on $\mathbb{C}$. After moving A in the linear system $|A|$, we may further assume that $\operatorname{Supp} A$ intersects $f(\mathbb{C})$. Then g vanishes somewhere, hence $g \equiv 0$ by Liouville's theorem, as expected.

The proof for the general case where f' is unbounded is slightly more subtle (cf. [Siu87]), and makes use of Nevanlinna theory, especially the logarithmic derivative lemma. Assume that $g = P(f', \ldots, f^{(k)})$ does not vanish identically. Fix a hermitian metric h on $\mathcal{O}(-A)$ such that $\omega := \Theta_{\mathcal{O}(A),h^{-1}} > 0$ is a Kähler metric. The starting point is the inequality

$$\frac{i}{2\pi}\partial\overline{\partial}\log\|g\|_h^2 = \frac{i}{2\pi}\partial\overline{\partial}\log\|P(f', \ldots, f^{(k)})\|_h^2 \geqslant f^*\omega.$$

In fact, as we are on $\mathbb{C}$, the Lelong-Poincaré equation shows that the left hand side is equal to the right hand side plus a certain linear combination of Dirac measures at points where $P(f', \ldots, f^{(k)})$ vanishes. Let us consider the growth and proximity functions

$$T_{f,\omega}(r) := \int_{r_0}^{r}\frac{d\rho}{\rho}\int_{D(0,\rho)} f^*\omega, \tag{8.16}$$

$$m_g(r) := \frac{1}{2\pi}\int_0^{2\pi}\log_+\|g(r\,e^{i\theta})\|_h^2\,d\theta. \tag{8.17}$$

We get

$$T_{f,\omega}(r) \leqslant \int_{r_0}^{r}\frac{d\rho}{\rho}\int_{D(0,\rho)}\frac{i}{2\pi}\partial\overline{\partial}\log\|g\|_h^2 = m_g(r) + \mathrm{Const} \tag{8.18}$$

thanks to the Jensen formula. Now, consider a (finite) family of rational functions (u_j) on X such that one can extract local coordinates from local determinations of the logarithms

$\log u_j$ at any point of X (if X is embedded in some projective space, it is sufficient to take rational functions of the form $u_j(z) = \ell_j(z)/\ell'_j(z)$ where ℓ_j, ℓ'_j are linear forms; we also view the u_j's as rational maps $u_j : X \dashrightarrow \mathbb{P}^1$). One can then express locally $P(f', \ldots, f^{(k)})$ as a polynomial Q in the logarithmic derivatives $D^p(\log u_j \circ f)$, with holomorphic coefficients in f, i.e.,

$$g = P(f', \ldots, f^{(k)}) = Q\big(f, D^p(\log u_j \circ f)_{p,j}\big), \qquad Q(z, v_{p,j}) = \sum a_\alpha(z) v^\alpha.$$

By compactness of X, we infer

$$m_g(r) = \frac{1}{2\pi}\int_0^{2\pi} \log_+ \|g(r\,e^{i\theta})\|_h^2\, d\theta \leqslant C_1 \sum_{j,\,1\leqslant p\leqslant k} m_{D^p(\log u_j\circ f)}(r) + C_2 \tag{8.19}$$

with suitable constants C_1, C_2. The logarithmic derivative lemma states that for every meromorphic function $h : \mathbb{C} \to \mathbb{P}^1$ we have

$$m_{D^p \log h}(r) \leqslant \log r + (1+\varepsilon)\log_+ T_{h,\omega_{\rm FS}}(r) + O(1) \quad /\!/,$$

where the notation // indicates as usual that the inequality holds true outside a set of finite Lebesgue measure in $[0, +\infty[$. We apply this to $h = u_j \circ f$ and use the standard fact that $T_{u_j\circ f,\omega_{\rm FS}}(r) \leqslant C_j T_{f,\omega}(r)$. We find in this way

$$m_{D^p(\log u_j\circ f)}(r) \leqslant C_3\big(\log r + \log_+ T_{f,\omega}(r)\big) \quad /\!/. \tag{8.20}$$

By putting (8.18–8.20) together, one obtains

$$T_{f,\omega}(r) \leqslant C\big(\log r + \log_+ T_{f,\omega}(r)\big) \quad /\!/.$$

We infer from here that $T_{f,\omega}(r) = O(\log r)$, hence $f(\mathbb{C})$ has a finite total area. By well known facts of Nevanlinna theory, we conclude that $C = \overline{f(\mathbb{C})}$ is a rational curve and that f extends as a rational map $\mathbb{P}^1 \to X$. In particular the derivative f' is bounded, but then the first case of the proof can be applied to conclude that $g = P(f', \ldots, f^{(k)}) \equiv 0$. $\square$

§8.E. Bloch theorem

The core of the result can be expressed as a characterization of the Zariski closure of an entire curve drawn on a complex torus. The proof is a simple consequence of the Ahlfors-Schwarz lemma (more specifically Theorem 8.8), combined with a jet bundle argument. We refer to [Och], [GrG80] (also [Dem95]) for a detailed proof.

8.21. Theorem. *Let Z be a complex torus and let $f : \mathbb{C} \to Z$ be a holomorphic map. Then the (analytic) Zariski closure $\overline{f(\mathbb{C})}^{\rm Zar}$ is a translate of a subtorus, i.e. of the form $a + Z'$, $a \in Z$, where $Z' \subset Z$ is a subtorus.*

The converse is of course also true: for any subtorus $Z' \subset Z$, we can choose a dense line $L \subset Z'$, and the corresponding map $f : \mathbb{C} \simeq a + L \hookrightarrow Z$ has Zariski closure $\overline{f(\mathbb{C})}^{\rm Zar} = a + Z'$.

§9. Morse inequalities and the Green-Griffiths-Lang conjecture

The goal of this section is to study the existence and properties of entire curves $f : \mathbb{C} \to X$ drawn in a complex irreducible n-dimensional variety X, and more specifically to show that they must satisfy certain global algebraic or differential equations as soon as X is projective of general type. By means of holomorphic Morse inequalities and a probabilistic analysis of the cohomology of jet spaces, we are able to prove a significant step of a generalized version of the Green-Griffiths-Lang conjecture on the algebraic degeneracy of entire curves. The use of holomorphic Morse inequalities was first suggested in [Dem07a], and then carried out in an algebraic context by S. Diverio in his PhD work ([Div08, Div09]). The general more analytic and more powerful results presented here first appeared in [Dem11]. We refer to [Dem12] for a more detailed exposition.

§9.A. Introduction

Our main target is the following deep conjecture concerning the algebraic degeneracy of entire curves, which generalizes the similar absolute statements given in § 4 (see also [GrGr79], [Lang86, Lang87]).

9.1. Generalized Green-Griffiths-Lang conjecture. *Let (X,V) be a projective directed manifold such that the canonical sheaf K_V is big (in the absolute case $V=T_X$, this means that X is a variety of general type, and in the relative case we will say that (X,V) is of general type). Then there should exist an algebraic subvariety $Y\subsetneq X$ such that every non constant entire curve $f:\mathbb{C}\to X$ tangent to V is contained in Y.*

The precise meaning of K_V and of its bigness will be explained below – our definition *does not coincide* with other frequently used definitions and is in our view better suited to the study of entire curves of (X,V). One says that (X,V) is Brody-hyperbolic when there are no entire curves tangent to V. According to (generalized versions of) conjectures of Kobayashi [Kob70, Kob76] the hyperbolicity of (X,V) should imply that K_V is big, and even possibly ample, in a suitable sense. It would then follow from conjecture (9.1) that (X,V) is hyperbolic if and only if for every irreducible variety $Y\subset X$, the linear subspace

$$V_{\widetilde Y}=\overline{T_{\widetilde Y\smallsetminus E}\cap\mu_*^{-1}V}\subset T_{\widetilde Y} \tag{9.2}$$

has a big canonical sheaf whenever $\mu:\widetilde Y\to Y$ is a desingularization and E is the exceptional locus.

By definition, proving the algebraic degeneracy means finding a non zero polynomial P on X such that all entire curves $f:\mathbb{C}\to X$ satisfy $P(f)=0$. As already explained in § 14, all known methods of proof are based on establishing first the existence of certain algebraic differential equations $P(f\,;f',f'',\ldots,f^{(k)})=0$ of some order k, and then trying to find enough such equations so that they cut out a proper algebraic locus $Y\subsetneq X$. We use for this global sections of $H^0(X,E^{\rm GG}_{k,m}V^*\otimes\mathcal{O}(-A))$ where A is ample, and apply the fundamental vanishing theorem 8.9. It is expected that the global sections of $H^0(X,E^{\rm GG}_{k,m}V^*\otimes\mathcal{O}(-A))$ are precisely those which ultimately define the algebraic locus $Y\subsetneq X$ where the curve f should lie. The problem is then reduced to (i) showing that there are many non zero sections of $H^0(X,E^{\rm GG}_{k,m}V^*\otimes\mathcal{O}(-A))$ and (ii) understanding what is their joint base locus. The first part of this program is the main result of this section.

9.3. Theorem. *Let (X,V) be a directed projective variety such that K_V is big and let A be an ample divisor. Then for $k\gg 1$ and $\delta\in\mathbb{Q}_+$ small enough, $\delta\leqslant c(\log k)/k$, the number of sections $h^0(X,E^{\rm GG}_{k,m}V^*\otimes\mathcal{O}(-m\delta A))$ has maximal growth, i.e. is larger that c_km^{n+kr-1} for some $m\geqslant m_k$, where c, $c_k>0$, $n=\dim X$ and $r=\operatorname{rank}V$. In particular, entire curves $f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)$ satisfy (many) algebraic differential equations.*

The statement is very elementary to check when $r=\operatorname{rank}V=1$, and therefore when $n=\dim X=1$. In higher dimensions $n\geqslant 2$, only very partial results were known at this point, concerning merely the absolute case $V = T_X$. In dimension 2, Theorem 9.3 is a consequence of the Riemann-Roch calculation of Green-Griffiths [GrGr79], combined with a vanishing theorem due to Bogomolov [Bog79] – the latter actually only applies to the top cohomology group H^n, and things become much more delicate when extimates of intermediate cohomology groups are needed. In higher dimensions, Diverio [Div08, Div09] proved the existence of sections of $H^0(X, E_{k,m}^{\rm GG}V^*\otimes\mathcal{O}(-1))$ whenever X is a hypersurface of $\mathbb{P}^{n+1}_{\mathbb{C}}$ of high degree $d\geqslant d_n$, assuming $k\geqslant n$ and $m\geqslant m_n$. More recently, Merker [Mer10] was able to treat the case of arbitrary hypersurfaces of general type, i.e. $d\geqslant n+3$, assuming this time k to be very large. The latter result is obtained through explicit algebraic calculations of the spaces of sections, and the proof is computationally very intensive. Bérczi [Ber10] also obtained related results with a different approach based on residue formulas, assuming $d\geqslant 2^{7n\log n}$.

All these approaches are algebraic in nature. Here, however, our techniques are based on more elaborate curvature estimates in the spirit of Cowen-Griffiths [CoGr76]. They require holomorphic Morse inequalities (see 9.10 below) – and we do not know how to translate our method in an algebraic setting. Notice that holomorphic Morse inequalities are essentially insensitive to singularities, as we can pass to non singular models and blow-up X as much as we want: if $\mu:\widetilde X\to X$ is a modification then $\mu_*\mathcal{O}_{\widetilde X}=\mathcal{O}_X$ and $R^q\mu_*\mathcal{O}_{\widetilde X}$ is supported on a codimension 1 analytic subset (even codimension 2 if X is smooth). It follows from the Leray spectral sequence that the cohomology estimates for L on X or for $\widetilde L=\mu^*L$ on $\widetilde X$ differ by negligible terms, i.e.

$$h^q(\widetilde X,\widetilde L^{\otimes m})-h^q(X,L^{\otimes m})=O(m^{n-1}).\tag{9.4}$$

Finally, singular holomorphic Morse inequalities (in the form obatined by L. Bonavero [Bon93]) allow us to work with singular Hermitian metrics h; this is the reason why we will only require to have big line bundles rather than ample line bundles. In the case of linear subspaces $V\subset T_X$, we introduce singular Hermitian metrics as follows.

9.5. Definition. *A singular Hermitian metric on a linear subspace $V\subset T_X$ is a metric h on the fibers of V such that the function* $\log h:\xi\mapsto\log|\xi|^2_h$ *is locally integrable on the total space of V.*

Such a metric can also be viewed as a singular Hermitian metric on the tautological line bundle $\mathcal{O}_{P(V)}(-1)$ on the projectivized bundle $P(V)=V\smallsetminus\{0\}/\mathbb{C}^*$, and therefore its dual metric h^* defines a curvature current $\Theta_{\mathcal{O}_{P(V)}(1),h^*}$ of type $(1,1)$ on $P(V)\subset P(T_X)$, such that

$$p^*\Theta_{\mathcal{O}_{P(V)}(1),h^*}=\frac{i}{2\pi}\partial\overline\partial\log h,\qquad\text{where } p:V\smallsetminus\{0\}\to P(V).$$

If $\log h$ is quasi-plurisubharmonic (or quasi-psh, which means psh modulo addition of a smooth function) on V, then $\log h$ is indeed locally integrable, and we have moreover

$$\Theta_{\mathcal{O}_{P(V)}(1),h^*}\geqslant -C\omega\tag{9.6}$$

for some smooth positive $(1,1)$-form on $P(V)$ and some constant $C>0$; conversely, if (9.6) holds, then $\log h$ is quasi-psh.

9.7. Definition. *We will say that a singular Hermitian metric h on V is* admissible *if h can be written as $h=e^\varphi h_{0|V}$ where h_0 is a smooth positive definite Hermitian on T_X and φ is a quasi-psh weight with analytic singularities on X, as in Definition* 9.5. *Then h can be seen as a singular Hermitian metric on $\mathcal{O}_{P(V)}(1)$, with the property that it induces a* *smooth positive definite metric on a Zariski open set* $X' \subset X \smallsetminus \mathrm{Sing}(V)$; *we will denote by* $\mathrm{Sing}(h) \supset \mathrm{Sing}(V)$ *the complement of the largest such Zariski open set* X'.

If h is an admissible metric, we define $\mathcal{O}_h(V^*)$ to be the sheaf of germs of holomorphic sections sections of $V^*_{|X\smallsetminus \mathrm{Sing}(h)}$ which are h^*-bounded near $\mathrm{Sing}(h)$; by the assumption on the analytic singularities, this is a coherent sheaf (as the direct image of some coherent sheaf on $P(V)$), and actually, since $h^* = e^{-\varphi}h_0^*$, it is a subsheaf of the sheaf $\mathcal{O}(V^*) := \mathcal{O}_{h_0}(V^*)$ associated with a smooth positive definite metric h_0 on T_X. If r is the generic rank of V and m a positive integer, we define similarly $K^m_{V,h}$ to be sheaf of germs of holomorphic sections of $(\det V^*_{|X'})^{\otimes m} = (\Lambda^r V^*_{|X'})^{\otimes m}$ which are $\det h^*$-bounded, and $K^m_V := K^m_{V,h_0}$.

If V is defined by $\alpha : X \dashrightarrow G_r(T_X)$, there always exists a modification $\mu : \widetilde{X} \to X$ such that the composition $\alpha \circ \mu : \widetilde{X} \to G_r(\mu^* T_X)$ becomes holomorphic, and then $\mu^* V_{|\mu^{-1}(X\smallsetminus \mathrm{Sing}(V))}$ extends as a locally trivial subbundle of $\mu^* T_X$ which we will simply denote by $\mu^* V$. If h is an admissible metric on V, then $\mu^* V$ can be equipped with the metric $\mu^* h = e^{\varphi\circ\mu}\mu^* h_0$ where $\mu^* h_0$ is smooth and positive definite. We may assume that $\varphi\circ\mu$ has divisorial singularities (otherwise just perform further blow-ups of $\widetilde{X}$ to achieve this). We then see that there is an integer m_0 such that for all multiples $m = pm_0$ the pull-back $\mu^* K^m_{V,h}$ is an invertible sheaf on $\widetilde{X}$, and $\det h^*$ induces a smooth non singular metric on it (when $h = h_0$, we can even take $m_0 = 1$). By definition we always have $K^m_{V,h} = \mu_*(\mu^* K^m_{V,h})$ for any $m \geqslant 0$. In the sequel, however, we think of $K_{V,h}$ not really as a coherent sheaf, but rather as the "virtual" $\mathbb{Q}$-line bundle $\mu_*(\mu^* K^{m_0}_{V,h})^{1/m_0}$, and we say that $K_{V,h}$ is big if $h^0(X, K^m_{V,h}) \geqslant cm^n$ for $m \geqslant m_1$, with $c > 0$, i.e. if the invertible sheaf $\mu^* K^{m_0}_{V,h}$ is big in the usual sense.

At this point, it is important to observe that "our" canonical sheaf K_V differs from the sheaf $\mathcal{K}_V := i_*\mathcal{O}(K_V)$ associated with the injection $i : X \smallsetminus \mathrm{Sing}(V) \hookrightarrow X$, which is usually referred to as being the "canonical sheaf", at least when V is the space of tangents to a foliation. In fact, $\mathcal{K}_V$ is always an invertible sheaf and there is an obvious inclusion $K_V \subset \mathcal{K}_V$. More precisely, the image of $\mathcal{O}(\Lambda^r T^*_X) \to \mathcal{K}_V$ is equal to $\mathcal{K}_V \otimes_{\mathcal{O}_X} \mathcal{J}$ for a certain coherent ideal $\mathcal{J} \subset \mathcal{O}_X$, and the condition to have h_0-bounded sections on $X \smallsetminus \mathrm{Sing}(V)$ precisely means that our sections are bounded by $\mathrm{Const}\sum|g_j|$ in terms of the generators (g_j) of $\mathcal{K}_V \otimes_{\mathcal{O}_X} \mathcal{J}$, i.e. $K_V = \mathcal{K}_V \otimes_{\mathcal{O}_X} \overline{\mathcal{J}}$ where $\overline{\mathcal{J}}$ is the integral closure of $\mathcal{J}$. More generally,

$$K^m_{V,h} = \mathcal{K}^m_V \otimes_{\mathcal{O}_X} \overline{\mathcal{J}}^{m/m_0}_{h,m_0} \tag{9.8}$$

where $\overline{\mathcal{J}}^{m/m_0}_{h,m_0} \subset \mathcal{O}_X$ is the (m/m_0)-integral closure of a certain ideal sheaf $\mathcal{J}_{h,m_0} \subset \mathcal{O}_X$, which can itself be assumed to be integrally closed; in our previous discussion, μ is chosen so that $\mu^*\mathcal{J}_{h,m_0}$ is invertible on $\widetilde{X}$.

The discrepancy already occurs e.g. with the rank 1 linear space $V \subset T_{\mathbb{P}^n_{\mathbb{C}}}$ consisting at each point $z \neq 0$ of the tangent to the line $(0z)$ (so that necessarily $V_0 = T_{\mathbb{P}^n_{\mathbb{C}},0}$). As a sheaf (and not as a linear space), $i_*\mathcal{O}(V)$ is the invertible sheaf generated by the vector field $\xi = \sum z_j\partial/\partial z_j$ on the affine open set $\mathbb{C}^n \subset \mathbb{P}^n_{\mathbb{C}}$, and therefore $\mathcal{K}_V := i_*\mathcal{O}(V^*)$ is generated over $\mathbb{C}^n$ by the unique 1-form u such that $u(\xi) = 1$. Since ξ vanishes at 0, the generator u is *unbounded* with respect to a smooth metric h_0 on $T_{\mathbb{P}^n_{\mathbb{C}}}$, and it is easily seen that K_V is the non invertible sheaf $K_V = \mathcal{K}_V \otimes \mathfrak{m}_{\mathbb{P}^n_{\mathbb{C}},0}$. We can make it invertible by considering the blow-up $\mu : \widetilde{X} \to X$ of $X = \mathbb{P}^n_{\mathbb{C}}$ at 0, so that $\mu^* K_V$ is isomorphic to $\mu^*\mathcal{K}_V \otimes \mathcal{O}_{\widetilde{X}}(-E)$ where E is the exceptional divisor. The integral curves C of V are of course lines through 0, and when a standard parametrization is used, their derivatives do not vanish at 0, while the sections of $i_*\mathcal{O}(V)$ do – another sign that $i_*\mathcal{O}(V)$ and $i_*\mathcal{O}(V^*)$ are the *wrong objects* to consider. Another standard example is obtained by taking a generic pencil of elliptic curves

$\lambda P(z) + \mu Q(z) = 0$ of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$, and the linear space V consisting of the tangents to the fibers of the rational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ defined by $z \mapsto Q(z)/P(z)$. Then V is given by

$$0 \longrightarrow i_*\mathcal{O}(V) \longrightarrow \mathcal{O}(T_{\mathbb{P}^2_{\mathbb{C}}}) \xrightarrow{PdQ-QdP} \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6) \otimes \mathcal{J}_S \longrightarrow 0$$

where $S = \mathrm{Sing}(V)$ consists of the 9 points $\{P(z) = 0\} \cap \{Q(z) = 0\}$, and $\mathcal{J}_S$ is the corresponding ideal sheaf of S. Since $\det \mathcal{O}(T_{\mathbb{P}^2}) = \mathcal{O}(3)$, we see that $\mathcal{K}_V = \mathcal{O}(3)$ is ample, which seems to contradict 9.1 since all leaves are elliptic curves. There is however no such contradiction, because $K_V = \mathcal{K}_V \otimes \mathcal{J}_S$ is not big in our sense (it has degree 0 on all members of the elliptic pencil). A similar example is obtained with a generic pencil of conics, in which case $\mathcal{K}_V = \mathcal{O}(1)$ and $\mathrm{card}\, S = 4$.

For a given admissible Hermitian structure (V, h), we define similarly the sheaf $E^{\mathrm{GG}}_{k,m} V^*_h$ to be the sheaf of polynomials defined over $X \smallsetminus \mathrm{Sing}(h)$ which are "h-bounded". This means that when they are viewed as polynomials $P(z\,;\,\xi_1, \ldots, \xi_k)$ in terms of $\xi_j = (\nabla^{1,0}_{h_0})^j f(0)$ where $\nabla^{1,0}_{h_0}$ is the $(1,0)$-component of the induced Chern connection on (V, h_0), there is a uniform bound

$$\big|P(z\,;\,\xi_1, \ldots, \xi_k)\big| \leqslant C\Big(\sum \|\xi_j\|_h^{1/j}\Big)^m \tag{9.9}$$

near points of $X \smallsetminus X'$ (see section 2 for more details on this). Again, by a direct image argument, one sees that $E^{\mathrm{GG}}_{k,m} V^*_h$ is always a coherent sheaf. The sheaf $E^{\mathrm{GG}}_{k,m} V^*$ is defined to be $E^{\mathrm{GG}}_{k,m} V^*_h$ when $h = h_0$ (it is actually independent of the choice of h_0, as follows from arguments similar to those given in section 2). Notice that this is exactly what is needed to extend the proof of the vanishing theorem 9.4 to the case of a singular linear space V ; the value distribution theory argument can only work when the functions $P(f\,;\,f', \ldots, f^{(k)})(t)$ do not exhibit poles, and this is guaranteed here by the boundedness assumption.

Our strategy can be described as follows. We consider the Green-Griffiths bundle of k-jets $X^{\mathrm{GG}}_k = J^k V \smallsetminus \{0\}/\mathbb{C}^*$, which by (9.3) consists of a fibration in *weighted projective spaces*, and its associated tautological sheaf

$$L = \mathcal{O}_{X^{\mathrm{GG}}_k}(1),$$

viewed rather as a virtual $\mathbb{Q}$-line bundle $\mathcal{O}_{X^{\mathrm{GG}}_k}(m_0)^{1/m_0}$ with $m_0 = \mathrm{lcm}(1, 2, \ldots, k)$. Then, if $\pi_k : X^{\mathrm{GG}}_k \to X$ is the natural projection, we have

$$E^{\mathrm{GG}}_{k,m} = (\pi_k)_* \mathcal{O}_{X^{\mathrm{GG}}_k}(m) \quad \text{and} \quad R^q(\pi_k)_* \mathcal{O}_{X^{\mathrm{GG}}_k}(m) = 0 \text{ for } q \geqslant 1.$$

Hence, by the Leray spectral sequence we get for every invertible sheaf F on X the isomorphism

$$H^q(X, E^{\mathrm{GG}}_{k,m} V^* \otimes F) \simeq H^q(X^{\mathrm{GG}}_k, \mathcal{O}_{X^{\mathrm{GG}}_k}(m) \otimes \pi_k^* F).$$

The latter group can be evaluated thanks to holomorphic Morse inequalities. Let us recall the main statement.

9.10. Holomorphic Morse inequalities ([Dem85]). *Let X be a compact complex manifolds, $E \to X$ a holomorphic vector bundle of rank r, and (L, h) a hermitian line bundle. The dimensions $h^q(X, E \otimes L^k)$ of cohomology groups of the tensor powers $E \otimes L^k$ satisfy the following asymptotic estimates as $k \to +\infty$:*

(WM) *Weak Morse inequalities*:

$$h^q(X, E\otimes L^k) \leqslant r\frac{k^n}{n!}\int_{X(L,h,q)} (-1)^q\Theta^n_{L,h} + o(k^n)\ .$$

(SM) *Strong Morse inequalities*:

$$\sum_{0\leqslant j\leqslant q}(-1)^{q-j}h^j(X, E\otimes L^k) \leqslant r\frac{k^n}{n!}\int_{X(L,h,\leqslant q)} (-1)^q\Theta^n_{L,h} + o(k^n)\ .$$

(RR) *Asymptotic Riemann-Roch formula*:

$$\chi(X, E\otimes L^k) := \sum_{0\leqslant j\leqslant n}(-1)^j h^j(X, E\otimes L^k) = r\frac{k^n}{n!}\int_X \Theta^n_{L,h} + o(k^n)\ .$$

Moreover (cf. Bonavero's PhD thesis [Bon93]), if $h = e^{-\varphi}$ is a singular hermitian metric with analytic singularities, the estimates are still true provided all cohomology groups are replaced by cohomology groups $H^q(X, E\otimes L^k\otimes \mathcal{I}(h^k))$ twisted with the multiplier ideal sheaves

$$\mathcal{I}(h^k) = \mathcal{I}(k\varphi) = \big\{f\in\mathcal{O}_{X,x},\ \ \exists V\ni x,\ \int_V |f(z)|^2e^{-k\varphi(z)}d\lambda(z) < +\infty\big\}.$$

The special case of 9.10 (SM) when $q = 1$ yields a very useful criterion for the existence of sections of large multiples of L.

9.11. Corollary. *Under the above hypotheses, we have*

$$h^0(X, E\otimes L^k) \geqslant h^0(X, E\otimes L^k) - h^1(X, E\otimes L^k) \geqslant r\frac{k^n}{n!}\int_{X(L,h,\leqslant 1)} \Theta^n_{L,h} - o(k^n)\ .$$

Especially L is big as soon as $\int_{X(L,h,\leqslant 1)}\Theta^n_{L,h} > 0$ for some hermitian metric h on L.

Now, given a directed manifold (X, V), we can associate with any admissible metric h on V a metric (or rather a natural family) of metrics on $L = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$. The space X_k^{GG} always possesses quotient singularities if $k\geqslant 2$ (and even some more if V is singular), but we do not really care since Morse inequalities still work in this setting thanks to Bonavero's generalization. As we will see, it is then possible to get nice asymptotic formulas as $k\to+\infty$. They appear to be of a *probabilistic nature* if we take the components of the k-jet (i.e. the successive derivatives $\xi_j = f^{(j)}(0)$, $1\leqslant j\leqslant k$) as random variables. This probabilistic behaviour was somehow already visible in the Riemann-Roch calculation of [GrGr79]. In this way, assuming K_V big, we produce a lot of sections $\sigma_j = H^0(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m)\otimes\pi_k^*F)$, corresponding to certain divisors $Z_j\subset X_k^{\mathrm{GG}}$. The hard problem which is left in order to complete a proof of the generalized Green-Griffiths-Lang conjecture is to compute the base locus $Z = \bigcap Z_j$ and to show that $Y = \pi_k(Z)\subset X$ must be a proper algebraic variety.

§9.B. Hermitian geometry of weighted projective spaces

The goal of this section is to introduce natural Kähler metrics on weighted projective spaces, and to evaluate the corresponding volume forms. Here we put $d^c = \frac{i}{4\pi}(\overline\partial-\partial)$ so that $dd^c = \frac{i}{2\pi}\partial\overline\partial$. The normalization of the d^c operator is chosen such that we have precisely

$(dd^c \log|z|^2)^n = \delta_0$ for the Monge-Ampère operator in $\mathbb{C}^n$. Given a k-tuple of "weights" $a = (a_1, \ldots, a_k)$, i.e. of integers $a_s > 0$ with $\gcd(a_1, \ldots, a_k) = 1$, we introduce the weighted projective space $P(a_1, \ldots, a_k)$ to be the quotient of $\mathbb{C}^k \smallsetminus \{0\}$ by the corresponding weighted $\mathbb{C}^*$ action:

$$P(a_1, \ldots, a_k) = \mathbb{C}^k \smallsetminus \{0\}/\mathbb{C}^*, \qquad \lambda \cdot z = (\lambda^{a_1} z_1, \ldots, \lambda^{a_k} z_k). \tag{9.12}$$

As is well known, this defines a toric $(k-1)$-dimensional algebraic variety with quotient singularities. On this variety, we introduce the possibly singular (but almost everywhere smooth and non degenerate) Kähler form $\omega_{a,p}$ defined by

$$\pi_a^* \omega_{a,p} = dd^c \varphi_{a,p}, \qquad \varphi_{a,p}(z) = \frac{1}{p} \log \sum_{1 \leqslant s \leqslant k} |z_s|^{2p/a_s}, \tag{9.13}$$

where $\pi_a : \mathbb{C}^k \smallsetminus \{0\} \to P(a_1, \ldots, a_k)$ is the canonical projection and $p > 0$ is a positive constant. It is clear that $\varphi_{p,a}$ is real analytic on $\mathbb{C}^k \smallsetminus \{0\}$ if p is an integer and a common multiple of all weights a_s, and we will implicitly pick such a p later on to avoid any difficulty. Elementary calculations give the following well-known formula for the volume

$$\int_{P(a_1, \ldots, a_k)} \omega_{a,p}^{k-1} = \frac{1}{a_1 \ldots a_k} \tag{9.14}$$

(notice that this is independent of p, as it is obvious by Stokes theorem, since the cohomology class of $\omega_{a,p}$ does not depend on p).

Our later calculations will require a slightly more general setting. Instead of looking at $\mathbb{C}^k$, we consider the weighted $\mathbb{C}^*$ action defined by

$$\mathbb{C}^{|r|} = \mathbb{C}^{r_1} \times \ldots \times \mathbb{C}^{r_k}, \qquad \lambda \cdot z = (\lambda^{a_1} z_1, \ldots, \lambda^{a_k} z_k). \tag{9.15}$$

Here $z_s \in \mathbb{C}^{r_s}$ for some k-tuple $r = (r_1, \ldots, r_k)$ and $|r| = r_1 + \ldots + r_k$. This gives rise to a weighted projective space

$$\begin{aligned} &P(a_1^{[r_1]}, \ldots, a_k^{[r_k]}) = P(a_1, \ldots, a_1, \ldots, a_k, \ldots, a_k), \\ &\pi_{a,r} : \mathbb{C}^{r_1} \times \ldots \times \mathbb{C}^{r_k} \smallsetminus \{0\} \longrightarrow P(a_1^{[r_1]}, \ldots, a_k^{[r_k]}) \end{aligned} \tag{9.16}$$

obtained by repeating r_s times each weight a_s. On this space, we introduce the degenerate Kähler metric $\omega_{a,r,p}$ such that

$$\pi_{a,r}^* \omega_{a,r,p} = dd^c \varphi_{a,r,p}, \qquad \varphi_{a,r,p}(z) = \frac{1}{p} \log \sum_{1 \leqslant s \leqslant k} |z_s|^{2p/a_s} \tag{9.17}$$

where $|z_s|$ stands now for the standard Hermitian norm $(\sum_{1 \leqslant j \leqslant r_s} |z_{s,j}|^2)^{1/2}$ on $\mathbb{C}^{r_s}$. This metric is cohomologous to the corresponding "polydisc-like" metric $\omega_{a,p}$ already defined, and therefore Stokes theorem implies

$$\int_{P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})} \omega_{a,r,p}^{|r|-1} = \frac{1}{a_1^{r_1} \ldots a_k^{r_k}}. \tag{9.18}$$

Using standard results of integration theory (Fubini, change of variable formula...), one obtains:

9.19. Proposition. *Let $f(z)$ be a bounded function on $P(a_1^{[r_1]},\ldots,a_k^{[r_k]})$ which is continuous outside of the hyperplane sections $z_s=0$. We also view f as a $\mathbb{C}^*$-invariant continuous function on $\prod(\mathbb{C}^{r_s}\smallsetminus\{0\})$. Then*

$$\begin{aligned}&\int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} f(z)\,\omega_{a,r,p}^{|r|-1}\\ &=\frac{(|r|-1)!}{\prod_s a_s^{r_s}}\int_{(x,u)\in\Delta_{k-1}\times\prod S^{2r_s-1}} f(x_1^{a_1/2p}u_1,\ldots,x_k^{a_k/2p}u_k)\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}}{(r_s-1)!}\,dx\,d\mu(u)\end{aligned}$$

where Δ_{k-1} is the $(k-1)$-simplex $\{x_s\geqslant 0,\ \sum x_s=1\}$, $dx=dx_1\wedge\ldots\wedge dx_{k-1}$ its standard measure, and where $d\mu(u)=d\mu_1(u_1)\ldots d\mu_k(u_k)$ is the rotation invariant probability measure on the product $\prod_s S^{2r_s-1}$ of unit spheres in $\mathbb{C}^{r_1}\times\ldots\times\mathbb{C}^{r_k}$. As a consequence

$$\lim_{p\to+\infty}\int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} f(z)\,\omega_{a,r,p}^{|r|-1}=\frac{1}{\prod_s a_s^{r_s}}\int_{\prod S^{2r_s-1}} f(u)\,d\mu(u).$$

Also, by elementary integrations by parts and induction on $k,r_1,\ldots,r_k$, it can be checked that

$$\int_{x\in\Delta_{k-1}}\prod_{1\leqslant s\leqslant k}x_s^{r_s-1}dx_1\ldots dx_{k-1}=\frac{1}{(|r|-1)!}\prod_{1\leqslant s\leqslant k}(r_s-1)!\,. \tag{9.20}$$

This implies that $(|r|-1)!\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}}{(r_s-1)!}\,dx$ is a probability measure on Δ_{k-1}.

§9.C. Probabilistic estimate of the curvature of k-jet bundles

Let (X,V) be a compact complex directed non singular variety. To avoid any technical difficulty at this point, we first assume that V is a holomorphic vector subbundle of T_X, equipped with a smooth Hermitian metric h.

According to the notation already specified in § 7, we denote by J^kV the bundle of k-jets of holomorphic curves $f:(\mathbb{C},0)\to X$ tangent to V at each point. Let us set $n=\dim_{\mathbb{C}}X$ and $r=\mathrm{rank}_{\mathbb{C}}V$. Then $J^kV\to X$ is an algebraic fiber bundle with typical fiber $\mathbb{C}^{rk}$, and we get a projectivized k-jet bundle

$$X_k^{\rm GG}:=(J^kV\smallsetminus\{0\})/\mathbb{C}^*,\qquad \pi_k:X_k^{\rm GG}\to X \tag{9.21}$$

which is a $P(1^{[r]},2^{[r]},\ldots,k^{[r]})$ weighted projective bundle over X, and we have the direct image formula $(\pi_k)_*\mathcal{O}_{X_k^{\rm GG}}(m)=\mathcal{O}(E_{k,m}^{\rm GG}V^*)$ (cf. Proposition 7.9). In the sequel, we do not make a direct use of coordinates, because they need not be related in any way to the Hermitian metric h of V. Instead, we choose a local holomorphic coordinate frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ of V on a neighborhood U of x_0, such that

$$\langle e_\alpha(z),e_\beta(z)\rangle=\delta_{\alpha\beta}+\sum_{1\leqslant i,j\leqslant n,\,1\leqslant\alpha,\beta\leqslant r}c_{ij\alpha\beta}z_i\overline{z}_j+O(|z|^3) \tag{9.22}$$

for suitable complex coefficients $(c_{ij\alpha\beta})$. It is a standard fact that such a normalized coordinate system always exists, and that the Chern curvature tensor $\frac{i}{2\pi}D_{V,h}^2$ of (V,h) at x_0 is then given by

$$\Theta_{V,h}(x_0)=-\frac{i}{2\pi}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}\,dz_i\wedge d\overline{z}_j\otimes e_\alpha^*\otimes e_\beta. \tag{9.23}$$

Cnsider a local holomorphic connection ∇ on $V_{|U}$ (e.g. the one which turns (e_α) into a parallel frame), and take $\xi_k = \nabla^k f(0) \in V_x$ defined inductively by $\nabla^1 f = f'$ and $\nabla^s f = \nabla_{f'}(\nabla^{s-1} f)$. This gives a local identification

$$J_k V_{|U} \to V_{|U}^{\oplus k}, \qquad f \mapsto (\xi_1, \dots, \xi_k) = (\nabla f(0), \dots, \nabla f^k(0))$$

and the weighted $\mathbb{C}^*$ action on $J_k V$ is expressed in this setting by

$$\lambda \cdot (\xi_1, \xi_2, \dots, \xi_k) = (\lambda \xi_1, \lambda^2 \xi_2, \dots, \lambda^k \xi_k).$$

Now, we fix a finite open covering $(U_\alpha)_{\alpha \in I}$ of X by open coordinate charts such that $V_{|U_\alpha}$ is trivial, along with holomorphic connections ∇_α on $V_{|U_\alpha}$. Let θ_α be a partition of unity of X subordinate to the covering (U_α). Let us fix $p > 0$ and small parameters $1 = \varepsilon_1 \gg \varepsilon_2 \gg \ldots \gg \varepsilon_k > 0$. Then we define a global weighted Finsler metric on $J^k V$ by putting for any k-jet $f \in J^k_x V$

$$\Psi_{h,p,\varepsilon}(f) := \Big(\sum_{\alpha \in I} \theta_\alpha(x) \sum_{1 \leqslant s \leqslant k} \varepsilon_s^{2p} \|\nabla^s_\alpha f(0)\|^{2p/s}_{h(x)} \Big)^{1/p} \tag{9.24}$$

where $\|\ \|_{h(x)}$ is the Hermitian metric h of V evaluated on the fiber V_x, $x = f(0)$. The function $\Psi_{h,p,\varepsilon}$ satisfies the fundamental homogeneity property

$$\Psi_{h,p,\varepsilon}(\lambda \cdot f) = \Psi_{h,p,\varepsilon}(f)\, |\lambda|^2 \tag{9.25}$$

with respect to the $\mathbb{C}^*$ action on $J^k V$, in other words, it induces a Hermitian metric on the dual L^* of the tautological $\mathbb{Q}$-line bundle $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ over X_k^{GG}. The curvature of L_k is given by

$$\pi_k^* \Theta_{L_k, \Psi^*_{h,p,\varepsilon}} = dd^c \log \Psi_{h,p,\varepsilon} \tag{9.26}$$

where $\pi_k : J^k V \smallsetminus \{0\} \to X_k^{\mathrm{GG}}$ is the canonical projection. Our next goal is to compute precisely the curvature and to apply holomorphic Morse inequalities to $L \to X_k^{\mathrm{GG}}$ with the above metric. It might look a priori like an untractable problem, since the definition of $\Psi_{h,p,\varepsilon}$ is a rather unnatural one. However, the "miracle" is that the asymptotic behavior of $\Psi_{h,p,\varepsilon}$ as $\varepsilon_s/\varepsilon_{s-1} \to 0$ is in some sense uniquely defined and very natural. It will lead to a computable asymptotic formula, which is moreover simple enough to produce useful results.

9.27. Lemma. *On each coordinate chart U equipped with a holomorphic connection ∇ of $V_{|U}$, let us define the components of a k-jet $f \in J^k V$ by $\xi_s = \nabla^s f(0)$, and consider the rescaling transformation*

$$\rho_{\nabla,\varepsilon}(\xi_1, \xi_2, \dots, \xi_k) = (\varepsilon_1^1 \xi_1, \varepsilon_2^2 \xi_2, \dots, \varepsilon_k^k \xi_k) \quad \text{on } J^k_x V,\ x \in U$$

(*it commutes with the $\mathbb{C}^*$-action but is otherwise unrelated and not canonically defined over X as it depends on the choice of ∇*). *Then, if p is a multiple of* $\mathrm{lcm}(1, 2, \dots, k)$ *and $\varepsilon_s/\varepsilon_{s-1} \to 0$ for all $s = 2, \dots, k$, the rescaled function $\Psi_{h,p,\varepsilon} \circ \rho^{-1}_{\nabla,\varepsilon}(\xi_1, \dots, \xi_k)$ converges towards*

$$\Big(\sum_{1 \leqslant s \leqslant k} \|\xi_s\|_h^{2p/s} \Big)^{1/p}$$

on every compact subset of $J^kV_{|U} \smallsetminus \{0\}$*, uniformly in* C^∞ *topology.*

Proof. Let $U \subset X$ be an open set on which $V_{|U}$ is trivial and equipped with some holomorphic connection ∇. Let us pick another holomorphic connection $\widetilde\nabla = \nabla + \Gamma$ where $\Gamma \in H^0(U, \Omega^1_X \otimes \mathrm{Hom}(V,V)$. Then $\widetilde\nabla^2 f = \nabla^2 f + \Gamma(f)(f')\cdot f'$, and inductively we get

$$\widetilde\nabla^s f = \nabla^s f + P_s(f\,;\,\nabla^1 f, \ldots, \nabla^{s-1} f)$$

where $P(x\,;\,\xi_1,\ldots,\xi_{s-1})$ is a polynomial with holomorphic coefficients in $x \in U$ which is of weighted homogeneous degree s in $(\xi_1,\ldots,\xi_{s-1})$. In other words, the corresponding change in the parametrization of $J^kV_{|U}$ is given by a $\mathbb{C}^*$-homogeneous transformation

$$\widetilde\xi_s = \xi_s + P_s(x\,;\,\xi_1,\ldots,\xi_{s-1}).$$

Let us introduce the corresponding rescaled components

$$(\xi_{1,\varepsilon},\ldots,\xi_{k,\varepsilon}) = (\varepsilon_1^1\xi_1,\ldots,\varepsilon_k^k\xi_k), \qquad (\widetilde\xi_{1,\varepsilon},\ldots,\widetilde\xi_{k,\varepsilon}) = (\varepsilon_1^1\widetilde\xi_1,\ldots,\varepsilon_k^k\widetilde\xi_k).$$

Then

$$\begin{aligned}\widetilde\xi_{s,\varepsilon} &= \xi_{s,\varepsilon} + \varepsilon_s^s\,P_s(x\,;\,\varepsilon_1^{-1}\xi_{1,\varepsilon},\ldots,\varepsilon_{s-1}^{-(s-1)}\xi_{s-1,\varepsilon})\\ &= \xi_{s,\varepsilon} + O(\varepsilon_s/\varepsilon_{s-1})^s\,O(\|\xi_{1,\varepsilon}\| + \ldots + \|\xi_{s-1,\varepsilon}\|^{1/(s-1)})^s\end{aligned}$$

and the error terms are thus polynomials of fixed degree with arbitrarily small coefficients as $\varepsilon_s/\varepsilon_{s-1} \to 0$. Now, the definition of $\Psi_{h,p,\varepsilon}$ consists of glueing the sums

$$\sum_{1\leqslant s\leqslant k} \varepsilon_s^{2p}\|\xi_k\|_h^{2p/s} = \sum_{1\leqslant s\leqslant k} \|\xi_{k,\varepsilon}\|_h^{2p/s}$$

corresponding to $\xi_k = \nabla^s_\alpha f(0)$ by means of the partition of unity $\sum\theta_\alpha(x) = 1$. We see that by using the rescaled variables $\xi_{s,\varepsilon}$ the changes occurring when replacing a connection ∇_α by an alternative one ∇_β are arbitrary small in C^∞ topology, with error terms uniformly controlled in terms of the ratios $\varepsilon_s/\varepsilon_{s-1}$ on all compact subsets of $V^k \smallsetminus \{0\}$. This shows that in C^∞ topology, $\Psi_{h,p,\varepsilon}\circ\rho^{-1}_{\nabla,\varepsilon}(\xi_1,\ldots,\xi_k)$ converges uniformly towards $(\sum_{1\leqslant s\leqslant k}\|\xi_k\|_h^{2p/s})^{1/p}$, whatever the trivializing open set U and the holomorphic connection ∇ used to evaluate the components and perform the rescaling are. □

Now, we fix a point $x_0 \in X$ and a local holomorphic frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ satisfying (9.22) on a neighborhood U of x_0. We introduce the rescaled components $\xi_s = \varepsilon_s^s\nabla^s f(0)$ on $J^kV_{|U}$ and compute the curvature of

$$\Psi_{h,p,\varepsilon}\circ\rho^{-1}_{\nabla,\varepsilon}(z\,;\,\xi_1,\ldots,\xi_k) \simeq \Bigg(\sum_{1\leqslant s\leqslant k}\|\xi_s\|_h^{2p/s}\Bigg)^{1/p}$$

(by Lemma 9.27, the errors can be taken arbitrary small in C^∞ topology). We write $\xi_s = \sum_{1\leqslant\alpha\leqslant r}\xi_{s\alpha}e_\alpha$. By (9.22) we have

$$\|\xi_s\|_h^2 = \sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}z_i\overline z_j\xi_{s\alpha}\overline\xi_{s\beta} + O(|z|^3|\xi|^2).$$

The question is to evaluate the curvature of the weighted metric defined by

$$\Psi(z\,;\,\xi_1,\ldots,\xi_k) = \Big(\sum_{1\leqslant s\leqslant k}\|\xi_s\|_h^{2p/s}\Big)^{1/p}$$
$$= \Big(\sum_{1\leqslant s\leqslant k}\Big(\sum_\alpha|\xi_{s\alpha}|^2+\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}z_i\overline{z}_j\xi_{s\alpha}\overline{\xi}_{s\beta}\Big)^{p/s}\Big)^{1/p}+O(|z|^3).$$

We set $|\xi_s|^2=\sum_\alpha|\xi_{s\alpha}|^2$. A straightforward calculation yields

$$\log\Psi(z\,;\,\xi_1,\ldots,\xi_k) =$$
$$=\frac{1}{p}\log\sum_{1\leqslant s\leqslant k}|\xi_s|^{2p/s}+\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}z_i\overline{z}_j\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}+O(|z|^3).$$

By (9.26), the curvature form of $L_k=\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is given at the central point x_0 by the following formula.

9.28. Proposition. *With the above choice of coordinates and with respect to the rescaled components $\xi_s=\varepsilon_s^s\nabla^s f(0)$ at $x_0\in X$, we have the approximate expression*

$$\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(x_0,[\xi])\simeq\omega_{a,r,p}(\xi)+\frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j$$

where the error terms are $O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$ uniformly on the compact variety X_k^{GG}. Here $\omega_{a,r,p}$ is the (degenerate) Kähler metric associated with the weight $a=(1^{[r]},2^{[r]},\ldots,k^{[r]})$ of the canonical $\mathbb{C}^*$ action on J^kV.

Thanks to the uniform approximation, we can (and will) neglect the error terms in the calculations below. Since $\omega_{a,r,p}$ is positive definite on the fibers of $X_k^{\mathrm{GG}}\to X$ (at least outside of the axes $\xi_s=0$), the index of the $(1,1)$ curvature form $\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(z,[\xi])$ is equal to the index of the $(1,1)$-form

$$\gamma_k(z,\xi):=\frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j \tag{9.29}$$

depending only on the differentials $(dz_j)_{1\leqslant j\leqslant n}$ on X. The q-index integral of $(L_k,\Psi^*_{h,p,\varepsilon})$ on X_k^{GG} is therefore equal to

$$\int_{X_k^{\mathrm{GG}}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}=$$
$$=\frac{(n+kr-1)!}{n!(kr-1)!}\int_{z\in X}\int_{\xi\in P(1^{[r]},\ldots,k^{[r]})}\omega^{kr-1}_{a,r,p}(\xi)\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n$$

where $\mathbb{1}_{\gamma_k,q}(z,\xi)$ is the characteristic function of the open set of points where $\gamma_k(z,\xi)$ has signature $(n-q,q)$ in terms of the dz_j's. Notice that since $\gamma_k(z,\xi)^n$ is a determinant, the product $\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n$ gives rise to a continuous function on X_k^{GG}. Formula 9.20 with $r_1=\ldots=r_k=r$ and $a_s=s$ yields the slightly more explicit integral

$$\int_{X_k^{\mathrm{GG}}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}=\frac{(n+kr-1)!}{n!(k!)^r}\quad\times$$
$$\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx\,d\mu(u),$$

where $g_k(z,x,u) = \gamma_k(z, x_1^{1/2p}u_1, \ldots, x_k^{k/2p}u_k)$ is given by

$$g_k(z,x,u) = \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j \tag{9.30}$$

and $\mathbb{1}_{g_k,q}(z,x,u)$ is the characteristic function of its q-index set. Here

$$d\nu_{k,r}(x) = (kr-1)!\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx \tag{9.31}$$

is a probability measure on Δ_{k-1}, and we can rewrite

$$\begin{aligned}\int_{X_k^{\rm GG}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} &= \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\ \times\\ &\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,d\nu_{k,r}(x)\,d\mu(u).\end{aligned} \tag{9.32}$$

Now, formula (9.30) shows that $g_k(z,x,u)$ is a "Monte Carlo" evaluation of the curvature tensor, obtained by averaging the curvature at random points $u_s\in S^{2r-1}$ with certain positive weights x_s/s; we should then think of the k-jet f as some sort of random variable such that the derivatives $\nabla^k f(0)$ are uniformly distributed in all directions. Let us compute the expected value of $(x,u)\mapsto g_k(z,x,u)$ with respect to the probability measure $d\nu_{k,r}(x)\,d\mu(u)$. Since $\int_{S^{2r-1}}u_{s\alpha}\overline{u}_{s\beta}d\mu(u_s) = \frac{1}{r}\delta_{\alpha\beta}$ and $\int_{\Delta_{k-1}}x_s\,d\nu_{k,r}(x) = \frac{1}{k}$, we find

$$\mathbf{E}(g_k(z,\bullet,\bullet)) = \frac{1}{kr}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\cdot\frac{i}{2\pi}\sum_{i,j,\alpha}c_{ij\alpha\alpha}(z)\,dz_i\wedge d\overline{z}_j.$$

In other words, we get the normalized trace of the curvature, i.e.

$$\mathbf{E}(g_k(z,\bullet,\bullet)) = \frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\Theta_{\det(V^*),\det h^*}, \tag{9.33}$$

where $\Theta_{\det(V^*),\det h^*}$ is the $(1,1)$-curvature form of $\det(V^*)$ with the metric induced by h. It is natural to guess that $g_k(z,x,u)$ behaves asymptotically as its expected value $\mathbf{E}(g_k(z,\bullet,\bullet))$ when k tends to infinity. If we replace brutally g_k by its expected value in (9.32), we get the integral

$$\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\frac{1}{(kr)^n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\int_X\mathbb{1}_{\eta,q}\eta^n,$$

where $\eta := \Theta_{\det(V^*),\det h^*}$ and $\mathbb{1}_{\eta,q}$ is the characteristic function of its q-index set in X. The leading constant is equivalent to $(\log k)^n/n!(k!)^r$ modulo a multiplicative factor $1+O(1/\log k)$. By working out a more precise analysis of the deviation, the following result has been proved in [Dem11] and [Dem12].

9.34. Probabilistic estimate. *Fix smooth Hermitian metrics h on V and $\omega = \frac{i}{2\pi}\sum\omega_{ij}dz_i\wedge d\overline{z}_j$ on X. Denote by $\Theta_{V,h} = -\frac{i}{2\pi}\sum c_{ij\alpha\beta}dz_i\wedge d\overline{z}_j\otimes e^*_\alpha\otimes e_\beta$ the curvature tensor of V with respect to an h-orthonormal frame (e_α), and put*

$$\eta(z) = \Theta_{\det(V^*),\det h^*} = \frac{i}{2\pi}\sum_{1\leqslant i,j\leqslant n}\eta_{ij}dz_i\wedge d\overline{z}_j,\qquad \eta_{ij} = \sum_{1\leqslant\alpha\leqslant r}c_{ij\alpha\alpha}.$$

Finally consider the k-jet line bundle $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \to X_k^{\mathrm{GG}}$ equipped with the induced metric $\Psi^*_{h,p,\varepsilon}$ (as defined above, with $1 = \varepsilon_1 \gg \varepsilon_2 \gg \ldots \gg \varepsilon_k > 0$). When k tends to infinity, the integral of the top power of the curvature of L_k on its q-index set $X_k^{\mathrm{GG}}(L_k, q)$ is given by

$$\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_X \mathbb{1}_{\eta,q}\eta^n + O((\log k)^{-1})\bigg)$$

for all $q = 0, 1, \ldots, n$, and the error term $O((\log k)^{-1})$ can be bounded explicitly in terms of Θ_V, η and ω. Moreover, the left hand side is identically zero for $q > n$.

The final statement follows from the observation that the curvature of L_k is positive along the fibers of $X_k^{\mathrm{GG}} \to X$, by the plurisubharmonicity of the weight (this is true even when the partition of unity terms are taken into account, since they depend only on the base); therefore the q-index sets are empty for $q > n$. It will be useful to extend the above estimates to the case of sections of

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big) \tag{9.35}$$

where $F \in \mathrm{Pic}_{\mathbb{Q}}(X)$ is an arbitrary $\mathbb{Q}$-line bundle on X and $\pi_k : X_k^{\mathrm{GG}} \to X$ is the natural projection. We assume here that F is also equipped with a smooth Hermitian metric h_F. In formulas (9.32–9.34), the renormalized curvature $\eta_k(z, x, u)$ of L_k takes the form

$$\eta_k(z, x, u) = \frac{1}{\frac{1}{kr}(1 + \frac{1}{2} + \ldots + \frac{1}{k})} g_k(z, x, u) + \Theta_{F,h_F}(z), \tag{9.36}$$

and by the same calculations its expected value is

$$\eta(z) := \mathbf{E}(\eta_k(z, \bullet, \bullet)) = \Theta_{\det V^*, \det h^*}(z) + \Theta_{F,h_F}(z). \tag{9.37}$$

Then the variance estimate for $\eta_k - \eta$ is unchanged, and the L^p bounds for η_k are still valid, since our forms are just shifted by adding the constant smooth term $\Theta_{F,h_F}(z)$. The probabilistic estimate 9.34 is therefore still true in exactly the same form, provided we use (9.35 – 9.37) instead of the previously defined L_k, η_k and η. An application of holomorphic Morse inequalities gives the desired cohomology estimates for

$$\begin{aligned} h^q\Big(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big)\Big) \\ = h^q(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big)\Big), \end{aligned}$$

provided m is sufficiently divisible to give a multiple of F which is a $\mathbb{Z}$-line bundle.

9.38. Theorem. *Let (X, V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V, h) and (F, h_F) smooth Hermitian structure on V and F respectively. We define*

$$\begin{aligned} L_k &= \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big), \\ \eta &= \Theta_{\det V^*, \det h^*} + \Theta_{F,h_F}. \end{aligned}$$

Then for all $q \geqslant 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have

$$\text{(a)} \qquad h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \leqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q)}(-1)^q\eta^n + O((\log k)^{-1})\Big),$$

$$\text{(b)} \qquad h^0(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,\leqslant 1)}\eta^n - O((\log k)^{-1})\Big),$$

$$\text{(c)} \qquad \chi(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) = \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\big(c_1(V^*\otimes F)^n + O((\log k)^{-1})\big).$$

Green and Griffiths [GrGr79] already checked the Riemann-Roch calculation (9.38 c) in the special case $V = T_X^*$ and $F = \mathcal{O}_X$. Their proof is much simpler since it relies only on Chern class calculations, but it cannot provide any information on the individual cohomology groups, except in very special cases where vanishing theorems can be applied; in fact in dimension 2, the Euler characteristic satisfies $\chi = h^0 - h^1 + h^2 \leqslant h^0 + h^2$, hence it is enough to get the vanishing of the top cohomology group H^2 to infer $h^0 \geqslant \chi$; this works for surfaces by means of a well-known vanishing theorem of Bogomolov which implies in general

$$H^n\Big(X, E_{k,m}^{\mathrm{GG}}T_X^* \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big) = 0$$

as soon as $K_X \otimes F$ is big and $m \gg 1$.

In fact, thanks to Bonavero's singular holomorphic Morse inequalities [Bon93], everything works almost unchanged in the case where $V \subset T_X$ has singularities and h is an admissible metric on V (see Definition 9.7). We only have to find a blow-up $\mu : \widetilde{X}_k \to X_k$ so that the resulting pull-backs μ^*L_k and μ^*V are locally free, and $\mu^*\det h^*$, $\mu^*\Psi_{h,p,\varepsilon}$ only have divisorial singularities. Then η is a $(1,1)$-current with logarithmic poles, and we have to deal with smooth metrics on $\mu^*L_k^{\otimes m} \otimes \mathcal{O}(-mE_k)$ where E_k is a certain effective divisor on X_k (which, by our assumption in 9.7, does not project onto X). The cohomology groups involved are then the twisted cohomology groups

$$H^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m}) \otimes \mathcal{J}_{k,m})$$

where $\mathcal{J}_{k,m} = \mu_*(\mathcal{O}(-mE_k))$ is the corresponding multiplier ideal sheaf, and the Morse integrals need only be evaluated in the complement of the poles, that is on $X(\eta,q) \smallsetminus S$ where $S = \mathrm{Sing}(V) \cup \mathrm{Sing}(h)$. Since

$$(\pi_k)_*\big(\mathcal{O}(L_k^{\otimes m}) \otimes \mathcal{J}_{k,m}\big) \subset E_{k,m}^{\mathrm{GG}}V^* \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)$$

we still get a lower bound for the H^0 of the latter sheaf (or for the H^0 of the un-twisted line bundle $\mathcal{O}(L_k^{\otimes m})$ on X_k^{GG}). If we assume that $K_V \otimes F$ is big, these considerations also allow us to obtain a strong estimate in terms of the volume, by using an approximate Zariski decomposition on a suitable blow-up of (X,V). The following corollary implies in particular Theorem 9.2.

9.39. Corollary. *If F is an arbitrary $\mathbb{Q}$-line bundle over X, one has*

$$\begin{aligned} h^0\Big(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big) \\ \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\mathrm{Vol}(K_V\otimes F) - O((\log k)^{-1})\Big) - o(m^{n+kr-1}),\end{aligned}$$

when $m \gg k \gg 1$, in particular there are many sections of the k-jet differentials of degree m twisted by the appropriate power of F if $K_V \otimes F$ is big.

Proof. The volume is computed here as usual, i.e. after performing a suitable modification $\mu : \widetilde{X} \to X$ which converts K_V into an invertible sheaf. There is of course nothing to prove if $K_V \otimes F$ is not big, so we can assume $\mathrm{Vol}(K_V \otimes F) > 0$. Let us fix smooth Hermitian metrics h_0 on T_X and h_F on F. They induce a metric $\mu^*(\det h_0^{-1} \otimes h_F)$ on $\mu^*(K_V \otimes F)$ which, by our definition of K_V, is a smooth metric. By the result of Fujita [Fuj94] on approximate Zariski decomposition, for every $\delta > 0$, one can find a modification $\mu_\delta : \widetilde{X}_\delta \to X$ dominating μ such that

$$\mu_\delta^*(K_V \otimes F) = \mathcal{O}_{\widetilde{X}_\delta}(A + E)$$

where A and E are $\mathbb{Q}$-divisors, A ample and E effective, with

$$\mathrm{Vol}(A) = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta.$$

If we take a smooth metric h_A with positive definite curvature form Θ_{A,h_A}, then we get a singular Hermitian metric $h_A h_E$ on $\mu_\delta^*(K_V \otimes F)$ with poles along E, i.e. the quotient $h_A h_E / \mu^*(\det h_0^{-1} \otimes h_F)$ is of the form $e^{-\varphi}$ where φ is quasi-psh with log poles $\log|\sigma_E|^2$ (mod $C^\infty(\widetilde{X}_\delta)$) precisely given by the divisor E. We then only need to take the singular metric h on T_X defined by

$$h = h_0 e^{\frac{1}{r}(\mu_\delta)^*\varphi}$$

(the choice of the factor $\frac{1}{r}$ is there to correct adequately the metric on $\det V$). By construction h induces an admissible metric on V and the resulting curvature current $\eta = \Theta_{K_V,\det h^*} + \Theta_{F,h_F}$ is such that

$$\mu_\delta^*\eta = \Theta_{A,h_A} + [E], \qquad [E] = \text{current of integration on } E.$$

Then the 0-index Morse integral in the complement of the poles is given by

$$\int_{X(\eta,0)\smallsetminus S} \eta^n = \int_{\widetilde{X}_\delta} \Theta_{A,h_A}^n = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta$$

and (9.39) follows from the fact that δ can be taken arbitrary small. □

9.40. Example. In some simple cases, the above estimates can lead to very explicit results. Take for instance X to be a smooth complete intersection of multidegree $(d_1, d_2, \ldots, d_s)$ in $\mathbb{P}_{\mathbb{C}}^{n+s}$ and consider the absolute case $V = T_X$. Then $K_X = \mathcal{O}_X(d_1 + \ldots + d_s - n - s - 1)$ and one can check via explicit bounds of the error terms (cf. [Dem11], [Dem12]) that a sufficient condition for the existence of sections is

$$k \geqslant \exp\Big(7.38\, n^{n+1/2}\Big(\frac{\sum d_j + 1}{\sum d_j - n - s - a - 1}\Big)^n\Big).$$

This is good in view of the fact that we can cover arbitrary smooth complete intersections of general type. On the other hand, even when the degrees d_j tend to $+\infty$, we still get a large lower bound $k \sim \exp(7.38\, n^{n+1/2})$ on the order of jets, and this is far from being optimal: Diverio [Div08, Div09] has shown e.g. that one can take $k = n$ for smooth hypersurfaces of high degree, using the algebraic Morse inequalities of Trapani [Tra95]. The next paragraph uses essentially the same idea, in our more analytic setting. □

§9.D. Non probabilistic estimate of the Morse integrals

We assume here that the curvature tensor $(c_{ij\alpha\beta})$ satisfies a lower bound

$$(9.41)\qquad \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\xi_i\overline{\xi}_j u_\alpha\overline{u}_\beta \geqslant -\sum \gamma_{ij}\xi_i\overline{\xi}_j\ |u|^2, \qquad \forall\xi\in T_X,\ u\in X$$

for some semipositive $(1,1)$-form $\gamma=\frac{i}{2\pi}\sum\gamma_{ij}(z)\,dz_i\wedge d\overline{z}_j$ on X. This is the same as assuming that the curvature tensor of (V^*,h^*) satisfies the semipositivity condition

$$(9.41')\qquad \Theta_{V^*,h^*}+\gamma\otimes\mathrm{Id}_{V^*}\geqslant 0$$

in the sense of Griffiths, or equivalently $\Theta_{V,h}-\gamma\otimes\mathrm{Id}_V\leqslant 0$. Thanks to the compactness of X, such a form γ always exists if h is an admissible metric on V. Now, instead of replacing Θ_V with its trace free part $\widetilde{\Theta}_V$ and exploiting a Monte Carlo convergence process, we replace Θ_V with $\Theta_V^\gamma=\Theta_V-\gamma\otimes\mathrm{Id}_V\leqslant 0$, i.e. $c_{ij\alpha\beta}$ by $c_{ij\alpha\beta}^\gamma=c_{ij\alpha\beta}+\gamma_{ij}\delta_{\alpha\beta}$. Also, we take a line bundle $F=A^{-1}$ with $\Theta_{A,h_A}\geqslant 0$, i.e. F seminegative. Then our earlier formulas (9.28), (9.35), (9.36) become instead

$$(9.42)\qquad g_k^\gamma(z,x,u)=\frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}^\gamma(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j\geqslant 0,$$

$$(9.43)\qquad L_k=\mathcal{O}_{X_k^{\mathrm{GG}}}(1)\otimes\pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big),$$

$$(9.44)\qquad \Theta_{L_k}=\eta_k(z,x,u)=\frac{1}{\frac{1}{kr}(1+\frac{1}{2}+\ldots+\frac{1}{k})}g_k^\gamma(z,x,u)-(\Theta_{A,h_A}(z)+r\gamma(z)).$$

In fact, replacing Θ_V by $\Theta_V-\gamma\otimes\mathrm{Id}_V$ has the effect of replacing $\Theta_{\det V^*}=\mathrm{Tr}\,\Theta_{V^*}$ by $\Theta_{\det V^*}+r\gamma$. The major gain that we have is that $\eta_k=\Theta_{L_k}$ is now expressed as a difference of semipositive $(1,1)$-forms, and we can exploit the following simple lemma, which is the key to derive algebraic Morse inequalities from their analytic form (cf. [Dem94], Theorem 12.3).

9.45. Lemma. *Let $\eta=\alpha-\beta$ be a difference of semipositive $(1,1)$-forms on an n-dimensional complex manifold X, and let $\mathbb{1}_{\eta,\leqslant q}$ be the characteristic function of the open set where η is non degenerate with a number of negative eigenvalues at most equal to q. Then*

$$(-1)^q\mathbb{1}_{\eta,\leqslant q}\ \eta^n\leqslant\sum_{0\leqslant j\leqslant q}(-1)^{q-j}\alpha^{n-j}\beta^j,$$

in particular

$$\mathbb{1}_{\eta,\leqslant 1}\ \eta^n\geqslant\alpha^n-n\alpha^{n-1}\wedge\beta\qquad \textit{for } q=1.$$

Proof. Without loss of generality, we can assume $\alpha>0$ positive definite, so that α can be taken as the base hermitian metric on X. Let us denote by

$$\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant 0$$

the eigenvalues of β with respect to α. The eigenvalues of $\eta=\alpha-\beta$ are then given by

$$1-\lambda_1\leqslant\ldots\leqslant 1-\lambda_q\leqslant 1-\lambda_{q+1}\leqslant\ldots\leqslant 1-\lambda_n,$$

hence the open set $\{\lambda_{q+1} < 1\}$ coincides with the support of $\mathbb{1}_{\eta,\leqslant q}$, except that it may also contain a part of the degeneration set $\eta^n = 0$. On the other hand we have

$$\binom{n}{j}\alpha^{n-j}\wedge\beta^j = \sigma_n^j(\lambda)\,\alpha^n,$$

where $\sigma_n^j(\lambda)$ is the j-th elementary symmetric function in the λ_j's. Thus, to prove the lemma, we only have to check that

$$\sum_{0\leqslant j\leqslant q}(-1)^{q-j}\sigma_n^j(\lambda) - \mathbb{1}_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\leqslant j\leqslant n}(1-\lambda_j)\geqslant 0.$$

This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda) = \sigma_{n-1}^j(\lambda) + \sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). $\square$

We apply here Lemma 9.45 with

$$\alpha = g_k^\gamma(z,x,u),\qquad \beta = \beta_k = \frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)(\Theta_{A,h_A}+r\gamma),$$

which are both semipositive by our assumption. The analogue of (9.32) leads to

$$\begin{aligned}
&\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}\\
&\quad= \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k^\gamma-\beta_k,\leqslant 1}\,(g_k^\gamma-\beta_k)^n\,d\nu_{k,r}(x)\,d\mu(u)\\
&\quad\geqslant \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}((g_k^\gamma)^n - n(g_k^\gamma)^{n-1}\wedge\beta_k)\,d\nu_{k,r}(x)\,d\mu(u).
\end{aligned}$$

The resulting integral now produces a "closed formula" which can be expressed solely in terms of Chern classes (assume that γ is the Chern form of some semipositive line bundle). It is just a matter of routine to find a sufficient condition for the positivity of the integral. One can first observe that g_k^γ is bounded from above by taking the trace of $(c_{ij\alpha\beta}$, in this way we get

$$0\leqslant g_k^\gamma\leqslant\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)(\Theta_{\det V^*}+r\gamma)$$

where the right hand side no longer depends on u. Also, since γ_k^γ is a sum of semipositive $(1,1)$-forms

$$g_k^\gamma = \sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\theta^\gamma(u_s),\qquad \theta^\gamma(u) = \sum_{i,j,\alpha,\beta}c^\gamma_{ij\alpha\beta}u_\alpha\overline{u}_\beta\,dz_i\wedge d\overline{z}_j,$$

hence for $k\geqslant n$ we have

$$(g_k^\gamma)^n\geqslant n!\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{x_{s_1}\ldots x_{s_n}}{s_1\ldots s_n}\,\theta^\gamma(u_{s_1})\wedge\theta^\gamma(u_{s_2})\wedge\ldots\wedge\theta^\gamma(u_{s_n}).$$

Since $\int_{S^{2r-1}}\theta^\gamma(u)\,d\mu(u) = \frac{1}{r}\,\mathrm{Tr}(\Theta_{V^*}+\gamma) = \frac{1}{r}\Theta_{\det V^*}+\gamma$, we infer from this

$$\begin{aligned}
&\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}(g_k^\gamma)^n\,d\nu_{k,r}(x)\,d\mu(u)\\
&\qquad\geqslant n!\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\Big(\int_{\Delta_{k-1}}x_1\ldots x_n\,d\nu_{k,r}(x)\Big)\Big(\frac{1}{r}\Theta_{\det V^*}+\gamma\Big)^n.
\end{aligned}$$

By putting everything together, we conclude:

9.46. Theorem. *Assume that* $\Theta_{V^*} \geqslant -\gamma \otimes \mathrm{Id}_{V^*}$ *with a semipositive* $(1,1)$*-form* γ *on* X*. Then the Morse integral of the line bundle*

$$L_k = \mathcal{O}_{X_k^{\rm GG}}(1) \otimes \pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big), \qquad A \geqslant 0$$

satisfies for $k \geqslant n$ *the inequality*

$$\begin{aligned} &\frac{1}{(n+kr-1)!}\int_{X_k^{\rm GG}(L_k,\leqslant 1)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} \\ (*)\quad &\geqslant \frac{1}{n!(k!)^r(kr-1)!}\int_X c_{n,r,k}\big(\Theta_{\det V^*}+r\gamma\big)^n - c'_{n,r,k}\big(\Theta_{\det V^*}+r\gamma\big)^{n-1}\wedge\big(\Theta_{A,h_A}+r\gamma\big)\end{aligned}$$

where

$$c_{n,r,k} = \frac{n!}{r^n}\Big(\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\Big)\int_{\Delta_{k-1}} x_1\ldots x_n\,d\nu_{k,r}(x),$$

$$c'_{n,r,k} = \frac{n}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^{n-1} d\nu_{k,r}(x).$$

Especially we have a lot of sections in $H^0(X_k^{\rm GG}, mL_k)$*,* $m \gg 1$*, as soon as the difference occurring in* $(*)$ *is positive.*

The statement is also true for $k < n$, but then $c_{n,r,k} = 0$ and the lower bound $(*)$ cannot be positive. By Corollary 9.11, it still provides a non trivial lower bound for $h^0(X_k^{\rm GG}, mL_k) - h^1(X_k^{\rm GG}, mL_k)$, though. For $k \geqslant n$ we have $c_{n,r,k} > 0$ and $(*)$ will be positive if $\Theta_{\det V^*}$ is large enough. By Formula 9.20 we have

$$c_{n,r,k} = \frac{n!\,(kr-1)!}{(n+kr-1)!}\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n} \geqslant \frac{(kr-1)!}{(n+kr-1)!}, \tag{9.47}$$

(with equality for $k = n$), and by ([Dem11], Lemma 2.20 (b)) we get the upper bound

$$\frac{c'_{n,k,r}}{c_{n,k,r}} \leqslant \frac{(kr+n-1)r^{n-2}}{k/n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\Big[1+\frac{1}{3}\sum_{m=2}^{n-1}\frac{2^m(n-1)!}{(n-1-m)!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^{-m}\Big].$$

The case $k = n$ is especially interesting. For $k = n \geqslant 2$ one can show (with $r \leqslant n$ and H_n denoting the harmonic sequence) that

$$\frac{c'_{n,k,r}}{c_{n,k,r}} \leqslant \frac{n^2+n-1}{3}n^{n-2}\exp\Big(\frac{2(n-1)}{H_n}+n\log H_n\Big) \leqslant \frac{1}{3}\big(n\log(n\log 24n)\big)^n. \tag{9.48}$$

We will also need the particular values

$$c_{2,2,2} = \frac{1}{20}, \qquad c'_{2,2,2} = \frac{9}{16}, \qquad \frac{c'_{2,2,2}}{c_{2,2,2}} = \frac{45}{4}, \tag{9.49$_2$}$$

$$c_{3,3,3} = \frac{1}{990}, \qquad c'_{3,3,3} = \frac{451}{4860}, \qquad \frac{c'_{3,3,3}}{c_{3,3,3}} = \frac{4961}{54}, \tag{9.49$_3$}$$

which can be obtained by direct calculations.

§10. Hyperbolicity properties of hypersurfaces of high degree

§10.A. Global generation of the twisted tangent space of the universal family

In [Siu02, Siu04], Y.T. Siu developed a new stategy to produce jet differentials, involving meromorphic vector fields on the total space of jet bundles – these vector fields are used to differentiate the sections of $E^{\mathrm{GG}}_{k,m}$ so as to produce new ones with less zeroes. The approach works especially well on universal families of hypersurfaces in projective space, thanks to the good positivity properties of the relative tangent bundle, as shown by L. Ein [Ein88, Ein91] and C. Voisin [Voi96]. This allows at least to prove the hyperbolicity of generic surfaces and generic 3-dimensional hypersurfaces of sufficiently high degree. We reproduce here the improved approach given by [Pau08] for the twisted global generation of the tangent space of the space of vertical two jets. The situation of k-jets in arbitrary dimension n is substantially more involved, details can be found in [Mer09].

Consider the universal hypersurface $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ of degree d given by the equation

$$\sum_{|\alpha|=d} A_\alpha\, Z^\alpha = 0,$$

where $[Z] \in \mathbb{P}^{n+1}$, $[A] \in \mathbb{P}^{N_d}$, $\alpha = (\alpha_0, \ldots, \alpha_{n+1}) \in \mathbb{N}^{n+2}$ and

$$N_d = \binom{n+d+1}{d} - 1.$$

Finally, we denote by $\mathcal{V} \subset \mathcal{X}$ the vertical tangent space, i.e. the kernel of the projection

$$\pi : \mathcal{X} \to U \subset \mathbb{P}^{N_d}$$

where U is the Zariski open set parametrizing smooth hypersurfaces, and by $J_k\mathcal{V}$ the bundle of k-jets of curves tangent to $\mathcal{V}$, i.e. curves contained in the fibers $X_s = \pi^{-1}(s)$. The goal is to describe certain meromorphic vector fields on the total space of $J_k\mathcal{V}$. In the special case $n = 2$, $k = 2$ considered by Păun [Pau08], one fixes the affine open set

$$\mathcal{U}_0 = \{Z_0 \neq 0\} \times \{A_{0d00} \neq 0\} \simeq \mathbb{C}^3 \times \mathbb{C}^{N_d}$$

in $\mathbb{P}^3 \times \mathbb{P}^{N_d}$ with the corresponding inhomogeneous coordinates $(z_j = Z_j/Z_0)_{j=1,2,3}$ and $(a_\alpha = A_\alpha/A_{0d00})_{|\alpha|=d,\alpha_1<d}$. Since α_0 is determined by $\alpha_0 = d - (\alpha_1 + \alpha_2 + \alpha_3)$, with a slight abuse of notation in the sequel, α will be seen as a multiindex $(\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{N}^3$, with moreover the convention that $a_{d00} = 1$. On this affine open set we have

$$\mathcal{X}_0 := \mathcal{X} \cap \mathcal{U}_0 = \Big\{ z_1^d + \sum_{|\alpha|\leqslant d,\alpha_1<d} a_\alpha\, z^\alpha = 0 \Big\}.$$

We now write down equations for the open variety $J_2\mathcal{V}_0$, where we indicated with $\mathcal{V}_0$ the restriction of $\mathcal{V} \subset T_{\mathcal{X}}$, the kernel of the differential of the second projection, to $\mathcal{X}_0$: elements in $J_2\mathcal{V}_0$ are therefore 2-jets of germs of "vertical" holomorphic curves in $\mathcal{X}_0$, that is curves tangent to vertical fibers. The equations, which live in a natural way in $\mathbb{C}^3_{z_j} \times \mathbb{C}^{N_d}_{a_\alpha} \times \mathbb{C}^3_{z'_j} \times \mathbb{C}^3_{z''_j}$, stand as follows.

$$\sum_{|\alpha|\leqslant d} a_\alpha\, z^\alpha = 0,$$

$$\sum_{1\leqslant j\leqslant 3}\ \sum_{|\alpha|\leqslant d} a_\alpha\, \frac{\partial z^\alpha}{\partial z_j}\, z'_j = 0,$$

$$\sum_{1\leqslant j\leqslant 3}\ \sum_{|\alpha|\leqslant d} a_\alpha\, \frac{\partial z^\alpha}{\partial z_j}\, z''_j + \sum_{1\leqslant j,k\leqslant 3}\ \sum_{|\alpha|\leqslant d} a_\alpha\, \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k}\, z'_j z'_k = 0.$$

Let $\mathcal{W}_0$ to be the closed algebraic subvariety of $J_2\mathcal{V}_0$ defined by

$$\mathcal{W}_0 = \{(z, a, z', z'') \in J_2\mathcal{V}_0 \mid z' \wedge z'' = 0\}$$

and let $\mathcal{W}$ be the Zariski closure of $\mathcal{W}_0$ in $J_2\mathcal{V}$: we call this set the *Wronskian locus* of $J_2\mathcal{V}$. Explicit calculations (cf. [Pau08]) then produce the following vector fields:

First family of tangent vector fields. For any multiindex α such that $\alpha_1 \geqslant 3$, consider the vector field

$$\theta_\alpha^{300} = \frac{\partial}{\partial a_\alpha} - 3z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} + 3z_1^2 \frac{\partial}{\partial a_{\alpha-2\delta_1}} - z_1^3 \frac{\partial}{\partial a_{\alpha-3\delta_1}},$$

where $\delta_j \in \mathbb{N}^4$ is the multiindex whose j-th component is equal to 1 and the others are zero. For the multiindexes α which verify $\alpha_1 \geqslant 2$ and $\alpha_2 \geqslant 1$, define

$$\begin{aligned}\theta_\alpha^{210} &= \frac{\partial}{\partial a_\alpha} - 2z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} - z_2 \frac{\partial}{\partial a_{\alpha-\delta_2}} + z_1^2 \frac{\partial}{\partial a_{\alpha-2\delta_1}} \\ &\quad + 2z_1z_2 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2}} - z_1^2z_2 \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_2}}.\end{aligned}$$

Finally, for those α for which $\alpha_1, \alpha_2, \alpha_3 \geqslant 1$, set

$$\begin{aligned}\theta_\alpha^{111} &= \frac{\partial}{\partial a_\alpha} - z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} - z_2 \frac{\partial}{\partial a_{\alpha-\delta_2}} - z_3 \frac{\partial}{\partial a_{\alpha-\delta_3}} \\ &\quad + z_1z_2 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2}} + z_1z_3 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_3}} + z_2z_3 \frac{\partial}{\partial a_{\alpha-\delta_2-\delta_3}} \\ &\quad - z_1z_2z_3 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2-\delta_3}}.\end{aligned}$$

Second family of tangent vector fields. We construct here the holomorphic vector fields in order to span the $\partial/\partial z_j$-directions. For $j = 1, 2, 3$, consider the vector field

$$\frac{\partial}{\partial z_j} - \sum_{|\alpha+\delta_j| \leqslant d} (\alpha_j + 1) a_{\alpha+\delta_j} \frac{\partial}{\partial a_\alpha}.$$

Third family of tangent vector fields. In order to span the jet directions, consider a vector field of the following form:

$$\theta_B = \sum_{|\alpha| \leqslant d, \alpha_1 < d} p_\alpha(z, a, b) \frac{\partial}{\partial a_\alpha} + \sum_{1 \leqslant j \leqslant 3} \sum_{k=1}^{2} \xi_j^{(k)} \frac{\partial}{\partial z_j^{(k)}},$$

where $\xi^{(k)} = B \cdot z^{(k)}$, $k = 1, 2$, and $B = (b_{jk})$ varies among 3×3 invertible matrices with complex entries. By studying more carefully these three families of vector fields, one obtains:

10.1. Theorem. *The twisted tangent space $T_{J_2\mathcal{V}} \otimes \mathcal{O}_{\mathbb{P}^3}(7) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(1)$ is generated over by its global sections over the complement $J_2\mathcal{V} \smallsetminus \mathcal{W}$ of the Wronskian locus $\mathcal{W}$. Moreover, one can choose generating global sections that are invariant with respect to the action of $\mathbb{G}_2$ on $J_2\mathcal{V}$.*

By similar, but more computationally intensive arguments [Mer09], one can investigate the higher dimensional case. The following result strengthens the initial announcement of [Siu04].

10.2. Theorem. *Let $J_k^{\rm vert}(\mathcal{X})$ be the space of vertical k-jets of the universal hypersurface*

$$\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$$

parametrizing all projective hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d. Then for $k = n$, there exist constants c_n and c'_n such that the twisted tangent bundle

$$T_{J_k^{\rm vert}(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(c_n) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(c'_n)$$

is generated by its global $\mathbb{G}_k$-invariant sections outside a certain exceptional algebraic subset $\Sigma \subset J_k^{\rm vert}(\mathcal{X})$. One can take either $c_n = \frac{1}{2}(n^2+5n)$, $c'_n = 1$ and Σ defined by the vanishing of certain Wronskians, or $c_n = n^2 + 2n$ and a smaller set $\widetilde{\Sigma} \subset \Sigma$ defined by the vanishing of the 1-jet part.

10.B. General strategy of proof

Let again $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ be the universal hypersurface of degree d in $\mathbb{P}^{n+1}$.

(10.3) *Assume that we can prove the existence of a non zero polynomial differential operator*

$$P \in H^0(\mathcal{X}, E_{k,m}^{\rm GG} T_{\mathcal{X}}^* \otimes \mathcal{O}(-A)),$$

where A is an ample divisor on $\mathcal{X}$, at least over some Zariski open set U in the base of the projection $\pi : \mathcal{X} \to U \subset \mathbb{P}^{N_d}$.

Observe that we now have a lot of techniques to do this; the existence of P over the family follows from lower semicontinuity in the Zariski topology, once we know that such a section P exists on a generic fiber $X_s = \pi^{-1}(s)$. Let $\mathcal{Y} \subset \mathcal{X}$ be the set of points $x \in \mathcal{X}$ where $P(x) = 0$, as an element in the fiber of the vector bundle $E_{k,m}^{\rm GG} T_{\mathcal{X}}^* \otimes \mathcal{O}(-A))$ at x. Then $\mathcal{Y}$ is a proper algebraic subset of $\mathcal{X}$, and after shrinking U we may assume that $Y_s = \mathcal{Y} \cap X_s$ is a proper algebraic subset of X_s for every $s \in U$.

(10.4) *Assume also, according to Theorems* 10.1 *and* 10.2*, that we have enough global holomorphic $\mathbb{G}_k$-invariant vector fields θ_i on $J_k\mathcal{V}$ with values in the pull-back of some ample divisor B on $\mathcal{X}$, in such a way that they generate $T_{J_k\mathcal{V}} \otimes p_k^* B$ over the dense open set $(J_k\mathcal{V})^{\rm reg}$ of regular k-jets, i.e. k-jets with non zero first derivative* (*here $p_k : J_k\mathcal{V} \to \mathcal{X}$ is the natural projection*)*.*

Considering jet differentials P as functions on $J_k\mathcal{V}$, the idea is to produce new ones by taking differentiations

$$Q_j := \theta_{j_1} \ldots \theta_{j_\ell} P, \qquad 0 \leqslant \ell \leqslant m,\ j = (j_1, \ldots, j_\ell).$$

Since the θ_j's are $\mathbb{G}_k$-invariant, they are in particular $\mathbb{C}^*$-invariant, thus

$$Q_j \in H^0(\mathcal{X}, E_{k,m}^{\rm GG} T_{\mathcal{X}}^* \otimes \mathcal{O}(-A + \ell B))$$

(and Q is in fact $\mathbb{G}'_k$ invariant as soon as P is). In order to be able to apply the vanishing theorems of § 8, we need $A - mB$ to be ample, so A has to be large compared to B. If $f : \mathbb{C} \to X_s$ is an entire curve contained in some fiber $X_s \subset \mathcal{X}$, its lifting $j_k(f) : \mathbb{C} \to J_k\mathcal{V}$ has to lie in the zero divisors of all sections Q_j. However, every non zero polynomial of degree m has at any point some non zero derivative of order $\ell \leqslant m$. Therefore, at any point where the θ_i generate the tangent space to $J_k\mathcal{V}$, we can find some non vanishing section Q_j. By the assumptions on the θ_i, the base locus of the Q_j's is contained in the union of $p_k^{-1}(\mathcal{Y}) \cup (J_k\mathcal{V})^{\rm sing}$; there is of course no way of getting a non zero polynomial at points of $\mathcal{Y}$ where P vanishes. Finally, we observe that $j_k(f)(\mathbb{C}) \not\subset (J_k\mathcal{V}^{\rm sing}$ (otherwise f is constant). Therefore $j_k(f)(\mathbb{C}) \subset p_k^{-1}(\mathcal{Y})$ and thus $f(\mathbb{C}) \subset \mathcal{Y}$, i.e. $f(\mathbb{C}) \subset Y_s = \mathcal{Y} \cap X_s$.

10.5. Corollary. *Let $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ be the universal hypersurface of degree d in $\mathbb{P}^{n+1}$. If $d \geqslant d_n$ is taken so large that conditions* (10.3) *and* (10.4) *are met with $A - mB$ ample, then the generic fiber X_s of the universal family $\mathcal{X} \to U$ satisfies the Green-Griffiths conjecture, namely all entire curves $f : \mathbb{C} \to X_s$ are contained in a proper algebraic subvariety $Y_s \subset X_s$, and the Y_s can be taken to form an algebraic subset $\mathcal{Y} \subset \mathcal{X}$.*

This is unfortunately not enough to get the hyperbolicity of X_s, because we would have to know that Y_s itself is hyperbolic. However, one can use the following simple observation due to Diverio and Trapani [DT10]. The starting point is the following general, straightforward remark. Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle let $\sigma \in H^0(\mathcal{X}, \mathcal{E}) \neq 0$; then, up to factorizing by an effective divisor D contained in the common zeroes of the components of σ, one can view σ as a section

$$\sigma \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-D)),$$

and this section now has a zero locus without divisorial components. Here, when $n \geqslant 2$, the very generic fiber X_s has Picard number one by the Noether-Lefschetz theorem, and so, after shrinking U if necessary, we can assume that $\mathcal{O}_{\mathcal{X}}(-D)$ is the restriction of $\mathcal{O}_{\mathbb{P}^{n+1}}(-p)$, $p \geqslant 0$ by the effectivity of D. Hence D can be assumed to be nef. After performing this simplification, $A - mB$ is replaced by $A - mB + D$, which is still ample if $A - mB$ is ample. As a consequence, we may assume $\operatorname{codim} \mathcal{Y} \geqslant 2$, and after shrinking U again, that all Y_s have $\operatorname{codim} Y_s \geqslant 2$.

10.6. Additional statement. *In corollary* 10.5*, under the same hypotheses* (10.3) *and* (10.4)*, one can take all fibers Y_s to have* $\operatorname{codim} Y_s \geqslant 2$.

This is enough to conclude that X_s is hyperbolic if $n = \dim X_s \leqslant 3$. In fact, this is clear if $n = 2$ since the Y_s are then reduced to points. If $n = 3$, the Y_s are at most curves, but we know by Ein and Voisin that a generic hypersurface $X_s \subset \mathbb{P}^4$ of degree $d \geqslant 7$ does not possess any rational or elliptic curve. Hence Y_s is hyperbolic and so is X_s, for s generic. □

10.7. Corollary. *Assume that $n = 2$ or $n = 3$, and that $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ is the universal hypersurface of degree $d \geqslant d_n \geqslant 2n + 1$ so large that conditions* (10.3) *and* (10.4) *are met with $A-mB$ ample. Then the very generic hypersurface $X_s \subset \mathbb{P}^{n+1}$ of degree d is hyperbolic.*

§10.C. Proof of the Green-griffiths conjecture for generic hypersurfaces in $\mathbb{P}^{n+1}$

The most striking progress made at this date on the Green-Griffiths conjecture itself is a recent result of Diverio, Merker and Rousseau [DMR10], confirming the statement when $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a generic hypersurface of large degree d, with a (non optimal) sufficient lower bound $d \geqslant 2^{n^5}$. Their proof is based in an essential way on Siu's strategy as developed in § 10.B, combined with the earlier techniques of [Dem95]. Using our improved bounds from § 9.D, we obtain here a better estimate (less than doubly exponential, actually of exponential order one $O(\exp(n^{1+\varepsilon}))$.

10.8. Theorem. *A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ with*

$$d_2 = 286, \quad d_3 = 7316, \quad d_n = \left\lfloor \frac{n^4}{3}\big(n\log(n\log(24n))\big)^n \right\rfloor \quad \textit{for } n \geqslant 4,$$

satisfies the Green-Griffiths conjecture.

Proof. Let us apply Theorem 9.46 with $V = T_X$, $r = n$ and $k = n$. The main starting point is the well known fact that $T^*_{\mathbb{P}^{n+1}} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(2)$ is semipositive (in fact, generated by its sections). Hence the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to T^*_{\mathbb{P}^{n+1}|X} \to T^*_X \to 0$$

implies that $T^*_X \otimes \mathcal{O}_X(2) \geqslant 0$. We can therefore take $\gamma = \Theta_{\mathcal{O}(2)} = 2\omega$ where ω is the Fubini-Study metric. Moreover $\det V^* = K_X = \mathcal{O}_X(d-n-2)$ has curvature $(d-n-2)\omega$, hence $\Theta_{\det V^*} + r\gamma = (d+n-2)\omega$. The Morse integral to be computed when $A = \mathcal{O}_X(p)$ is

$$\int_X \Big(c_{n,n,n}(d+n-2)^n - c'_{n,n,n}(d+n-2)^{n-1}(p+2n)\Big)\omega^n,$$

so the critical condition we need is

$$d+n-2 > \frac{c'_{n,n,n}}{c_{n,n,n}}(p+2n).$$

On the other hand, Siu's differentiation technique requires $\frac{m}{n^2}(1+\frac{1}{2}+\ldots+\frac{1}{n})A - mB$ to be ample, where $B = \mathcal{O}_X(n^2+2n)$ by Merker's result 10.2. This ampleness condition yields

$$\frac{1}{n^2}\Big(1+\frac{1}{2}+\ldots+\frac{1}{n}\Big)p - (n^2+2n) > 0,$$

so one easily sees that it is enough to take $p = n^4 - 2n$ for $n \geqslant 3$. Our estimates (9.48) and (9.49) give the expected bound d_n. ☐

Thanks to 10.6, one also obtains the generic hyperbolicity of 2 and 3-dimensional hypersurfaces of large degree.

10.9. Theorem. *For $n = 2$ or $n = 3$, a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ is Kobayashi hyperbolic.*

Using more explicit calculations of Chern classes (and invariant jets rather than Green-Griffiths jets) Diverio-Trapani [DT10] obtained the better lower bound $d \geqslant d_3 = 593$ in dimension 3. In the case of surfaces, Paun [Pau08] obtained $d \geqslant d_2 = 18$, using deep results of McQuillan [McQ98].

One may wonder whether it is possible to use jets of order $k < n$ in the proof of 10.8 and 10.9. Diverio [Div08] showed that the answer is nagative (his proof is based on elementary facts of representation theory and a vanishing theorem of Brückmann-Rackwitz [BR90]):

10.10. Proposition ([Div08]). *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. Then*

$$H^0(X, E^{\mathrm{GG}}_{k,m}T^*_X) = 0$$

for $m \geqslant 1$ and $1 \leqslant k < n$. More generally, if $X \subset \mathbb{P}^{n+s}$ is a smooth complete intersection of codimension s, there are no global jet differentials for $m \geqslant 1$ and $k < n/s$.

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Tits alternative for automorphism groups of compact Kähler manifolds

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(Preliminary version)

Abstract

We present a proof of the Tits type alternative for automorphism groups of compact Kähler manifolds which is recently obtained by De-Qi Zhang. Several related results will be discussed.

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1 Introduction

In this paper, we will present a proof of the Tits type alternative obtained by De-Qi Zhang in [50] which confirms a conjecture by Keum-Oguiso-Zhang [27]. This nice result is intimately connected to developments in complex dynamics of several variables. So we will survey some results from holomorphic dynamics which are related to the Tits type alternative and to its proof.

Several statements in the paper are purely algebraic and one can ask if they can be obtained by purely algebraic methods and also if they can be extended to automorphisms of algebraic manifolds over a finite field. The question is far from the author's competence. So only analytical tools will be discussed here and we refer the reader to a recent paper by Esnault-Srinivas [20] which is a first step to study similar questions for finite fields.

Let X be a compact Kähler manifold, e.g. a complex projective manifold, of dimension k. Denote by $\mathrm{Aut}(X)$ the group of all holomorphic automorphisms of X. Following a result by Bochner-Montgomery, $\mathrm{Aut}(X)$ is a complex Lie group of finite dimension [4], see also [1, 4, 40] and the references therein for results on upper bounds of the dimension.

The group $\mathrm{Aut}(X)$ may have an infinite number of connected components. Let $\mathrm{Aut}_0(X)$ denote the connected component of the identity. Elements in this subgroup are those induced by global holomorphic vector fields on X. They are almost characterized by the property that the associated actions on cohomology preserve a Kähler class. More precisely, Fujiki and Lieberman proved that the group of all automorphisms preserving a given Kähler class is a finite union of connected components of $\mathrm{Aut}(X)$, see [22, 30] and Theorem 5.1 below.

Quite recently, ideas from complex dynamics allowed to study the "discrete part" of $\mathrm{Aut}(X)$. Inspired by results in [16], a Tits type alternative was conjectured and proved in some cases by Keum-Oguiso-Zhang in [27], see also [36, 49]. The conjecture was fully obtained by De-Qi Zhang in [50]. His theorem corresponds to the second assertion of the following theorem.

Theorem 1.1. *Let X be a compact Kähler manifold of dimension k and of Kodaira dimension κ_X. Define $\kappa := \max(\kappa_X, 0)$ if $\kappa_X < k$ and $\kappa := k-1$ otherwise. Let G be a group of holomorphic automorphisms of X which does not contain any free non-abelian subgroup. Then G admits a finite index subgroup G' satisfying the following properties:*

1. *G' is solvable; in other words, $\mathrm{Aut}(X)$ satisfies the Tits alternative, see also Theorem 5.3;*

2. *The set N' of zero entropy elements of G' is a normal subgroup of G' and G'/N' is a free abelian group of rank at most $k - \kappa - 1$.*

The entropy of an automorphism was originally introduced as a dynamical invariant. However, thanks to results by Gromov and Yomdin [25, 47], it can be also considered as an algebraic invariant. The notion, its properties and its relations with the dynamical degrees of automorphism will be presented in Section 3.

A main tool in the proof of Theorem 1.1 is a mixed version of the classical Hodge-Riemann theorem. It will be discussed in Section 2. In Section 4, we will survey some results on meromorphic fibrations which are preserved by an automorphism. These results will be applied to the Iitaka's fibration of X and are used to obtain the rank estimate in Theorem 1.1. Finally, the proof of Theorem 1.1 will be given in Section 5.

We deduce from Theorem 1.1 the following important consequence, see [16, 50]. We say that a group G of automorphisms of X has *positive entropy* if all elements of g, except the identity, have positive entropy.

Corollary 1.2. *Let G be a group of automorphisms of X. Assume that G is abelian and has positive entropy. Then, G is a free abelian group of rank at most $k - \kappa - 1$.*

The rank estimates in Theorem 1.1 and Corollary 1.2 are optimal as shown in the following example.

Example 1.3. Consider the natural action of $\mathrm{SL}(k, \mathbb{Z})$ on the complex torus $\mathbb{C}^k/(\mathbb{Z}^k + i\mathbb{Z}^k)$. The Kodaira dimension of the torus is zero. By a theorem of

Prasad-Raghunathan [38], the action on the right of $\mathrm{SL}(k,\mathbb{Z})\setminus\mathrm{SL}(k,\mathbb{R})$ of the group of diagonal matrices in $\mathrm{SL}(k,\mathbb{R})$ admits compact orbits. These compact orbits can be identified to quotients of $\mathbb{R}^{k-1}$ by subgroups of $\mathrm{SL}(k,\mathbb{Z})$. We deduce that $\mathrm{SL}(k,\mathbb{Z})$ admits free commutative subgroups of rank $k-1$ which are diagonalizable. The elements of these subgroups, except the identity, admit eigenvalues with modulus larger than 1. They can be identified to rank $k-1$ free commutative groups of automorphisms on the considered complex torus. It is then not difficult to check that such groups have positive entropy.

This example suggests the following open problem which was stated in the arXiv version of [16].

Problem 1.4. Classify compact Kähler manifolds of dimension $k\geq 3$ admitting a free commutative group of automorphisms of rank $k-1$ which is of positive entropy.

In dimension $k=2$, many K3 surfaces and rational surfaces admit positive entropy automorphisms, see e.g. [2, 3, 8, 12, 32, 33, 34, 41] and also [19, 26, 29, 37]. In higher dimension, some partial results on the above problem were obtained in [51]. We can also ask the same question for groups of rank $k-p$ and for manifolds of dimension large enough, e.g. for groups of rank $k-2$ with $k\geq 4$.

Corollary 1.2 and the Margulis super-rigidity theorem [31] play a crucial role in a result of Cantat on the action of a simple lattice on compact Kähler manifolds. His result gives an affirmative answer to a version of Zimmer's problem in the case of holomorphic group actions, see also [48]. The following statement is slightly stronger than the one given in [5] where the rank bound was k instead of $k-\max(\kappa_X,0)$.

Theorem 1.5. *Let Γ be a lattice of a simple algebraic Lie $\mathbb{R}$-group G. Assume that Γ admits a representation in $\mathrm{Aut}(X)$ with infinite image. Then the real rank of G is at most $k-\max(\kappa_X,0)$.*

We refer to the paper by Cantat [5] for the proof, see also [6, 7] for more results in this direction.

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2 Mixed Hodge-Riemann theorem

In this section, we will recall the mixed version of the classical Hodge-Riemann theorem which is used for the main results in this paper. We refer to the books by Demailly [9] and Voisin [43] for basic notions and results on Hodge theory.

Let X be a compact Kähler manifold of dimension k as above. For $0 \leq p, q \leq k$, denote by $H^{p,q}(X, \mathbb{C})$ the Hodge cohomology group of bidegree (p, q). We often identify $H^{k,k}(X, \mathbb{C})$ with $\mathbb{C}$ via the integration of maximal bidegree forms on X. Define for all $0 \leq p \leq k$

$$H^{p,p}(X, \mathbb{R}) := H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{R}).$$

As a consequence of the Hodge theory, we have

$$H^{p,p}(X, \mathbb{C}) = H^{p,p}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Denote by $\mathscr{K}$ *the Kähler cone* of X, i.e. the set of all classes of Kähler forms on X, see [11] for a characterization of $\mathscr{K}$. This is a strictly convex open cone in $H^{1,1}(X, \mathbb{R})$. The closed cone $\overline{\mathscr{K}}$ is called *the nef cone* and its elements are *the nef classes* of X.

To each class $\Omega \in H^{k-2,k-2}(X, \mathbb{R})$, we associate the quadratic form Q_Ω on $H^{1,1}(X, \mathbb{R})$ defined by

$$Q_\Omega(\alpha, \beta) := -\alpha \smile \beta \smile \Omega \quad \text{for} \quad \alpha, \beta \in H^{1,1}(X, \mathbb{R}).$$

For any non-zero class $\Omega' \in H^{k-1,k-1}(X, \mathbb{R})$, define *the primitive subspace* $P_{\Omega'}$ of $H^{1,1}(X, \mathbb{R})$ associated to Ω' by

$$P_{\Omega'} := \big\{ \alpha \in H^{1,1}(X, \mathbb{R}) : \ \alpha \smile \Omega' = 0 \big\}.$$

By Poincaré duality, this is a hyperplane of $H^{1,1}(X, \mathbb{R})$. We have the following result, see [14].

Theorem 2.1. *Let $c_1, \ldots, c_{k-1}$ be Kähler classes on X. Define $\Omega := c_1 \smile \cdots \smile c_{k-2}$ and $\Omega' := \Omega \smile c_{k-1}$. Let $h^{1,1}$ denote the dimension of $H^{1,1}(X, \mathbb{R})$. Then, the quadratic form Q_Ω has signature $(h^{1,1} - 1, 1)$ and is positive definite on $P_{\Omega'}$.*

When the classes c_j are equal, we obtain the classical Hodge-Riemann theorem. The Hodge-Riemann theorem for higher bidegree cohomology groups, the hard Lefschetz's theorem and the Lefschetz decomposition theorem can also be generalized in the same way.

Note that all these results are due to Khovanskii [28] and Teissier [44] when X is a projective manifold and the c_j's are integral classes. A linear version was obtained by Timorin in [45]. Gromov proved in [24] that Q_Ω is semi-positive on $P_{\Omega'}$. His result is in fact enough for our purpose.

Corollary 2.2. *Let α, β and c_j be nef classes. Define $\Omega := c_1 \smile \cdots \smile c_{k-2}$. Then, we have*

$$|Q_\Omega(\alpha, \alpha)||Q_\Omega(\beta, \beta)| \leq |Q_\Omega(\alpha, \beta)|^2.$$

Proof. Note that since α, β and the c_j's are nef classes, $Q_\Omega(\alpha, \alpha)$, $Q_\Omega(\beta, \beta)$ and $Q_\Omega(\alpha, \beta)$ are negative. By continuity, we can assume that α, β and the c_j's are Kähler classes. Define $A := |Q(\alpha, \alpha)|$, $B := |Q(\alpha, \beta)|$, $C := |Q(\beta, \beta)|$ and $\Omega' := \Omega \smile c_{k-1}$ for some Kähler class c_{k-1}.

In order to obtain the corollary, we only have to consider the case where α and β are not collinear. So the plane generated by α and β intersects $P_{\Omega'}$ along a real line. Let $a\alpha + b\beta$ be a point in this real line with $(a, b) \neq (0, 0)$. Since Q_Ω is positive defined on $P_{\Omega'}$, we deduce that

$$Aa^2 + 2Bab + Cb^2 = -Q_\Omega(a\alpha + b\beta, a\alpha + b\beta) \leq 0.$$

It follows that $B^2 \geq AC$. □

Let $\mathscr{K}_p$ be the set of all the classes of strictly positive closed (p, p)-forms in $H^{p,p}(X, \mathbb{R})$. Denote by $\mathscr{K}_p^*$ the dual cone of $\mathscr{K}_{k-p}$ with respect to the Poincaré duality. These cones are strictly convex and open. We also have $\mathscr{K} = \mathscr{K}_1$.

Definition 2.3. Let Θ be a class in $\overline{\mathscr{K}}_p^* \setminus \{0\}$ with $p \leq k-2$. We say that Θ is a *weak Hodge-Riemann class* (wHR-class for short) if for all Kähler classes c_j, the quadratic form Q_Ω is semi-positive on $P_{\Omega'}$, where $\Omega := \Theta \smile c_1 \smile \cdots \smile c_{k-p-2}$ and $\Omega' := \Omega \smile c_{k-p-1}$.

By continuity, if Θ is wHR and the c_j's are nef classes such that $\Omega' \neq 0$, then the quadratic form Q_Ω is still semi-positive on the hyperplane $P_{\Omega'}$. Observe also that the set of wHR-classes is closed in $\overline{\mathscr{K}}_p^* \setminus \{0\}$. By Theorem 2.1, if Θ is a product of nef classes or a limit of such products, then it is a wHR-class. The following result is obtained exactly as in Corollary 2.2.

Proposition 2.4. *Let Θ be a wHR class in $\overline{\mathscr{K}}_p^* \setminus \{0\}$. Let c_j be nef classes and define $\Omega := \Theta \smile c_1 \smile \cdots \smile c_{k-p-2}$. Then, we have*

$$|Q_\Omega(\alpha, \alpha)||Q_\Omega(\beta, \beta)| \leq |Q_\Omega(\alpha, \beta)|^2.$$

For any class $\Theta \in \overline{\mathscr{K}}_p^*$, denote by $\overline{\mathscr{K}}(\Theta)$ the closure of $\Theta \smile \mathscr{K}$ in $\overline{\mathscr{K}}_{p+1}^*$. We call it *the nef cone relative to* Θ or *the* Θ*-nef cone.* It is closed, strictly convex and contained in the linear space $\Theta \smile H^{1,1}(X, \mathbb{R})$. It contains the cone $\Theta \smile \overline{\mathscr{K}}$ and can be strictly larger than this cone; in other words, $\Theta \smile \overline{\mathscr{K}}$ is not always closed. Observe that if Θ is a wHR-class, so are the classes in $\overline{\mathscr{K}}(\Theta) \setminus \{0\}$.

Proposition 2.5. *Let $\pi : X' \to X$ be a holomorphic map between compact Kähler manifolds. If Θ' is a wHR-class on X' such that $\pi_*(\Theta') \neq 0$, then $\pi_*(\Theta')$ is a wHR-class on X. In particular, if Θ is the class of an irreducible analytic subset of X then Θ is wHR.*

Proof. If c is a nef class on X then $\pi^*(c)$ is a nef class on X'. Therefore, the first assertion is a direct consequence of Definition 2.3. For the second assertion, assume that Θ is the class of an irreducible analytic subset of X. If the analytic set is smooth, it is enough to apply Theorem 2.1 to this manifold. Otherwise, we use a resolution of singularities and the first assertion allows us to reduce the problem to the smooth case. □

Lemma 2.6. *Let Θ be a class in $\overline{\mathscr{K}}^*_p$, M a class in $H^{q,q}(X,\mathbb{R})$ and Θ' a class in $\overline{\mathscr{K}}(\Theta)$. Assume that $\Theta \smile M$ is in $\overline{\mathscr{K}}^*_{p+q}$ and write $\Theta' = \Theta \smile L$ with $L \in H^{1,1}(X,\mathbb{R})$. Then $\Theta \smile M \smile L$ is a class in $\overline{\mathscr{K}}(\Theta \smile M)$ and it does not depend on the choice of L.*

Proof. The independence of the choice of L is obvious. Since Θ' is in $\overline{\mathscr{K}}(\Theta)$ there is a sequence of Kähler classes L_n such that $\Theta \smile L_n \to \Theta \smile L$. We deduce that $\Theta \smile M \smile L_n$ converge to $\Theta \smile M \smile L$. Therefore, the last class belongs to $\overline{\mathscr{K}}(\Theta \smile M)$. □

Definition 2.7. Let Θ and Θ' be two classes in $H^{p,p}(X,\mathbb{R})$. We say that they are *numerically almost equivalent* and we write $\Theta \simeq_n \Theta'$ if

$$(\Theta - \Theta') \smile c_1 \smile \ldots \smile c_{k-p} = 0$$

for all classes $c_j \in H^{1,1}(X,\mathbb{R})$.

Note that for Θ in $\overline{\mathscr{K}}^*_p$ we have $\Theta \not\simeq_n 0$ if and only if $\Theta \neq 0$. We will need the following proposition which uses the ideas from [16, 18, 50].

Proposition 2.8. *Let Θ be a wHR-class in $\overline{\mathscr{K}}^*_p$. Let Θ_1 and Θ_2 be two classes in $\overline{\mathscr{K}}(\Theta)$. Write $\Theta_j = \Theta \smile L_j$ with $L_j \in H^{1,1}(X,\mathbb{R})$. Assume that $\Theta \smile L_1 \smile L_2 = 0$. Then, we also have $\Theta \smile L_1^2 = \Theta \smile L_2^2 = 0$. Moreover, there is a pair of real numbers $(t_1,t_2) \neq (0,0)$ such that $\Theta \smile (t_1L_1 + t_2L_2) \simeq_n 0$.*

Proof. By Lemma 2.6, the classes $\Theta \smile L_1 \smile L_2$ and $\Theta \smile L_j^2$ belong to $\overline{\mathscr{K}}^*_{p+2}$ and depend only on Θ_j but not on the choice of L_j. Observe also that we only need to consider the case where Θ_1 and Θ_2 are linearly independent. So Θ_1, Θ_2 belong to $\overline{\mathscr{K}}^*_{p+1} \setminus \{0\}$ and L_1, L_2 are linearly independent. Denote by H the real plane in $H^{1,1}(X,\mathbb{R})$ generated by L_1 and L_2.

Let $c_1, \ldots, c_{k-p-1}$ be Kähler classes. Define $\Omega := \Theta \smile c_1 \smile \cdots \smile c_{k-p-2}$ and $\Omega' := \Omega \smile c_{k-p-1}$. We deduce from the hypothesis on $\Theta \smile L_1 \smile L_2$ and the definition of Q_Ω that Q_Ω is semi-negative on H. Since Θ is a wHR-class, Q_Ω is semi-positive on $P_{\Omega'}$. It follows that Q_Ω vanishes on $H \cap P_{\Omega'}$.

Consider a pair of real numbers $(t_1,t_2) \neq (0,0)$ such that $t_1L_1 + t_2L_2$ belongs to $P_{\Omega'}$. We have

$$\Theta \smile (t_1L_1 + t_2L_2)^2 \smile c_1 \smile \ldots \smile c_{k-p-2} = 0.$$

Hence,

$$\Theta \smile (t_1^2L_1^2 + t_2^2L_2^2) \smile c_1 \smile \ldots \smile c_{k-p-2} = 0.$$

Since $\Theta \smile L_j^2 \in \overline{\mathscr{K}}^*_{p+2}$, we conclude that $\Theta \smile L_j^2 = 0$ if $t_j \neq 0$.

Recall that we suppose $\Theta_j \in \overline{\mathscr{K}}^*_{p+1} \setminus \{0\}$. Therefore, L_j does not belong to $P_{\Omega'}$. Thus, $t_j \neq 0$ and $\Theta \smile L_j^2 = 0$. We can assume that $t_1 = 1$ and write $t := t_2$. The number t is the unique real number such that $L_1 + tL_2 \in H \cap P_{\Omega'}$. By Cauchy-Schwarz's inequality applied to the restriction of Q_Ω to $P_{\Omega'}$, we have $Q_\Omega(L_1 + tL_2, c) = 0$ for every c in the hyperplane $P_{\Omega'}$. This together with the first assertion in the proposition implies that $Q_\Omega(L_1 + tL_2, c) = 0$ for every $c \in H^{1,1}(X, \mathbb{R})$. Then, it follows from the Poincaré duality that

$$\Theta \smile (L_1 + tL_2) \smile c_1 \smile \ldots \smile c_{k-p-2} = 0.$$

In order to obtain the last assertion in the proposition, it remains to check that t is independent of c_j. By definition, t depends symmetrically on the c_j's. However, the last identity, which is stronger than the property $L_1 + tL_2 \in P_{\Omega'}$, shows that it does not depend on c_{k-p-1}. We conclude that t does not depend on c_j for every j. This completes the proof of the proposition. $\square$

3 Topological entropy and dynamical degrees

The topological entropy of a map is a dynamical invariant. In the case of holomorphic automorphisms of a compact Kähler manifold, results by Gromov and Yomdin imply that the topological entropy is in fact an algebraic invariant.

Let f be a holomorphic automorphism of X. It defines a dynamical system on X. Denote by $f^n := f \circ \cdots \circ f$, n times, the iterate of order n of f. If x is a point of X, the orbit of x is the sequence of points

$$x, f(x), f^2(x), \ldots, f^n(x), \ldots$$

The topological entropy of f measures the divergence of the orbits or in some sense it measures the rate of expansion of the "number" of orbits one can distinguish when the time n goes to infinity. The formal definition is given below.

Definition 3.1. Let $\epsilon > 0$ and $n \in \mathbb{N}$. Two points x and y in X are said to be (n, ϵ)*-separated* if we have for some integer $0 \leq j \leq n-1$

$$\mathrm{dist}(f^j(x), f^j(y)) > \epsilon.$$

Definition 3.2. Let $N(\epsilon, n)$ denote the maximal number of points mutually (n, ϵ)-separated. *The (topological) entropy* of f is given by the formula

$$h_t(f) := \sup_{\epsilon > 0} \limsup_{n \to \infty} \frac{\log N(\epsilon, n)}{n} = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log N(\epsilon, n)}{n} \cdot$$

Note that the notion of separated points depends on the metric on X but the topological entropy is independent of the choice of the metric. So the topological entropy is a topological invariant.

The pull-back operator f^* on differential forms induces a graded automorphism of the Hodge cohomology ring of X

$$f^* : \bigoplus_{0\leq p,q\leq k} H^{p,q}(X,\mathbb{C}) \to \bigoplus_{0\leq p,q\leq k} H^{p,q}(X,\mathbb{C}).$$

A similar property holds for the de Rham cohomology ring which can be identified to the real part of the Hodge cohomology ring. We have a graded automorphism

$$f^* : \bigoplus_{0\leq m\leq 2k} H^m(X,\mathbb{R}) \to \bigoplus_{0\leq m\leq 2k} H^m(X,\mathbb{R}).$$

The last operator preserves a lattice on which it is identified to

$$f^* : \bigoplus_{0\leq m\leq 2k} \frac{H^m(X,\mathbb{Z})}{\text{torsion}} \to \bigoplus_{0\leq m\leq 2k} \frac{H^m(X,\mathbb{Z})}{\text{torsion}}.$$

In particular, on a suitable basis, the map $f^* : H^m(X,\mathbb{R}) \to H^m(X,\mathbb{R})$ is given by a square matrix with integer entries. When X is a projective manifold, we can also consider the action of f on Néron-Severi groups.

The following algebraic invariants were implicitly considered in Gromov [25], see also [17, 21, 39].

Definition 3.3. We call *dynamical degree of order p of f* the spectral radius $d_p(f)$ of the linear morphism $f^* : H^{p,p}(X,\mathbb{C}) \to H^{p,p}(X,\mathbb{C})$ and we call *algebraic entropy of f* the number

$$h_a(f) := \max_{0\leq p\leq k} \log d_p(f).$$

It follows from the above discussion that each $d_p(f)$ is a root of a monic polynomial with integer coefficients. In particular, it is an algebraic number.

If ω is a Kähler form on X, it is not difficult to see that the dynamical degrees can by computed with the formula

$$\begin{aligned} d_p(f) &= \lim_{n\to\infty} \Big[\int_X (f^n)^*(\omega^p)\wedge\omega^{k-p}\Big]^{1/n} = \lim_{n\to\infty} \Big[(f^n)^*\{\omega\}^p \smile \{\omega\}^{k-p}\Big]^{1/n} \\ &= \lim_{n\to\infty} \Big[\int_X \omega^p\wedge(f^n)_*(\omega^{k-p})\Big]^{1/n} = \lim_{n\to\infty} \Big[\{\omega\}^p \smile (f^n)_*\{\omega\}^{k-p}\Big]^{1/n}. \end{aligned} \tag{3.1}$$

We see that $d_0(f) = d_k(f) = 1$ and $d_p(f) = d_{k-p}(f^{-1})$.

The following result is a consequence of a theorem by Gromov [25] and another theorem by Yomdin [47]. It shows that the topological entropy of a holomorphic automorphism can be computed algebraically.

Theorem 3.4. *We have*

$$h_t(f) = h_a(f).$$

Gromov theorem implies that $h_t(f) \leq h_a(f)$. A similar property holds for all meromorphic self-maps on compact Kähler manifolds, see [17].

Yomdin theorem says that if $V \subset X$ is a real manifold, smooth up to the boundary, then the volume growth of the sequence $f^n(V)$, $n \geq 0$, is bounded by $h_t(f)$. More precisely, we have

$$h_t(f) \geq \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{vol} f^n(V),$$

where we use the m-dimensional volume $\mathrm{vol}(\cdot)$ with $m := \dim_{\mathbb{R}} V$. Yomdin's theorem holds for all smooth maps on compact real manifolds.

Applying Yomdin theorem to real compact manifolds without boundary in X, we obtain that

$$h_t(f) \geq \log \rho_m(f)$$

if $\rho_m(f)$ is the spectral radius of $f^* : H^m(X, \mathbb{R}) \to H^m(X, \mathbb{R})$. In particular, we obtain the reverse of the above Gromov's inequality.

Let $\rho_{p,q}(f)$ denote the spectral radius of $f^* : H^{p,q}(X, \mathbb{C}) \to H^{p,q}(X, \mathbb{C})$. Arguing as above, we obtain

$$h_t(f) \geq \log \rho_{p,q}(f).$$

This together with Gromov's inequality yields

$$\rho_{p,q}(f) \leq \max_{0\leq p\leq k} d_p(f).$$

In fact, the following more general result holds, see [13].

Proposition 3.5. *We have for* $0 \leq p, q \leq k$

$$\rho_{p,q}(f) \leq \sqrt{d_p(f) d_q(f)}.$$

This inequality also explains why the dynamical degrees $d_p(f)$ play a more important role than the degrees $\rho_{p,q}(f)$, $p \neq q$, in the dynamical study of f. A similar property holds for general meromorphic self-maps on compact Kähler manifolds.

The following result is a direct consequence of Corollary 2.2 and the identity (3.1).

Proposition 3.6. *The function* $p \mapsto \log d_p(f)$ *is concave in* p*, that is,*

$$d_p(f)^2 \geq d_{p-1}(f) d_{p+1}(f) \quad \textit{for} \quad 1 \leq p \leq k-1.$$

In particular, there are two integers $0 \leq r \leq s \leq k$ *such that*

$$1 = d_0(f) < \cdots < d_r(f) = \cdots = d_s(f) > \cdots > d_k(f) = 1.$$

We obtain the following corollary, see [16].

Corollary 3.7. *The automorphism f has positive entropy if and only if $d_1(f) > 1$ (resp. $d_{k-1}(f) > 1$). Moreover, in this case, there is a number $A > 1$ depending only on the second Betti number of X such that*

$$h_t(f) \geq \log A \quad \textit{and} \quad d_p(f) \geq A \quad \textit{for} \quad 1 \leq p \leq k-1.$$

Proof. The first assertion is a consequence of Theorem 3.4 and Proposition 3.6. Assume now that f has positive entropy. We have $d_p(f) > 1$ for $1 \leq p \leq k-1$. It suffices to prove that $d_1(f) \geq A$ and $d_{k-1}(f) \geq A$. We only have to check the first inequality since $d_{k-1}(f) = d_1(f^{-1})$.

It follows from Proposition 3.5 that $d_1(f)$ is the spectral radius of $f^* : H^2(X, \mathbb{R}) \to H^2(X, \mathbb{R})$. So it is the largest root of a monic polynomial of degree b_2 with integer coefficients, where $b_2 := \dim H^2(X, \mathbb{R})$ denotes the second Betti number of X. If this polynomial admits a coefficient of absolute value larger than $2^{b_2} b_2!$ then it admits a root of modulus larger than 2. In this case, we have $d_1(f) \geq 2$. Otherwise, the polynomial belongs to a finite family and hence $d_1(f)$ belongs to a finite set depending only on b_2. The result follows. □

4 Fibrations and relative dynamical degrees

We will consider the restriction of an automorphism to analytic sets which may be singular. In general, a resolution of singularities gives us maps which are no more holomorphic. So it is useful to extend the notion of dynamical degrees to meromorphic maps.

For the moment, let (X, ω) be a compact Kähler manifold. Let $f : X \to X$ be a meromorphic map which is dominant, i.e. its image contains a Zariski open subset of X. *The dynamical degree of order* p of f is defined by

$$d_p(f) = \lim_{n\to\infty} \Big[\int_X (f^n)^*(\omega^p) \wedge \omega^{k-p} \Big]^{1/n}. \tag{4.1}$$

It is not difficult to see that the definition does not depend on the choice of ω. The existence of the above limit is not obvious. It is based on some result on the regularization of positive closed currents [17], see also [10]. Dynamical degrees are bi-meromorphic invariants. More precisely, we have the following result, see [17].

Theorem 4.1. *Let f and g be dominant meromorphic self-maps on compact Kähler manifolds X and Y respectively, of the same dimension k. Let $\pi : X \to Y$ be a bi-meromorphic map. Assume that $g \circ \pi = \pi \circ f$. Then, we have $d_p(f) = d_p(g)$ for $0 \leq p \leq k$.*

So we can extend the notion of dynamical degrees to meromorphic maps on varieties by using a resolution of singularities. Note that Proposition 3.6 still holds in this case except we only have $d_k(f) \geq 1$ with equality when f is a bi-meromorphic map.

The last theorem can be viewed also as a consequence of Theorem 4.2 below. Consider now a dominant meromorphic map $g : Y \to Y$, where Y is a compact Kähler manifold of dimension $l \leq k$. Let $\pi : X \to Y$ a dominant meromorphic map and assume as above that $g \circ \pi = \pi \circ f$. So f preserves the meromorphic fibration defined by π.

We can define *the dynamical degree of f relative to the fibration* by

$$d_p(f|\pi) := \lim_{n\to\infty} \Big[\int_{\pi^{-1}(y)} (f^n)^*(\omega^p) \wedge \omega^{k-l-p} \Big]^{1/n},$$

where y is a generic point in Y. The definition does not depend on the generic choice of y and the function $p \mapsto \log d_p(f|\pi)$ is concave on p. The following result relates the dynamical degrees of f and the ones of g, see [15].

Theorem 4.2. *Let f, g, π be as above. Then, we have for $0 \leq p \leq k$*

$$d_p(f) = \max_{\max(0,p-k+l)\leq s\leq \min(p,l)} d_s(g) d_{p-s}(f|\pi).$$

Note that the domain of s in the last formula is exactly the set of s such that $d_s(g)$ and $d_{p-s}(f|\pi)$ are meaningful. In the case where $k = l$, we necessarily have $s = p$ and $d_0(f|\pi) = 1$. So the last formula implies Theorem 4.1. We will also apply the last theorem to the case of pluricanonical fibrations of X.

Let K_X denote the canonical line bundle of X. Let $H^0(X, K_X^N)$ denote the space of holomorphic sections of K_X^N and $H^0(X, K_X^N)^*$ its dual space. Assume that $H^0(X, K_X^N)$ has a positive dimension. If x is a generic point in X, the family H_x of sections which vanish at x is a hyperplane of $H^0(X, K_X^N)$ passing through 0. So the correspondence $x \mapsto H_x$ defines a meromorphic map

$$\pi_N : X \to \mathbb{P}H^0(X, K_X^N)^*$$

from X to the projectivization of $H^0(X, K_X^N)^*$ which is called a *pluricanonical fibration* of X. Let Y_N denote the image of X by π_N. The *Kodaira dimension* of X is $\kappa_X := \max_{N\geq 1} \dim Y_N$. When $H^0(X, K_X^N) = 0$ for every $N \geq 1$, the Kodaira dimension of X is defined to be $-\infty$. We have the following result, see [35, 42].

Theorem 4.3. *Let $f : X \to X$ be a dominant meromorphic map. Assume that $\kappa_X \geq 1$. Then f preserves the pluricanonical fibration $\pi_N : X \to Y_N$. Moreover, the map $g_N : Y_N \to Y_N$ induced by f is periodic, i.e. $g_N^m = \mathrm{id}$ for some integer $m \geq 1$.*

We deduce that $d_p(g_N) = 1$ for every p. This property can also be deduced from a weaker property that g_N is the restriction to Y_N of a linear map on $\mathbb{P}H^0(X, K_X^N)^*$ which is a consequence of the definition of g_N. The following result is a consequence of Corollary 3.7 and Theorem 4.2.

Corollary 4.4. *Let f be a holomorphic automorphism of X. Assume that $0 \leq \kappa_X \leq k-1$. Let Y_N, π_N, g_N be as above. Then*

$$h_t(f) = \max_{1\leq p\leq k-\dim Y_N -1} d_p(f|\pi_N) \quad \textit{and} \quad d_1(f) = d_1(f|\pi_N).$$

In particular, f has positive entropy if and only if $d_1(f|\pi_N) \geq A$, where $A > 1$ is the constant given in Corollary 3.7.

5 Tits alternative for automorphism groups

We are now ready to give the proof of Theorem 1.1. We first recall two important results due to Fujiki and Lieberman [22, 30].

Theorem 5.1. *Let c be a Kähler class on X. Denote by $\mathrm{Aut}_c(X)$ the group of elements g of $\mathrm{Aut}(X)$ such that $g^*(c) = c$. Then $\mathrm{Aut}_c(X)$ is a finite union of connected components of $\mathrm{Aut}(X)$.*

Let $\mathrm{Alb}(X)$ denote the Albanese torus of X and $\phi : X \to \mathrm{Alb}(X)$ the Albanese map. The identity component of the automorphism group of $\mathrm{Alb}(X)$ is denoted by $A(X)$. This is the group of translations on $\mathrm{Alb}(X)$ which is isomorphic to $\mathrm{Alb}(X)$ as complex Lie groups. It is not difficult to see that any automorphism g of X induces an automorphism h of $\mathrm{Alb}(X)$ such that $h \circ \phi = \phi \circ g$. So we have a natural Lie group morphism

$$\psi : \mathrm{Aut}_0(X) \to A(X).$$

Theorem 5.2. *The kernel $\ker(\psi)$ of ψ is a linear algebraic $\mathbb{C}$-group.*

We recall also the following version of Tits' theorem [46].

Theorem 5.3. *Let G be a linear $\mathbb{R}$-group. Then, it satisfies the Tits alternative, that is, any subgroup of G either has a free non-abelian subgroup or virtually solvable, i.e. possesses a solvable subgroup of finite index.*

The following result gives us the first assertion of Theorem 1.1.

Theorem 5.4. *The group $\mathrm{Aut}(X)$ satisfies the Tits alternative.*

We first prove a preliminary lemma.

Lemma 5.5. *Let A be a group and B a normal subgroup of A. Assume that B and A/B are virtually solvable. Then A is virtually solvable.*

Proof. Let $\pi : A \to A/B$ be the canonical group morphism. If D is a solvable finite index subgroup of A/B, we can replace A by $\pi^{-1}(D)$ in order to assume that A/B is solvable. Let

$$\{1\} = D_0 \lhd D_1 \lhd \cdots \lhd D_{m-1} \lhd D_m = A/B$$

be a subnormal series such that D_j is normal in A/B and D_{j+1}/D_j is abelian for every $0 \leq j \leq m-1$. We can use here the derived series of A/B.

By lattice theorem (correspondence theorem), there is a subnormal series

$$B = B_0 \lhd B_1 \lhd \cdots \lhd B_{m-1} \lhd B_m = A$$

such that B_j is normal in A and $B_j/B_{j-1} = D_j/D_{j-1}$ for $1 \leq j \leq m$. Recall that B is virtually solvable. We will show that B_1 satisfies the same property and then using a simple induction, we obtain that A is virtually solvable. So in order to simplify the notation, we can assume that $m = 1$ or equivalently A/B is abelian.

Let C be a solvable finite index subgroup of B. We can replace C by the intersection of bCb^{-1} with $b \in B$ in order to assume that C is normal in B. Without loss of generality, we can also assume that C is a maximal normal solvable subgroup of B with finite index. The maximality and the lattice theorem imply that B/C admits no solvable normal subgroup different from $\{1\}$. Since B is normal in A, we have $a^{-1}Ca \subset B$ for every $a \in A$. We claim that C is normal in A, i.e. $a^{-1}Ca = C$ for every $a \in A$.

Taking into account this property, we first complete the proof of the lemma. Observe that if a is an element of A then $b \mapsto a^{-1}ba$ induces an automorphism of the group B/C. Let A' denote the set of all elements $a \in A$ such that the above automorphism is identity. Since B/C is a finite group, A' is a finite index subgroup of A.

Since A/B is abelian, $A'' := [A', A']$ is a subgroup of B. By construction, if a' is an element of A' and b an element of B, then $[a', b]$ is an element of C. We deduce that $[A'', A'']$ is a subgroup of C; in particular, it is solvable. Thus, A' is solvable. It remains to prove the above claim.

Define $D := a^{-1}Ca$. Since $b \mapsto a^{-1}ba$ is an automorphism of B, D is a maximal normal solvable subgroup of B with finite index and B/D is isomorphic to B/C. So it suffices to check that $D \subset C$. The natural short exact sequence

$$\{1\} \longrightarrow D \longrightarrow B \longrightarrow B/D \longrightarrow \{1\}$$

induces the following one

$$\{1\} \longrightarrow \frac{D}{C \cap D} \xrightarrow{\pi_1} \frac{B}{C \cap D} \longrightarrow \frac{B}{D} \longrightarrow \{1\}.$$

Similarly, we have

$$\{1\} \longrightarrow \frac{C}{C \cap D} \longrightarrow \frac{B}{C \cap D} \xrightarrow{\pi_2} \frac{B}{C} \longrightarrow \{1\}.$$

Since π_1 is injective and π_2 is surjective, the image of $\pi_2 \circ \pi_1$ is a normal subgroup of B/C. On the other hand, this subgroup should be solvable since D is solvable. We deduce from the maximality of C that the image of $\pi_2 \circ \pi_1$ is equal to $\{1\}$. Hence, $D \subset C$. This completes the proof of the lemma. $\square$

Proof of Theorem 5.4. Let G be a subgroup of $\mathrm{Aut}(X)$ which does not contain any free non-abelian subgroup. We have to show that G admits a solvable subgroup of finite index. Consider the natural group morphism

$$\rho : \mathrm{Aut}(X) \to \mathrm{GL}(H^2(X, \mathbb{R})).$$

By Theorem 5.3, $\rho(G)$ is virtually solvable. By Lemma 5.5, we only have to check that $G \cap \ker \rho$ is virtually solvable.

By Theorem 5.1, $G \cap \ker(\rho)$ is a finite extension of $G \cap \mathrm{Aut}_0(X)$. So we only have to check that $G \cap \mathrm{Aut}_0(X)$ is virtually solvable. Since $\psi(G \cap \mathrm{Aut}_0(X))$ is abelian, by Lemma 5.5, it suffices to show that $\ker \psi \cap G$ is virtually solvable. But this is a consequence of Theorems 5.2 and 5.3. □

We now turn to the proof of the second assertion in Theorem 1.1. Let G be a group as in this theorem. By Theorem 5.4, G is virtually solvable. We will need the following version of the Lie-Kolchin theorem due to Keum-Oguiso-Zhang [27].

Theorem 5.6. *Let H be a virtually solvable group acting linearly on a strictly convex closed cone C of finite dimension. Then, H admits a finite index subgroup H' and a non-zero vector $v \in C$ such that the half-line $\mathbb{R}_+ v$ is invariant by H'.*

This result was obtained by induction on the derived length of a suitable finite index solvable subgroup of G. The case where G is abelian is a consequence of the classical Perron-Frobenius theorem.

Observe that in the case $\kappa_X = k$ the second assertion in Theorem 1.1 is a direct consequence of Theorem 4.3 since in this case every automorphism has zero entropy. Assume now that $\kappa_X \leq k-1$. Fix now an integer N such that $\dim Y_N = \kappa_X$, where Y_N is as defined in Section 4. In order to simplify the notation, define $\pi := \pi_N$, $Y := Y_N$ and $\kappa := \max(\kappa_X, 0)$. If $\kappa_X = -\infty$, we consider that Y is a point. Let Θ_κ denote the class of a generic fiber of π. In general, the generic fibers of π are not necessarily irreducible. However, by Stein's factorization theorem [23, Ch. 10.6], their irreducible components have the same cohomology class. Therefore, by Proposition 2.5, Θ_κ is a wHR-class in $\overline{\mathscr{K}}^*_\kappa \setminus \{0\}$. By Theorem 4.3, this class is fixed under the action of $\mathrm{Aut}(X)$.

Lemma 5.7. *There is a finite index subgroup G' of G such that for every $\kappa \leq p \leq k-1$, there exists a wHR-class Θ_p in $\overline{\mathscr{K}}^*_p \setminus \{0\}$ and a character $\chi_p : G' \to \mathbb{R}^*$ of G' such that $g^*(\Theta_p) = \chi_p(g)\Theta_p$ for $g \in G'$. Moreover, we have $\Theta_p \in \overline{\mathscr{K}}(\Theta_{p-1})$ when $p \geq \kappa + 1$.*

Proof. We construct Θ_p by induction on p. The class Θ_κ was already constructed above and we can take $G' = G$. Assume that Θ_{p-1} was constructed. Then, G' induces an affine action on a basis of the strictly convex cone $\overline{\mathscr{K}}(\Theta_{p-1})$. By Theorem 5.6, replacing G' by a suitable finite index subgroup, we can find a class $\Theta_p \in \overline{\mathscr{K}}(\Theta_{p-1}) \setminus \{0\}$ whose direction is invariant by G'. Since Θ_{p-1} is a wHR-class, Θ_p is also a wHR-class. □

Consider the group morphism $\phi : G' \to \mathbb{R}^{k-\kappa-1}$ given by

$$\phi(g) := \big(\log \chi_{\kappa+1}(g), \ldots, \log \chi_{k-1}(g)\big).$$

The following lemma will permit to show that $\mathrm{Im}(\phi)$ is discrete.

Lemma 5.8. *We have $\|\phi(g)\| \geq \frac{1}{2}\log d_{k-1}(g)$ for all $g \in G'$.*

Proof. Assume that $\|\phi(g)\| < \frac{1}{2}\log d_{k-1}(g)$ for some $g \in G'$. Then, we have

$$d_{k-1}(g)^{-1/2} < \chi_p(g) < d_{k-1}(g)^{1/2} \tag{5.1}$$

for every p. Recall that $d_{k-1}(g) = d_1(g^{-1}) = d_1(g^{-1}|\pi)$, see Corollary 4.4. Let Θ_p be as in Lemma 5.7 and write $\Theta_p = \Theta_{p-1} \smile L_p$ with some class $L_p \in H^{1,1}(X,\mathbb{R})$.

Since g^{-1} preserves $\overline{\mathscr{K}}(\Theta_\kappa)$, it follows from the classical Perron-Frobenius theorem that there is a class $\Theta \in \overline{\mathscr{K}}(\Theta_\kappa) \setminus \{0\}$ depending on g such that

$$(g^{-1})^*(\Theta) = d_1(g^{-1}|\pi)\Theta = d_{k-1}(g)\Theta$$

or equivalently

$$g^*(\Theta) = d_{k-1}(g)^{-1}\Theta.$$

Write $\Theta = \Theta_\kappa \smile L$ with $L \in H^{1,1}(X,\mathbb{R})$.

By Lemma 2.6, $\Theta_p \smile L$ does not depend on the choice of L and it is not difficult to see that

$$g^*(\Theta_p \smile L) = \chi_p(g) d_{k-1}(g)^{-1}\Theta_p \smile L. \tag{5.2}$$

Since $g^* = \mathrm{id}$ on $H^{k,k}(X,\mathbb{R})$ and $\chi_{k-1}(g)d_{k-1}(g)^{-1} \neq 1$, we deduce that $\Theta_{k-1} \smile L = 0$. Let $q \leq k-1$ be the smallest integer such that $\Theta_q \smile L = 0$.

Since Θ belongs to $\overline{\mathscr{K}}^*_{\kappa+1} \setminus \{0\}$, we have $\Theta \not\simeq_n 0$. Therefore, we have $q \geq \kappa+1$. We have $\Theta_{q-1} \smile L_q \smile L = 0$. By Proposition 2.8, there is a pair of real numbers $(t_1, t_2) \neq (0,0)$ such that

$$\Theta_{q-1} \smile (t_1 L_q + t_2 L) \simeq_n 0.$$

Using the action of g^* and the relation (5.2), we obtain that

$$\Theta_{q-1} \smile \big(t_1\chi_q(g)L_q + t_2\chi_{q-1}(g)d_{k-1}(g)^{-1}\big) \simeq_n 0.$$

The last two identities together with (5.1) yield $\Theta_{q-1} \smile L \simeq_n 0$. By Lemma 2.6, $\Theta_{q-1} \smile L$ belongs to $\overline{\mathscr{K}}^*_q$. Thus, $\Theta_{q-1} \smile L = 0$. This contradicts the minimality of q. The lemma follows. □

End of the proof of Theorem 1.1. When $h_t(g) = 0$, the spectral radius of g^* on $\oplus H^m(X,\mathbb{R})$ is equal to 1. Since g^* is given by a matrix with integer entries, we deduce that all eigenvalues of g^* have modulus 1. It follows that $\phi(g) = 0$. Conversely, if $\phi(g) = 0$, by Lemma 5.8, $d_{k-1}(g) = 1$; thus, by Corollary 3.7, we get $h_t(g) = 0$. So we have $N' = \ker\phi$. In particular, N' is a normal subgroup of G'. The group G'/N' is isomorphic to $\phi(G')$. By Corollary 3.7 and Lemma 5.8, $\phi(G')$ is a discrete subset of $\mathbb{R}^{k-\kappa-1}$. So G'/N' is a free abelian group of rank $\leq k-\kappa-1$. This finishes the proof of the theorem. □

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A FINITENESS THEOREM FOR GALOIS REPRESENTATIONS OF FUNCTION FIELDS OVER FINITE FIELDS (AFTER DELIGNE)

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Abstract. We give a detailed account of Deligne's letter [13] to Drinfeld dated June 18, 2011, in which he shows that there are finitely many irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaves with bounded ramification, up to isomorphism and up to twist, on a smooth variety defined over a finite field. The proof relies on Lafforgue's Langlands correspondence over curves [27]. In addition, Deligne shows the existence of affine moduli of finite type over $\mathbb{Q}$. A corollary of Deligne's finiteness theorem is the existence of a number field which contains all traces of the Frobenii at closed points, which was the main result of [12] and which answers positively his own conjecture [9, Conj. 1.2.10 (ii)].

1. Introduction

In Weil II [9, Conj. 1.2.10] Deligne conjectured that if X is a normal connected scheme of finite type over a finite field, and V is an irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaf of rank r, with finite determinant, then

(i) V has weight 0,

(ii) there is a number field $E(V) \subset \bar{\mathbb{Q}}_\ell$ containing all the coefficients of the local characteristic polynomials $\det(1 - tF_x|_{V_x})$, where x runs through the closed points of X and F_x is the geometric Frobenius at the point x,

(iii) V admits ℓ'-companions for all prime numbers $\ell' \neq p$.

As an application of his Langlands correspondence for GL_r, Lafforgue [27] proved (i), (ii), (iii) for X a smooth curve, out of which one deduces (i) in general. Using Lafforgue's results, Deligne showed (ii) in [12]. Using (ii) and ideas of Wiesend, Drinfeld [15] showed (iii) assuming in addition X to be smooth. A slightly more elementary variant of Deligne's argument for (ii) was given in [18].

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Those conjecture were formulated with the hope that a more motivic statement could be true, which would say that those lisse sheaves come from geometry. On the other hand, over smooth varieties over the field of complex numbers, Deligne in [11] showed finiteness of geometric variations of pure Hodge structures of bounded rank, a theorem which, in weight one, is due to Faltings [19]. Those are always regular singular, while lisse $\bar{\mathbb{Q}}_\ell$-sheaves are not necessarily tame. However, any lisse sheaf has bounded ramification (see Proposition 3.9 for details). Furthermore, one may twist a lisse $\bar{\mathbb{Q}}_\ell$-sheaf by a character coming from the ground field. Thus it is natural to expect:

Theorem 1.1 (Deligne)**.** *There are only finitely many irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaves up to twist on X with suitably bounded ramification at infinity.*

Deligne shows this theorem in [13] by extending his arguments from [12]. A precise formulation is given in Theorem 2.1 based on the ramification theory explained in Section 3.3.

Our aim in this note is to give a detailed account of Deligne's proof of this finiteness theorem for lisse $\bar{\mathbb{Q}}_\ell$-sheaves and consequently of his proof of (ii). For some remarks on the difference between our method and Deligne's original argument for proving (ii) in [12] see Section 2.4.

In fact Deligne shows a stronger finiteness theorem which comprises finiteness of the number of what we call *generalized sheaves* on X. A generalized sheaf consists of an isomorphism class of a semi-simple lisse $\bar{\mathbb{Q}}_\ell$-sheaf on every smooth curve mapping to X, which are assumed to be compatible in a suitable sense. These generalized sheaves were first studied by Drinfeld [15]. His main theorem roughly says that if a generalized sheaf is tame at infinity along each curve then it comes from a lisse sheaf on X, extending the rank one case treated in [35], [36]. Deligne suggests that a more general statement should be true:

Question 1.2. Does any generalized sheaf with bounded ramification come from a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X?

For a precise formulation of the question see Question 2.3. The answer to this question is not even known for rank one sheaves, in which case the problem has been suggested already earlier in higher dimensional class field theory. On the other hand Deligne's finiteness for generalized sheaves has interesting consequences for relative Chow groups of 0-cycles over finite fields, see Section 2.3.

Some comments on the proof of the finiteness theorem: Deligne uses in a crucial way his key theorem [12, Prop. 2.5] on curves asserting that a semi-simple lisse $\bar{\mathbb{Q}}_\ell$-sheaf is uniquely determined by its characteristic polynomials of the Frobenii at all closed points of some explicitly bounded degree, see Theorem 5.1. This enables him to construct a coarse moduli space of generalized sheaves $L_r(X, D)$ as an affine scheme of finite type over $\mathbb{Q}$, such that its $\bar{\mathbb{Q}}_\ell$-points correspond to the generalized sheaves of rank r and bounded ramification by the given divisor D at infinity.

We simplify Deligne's construction of the moduli space slightly. Our method yields less information on the resulting moduli, yet it is enough to deduce the finiteness theorem. In fact finiteness is seen by showing that irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaves up to twist are in bijection with (some of) the one-dimensional irreducible components of the moduli space (Corollary 7.2).

We give some applications of Deligne's finiteness theorem.

Firstly, it implies the existence of a number field $E(V)$ as in (ii) above, see Theorem 2.6. This number field is in fact stable by an ample hyperplane section if X is projective, see Proposition 7.4.

Secondly, as mentioned above the degree zero part of the relative Chow group of 0-cycles with bounded modulus is finite (Theorem 2.5).

Deligne addresses the question of the number of irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaves with bounded ramification. In [14] some concrete examples on the projective line minus a divisor of degree ≤ 4 are computed. In Section 8 we formulate Deligne's qualitative conjecture. This formulation rests on emails he sent us and on his lecture in June 2012 in Orsay on the occasion of the Laumon conference.

Acknowledgment: Our note gives an account of the 9 dense pages written by Deligne to Drinfeld [13]. They rely on [12] and [15] and contain a completely new idea of great beauty, to the effect of showing finiteness by constructing moduli of finite type and equating the classes of the sheaves one wants to count with some of the irreducible components. We thank Pierre Deligne for his willingness to read our note and for his many enlightening comments.

Parts of the present note are taken from our seminar note [18]. They grew out of discussions at the Forschungsseminar at Essen during summer 2011. We thank all participants of the seminar.

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2. The finiteness theorem and some consequences

2.1. **Deligne's finiteness theorem (weak form).** We begin by formulating a version of Deligne's finiteness theorem for ℓ-adic Galois representations of functions fields. Later in this section we introduce the notion of a generalized ℓ-adic representation, which is necessary in order to state a stronger form of Deligne's finiteness result. We also explain applications to a conjecture from Weil II [9, Conj. 1.2.10 (ii)] and to Chow groups of 0-cycles.

Let $\mathrm{Sm}_{\mathbb{F}_q}$ be the category of smooth separated schemes $X/\mathbb{F}_q$ of finite type over the finite field $\mathbb{F}_q$. We fix once for all an algebraic closure $\mathbb{F} \supset \mathbb{F}_q$. To $X \in \mathrm{Sm}_{\mathbb{F}_q}$ connected one associates functorially the *Weil group* $W(X)$ [9, 1.1.7], a topological group, well-defined up to an inner automorphism by $\pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F})$ when X is geometrically connected over $\mathbb{F}_q$. If so, then it sits in an exact sequence

$$0 \to \pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F}) \to W(X) \to W(\mathbb{F}_q) \to 0.$$

There is a canonical identification $W(\mathbb{F}_q) = \mathbb{Z}$.

We fix a prime number ℓ with $(\ell, q) = 1$. Let $\mathcal{R}_r(X)$ be the set of lisse $\bar{\mathbb{Q}}_\ell$-Weil sheaves on X of dimension r up to isomorphism and up to semi-simplification. For X connected, a lisse $\bar{\mathbb{Q}}_\ell$-Weil sheaf on X is the same as a continuous representations $W(X) \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$. As we do not want to talk about a topology on $\bar{\mathbb{Q}}_\ell$ we define the latter continuous representations ad hoc as the homomorphisms which factor through a continuous homomorphism $W(X) \to \mathrm{GL}_r(E)$ for some finite extension E of $\mathbb{Q}_\ell$, see [9, (1.1.6)].

The weak form of the finiteness theorem says that the number of classes of irreducible sheaves in $\mathcal{R}_r(X)$ with bounded wild ramification is finite up to twist. Let us give some more details. Let $X \subset \bar{X}$ be a normal compactification of the connected scheme X such that $\bar{X} \setminus X$ is the support of an effective Cartier divisor on $\bar{X}$. Let $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor with support in $\bar{X} \setminus X$. In Section 3.3 we will define a subset $\mathcal{R}_r(X, D)$ of representations whose Swan conductor along any smooth curve mapping to $\bar{X}$ is bounded by the pullback of D to the completed curve. We show that for any $V \in \mathcal{R}_r(X)$ there is a divisor D with $V \in \mathcal{R}_r(X, D)$, see Proposition 3.9.

For $V \in \mathcal{R}_r(X, D)$ we have the notion of twist $\chi \cdot V$ by an element $\chi \in \mathcal{R}_1(\mathbb{F}_q)$.

Theorem 2.1 (Deligne)**.** *Let $X \in \mathrm{Sm}_{\mathbb{F}_q}$ be connected and $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor with support in $\bar{X} \setminus X$. The set of irreducible sheaves $V \in \mathcal{R}_r(X, D)$ is finite up to twist by elements of $\mathcal{R}_1(\mathbb{F}_q)$.*

In particular the theorem implies that for any integer $N > 0$ there are only finitely many irreducible $V \in \mathcal{R}_r(X, D)$ with $\det(V)^{\otimes N} = 1$. Theorem 2.1 is a consequence of the stronger Finiteness Theorem 2.4.

Remark 2.2. Any irreducible lisse Weil sheaf on X is a twist of an étale sheaf, Proposition 4.3. So the theorem could also be stated with étale sheaves instead of Weil sheaves.

2.2. **Existence problem and strong finiteness.** By $\mathrm{Cu}(X)$ we denote the set of normalizations of closed integral subschemes of X of dimension one.

We say that a family $(V_C)_{C\in \mathrm{Cu}(X)}$ with $V_C \in \mathcal{R}_r(C)$ is *compatible* if for all pairs (C, C') we have

$$V_C|_{(C\times_X C')_{\mathrm{red}}} = V_{C'}|_{(C\times_X C')_{\mathrm{red}}} \in \mathcal{R}_r((C\times_X C')_{\mathrm{red}}).$$

We write $\mathcal{V}_r(X)$ for the set of compatible families – also called *generalized sheaves*.

It is not difficult to see that the canonical map $\mathcal{R}_r(X) \to \mathcal{V}_r(X)$ is injective, Proposition 4.1. One might ask, what is the image of $\mathcal{R}_r(X)$ in $\mathcal{V}_r(X)$.

With the notation as above we can also define the set $\mathcal{V}_r(X, D)$ of generalized sheaves with bounded wild ramification, see Definition 3.6. Deligne expresses the hope that the following question about existence of ℓ-adic sheaves might have a positive answer.

Question 2.3. Is the map $\mathcal{R}_r(X, D) \to \mathcal{V}_r(X, D)$ bijective for any Cartier divisor $D \in \mathrm{Div}^+(\bar{X})$ with support in $\bar{X} \setminus X$?

To motivate the question one should think of the set of curves $\mathrm{Cu}(X)$ together with the systems of intersections of curves as the 2-skeleton of X. To be more precise, the analogy is as follows: For a CW-complex S let $S_{\le d}$ be the union of i-cells of S ($i \le d$), i.e. its d-skeleton. Assume that $S_{\le 0}$ consists of just one point.

CW-complex S (with $S_{\le 0} = \{*\}$)	Variety $X/\mathbb{F}_q$
1-sphere S^1 with topological fundamental group $\pi_1(S^1) = \mathbb{Z}$	Finite field $\mathbb{F}_q$ with Weil group $W(\mathbb{F}_q) = \mathbb{Z}$
S^1-bouquet $S_{\le 1}$	Set of closed points $\lvert X\rvert$
2-cell in S	Curve in $\mathrm{Cu}(X)$
Relation in $\pi_1(S)$ coming from 2-cell	Reciprocity law on curve
2-skeleton $S_{\le 2}$	System of curves $\mathrm{Cu}(X)$
Local system on S	Lisse $\bar{\mathbb{Q}}_\ell$-Weil sheaf on X

In the sense of this analogy, Deligne's Question 2.3 is the analog of the fact that the fundamental groups of S and $S_{\le 2}$ are the same [23, Thm. 4.23], except that we consider only the information contained in ℓ-adic representations, in addition only modulo semi-simplification, and that there is no analog of wild ramification over CW-complexes.

For $D = 0$ a positive answer to Deligne's question is given by Drinfeld [15, Thm 2.5]. His proof uses a method developed by Wiesend [36] to reduce the problem to Lafforgue's theorem. For $r = 1$ and $D = 0$ it was first shown by Schmidt–Spiess [35] using motivic cohomology, and later by Wiesend [37] using more elementary methods.

The strong form of Deligne's finiteness theorem says that Theorem 2.1 remains true for generalized sheaves. We say that a *generalized sheaf* $V \in \mathcal{V}_r(X)$ *on a connected scheme* X is *irreducible* if it cannot be written in the from $V_1 \oplus V_2$ with $V_i \in \mathcal{V}_{r_i}(X)$ and $r_1, r_2 > 0$. In Appendix B, Proposition B.1, we give a proof of the well known fact that a sheaf $V \in \mathcal{R}_r(X)$ is irreducible if and only if its image in $\mathcal{V}_r(X)$ is irreducible.

The main result of this note now says:

Theorem 2.4 (Deligne)**.** *Let $X \in \mathrm{Sm}_{\mathbb{F}_q}$ be connected and $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor supported in $\bar{X}\setminus X$. The set of irreducible generalized sheaves $V \in \mathcal{V}_r(X, D)$ is finite up to twist by elements from $\mathcal{R}_1(\mathbb{F}_q)$.*

The theorem implies in particular that for a given integer $N > 0$ there are only finitely many $V \in \mathcal{V}_r(X, D)$ with $\det(V)^{\otimes N} = 1$. Following Deligne we will reduce the theorem to the one-dimensional case, where it is a well known consequence of the Langlands correspondence of Drinfeld–Lafforgue. Some hints how the one-dimensional case is related to the theory of automorphic forms are given in Section 4.3. The proof of Theorem 2.4 is completed in Section 7.

Idea of proof. The central idea of Deligne is to define an algebraic moduli space structure on the set $\mathcal{V}_r(X, D)$, such that it becomes an affine scheme of finite type over $\mathbb{Q}$. In fact $\mathcal{V}_r(X, D)$ will be the $\bar{\mathbb{Q}}_\ell$-points of this moduli space. One shows that the irreducible components of the moduli space over $\bar{\mathbb{Q}}_\ell$ are 'generated' by certain twists of generalized sheaves, which implies the finiteness theorem, because there are only finitely many irreducible components.

Firstly, one constructs the moduli space structure of finite type over $\mathbb{Q}$ for $\dim(X) = 1$. Then one immediately gets an algebraic structure on $\mathcal{V}_r(X, D)$ in the higher dimensional case and the central point is to show that $\mathcal{V}_r(X, D)$ is of finite type over $\mathbb{Q}$ for higher dimensional X too.

The main method to show the finite type property is a result of Deligne (Theorem 5.1), relying on Weil II and the Langlands correspondence, which says that for one-dimensional X there is a natural number N depending logarithmically on the genus of $\bar{X}$ and the degree of D such that $V \in \mathcal{V}_r(X, D)$ is determined by the polynomials $f_V(x)$ with $\deg(x) \le N$. Here for $V \in \mathcal{V}_r(X, D)$ we denote by $f_V(x)$ the characteristic polynomial of the Frobenius at the closed point $x \in |X|$, see Section 4.1 for a precise definition.

2.3. **Application: Finiteness of relative Chow group of 0-cycles.** It was shown by Colliot-Thélène–Sansuc–Soulé [8] and by Kato–Saito [24] that over a finite field, the Chow group of 0-cycles of degree 0 of a proper variety is finite.

Assume now that $X \subset \bar{X}$ is a compactification as above and let $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor with support in $\bar{X} \setminus X$. For a curve $C \in \mathrm{Cu}(X)$ and an effective divisor $E \in \mathrm{Div}^+(\bar{C})$ with support in $\bar{C} \setminus C$, where $\bar{C}$ is the smooth compactification of C, let

$$P_{k(C)}(E) = \{g \in k(C)^\times | \mathrm{ord}_x(1-g) \ge \mathrm{mult}_x(E) + 1 \text{ for } x \in \bar{C} \setminus C\}$$

be the unit group with modulus well known from the ideal theoretic version of global class field theory. Set

$$\mathrm{CH}_0(X, D) = Z_0(X)/\mathrm{im}[\bigoplus_{C \in \mathrm{Cu}(X)} P_{k(C)}(\bar{\phi}^* D)].$$

Here $\bar{\phi} : \bar{C} \to \bar{X}$ is the extension of the natural morphism $\phi : C \to X$. A similar Chow group of 0-cycles is used in [17], [31] to define generalized Albanese varieties. For $D = 0$ and $\bar{X} \setminus X$ a simple normal crossing divisor it is isomorphic to the Suslin homology group $H_0(X)$ [34]. For $\dim(X) = 1$ it is the classical ideal class group with modulus $D + E$, where E is the reduced divisor with support $\bar{X} \setminus X$.

From Deligne's finiteness Theorem 2.4 and class field theory one immediately obtains a finiteness result which was expected to hold in higher dimensional class field theory.

Theorem 2.5. *For any $D \in \mathrm{Div}^+(\bar{X})$ as above the kernel of the degree map from $\mathrm{CH}_0(X, D)$ to $\mathbb{Z}$ is finite.*

2.4. **Application: Coefficients of characteristic polynomial of the Frobenii at closed points.** In [9, Conjecture 1.2.10] Deligne conjectured that sheaves $V \in \mathcal{R}_r(X)$ with certain obviously necessary properties should behave as if they all came from geometry, i.e. as if they were ℓ-adic realizations of pure motives over X. In particular they should not only be 'defined over' $\bar{\mathbb{Q}}_\ell$, but over $\bar{\mathbb{Q}}$. In this section we explain how this latter conjecture of Deligne (for the precise formulation see Corollary 2.7 below), follows from Theorem 2.4.

In fact Corollary 2.7 is the main result of Deligne's article [12]. His proof uses Bombieri's upper estimates for the ℓ-adic Euler characteristic of an affine variety defined over a finite field, (and Katz' improvement for the Betti numbers) in terms of the embedding dimension, the number and the degree of the defining equations, which rests, aside of Weil II, on Dwork's p-adic methods. In [18] it was observed that one could replace the use of p-adic cohomology theory by some more elementary ramification theory. After this Deligne extended his methods in [13] to obtain the Finiteness Theorem 2.4.

For $V \in \mathcal{V}_r(X)$ and $x \in |X|$ one defines the characteristic polynomial of Frobenius $f_V(x) \in \bar{\mathbb{Q}}_\ell[t]$ at the point x, see Section 4.1. Let $E(V)$ be the subfield of $\bar{\mathbb{Q}}_\ell$ generated by all coefficients of all the polynomials $f_V(x)$ where $x \in |X|$ runs through the closed points.

Theorem 2.6. *Let $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor with support in $\bar{X} \setminus X$. For $V \in \mathcal{V}_r(X, D)$ irreducible with $\det(V)$ of finite order, the field $E(V)$ is a number field.*

In Section 7 we deduce Theorem 2.6 from Theorem 2.4. By associating to $V \in \mathcal{R}_r(X)$ its generalized sheaf in $\mathcal{V}_r(X)$, one finally obtains Deligne's conjecture [12, Conj. 1. 2.10(ii)] from Weil II.

Corollary 2.7. *For $V \in \mathcal{R}_r(X)$ irreducible with $\det(V)$ of finite order the field $E(V)$ is a number field.*

In fact by Proposition 3.9 there is a divisor D such that $V \in \mathcal{R}_r(X, D)$. Then apply Theorem 2.6 to the induced generalized sheaf in $\mathcal{V}_r(X, D)$.

3. Ramification theory

In this section we review some facts from ramification theory. We work over the finite field $\mathbb{F}_q$. In fact all results remain true over a perfect base field of positive characteristic and for lisse étale ℓ-adic sheaves.

3.1. **Local ramification.** We follow [28, Sec. 2.2]. Let K be a complete discretely valued field with perfect residue field of characteristic $p > 0$. Let $G = \mathrm{Gal}(\bar{K}/K)$, where $\bar{K}$ is a separable closure of K. There is a descending filtration $(I^{(\lambda)})_{0 \le \lambda \in \mathbb{R}}$ by closed normal subgroups of G with the following properties:

- $\bigcap_{\lambda' < \lambda} I^{(\lambda')} = I^{(\lambda)}$,
- $\bigcap_{\lambda \in \mathbb{R}} I^{(\lambda)} = 0$,
- $I^{(0+)}$ is the unique maximal pro-p subgroup of the inertia group $I^{(0)}$, where $I^{(\lambda+)}$ is defined as $\overline{\bigcup_{\lambda' > \lambda} I^{(\lambda')}}$.

Let $G \to \mathrm{GL}(V)$ be a continuous representation on a finite dimensional $\bar{\mathbb{Q}}_\ell$-vector space V with $\ell \neq p$.

Definition 3.1. The *Swan conductor* of V is defined as

$$\mathrm{Sw}(V) = \sum_{\lambda > 0} \lambda \dim(V^{I^{(\lambda+)}}/V^{I^{(\lambda)}}).$$

The Swan conductor is additive with respect to extensions of ℓ-adic Galois representations, it does not change if we replace V by its semi-simplification.

For later reference we recall the behavior of the Swan conductor with respect to direct sum and tensor product. If V, V' are two $\bar{\mathbb{Q}}_\ell$-G-modules as above and $V^\vee$ denotes the dual representation, then

$$\mathrm{Sw}(V \oplus V') = \mathrm{Sw}(V) + \mathrm{Sw}(V') \tag{3.1}$$

$$\frac{\mathrm{Sw}(V \otimes V')}{\mathrm{rank}(V)\mathrm{rank}(V')} \le \frac{\mathrm{Sw}(V)}{\mathrm{rank}(V)} + \frac{\mathrm{Sw}(V')}{\mathrm{rank}(V')} \tag{3.2}$$

$$\mathrm{Sw}(V^\vee) = \mathrm{Sw}(V) \tag{3.3}$$

3.2. **Global ramification** (dim = 1)**.** Let $X/\mathbb{F}_q$ be a smooth connected curve with smooth compactification $X \subset \bar{X}$. Let V be in $\mathcal{R}_r(X)$.

The *Swan conductor* $\mathrm{Sw}(V)$ is defined to be the effective Cartier divisor

$$\sum_{x \in |\bar{X}|} \mathrm{Sw}_x(V) \cdot [x] \in \mathrm{Div}^+(\bar{X}).$$

Here $\mathrm{Sw}_x(V)$ is the Swan conductor of the restriction of the representation class V to the complete local field $\mathrm{frac}(\widehat{\mathcal{O}_{\bar{X},x}})$. We say that V is *tame* if $\mathrm{Sw}(V) = 0$.

Clearly the Swan conductor of V is the same as the Swan conductor of any twist $\chi \cdot V$, $\chi \in \mathcal{R}_1(\mathbb{F}_q)$.

Let $\phi : X' \to X$ be an étale covering of smooth curves with compactification $\bar{\phi} : \bar{X}' \to \bar{X}$. By $\mathrm{D}_{\bar{X}'/\bar{X}} \in \mathrm{Div}^+(\bar{X})$ we denote the discriminant [32] of $\bar{X}'$ over $\bar{X}$, cf. Section 3.3.

Lemma 3.2 (Conductor-discriminant-formula). *For $V \in \mathcal{R}_r(X)$ with $\phi^*(V)$ tame the inequality of divisors*

$$\mathrm{Sw}(V) \leq \mathrm{rank}(V)\, \mathrm{D}_{\bar{X}'/\bar{X}}$$

holds on $\bar{X}$.

Proof. By abuse of notation we write V also for a sheaf representing V. There is an injective map of sheaves on X

$$V \to \phi_* \circ \phi^*(V)$$

For any $x \in |X|$

$$\mathrm{Sw}_x(V) \leq \mathrm{Sw}_x(\phi_* \circ \phi^*(V)) \leq \mathrm{rank}(V)\, \mathrm{mult}_x(\mathrm{D}_{\bar{X}'/\bar{X}}).$$

The second inequality follows from [30, Prop. 1(c)]. □

Definition 3.3. Let $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor. The subset $\mathcal{R}_r(X, D) \subset \mathcal{R}_r(X)$ is defined by the condition $\mathrm{Sw}(V) \leq D$. If $V \in \mathcal{R}(X)$ lies in $\mathcal{R}_r(X, D)$, we say that its *ramification is bounded by* D.

Let $\mathbb{F}_{q^n}$ be the algebraic closure of $\mathbb{F}_q$ in $k(X)$.

Definition 3.4. For a divisor $D \in \mathrm{Div}^+(\bar{X})$ we define the *complexity* of D to be

$$\mathcal{C}_D = 2g(\bar{X}) + 2\deg_{\mathbb{F}_{q^n}}(D) + 1,$$

where $g(\bar{X})$ is the genus of $\bar{X}$ over $\mathbb{F}_{q^n}$ and $\deg_{\mathbb{F}_{q^n}}$ is the degree over $\mathbb{F}_{q^n}$. Here we assume that X is geometrically connected.

Proposition 3.5. *Assume $X/\mathbb{F}_q$ is geometrically connected. For $D \in \mathrm{Div}^+(\bar{X})$ with $\mathrm{supp}(D) = \bar{X} \setminus X$ and for $V \in \mathcal{R}_r(X, rD)$, the inequality*

$$\dim_{\bar{\mathbb{Q}}_\ell} H^0_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, V) + \dim_{\bar{\mathbb{Q}}_\ell} H^1_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, V) \leq \mathrm{rank}(V)\, \mathcal{C}_D$$

holds.

Proof. Grothendieck-Ogg-Shafarevich theorem says that

$$\chi_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, V) = (2 - 2g(\bar{X}))\,\mathrm{rank}(V) - \sum_{x \in \bar{X} \setminus X} (\mathrm{rank}(V) + s_x(V)),$$

see [28, Théorème 2.2.1.2]. Furthermore

$$\dim\, H^0_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, V) \le r \text{ and}$$
$$\dim\, H^2_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, V) = \dim\, H^0(X \otimes_{\mathbb{F}_q} \mathbb{F}, V^\vee) \le r.$$

□

3.3. **Global ramification** (dim ≥ 1)**.** We follow in idea of Alexander Schmidt for the definition of the discriminant for higher dimensional schemes.

Let X be a connected scheme in $\mathrm{Sm}_{\mathbb{F}_q}$. Let $X \subset \bar{X}$ be a normal compactification of X over k such that $\bar{X} \setminus X$ is the support of an effective Cartier divisor on $\bar{X}$. Clearly, such a compactification always exists.

Let $\mathrm{Cu}(X)$ be the set of normalizations of closed integral subschemes of X of dimension one. For C in $\mathrm{Cu}(X)$ denote by $\phi : C \to X$ the natural morphism. By $\bar{C}$ we denote the smooth compactification of C over $\mathbb{F}_q$ and by $\bar{\phi} : \bar{C} \to \bar{X}$ we denote the canonical extension.

Recall that in Section 2 we introduced the set of lisse $\bar{\mathbb{Q}}_\ell$-Weil sheaves $\mathcal{R}_r(X)$ and of generalized sheaves $\mathcal{V}_r(X)$ on X of rank r.

Definition 3.6. For $V \in \mathcal{R}_r(X)$ or $V \in \mathcal{V}_r(X)$ and $D \in \mathrm{Div}^+(\bar{X})$ an effective Cartier with support in $\bar{X} \setminus X$ we (formally) write $\mathrm{Sw}(V) \le D$ and say that the *ramification of V is bounded by D* if for every curve $C \subset \mathrm{Cu}(X)$ we have

$$\mathrm{Sw}(\phi^*(V)) \le \bar{\phi}^*(D)$$

in the sense of Section 3.2. The subsets $\mathcal{R}_r(X, D) \subset \mathcal{R}_r(X)$ and $\mathcal{V}(X, D) \subset \mathcal{V}(X)$ are defined by the condition $\mathrm{Sw}(V) \le D$.

In the rest of this section we show that for any $V \in \mathcal{R}_r(X)$ there is an effective divisor D with $\mathrm{Sw}(V) \le D$.

Let $\psi : X' \to X$ be an étale covering (thus finite) and let $\bar{\psi} : \bar{X}' \to \bar{X}$ be the finite, normal extension of X' over $\bar{X}$.

Definition 3.7 (A. Schmidt)**.** The *discriminant* $\mathcal{I}(\mathrm{D}_{\bar{X}'/\bar{X}})$ is the coherent ideal sheaf in $\mathcal{O}_{\bar{X}}$ locally generated by all elements

$$\det(\mathrm{Tr}_{K'/K}(x_i\, x_j))_{i,j}$$

where $x_1, \ldots, x_n \in \psi_*(\mathcal{O}_{\bar{X}'})$ are local sections restricting to a basis of K' over K. Here $K = k(X)$ and $K' = k(X')$.

Clearly, $\mathcal{I}(\mathrm{D}_{\bar{X}'/\bar{X}})|_X = \mathcal{O}_X$. This definition extends the classical definition for curves [32], in which case $\mathcal{I}(\mathrm{D}_{\bar{X}'/\bar{X}}) = \mathcal{O}_{\bar{X}}(-\mathrm{D}_{\bar{X}'/\bar{X}})$, where $X \subset \bar{X}$ and $X' \subset \bar{X}'$ are the smooth compactifications.

The following lemma is easy to show.

Lemma 3.8 (Semi-continuity). *In the situation of Definition 3.7 let $\bar{\phi} : \bar{C} \to \bar{X}$ be a smooth curve mapping to $\bar{X}$ with $C = \bar{\phi}^{-1}(X)$ non-empty. Let C' be a connected component of $C \times_X X'$ and let $C' \hookrightarrow \bar{C}'$ be the smooth compactification. Then*

$$\bar{\phi}^{-1}(\mathcal{I}(\mathrm{D}_{\bar{X}'/\bar{X}})) \subset \mathcal{O}_{\bar{C}}(-\mathrm{D}_{\bar{C}'/\bar{C}}).$$

Proposition 3.9. *For $V \in \mathcal{R}_r(X)$ there is an effective Cartier divisor $D \in \mathrm{Div}^+(\bar{X})$ such that $\mathrm{Sw}(V) \leq D$.*

Proof. By Remark 2.2 we can assume that V is an étale sheaf on X. Then there is a local field $E \subset \bar{\mathbb{Q}}_\ell$ finite over $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_E$ such that V comes form an ℓ-adic $\mathcal{O}_E$-sheaf V_1. Let $\hat{E}$ be the finite residue field of $\mathcal{O}_E$. There is a connected étale covering $\psi : X' \to X$ such that $\psi^*(V_1 \otimes_{\mathcal{O}_E} \hat{E})$ is trivial. This implies that $\psi^*(V)$ is tame. Let $D_1 \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor with support in $\bar{X} \setminus X$ such that $\mathcal{O}_{\bar{X}}(-D_1) \subset \mathcal{I}(\mathrm{D}_{\bar{X}'/\bar{X}})$ and set $D = \mathrm{rank}(V) D_1$. With the notation of Lemma 3.8 we obtain

$$\bar{\phi}^*(D_1) \geq \mathrm{D}_{\bar{C}'/\bar{C}}$$

As the pullback of V to C' is tame we obtain from Lemma 3.2 the first inequality in

$$\mathrm{Sw}(\phi^*(V)) \leq \mathrm{rank}(V)\mathrm{D}_{\bar{C}'/\bar{C}} \leq \bar{\phi}^*(D).$$

□

Remark 3.10. We do not know any example for a $V \in \mathcal{V}_r(X)$ for which there does *not* exist a divisor D with $\mathrm{Sw}(V) \leq D$. If such an example existed, it would in particular show, in view of Proposition 3.9, that not all generalized sheaves are actual sheaves.

We conclude this section by a remark on the relation of our ramification theory with the theory of Abbes-Saito [4]. We expect that for $V \in \mathcal{R}_r(X)$, $\mathrm{Sw}(V) \leq D$ is equivalent to the following: For every open immersion $X \subset X_1$ over $\mathbb{F}_q$ with the property that $X_1 \setminus X$ is a simple normal crossing divisor and for any morphism $h : X_1 \to \bar{X}$, the Abbes-Saito log-ramification Swan conductor of $h^*(V)$ at a maximal point of $X_1 \setminus X$ is $\leq$ the multiplicity of $h^*(D)$ at the maximal point.

For $D = 0$ this equivalence is shown in [26] relying on [36]. For $r = 1$ it is known modulo resolution of singularities by work of I. Barrientos (forthcoming PhD thesis, Universität Regensburg).

4. ℓ-adic sheaves

4.1. **Basics.** For $X \in \mathrm{Sm}_{\mathbb{F}_q}$ we defined in Section 2 the set $\mathcal{R}_r(X)$ of lisse $\bar{\mathbb{Q}}_\ell$-Weil sheaves on X of rank r up to isomorphism and up to semi-simplification and the set $\mathcal{V}_r(X)$ of generalized sheaves. Clearly, $\mathcal{R}_r$ and $\mathcal{V}_r$ form contravariant functors from $\mathrm{Sm}_{\mathbb{F}_q}$ to the category of sets.

For $V \in \mathcal{R}_r(X)$ taking characteristic polynomial of Frobenius defines a function

$$f_V : |X| \to \bar{\mathbb{Q}}_\ell[t], \quad f_V(x) = \det(1 - t\, F_x, V_{\bar{x}}).$$

For $V \in \mathcal{V}_r(X)$ we can still define $f_V(x)$ by choosing a curve $C \in \mathrm{Cu}(X)$ such that $C \to X$ is a closed immersion in a neighborhood of x and we set $f_V(x) = f_{V_C}(x)$. It follows from the definition that $f_V(x)$ does not depend on the choice of C.

We define the trace

$$t_V^n : X(\mathbb{F}_{q^n}) \to \bar{\mathbb{Q}}_\ell, \quad t_V^n(x) = \mathrm{tr}(F_x, V_{\bar{x}})$$

for $V \in \mathcal{R}_r(X)$ and similarly for $V \in \mathcal{V}_r(X)$.

We define $\mathcal{P}_r$ to be the affine scheme over $\mathbb{Q}$ whose points $\mathcal{P}_r(A)$ with values in a $\mathbb{Q}$-algebra A consist of the set of polynomials $1 + a_1 t + \cdots + a_r t^r \in A[t]$ with $a_r \in A^\times$. Mapping $(\alpha_i)_{1 \le i \le r}$ with $\alpha_i \in A^\times$ to

$$(1 - \alpha_1 t) \cdots (1 - \alpha_r t) \in A[t]$$

defines a scheme isomorphism

$$\mathbb{G}_m^r / S_r \xrightarrow{\simeq} \mathcal{P}_r, \tag{4.1}$$

where S_r is the permutation group of r elements.

For $d \ge 1$ the finite morphism $\mathbb{G}_m^r \to \mathbb{G}_m^r$ which sends $(\alpha_1, \ldots, \alpha_r)$ to $(\alpha_1^d, \ldots, \alpha_r^d)$ descends to $\mathcal{P}_r$ to define the finite scheme homomorphism $\psi_d : \mathcal{P}_r \to \mathcal{P}_r$.

Let $\mathcal{L}_r(X)$ be the product $\prod_{|X|} \mathcal{P}_r$ with one copy of $\mathcal{P}_r$ for every closed point of X. It is an affine scheme over $\mathbb{Q}$ which if $\dim(X) \ge 1$ is not of finite type over $\mathbb{Q}$. Denote by $\pi_x : \mathcal{L}_r(X) \to \mathcal{P}_r$ the projection onto the factor corresponding to $x \in |X|$. We make $\mathcal{L}_r$ into a contravariant functor from $\mathrm{Sm}_{\mathbb{F}_q}$ to the category of affine schemes over $\mathbb{Q}$

as follows: Let $f : Y \to X$ be a morphism of schemes in $\mathrm{Sm}_{\mathbb{F}_q}$. The image of $(P_x)_{x\in|X|} \in \mathcal{L}_r(A)$ under pullback by f is defined to be

$$\left(\psi_{[k(y):k(f(y))]} P_{f(x)}\right)_{y\in|Y|} \in \mathcal{L}_r(A).$$

For $N > 0$ we similarly define $\mathcal{L}_r^{\le N}(X)$ to be the product over all $x \in |X|$ with $\deg(x) \le N$ over $\mathbb{F}_q$, with the corresponding forgetful morphism $\mathcal{L}_r(X) \to \mathcal{L}_r^{\le N}(X)$.

Putting things together we get morphisms of contravariant functors

$$\mathcal{R}_r(X) \longrightarrow \mathcal{V}_r(X) \xrightarrow{\kappa:V\mapsto f_V} \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell). \tag{4.2}$$

Proposition 4.1. *For $X \in \mathrm{Sm}_{\mathbb{F}_r}$ the maps $\mathcal{R}_r(X) \to \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell)$ and $\mathcal{V}_r(X) \xrightarrow{\kappa} \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell)$ are injective.*

Proof. We only have to show the injectivity for $\mathcal{R}_r(X)$, since the curve case for $\mathcal{R}_r(X)$ implies already the general case for $\mathcal{V}_r(X)$. We can easily recover the trace functions t_V^n from the characteristic polynomials f_V. The Chebotarev density theorem [20, Ch. 6] implies that the traces of Frobenius determine semi-simple sheaves, see [28, Thm. 1.1.2]. □

In Section 5 we will prove a much stronger result, saying that a finite number of characteristic polynomials $f_V(x)$ are sufficient to recover V up to twist, as long as V runs over ℓ-adic sheaves with some fixed bounded ramification and fixed rank.

For later reference we recall the relation between Weil sheaves and étale sheaves from Weil II [9, Prop. 1.3.4]. We say that $V \in \mathcal{R}_r(X)$ is *étale* if it comes from a lisse étale $\bar{\mathbb{Q}}_\ell$-sheaf on X.

Proposition 4.2. *For X connected and $V \in \mathcal{R}_1(X)$, which we consider as a continuous homomorphism $V : W(X) \to \bar{\mathbb{Q}}_\ell^\times$, the geometric monodromy group $\mathrm{im}(\pi_1(X_{\bar{\mathbb{F}}})) \subset W(X)/\ker(V)$ is finite, in particular the monodromy group $W(X)/\ker(V)$ is discrete. The sheaf V extends to a continuous homomorphism $\pi_1(X) \to \bar{\mathbb{Q}}_\ell^\times$, i.e. V is étale, if and only if $\mathrm{im}(V) \subset \bar{\mathbb{Z}}_\ell^\times$.*

Proposition 4.3. *For X connected an irreducible $V \in \mathcal{R}_r(X)$ is étale if and only if its determinant $\det(V)$ is étale. In particular there is always a twist $\chi \cdot V$ with $\chi \in \mathcal{R}_1(\mathbb{F}_q)$ which is étale.*

4.2. **Implications of Langlands.** In this section we recall some consequences of the Langlands correspondence of Drinfeld and Lafforgue [27] for the theory of ℓ-adic sheaves.

The following theorem is shown in [27, Théorème VII.6].

Theorem 4.4. *For $X \in \mathrm{Sm}_{\mathbb{F}_q}$ connected of dimension one and for $V \in \mathcal{R}_r(X)$ irreducible with determinant of finite order the following holds:*

(i) *For an arbitrary, not necessarily continuous, automorphism $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$, there is a $V_\sigma \in \mathcal{R}_r(X)$, called σ-companion, such that*
$$f_{V_\sigma} = \sigma(f_V),$$
where σ acts on the polynomial ring $\bar{\mathbb{Q}}_\ell[t]$ by σ on $\bar{\mathbb{Q}}_\ell$ and by $\sigma(t) = t$.

(ii) *V is pure of weight 0.*

Later, we deduce from the theorem that σ-companions exist for arbitrary $V \in \mathcal{R}_r(X)$ in dimension one, not necessarily of finite determinant, see Corollary 4.7.

For $\dim(X)$ arbitrary and $V \in \mathcal{R}_1(X)$, which we consider as a continuous homomorphism $V : W(X) \to \bar{\mathbb{Q}}_\ell^\times$, the σ-companion V_σ simply corresponds to the continuous map $\sigma \circ V : W(X) \to \bar{\mathbb{Q}}_\ell^\times$. In fact $\sigma \circ V$ is continuous, because $W(X)/\ker(V)$ is discrete by Proposition 4.2.

From Lafforgue's theorem one can deduce certain results on higher dimensional schemes.

Corollary 4.5. *Let X be a connected scheme in $\mathrm{Sm}_{\mathbb{F}_q}$ of arbitrary dimension. For an irreducible $V \in \mathcal{R}_r(X)$ the following are equivalent:*

(i) *V is pure of weight 0,*
(ii) *there is a closed point $x \in X$ such that $V_{\bar{x}}$ is pure of weight 0,*
(iii) *there is $\chi \in \mathcal{R}_1(\mathbb{F}_q)$ pure of weight 0 such that the determinant $\det(\chi \cdot V)$ is of finite order.*

Proof. (iii) $\Rightarrow$ (i):
For a closed point $x \in X$ choose a curve C/k and a morphism $\phi : C \to X$ such that x is in the set theoretic image of ϕ and such that $\phi^* V$ is irreducible. A proof of the existence of such a curve is given in an appendix, Proposition B.1. Then by Theorem 4.4 the sheaf $\phi^* V$ is pure of weight 0 on C, so $V_{\bar{x}}$ is also pure of weight 0.

(i) $\Rightarrow$ (ii): Trivially.

(ii) $\Rightarrow$ (iii):
Choose $\chi \in \mathcal{R}_1(\mathbb{F}_q)$ such that $(\chi|_{k(x)})^{\otimes r} = \det(V_{\bar{x}})^\vee$. By Proposition 4.2 it follows that the determinant $\det(\chi \cdot V)$ has finite order. □

Let $\mathcal{W}$ be the quotient of $\bar{\mathbb{Q}}_l^\times$ modulo the numbers of weight 0 in the sense of [9, Def. 1.2.1] (algebraic numbers all complex conjugates of which have absolute value 1).

Corollary 4.6. *A sheaf $V \in \mathcal{R}_r(X)$, resp. a generalized sheaf $V \in \mathcal{V}_r(X)$, can be decomposed uniquely as a sum*

$$V = \bigoplus_{w \in \mathcal{W}} V_w$$

with the property that $V_w \in \mathcal{R}_r(X)$, resp. $V_w \in \mathcal{V}_r(X)$, such that for each point $x \in |X|$, all eigenvalues of the Frobenius F_x on V_w lie in the class w.

Corollary 4.7. *Assume $\dim(X) = 1$. For $V \in \mathcal{R}_r(X)$ and an automorphism $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$, there is a σ-companion to V, i.e. $V_\sigma \in \mathcal{R}_r(X)$ such that*

$$f_{V_\sigma} = \sigma(f_V).$$

Proof. Without loss of generality we may assume that V is irreducible. In the same way as in the proof of Corollary 4.5 we find $\chi \in \mathcal{R}_1(\mathbb{F}_q)$ such that $\chi \cdot V$ has determinant of finite order. A σ-companion of $\chi \cdot V$ exists by Theorem 4.4 and a σ-companion of χ exists by the remarks below Theorem 4.4. As the formation of σ-companions is compatible with tensor products, $V_\sigma = (V \cdot \chi)_\sigma \cdot (\chi_\sigma)^\vee$ is a σ-companion of V. $\square$

Deligne showed a compatibility result [10, Thm. 9.8] for the Swan conductor of σ-companions.

Proposition 4.8. *Let V and V_σ be σ-companions on a one-dimensional $X \in \mathrm{Sm}_{\mathbb{F}_q}$ as in as in Corollary 4.7. Then $\mathrm{Sw}(V) = \mathrm{Sw}(V_\sigma)$.*

Recall from (4.2) that there is a canonical injective map of sets $\mathcal{V}_r(X) \xrightarrow{\kappa} \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell)$. In the following corollary we use the notation of Section 3.3.

Corollary 4.9. *For $X \in \mathrm{Sm}_{\mathbb{F}_q}$ and an effective Cartier divisor $D \in \mathrm{Div}^+(\bar{X})$ with support in $\bar{X} \setminus X$ the action of $\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ on $\mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell)$ stabilizes $\alpha(\mathcal{V}_r(X))$ and $\alpha(\mathcal{V}_r(X, D))$.*

Remark 4.10. Drinfeld has shown [15] that Corollary 4.7 remains true for higher dimensional $X \in \mathrm{Sm}_{\mathbb{F}_q}$. His argument relies on Deligne's Theorem 2.6.

4.3. **Proof of Thm. 2.1** (dim = 1)**.** Theorem 2.1 for one-dimensional schemes is a well-known consequence of Lafforgue's Langlands correspondence for GL_r [27]. Let $X \in \mathrm{Sm}_{\mathbb{F}_q}$ be of dimension one with smooth compactification $\bar{X}$, $L = k(X)$. The Langlands correspondence says that there is a natural bijective equivalence between cuspidal automorphic irreducible representations π of $\mathrm{GL}_r(\mathbb{A}_L)$ (with values in $\bar{\mathbb{Q}}_\ell$) and continuous irreducible representations of the Weil group $\sigma_\pi : W(L) \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$, which are unramified almost everywhere. For such an automorphic π one defines an (Artin) conductor $\mathrm{Ar}(\pi) \in \mathrm{Div}^+(X)$ and one constructs an open compact subgroup $K \subset \mathrm{GL}_r(\mathbb{A}_L)$ depending only on $\mathrm{Ar}(\pi)$ such that the space of K invariant vectors of π has dimension one, see [22].

The divisor $\mathrm{Ar}(\pi)$ has support in $\bar{X} \setminus X$ if and only if σ_π is unramified over X. Moreover

$$\mathrm{Sw}_x(\sigma_\pi) + r \geq \mathrm{Ar}_x(\pi)$$

for $x \in |\bar{X}|$.

For an arbitrary compact open subgroup $K \subset \mathrm{GL}_r(\mathbb{A}_L)$ the number of cuspidal automorphic irreducible representations π with fixed central character and which have a non-trivial K-invariant vector is finite by work of Harder, Gelfand and Piatetski-Shapiro, see [29, Thm. 9.2.14].

Via the Langlands correspondence this implies that for given $D \in \mathrm{Div}^+(\bar{X})$ with support in $\bar{X} \setminus X$ and for given $W \in \mathcal{R}_1(X)$ the number of irreducible $V \in \mathcal{R}_r(X)$ with $\det(V) = W$ and with $\mathrm{Sw}(V) \leq D$ is finite. Recall that the determinant of σ_π corresponds to the central character of π via class field theory.

4.4. **Structure of a lisse $\bar{\mathbb{Q}}_\ell$-sheaf over a scheme over a finite field.** Let the notation be as above. The following proposition is shown in [5, Prop. 5.3.9].

Proposition 4.11. *Let V be irreducible in $\mathcal{R}_r(X)$.*

(i) *Let m be the number of irreducible constituents of $V_{\mathbb{F}}$. There is a unique irreducible $V^\flat \in \mathcal{R}_{r/m}(X_{\mathbb{F}_{q^m}})$ such that*
  - *the pullback of $V^\flat$ to $X \otimes_{\mathbb{F}_q} \mathbb{F}$ is irreducible,*
  - *$V = b_{m,*}V^\flat$, where b_m is the natural map $X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m} \to X$.*

(ii) *V is pure of weight 0 if and only if $V^\flat$ is pure of weight 0.*

(iii) *If $V' \in \mathcal{R}_r(X)$ is another sheaf on X with $V'_{\mathbb{F}} = V_{\mathbb{F}}$, then there is a unique sheaf $W \in \mathcal{R}_1(\mathbb{F}_{q^m})$ with*

$$V' = b_{m,*}(V^\flat \otimes W).$$

A special case of the Grothendieck trace formula [28, (1.1.1.3)] says:

Proposition 4.12. *Let V and m be as in Proposition 4.11. For $n \geq 1$ and $x \in X(\mathbb{F}_{q^n})$*

$$t^n_V(x) = \sum_{\substack{y \in X_{\mathbb{F}_{q^m}}(\mathbb{F}_{q^n}) \\ y \mapsto x}} t^n_{V^\flat}(y).$$

Concretely, $t_V^n(x) = 0$ if m does not divide n.

5. Frobenius on curves

We now present Deligne's key technical method for proving his finiteness theorems. It strengthens Proposition 4.1 on curves by allowing us to recover an ℓ-adic sheaf from an effectively determined finite number of characteristic polynomials of Frobenius.

Our notation is explained in Section 2 and Section 4.1. Throughout this section X is a geometrically connected scheme in $\mathrm{Sm}_{\mathbb{F}_q}$ with $\dim(X) = 1$.

Theorem 5.1 (Deligne)**.** *The natural map*

$$\mathcal{R}_r(X, D) \xrightarrow{\kappa_N} \mathcal{L}_r^{\leq N}(X)(\bar{\mathbb{Q}}_\ell)$$

is injective if

$$N \geq 4r^2 \lceil \log_q(2r^2 \mathcal{C}_D) \rceil \tag{5.1}$$

Here for a real number w we let $\lceil w \rceil$ be the smallest integer larger or equal to w. Theorem 5.1 relies on the Langlands correspondence and weight arguments form Weil II. The Langlands correspondence enters via Corollary 4.6.

We deduce Theorem 5.1 from the following trace version, which does not rely on the Langlands correspondence.

Proposition 5.2. *If $V, V' \in \mathcal{R}_r(X, D)$ are pure of weight 0 and satisfy $t_V^n = t_{V'}^n$ for all*

$$n \leq 4r^2 \lceil \log_q(2r^2 \, \mathcal{C}_D) \rceil, \tag{5.2}$$

then $V = V'$.

Prop. 5.2 $\Rightarrow$ Thm. 5.1. Let $V, V' \in \mathcal{R}_r(X, D)$. We write

$$V = \bigoplus_{w \in \mathcal{W}} V_w \quad \text{and} \quad V' = \bigoplus_{w \in \mathcal{W}} V'_w$$

as in Corollary 4.6. The condition $\alpha_N(V) = \alpha_N(V')$ implies $\alpha_N(V_w) = \alpha_N(V'_w)$, thus $t^n_{V_w} = t^n_{V'_w}$ for all $w \in \mathcal{W}$ and all n as in (5.2). By Proposition 5.2, applied to some twist of weight 0 of V_w and V'_w by the same χ, this implies $V_w = V'_w$ for all $w \in \mathcal{W}$. $\square$

5.1. **Proof of Proposition 5.2.** Let J be the set of irreducible $W \in \mathcal{R}_s(X)$, $1 \le s \le r$, which are twists of direct summands of $V \oplus V'$. Set $I = J/\text{twist}$. Choose representative sheaves $S_i \in \mathcal{R}(X)$ which are pure of weight 0 ($i \in I$). In particular this implies that $\mathrm{Hom}_{X \otimes_{\mathbb{F}_q} \mathbb{F}}(S_{i_1}, S_{i_2}) = 0$ for $i_1 \neq i_2 \in I$ by Proposition 4.11. Also for each $i \in I$ we have

$$S_i = b_{m_i,*} S_i^\flat$$

for positive integers m_i and irreducible $S_i^\flat \in \mathcal{R}(X_{\mathbb{F}_{q^{m_i}}})$ with the notation of Proposition 4.11.

It follows from Proposition 4.11 that there are $W_i, W_i' \in \mathcal{R}(\mathbb{F}_{q^{m_i}})$ pure of weight 0 such that

$$V = \bigoplus_{i \in I} b_{m_i,*}(S_i^\flat \otimes_{\bar{\mathbb{Q}}_\ell} W_i)$$

and

$$V' = \bigoplus_{i \in I} b_{m_i,*}(S_i^\flat \otimes_{\bar{\mathbb{Q}}_\ell} W_i').$$

For $n > 0$ set

$$I_n = \{i \in I,\ m_i | n\}.$$

Lemma 5.3. *The functions*

$$t^n_{S_i} : X(\mathbb{F}_{q^n}) \to \bar{\mathbb{Q}}_\ell \qquad (i \in I_n)$$

are linearly independent over $\bar{\mathbb{Q}}_\ell$ for $n \ge 2 \log_q(2r^2 \mathcal{C}_D)$.

Proof. Fix an isomorphism $\iota : \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. Assume we have a linear relation

$$\sum_{i \in I_n} \lambda_i\, t^n_{S_i} = 0,\ \ \lambda_i \in \bar{\mathbb{Q}}_\ell, \tag{5.3}$$

such that not all λ_i are 0. Multiplying by a constant in $\bar{\mathbb{Q}}_\ell^\times$, we may assume that $|\iota(\lambda_{i_\circ})| = 1$ for one $i_\circ \in I_n$ and $|\iota(\lambda_i)| \le 1$ for all $i \in I_n$. Set

$$\langle S_{i_1}, S_{i_2} \rangle_n = \sum_{x \in X(\mathbb{F}_{q^n})} t^n_{Hom(S_{i_1}, S_{i_2})}(x)$$

for $i_1, i_2 \in I_n$. Observe that

$$t^n_{Hom(S_{i_1}, S_{i_2})} = t^n_{S^\vee_{i_1}} \cdot t^n_{S_{i_2}}.$$

Multiplying (5.3) by $t^n_{S^\vee_{i_\circ}}$ and summing over all $x \in X(\mathbb{F}_{q^n})$ one obtains

$$\sum_{i \in I_n} \lambda_i \, \langle S_{i_\circ}, S_i \rangle_n = 0. \tag{5.4}$$

Claim 5.4. One has

(i)
$$|\iota\langle S_{i_\circ}, S_i\rangle_n| \leq \mathrm{rank}(S_{i_\circ})\mathrm{rank}(S_i)\, \mathcal{C}_D\, q^{n/2}$$
for $i \neq i_\circ$,

(ii)
$$|m_{i_\circ}\, q^n - \iota\langle S_{i_\circ}, S_{i_\circ}\rangle_n| \leq \mathrm{rank}(S_{i_\circ})^2\, \mathcal{C}_D\, q^{n/2}.$$

Proof of (i):
By [9, Théorème 3.3.1] the eigenvalues α of F^n on $H^k_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, Hom(S_{i_\circ}, S_{i_\circ}))$ for $k \leq 1$ fulfill
$$|\iota\alpha| \leq q^{n/2}.$$
On the other hand
$$\dim_{\bar{\mathbb{Q}}_\ell}(H^0_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, Hom(S_{i_\circ}, S_i))) + \dim_{\bar{\mathbb{Q}}_\ell}(H^1_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, Hom(S_{i_\circ}, S_i))) \leq \mathrm{rank}(S_{i_\circ})\mathrm{rank}(S_i)\, \mathcal{C}_D$$
by Proposition 3.5. In fact the we have
$$\mathrm{Sw}(Hom(S_{i_\circ}, S_i)) \leq \mathrm{rank}(S_{i_\circ})\mathrm{rank}(S_i) D$$
by (3.1) - (3.3). Under the assumption $i \neq i_\circ$ one has
$$H^2_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, Hom(S_{i_\circ}, S_i)) = \mathrm{Hom}_{X \otimes_{\mathbb{F}_q} \mathbb{F}}(S_i, S_{i_\circ}) \otimes \bar{\mathbb{Q}}_\ell(-1) = 0$$
by Poincaré duality. Putting this together and using Grothendieck's trace formula [28, 1.1.1.3] one obtains (i).

Proof of (ii):
It is similar to (i) but this time we have
$$\dim_{\bar{\mathbb{Q}}_\ell} H^2_c(X \otimes_{\mathbb{F}_q} F, Hom(S_{i_\circ}, S_i)) = m_{i_\circ}$$
and for an eigenvalue α of F^n on
$$H^2_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, Hom(S_{i_\circ}, S_i)) = \mathrm{Hom}_{X \otimes_{\mathbb{F}_q} \mathbb{F}}(S_i, S_{i_\circ}) \otimes \bar{\mathbb{Q}}_\ell(-1)$$
we have $\alpha = q^n$. This finishes the proof of the claim.

Since under the assumption on n from Lemma 5.3
$$\mathcal{C}_D\, \mathrm{rank}(S_{i_\circ}) \sum_{i \in I_n} \mathrm{rank}(S_i) < q^{n/2},$$
we get a contradiction to the linear dependence (5.3).

□

By Proposition 4.12 for any $n \geq 0$ we have

$$t_V^n = \sum_{i \in I_n} t_{W_i}^n \, t_{S_i}^n$$

and

$$t_{V'}^n = \sum_{i \in I_n} t_{W_i'}^n \, t_{S_i}^n .$$

Under the assumption of equality of traces from Theorem 5.2 and using Lemma 5.3 we get

$$\mathrm{Tr}(F^n, W_i) = \mathrm{Tr}(F^n, W_i') \qquad i \in I_n \tag{5.5}$$

for

$$2 \log_q(2r^2 \mathcal{C}_D) \leq n \leq 4r^2 \lceil \log_q(2r^2 \, \mathcal{C}_D) \rceil .$$

In particular this means that equality (5.5) holds for

$$n \in \{m_i \, A, m_i \, (A+1), \ldots, m_i \, (A+2r-1)\},$$

where $A = \lceil 2 \log_q(2r^2 \mathcal{C}_D) \rceil$. So Lemma 5.5 applied to the set $\{b_1, \ldots, b_w\}$ of eigenvalues of F^{m_i} of W_i and W_i' (so $w \leq 2r$) shows that $W_i = W_i'$ for all $i \in I$.

Lemma 5.5. *Let k be a field and consider elements $a_1, \ldots, a_w \in k$, $b_1, \cdots b_w \in k^\times$ such that*

$$F(n) := \sum_{1 \leq j \leq w} a_j \, b_j^n = 0$$

for $1 \leq n \leq w$. Then $F(n) = 0$ for all $n \in \mathbb{Z}$.

Proof. Without loss of generality we can assume that the b_j are pairwise different for $1 \leq j \leq w$. Then the Vandermonde matrix

$$(b_j^n)_{1 \leq j, n \leq w}$$

has non-vanishing determinant, which implies that $a_j = 0$ for all j. $\square$

6. Moduli space of ℓ-adic sheaves

In Section 4.1 we introduced an injective map

$$\kappa : \mathcal{V}_r(X) \to \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell)$$

from the set of generalized ℓ-adic sheaves to the $\bar{\mathbb{Q}}_\ell$-points of an affine scheme $\mathcal{L}_r(X)$ defined over $\mathbb{Q}$, which is not of finite type over $\mathbb{Q}$ if $\dim(X) \geq 1$. Assume that there is a connected normal projective compactification $X \subset \bar{X}$ such that $\bar{X} \setminus X$ is the support of an effective Cartier divisor on $\bar{X}$. We use the notation of Section 4.1.

The existence of the moduli space of ℓ-adic sheaves on X is shown in the following theorem of Deligne.

Theorem 6.1. *For any effective Cartier divisor $D \in \mathrm{Div}^+(\bar{X})$ with support in $\bar{X} \setminus X$ there is a unique reduced closed subscheme $L_r(X, D)$ of $\mathcal{L}_r(X)$ which is of finite type over $\mathbb{Q}$ and such that*

$$L_r(X, D)(\bar{\mathbb{Q}}_\ell) = \kappa(\mathcal{V}_r(X, D)).$$

Uniqueness is immediate from Proposition A.1. In Section 6.2 we construct $L_r(X, D)$ for $\dim(X) = 1$. In Section 6.3 we construct $L_r(X, D)$ for general X. Before we begin the proof we introduce some elementary constructions on $\mathcal{L}_r(X)$.

6.1. **Direct sum and twist as scheme morphisms.** For $r = r_1 + r_2$ the isomorphism

$$\mathbb{G}_m^{r_1} \times_{\mathbb{Q}} \mathbb{G}_m^{r_2} \xrightarrow{\simeq} \mathbb{G}_m^r$$

together with the embedding of groups $\Sigma_{r_1} \times \Sigma_{r_2} \subset \Sigma_{r_1+r_2}$ induces a finite surjective map

$$- \oplus - : \mathcal{P}_{r_1} \times_{\mathbb{Q}} \mathcal{P}_{r_2} \to \mathcal{P}_r, \quad (P, Q) \mapsto PQ \tag{6.1}$$

via the isomorphism (4.1). We call it the direct sum.

There is a twisting action by $\mathbb{G}_m$

$$\mathbb{G}_m \times_{\mathbb{Q}} \mathcal{P}_r \to \mathcal{P}_r, \quad (\alpha, P) \mapsto \alpha \cdot P$$

defined by the diagonal action of $\mathbb{G}_m$ on $\mathbb{G}_m^r$

$$(\alpha, (\alpha_1, \ldots, \alpha_r)) \mapsto (\alpha \cdot \alpha_1, \ldots, \alpha \cdot \alpha_r)$$

and the isomorphism (4.1).

We now extend the direct sum and twist morphisms to $\mathcal{L}(X)$.

By taking direct sum on any factor of $\mathcal{L}(X)$ we get for $r_1 + r_2 = r$ a morphism of schemes over $\mathbb{Q}$

$$- \oplus - : \mathcal{L}_{r_1}(X) \times \mathcal{L}_{r_2}(X) \to \mathcal{L}_r(X) \tag{6.2}$$

Note that the direct sum is not a finite morphism in general, since we have an infinite product over closed points of X.

The twist is an action of $\mathbb{G}_m$

$$\mathbb{G}_m \times_{\mathbb{Q}} \mathcal{L}_r(X) \to \mathcal{L}_r(X) \tag{6.3}$$

given by

$$(\alpha, (P_x)_{x \in |X|}) \mapsto \alpha \cdot (P_x)_{x \in |X|} = (\alpha^{\deg(x)} \cdot P_x)_{x \in |X|}$$

where we take the degree of a point x over $\mathbb{F}_q$.

Let k be a field containing $\mathbb{Q}$ and $P_i \in \mathcal{L}_{r_i}(k)$, $i = 1, \ldots, n$. Assume $r_i > 0$ for all i and set $r = r_1 + \cdots + r_n$.

Lemma 6.2. *The morphism of schemes over the field k*

$$\rho : \mathbb{G}_m^n \to \mathcal{L}_r(X), \quad (\alpha_i)_{i=1,\dots,n} \mapsto \alpha_1 \cdot P_1 \oplus \cdots \oplus \alpha_n \cdot P_n$$

is finite.

Proof. In fact already the composition of ρ with the projection to one factor $\mathcal{P}_r$ of $\mathcal{L}_r(X)$, corresponding to a point $x \in |X|$, is finite. To see this write this morphism as the composition of finite morphisms over k

$$\mathbb{G}_m^n \xrightarrow{\psi_{\deg(x)}} \mathbb{G}_m^n \xrightarrow{\cdot(P_1,\dots,P_n)} \mathcal{P}_{r_1} \times \cdots \times \mathcal{P}_{r_n} \xrightarrow{\oplus} \mathcal{P}_r.$$

□

6.2. **Moduli over curves.** In this section we prove Theorem 6.1 for $\dim(X) = 1$. The dimension one case of Theorem 2.1 was shown in Section 4.3. In particular we get:

Lemma 6.3. *There are up to twist only* finitely many *irreducible direct summands of the sheaves $V \in \mathcal{R}_r(X, D) = \mathcal{V}_r(X, D)$.*

Step 1:
Consider $V_1 \oplus \cdots \oplus V_n \in \mathcal{R}_r(X, D)$ and the map
(6.4)
$$(\mathcal{R}_1(\mathbb{F}_q))^n \to \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell), \quad (\chi_1, \dots, \chi_n) \mapsto \kappa(\chi_1 \cdot V_1 \oplus \cdots \oplus \chi_n \cdot .V_n)$$
This map is just the induced map on $\bar{\mathbb{Q}}_\ell$-points of the finite scheme morphism over $k = \bar{\mathbb{Q}}_\ell$ from Lemma 6.2, where we take $P_i = \kappa(V_i)$. By Proposition A.3 there is a unique reduced closed subscheme $L(V_i)$ of $\mathcal{L}_r(X) \otimes \bar{\mathbb{Q}}_\ell$ of finite type over $\bar{\mathbb{Q}}_\ell$ such that $L(V_i)(\bar{\mathbb{Q}}_\ell)$ is the image of the map (6.4).

Step 2:
By Lemma 6.3 there are only finitely many direct sums

$$V_1 \oplus \cdots \oplus V_n \in \mathcal{R}_r(X, D) \tag{6.5}$$

with V_i irreducible up to twists $\chi_i \mapsto \chi_i \cdot V_i$ with $\chi_i \in \mathcal{R}_1(\mathbb{F}_q)$. Let

$$L_r(X, D)_{\bar{\mathbb{Q}}_\ell} \hookrightarrow \mathcal{L}_r(X) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$$

be the reduced scheme, which is the union of the finitely many closed subschemes $L(V_i) \hookrightarrow \mathcal{L}_r(X) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$ corresponding to representatives of the finitely many twisting classes of direct sums (6.5). Clearly $L_r(X, D)_{\bar{\mathbb{Q}}_\ell}(\bar{\mathbb{Q}}_\ell) = \kappa(\mathcal{R}_r(X, D))$ and $L_r(X, D)_{\bar{\mathbb{Q}}_\ell}$ is of finite type over $\bar{\mathbb{Q}}_\ell$.

Step 3:
By Corollary 4.9 the automorphism group $\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ acting on $\mathcal{L}_r(X)$ stabilizes $\kappa(\mathcal{R}_r(X, D))$. Therefore by the descent Proposition A.2 the

scheme $L_r(X,D)_{\bar{\mathbb{Q}}_\ell} \hookrightarrow \mathcal{L}_r(X) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$ over $\bar{\mathbb{Q}}_\ell$ descends to a closed subscheme $L_r(X,D) \hookrightarrow \mathcal{L}_r(X)$. This is the *moduli space of ℓ-adic sheaves* on curves, the existence of which was claimed in Theorem 6.1.

From the proof of Lemma 6.2 and the above construction we deduce:

Lemma 6.4. *For any $x \in |X|$ the composite map*

$$L_r(X,D) \to \mathcal{L}_r(X) \xrightarrow{\pi_x} \mathcal{P}_r$$

is a finite morphism of schemes.

6.3. **Higher dimension.** Now the dimension $d = \dim(X)$ of $X \in \mathrm{Sm}_{\mathbb{F}_q}$ is allowed to be arbitrary. In order to prove Theorem 6.1 in general we first construct a closed subscheme $L_r(X,D) \hookrightarrow \mathcal{L}_r(X)$ such that

$$L_r(X,D)(\bar{\mathbb{Q}}_\ell) = \kappa(\mathcal{V}_r(X,D))$$

relying on Theorem 6.1 for curves. However from this construction it is not clear that $L_r(X,D)$ is of finite type over $\mathbb{Q}$. The main step is to show that it is of finite type using Theorem 5.1.

Step 1:
We define the reduced closed subscheme $L_r(X,D) \hookrightarrow \mathcal{L}_r(X)$ by the Cartesian square (in the category of reduced schemes)

$$\begin{tikzcd}
L_r(X,D) \arrow[r] \arrow[d] & \mathcal{L}_r(X) \arrow[d] \\
\prod_{C \in \mathrm{Cu}(X)} L_r(C, \bar{\phi}^*(D)) \arrow[r] & \prod_{C \in \mathrm{Cu}(X)} \mathcal{L}_r(C)
\end{tikzcd}$$

where $\mathrm{Cu}(X)$ is defined in Section 2.2. Clearly, from the curve case of Theorem 6.1 and the definition of $\mathcal{V}_r(X,D)$ we get

$$L_r(X,D)(\bar{\mathbb{Q}}_\ell) = \kappa(\mathcal{V}_r(X,D)).$$

Step 2:
Let C be a purely one-dimensional scheme which is separated and of finite type over $\mathbb{F}_q$. Let $\phi_i : E_i \to C$ $(i = 1, \ldots, m)$ be the normalizations of the irreducible components of C and let $\phi : E = \coprod_i E_i \to C$ be the disjoint union. Let $D \in \mathrm{Div}^+(\bar{E})$ be an effective divisor with supports in $\bar{E} \setminus E$. Here $\bar{E}$ is the canonical smooth compactification of E. Define the reduced scheme $L_r(C,D)$ by the Cartesian square (in

the category of reduced schemes)

$$\begin{tikzcd}
L_r(C,D) \arrow[r] \arrow[d] & \prod_{j=1,\dots,m} L_r(E_j,D_j) \arrow[d] \\
\prod_{i=1,\dots,m} L_r(E_i,D_i) \arrow[r] & \prod_{i\neq j} L_r((E_i\times_C E_j)_{\mathrm{red}})
\end{tikzcd}$$

Step 3:

By an *exhaustive system of curves* on X we mean a sequence $(C_n)_{n\geq 0}$ of purely one-dimensional closed subschemes $C_n \hookrightarrow X$ with the properties (a) – (d) listed below. We write $\phi : E_n \to X$ for the normalization of C_n. For a divisor $D' \in \mathrm{Div}^+(\bar{E}_n)$ we let $\mathcal{C}_{D'}$ be the maximum of the complexities of the irreducible components of $E_n \otimes \mathbb{F}$, see Definition 3.4.

(a) $C_n \hookrightarrow C_{n+1}$ for $n \geq 0$,
(b) $E_n(\mathbb{F}_{q^n}) \to X(\mathbb{F}_{q^n})$ is surjective,
(c) the fields of constants of the irreducible components of E_n ($n \geq 0$) are bounded,
(d) the complexity $\mathcal{C}_{\bar{\phi}_n^*(D)}$ of E_n satisfies
$$\mathcal{C}_{\bar{\phi}_n^*(D)} = \mathcal{O}(n).$$

Lemma 6.5. *Any $X \in \mathrm{Sm}_{\mathbb{F}_q}$ admits an exhaustive system of curves.*

The proof of the lemma is given below.

Let now (C_n) be an exhaustive system of curves on X. Set $D_n = \bar{\phi}_n^*(D) \in \mathrm{Div}^+(\bar{E}_n)$. An immediate consequence of (a)–(d) and the Riemann hypothesis for curves is that for $n \gg 0$ any irreducible component of C_{n+1} meets C_n. This implies by Lemma 6.4 that the tower of affine schemes of finite type over $\mathbb{Q}$

$$\cdots \to L_r(C_{n+1}, D_{n+1}) \xrightarrow{\tau} L_r(C_n, D_n) \to \cdots$$

has finite transition morphisms. Clearly, $L_r(X,D)$ maps to this tower. Since the complexities of the irreducible curves grow linearly in n and the fields of constants are bounded, Theorem 5.1 implies that there is $N \geq 0$ such that the map

$$L_r(C_{n+1}, D_{n+1})(\bar{\mathbb{Q}}_\ell) \to \mathcal{L}_r^{\leq n}(E_{n+1})$$

is injective for $n \geq N$. As this map factors through

$$\tau : L_r(C_{n+1}, D_{n+1})(\bar{\mathbb{Q}}_\ell) \to L_r(C_n, D_n)(\bar{\mathbb{Q}}_\ell)$$

by (b), we get injectivity of τ on $\bar{\mathbb{Q}}_\ell$-points for $n \geq N$. Consider the intersection of the images

$$I_n = \bigcap_{i\geq 0} \tau^i(L_r(C_{n+i}, D_{n+i})) \hookrightarrow L_r(C_n, D_n),$$

endowed with the reduced closed subscheme structure. Then the transition maps in the tower

$$\cdots \to I_{n+1} \to I_n \to \cdots$$

are finite and induce bijections on $\bar{\mathbb{Q}}_\ell$-points for $n \geq N$. By Proposition A.4 we get an $N' \geq 0$ such that $I_{n+1} \to I_n$ is an isomorphism of schemes for $n \geq N'$. So we get a closed immersion

$$L_r(X, D) \to \varprojlim_n L_r(C_n, D_n) \cong \varprojlim_n I_n \xrightarrow{\simeq} I_{N'},$$

and therefore $L_r(X, D)$ is of finite type over $\mathbb{Q}$.

Proof of Lemma 6.5. Using Noether normalization we find a finite number of finite surjective morphisms

$$\bar{\eta}_s : \bar{X} \to \mathbb{P}^d, \quad s = 1, \ldots, w$$

with the property that every point $x \in |X|$ is in the étale locus of one of the $\eta_s = \bar{\eta}_s|_X$. See [25, Theorem 1] for more details.

Claim 6.6. For a point $y \in \mathbb{P}^d(\mathbb{F}_{q^n})$ there is a morphism $\phi_y : \mathbb{P}^1 \to \mathbb{P}^d$ of degree $< n$ with $y \in \phi_y(\mathbb{P}^1(\mathbb{F}_{q^n}))$.

Proof of Claim. The closed point y lies in an affine chart

$$\mathbb{A}^d_{\mathbb{F}_q} = \mathrm{Spec}\,(\mathbb{F}_q[T_1, \ldots, T_d]) \hookrightarrow \mathbb{P}^d_{\mathbb{F}_q}$$

and gives rise to a homomorphism $\mathbb{F}_q[T_1, \ldots, T_d] \to \mathbb{F}_{q^n}$. We choose an embedding $\mathrm{Spec}\,\mathbb{F}_{q^n} \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q} = \mathrm{Spec}\,(\mathbb{F}_q[T])$ and a lifting

$$\mathbb{F}_q[T_1, \ldots, T_d] \to \mathbb{F}_q[T]$$

with $\deg(\phi(T_i)) < n$ $(1 \leq i \leq d)$. By projective completion we obtain a morphism $\phi_y : \mathbb{P}^1_{\mathbb{F}_q} \to \mathbb{P}^d_{\mathbb{F}_q}$ of degree less than n factoring the morphism $y \to \mathbb{P}^d$.

□

For $x \in |X|$ of degree n choose a lift $x \in X(\mathbb{F}_{q^n})$ and an s such that x is in the étale locus of η_s. Furthermore choose $\phi_y : \mathbb{P}^1 \to \mathbb{P}^d$ as in the claim with $y = \eta_s(x)$. Clearly x lifts to a smooth point of $(\mathbb{P}^1 \times_{\mathbb{P}^d} X)(\mathbb{F}_{q^n})$ contained in an irreducible component which we call Z. Let $\phi_x : C_x \to X$ be the normalization of the image of Z in X. Then $x \in \phi_x(C_x(\mathbb{F}_{q^n}))$.

We assume now that we have made the choice of the curve $\phi_x : C_x \to X$ above for any point $x \in |X|$. As usual $\bar{\phi}_x : \bar{C}_x \to \bar{X}$ denotes the smooth compactification of C_x. From the Riemann-Hurwitz formula [21, Cor. 2] we deduce the growth property

$$\mathcal{C}_{\bar{\phi}_x^*(D)} = \mathcal{O}(\deg(x))$$

for the complexity of $\bar{C}_x$. Furthermore it is clear that the fields of constants of the curves C_x are bounded. Therefore the subschemes

$$C_n = \bigcup_{\deg(x)\le n} \phi_x(C_x) \hookrightarrow X$$

satisfy the conditions (a)–(d) above. $\square$

7. Irreducible components and proof of finiteness theorems

Recall that we defined irreducible generalized sheaves in Section 2 and that in Section 6 we constructed an affine scheme $L_r(X, D)$ of finite type over $\mathbb{Q}$, the $\bar{\mathbb{Q}}_\ell$-points of which are in bijection with generalized sheaves of rank r with ramification bounded by D. For this we had to assume that $\bar{X}$ is a normal projective variety defined over $\mathbb{F}_q$ and D is an effective Cartier divisor supported in $\bar{X} \setminus X$.

The following theorem describes the irreducible components of $L_r(X, D)$ over $\bar{\mathbb{Q}}$ or, what is the same, over $\bar{\mathbb{Q}}_\ell$.

Theorem 7.1. A) *Given $V_1, \ldots, V_m$ irreducible in $\mathcal{V}(X)$ such that $V_1 \oplus \ldots \oplus V_m \in \mathcal{V}_r(X, D)$, there is a unique irreducible component $Z \hookrightarrow L_r(X, D) \otimes \bar{\mathbb{Q}}$ such that*

$$Z(\bar{\mathbb{Q}}_\ell) = \{\kappa(\chi_1 \cdot V_1 \oplus \ldots \oplus \chi_m \cdot V_m) \,|\, \chi_i \in \mathcal{R}_1(\mathbb{F}_q)\} \tag{7.1}$$

B) *If $Z \hookrightarrow L_r(X, D) \otimes \bar{\mathbb{Q}}$ is an irreducible component, then there are $V_1, \ldots, V_m$ irreducible in $\mathcal{V}_r(X, D)$ such that* (7.1) *holds true.*

Proof. We first prove B). Let d be the dimension of Z, so $\bar{\mathbb{Q}}(Z)$ has transcendence degree d over $\bar{\mathbb{Q}}$. Let $\kappa(V) \in Z(\bar{\mathbb{Q}}_\ell)$ be a geometric generic point, corresponding to $\iota : \bar{\mathbb{Q}}(Z) \hookrightarrow \bar{\mathbb{Q}}_\ell$.

By definition, the coefficients of the local polynomials $f_V(x)$, $x \in |X|$ span $\iota(\bar{\mathbb{Q}}(Z))$. The subfield K of $\bar{\mathbb{Q}}_\ell$ spanned by the (inverse) roots of the $f_V(x)$ is algebraic over $\iota(\bar{\mathbb{Q}}(Z))$, and thus has transcendence degree d over $\bar{\mathbb{Q}}$ as well.

Writing

$$V = \oplus_{w \in \mathcal{W}} V_w \tag{7.2}$$

thanks to Corollary 4.6, the number m of such w with $V_w \neq 0$ is $\geq d$. Indeed those w have the property that they span K.

On the other hand, the map (6.4) corresponding to the decomposition (7.2) is the $\bar{\mathbb{Q}}_\ell$-points of a finite map with source $\mathbb{G}_m^m$, which is irreducible, and has image contained in Z. So we conclude $m = d$ and that the morphism $\mathbb{G}_m^m \to Z$ is finite surjective.

We prove A). By Corollary 4.6, the V_i have the property that there is a $w_i \in \mathcal{W}$ such that all the inverse eigenvalues of the Frobenius F_x on V_i lie in the class of w_i. Replacing V_i by $\chi_i \cdot V_i$ for adequately chosen $\chi_i \in \mathcal{R}_1(\mathbb{F}_q)$, we may assume that $w_i \neq w_j$ in $\mathcal{W}$ if $i \neq j$. We consider the irreducible reduced closed subscheme $Z \hookrightarrow L_r(X, D) \otimes \bar{\mathbb{Q}}_\ell$ defined by its $\bar{\mathbb{Q}}_\ell$-points $\{\kappa(\chi_1 \cdot V_1 \oplus \ldots \oplus \chi_m \cdot V_m) \mid \chi_i \in \mathcal{R}_1(\mathbb{F}_q)\}$. Let Z' be an irreducible component of $L_r(X, D) \otimes \bar{\mathbb{Q}}_\ell$ containing Z. Thus by B),

$$Z'(\bar{\mathbb{Q}}_\ell) = \{\kappa(\chi'_1 \cdot V'_1 \oplus \ldots \chi'_{m'} \cdot V'_{m'}) \mid \chi'_i \in \mathcal{R}_1(\mathbb{F}_q)\}.$$

So there are χ'_i such that

$$V_1 \oplus \ldots \oplus V_m = \chi'_1 V'_1 \oplus \ldots \oplus \chi'_{m'} V'_{m'} \tag{7.3}$$

As V'_j is irreducible for any $j \in \{1, \ldots, m'\}$, it is of class w for some $w \in \mathcal{W}$ in the sense of Corollary 4.6. So for each $j \in \{1, \ldots, m'\}$, there is a $i \in \{1, \ldots, m\}$ with $\chi'_j \cdot V'_j \subset V_i$, and thus $\chi'_j \cdot V'_j = V_i$ as V_i is irreducible. This implies $m = m'$ and the decompositions (7.3) are the same, up to ordering. So $Z = Z'$. □

Corollary 7.2. *A generalized sheaf $V \in \mathcal{V}_r(X, D)$ is irreducible if and only if $\kappa(V)$ lies on a one-dimensional irreducible component of $L_r(X, D) \otimes \bar{\mathbb{Q}}_\ell$. In this case $\kappa(V)$ lies on a unique irreducible component $Z/\bar{\mathbb{Q}}_\ell$. The component Z has the form*

$$Z(\bar{\mathbb{Q}}_\ell) = \{\kappa(\chi \cdot V) \mid \chi \in \mathcal{R}_1(\mathbb{F}_q)\}$$

and it does not meet any other irreducible component.

Remark 7.3. If Question 2.3 had a positive answer and using a more refined analysis of Deligne [13] one could deduce that the moduli space $L_r(X, D)$ is smooth and any irreducible component is of the from $\mathbb{G}_m^{s_1} \times \mathbb{A}^{s_2}$ $(s_1, s_2 \geq 0)$.

Proof of Theorem 2.4. Using the Chow lemma [1, Sec. 5.6] we can assume without loss of generality that $\bar{X}$ is projective. By Corollary 7.2, the set of one-dimensional irreducible components of $L_r(X, D) \otimes \bar{\mathbb{Q}}$ is in bijection with the set of irreducible generalized sheaves on X up to twist by $\mathcal{R}_1(\mathbb{F}_q)$. Since $L_r(X, D)$ is of finite type, there are only finitely many irreducible components. □

Proof of Theorem 2.6. By Corollary 4.9 and there is a natural action of $\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ on $\mathcal{V}_r(X, D)$ compatible via f_V with the action on $\bar{\mathbb{Q}}_\ell[t]$ which fixes t. Let $N > 0$ be an integer such that $\det(V)^{\otimes N} = 1$. For $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ we then have

$$1 = \sigma(\det(V)^{\otimes N}) = \det(\sigma(V))^{\otimes N},$$

Then Theorem 2.4, (see also the remark following the theorem), implies that the orbit of V under $\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ is finite. Let $H \subset \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ be the stabilizer group of V. As $[\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}) : H] < \infty$ we get that $E(V) = \bar{\mathbb{Q}}_\ell^H$ is a number field. □

In order to effectively determine the field $E(V)$ for $V \in \mathcal{R}_r(X)$ with $X \in \mathrm{Sm}_{\mathbb{F}_q}$ projective one can use the following simple consequence of a theorem of Drinfeld [15], which itself relies on Deligne's Theorem 2.6.

Proposition 7.4. *For $X/\mathbb{F}_q$ a smooth projective geometrically connected scheme and $H \hookrightarrow X$ a smooth hypersurface section with* $\dim(H) >$ 0 *consider $V \in \mathcal{R}_r(X)$. Then $E(V) = E(V|_H)$.*

Proof. Observe that the Weil group of H surjects onto the Weil group of X, so we get an injection $\mathcal{R}_r(X) \to \mathcal{R}_r(H)$. By [15] Corollary 4.7 remains true for higher dimensional smooth schemes $X/\mathbb{F}_q$, i.e. for any $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ there exists a σ-companion V_σ to V. By the above injectivity, the sheaves V and $V|_H$ have the same stabilizer G in $\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$. We get

$$E(V) = \bar{\mathbb{Q}}_\ell^G = E(V|_H).$$

□

8. Deligne's conjecture on the number of irreducible lisse sheaves of rank r over a smooth curve with prescribed local monodromy at infinity

Let C be a smooth quasi-projective geometrically irreducible curve over $\mathbb{F}_q$, $C \hookrightarrow \bar{C}$ be a smooth compactification. One fixes an algebraic closure $\mathbb{F} \supset \mathbb{F}_q$ of $\mathbb{F}_q$. For each point $s \in (\bar{C} \setminus C)(\mathbb{F})$, one fixes a $\bar{\mathbb{Q}}_\ell$-representation V_s of the inertia

$$I(s) = \mathrm{Gal}(K_s^{\mathrm{sep}}/K_s)$$

where K_s is the completion of the function field $K = k(C)$ at s. We write

$$I(\bar{s}) = P \rtimes \prod_{\ell' \neq p} \mathbb{Z}_\ell(1),$$

where P is the wild inertia, a pro-p-group. A generator $\xi_{\ell'}$ of $\mathbb{Z}_{\ell'}(1)$, $\ell' \neq p$, acts on V_s for all $s \in (\bar{C} \setminus C)(\mathbb{F})$. Since the open immersion $j : C \hookrightarrow \bar{C}$ is defined over $\mathbb{F}_q$, if $s \in (\bar{C} \setminus C)(\mathbb{F})$ is defined over $\mathbb{F}_{q^n}$, for any conjugate point $s' \in (\bar{C} \setminus C)(\mathbb{F})$, the group $I(s')$ is conjugate to $I(s)$ by $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. One requires the following condition to be fulfilled.

i) If $s' \in (\bar{C} \setminus C)(\mathbb{F})$ is $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$-conjugate to s, the conjugation which identifies $I(s')$ and $I(s)$ identifies $V_{s'}$ and V_s.

Let V be an irreducible lisse $\bar{\mathbb{Q}}_\ell$ sheaf of rank r on $C \otimes_{\mathbb{F}_q} \mathbb{F}$ such that the set of isomorphism classes of restrictions $\{V \otimes K_s\}$ to $\operatorname{Spec} K_s$ is the set $\{V_s\}$ defined above with the condition i). Then if for a natural number $n \geq 1$, V is F^n invariant, V descends to a Weil sheaf on $C \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$. By Weil II, (1.3.3), $\det(V)$ is torsion. Thus by the dimension one case of Theorem 2.1 the cardinality of the set of such F^n-invariant sheaves V is finite.

If such a V exists, then the set $\{V_{\bar{s}}\}$ satisfies automatically

ii) For any $\ell' \neq p$, $\xi_{\ell'}$ acts trivially on $\otimes_{s \in (\bar{C} \setminus C)(\mathbb{F})} \det(V_s)$.

Indeed, as $\det(V)$ is torsion, a p power $\det(V)^{p^N}$ has torsion t prime to p, thus defines a class in $H^1(C \otimes_{\mathbb{F}_q} \mathbb{F}, \mu_t)$. The exactness of the localization sequence $H^1(C \otimes_{\mathbb{F}_q} \mathbb{F}, \mu_t) \xrightarrow{\text{res}} \oplus_{s \in (\bar{C} \setminus C)(\mathbb{F})} \mathbb{Z}/t \xrightarrow{\text{sum}} H^2(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, \mu_t) = \mathbb{Z}/t$ implies that the sum of the residues is 0. This shows ii).

Furthermore, if such a V exists, then the set $\{V_{\bar{s}}\}$ satisfies automatically

iii) The action of $\xi_{\ell'}$ on V_s is quasi-unipotent for all $\ell' \neq p$ and all $s \in (\bar{C} \setminus C)(\mathbb{F})$.

Indeed, this is Grothendieck's theorem, see [33, Appendix].

Given a set $\{V_s\}$ for all $s \in (\bar{C} \setminus C)(\mathbb{F})$, satisfying the conditions i), ii), iii), Conjecture 8.1 predicts a qualitative shape for the cardinality of the F^n invariants of the set M of irreducible lisse $\bar{\mathbb{Q}}_\ell$ sheaves on $C \otimes_{\mathbb{F}_q} \mathbb{F}$ of rank r with $V \otimes K_s$ isomorphic to V_s.

If V is an element of M, then $H^0(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd(V)) = \bar{\mathbb{Q}}_\ell$, spanned by the identity. Indeed, a global section is an endomorphism $V \xrightarrow{f} V$ on $C \otimes_{\mathbb{F}_q} \mathbb{F}$. f is defined by an endomorphism of the $\bar{\mathbb{Q}}_\ell$ vector space V_a which commutes with the action of $\pi_1(\bar{C}, a)$, where $a \in C(\mathbb{F})$ is a given closed geometric point. Since this action is irreducible, the endomorphism is a homothety. We write $\mathcal{E}nd(V) = \mathcal{E}nd(V)^0 \oplus \bar{\mathbb{Q}}_\ell$, where $\mathcal{E}nd(V)^0$ is the trace-free part, thus $j_*\mathcal{E}nd(V) = j_*\mathcal{E}nd(V)^0 \oplus \bar{\mathbb{Q}}_\ell$. Thus $H^0(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd^0(V)) = 0$. The cup-product

$$j_*\mathcal{E}nd(V)) \times j_*\mathcal{E}nd(V) \to j_*\bar{\mathbb{Q}}_\ell = \bar{\mathbb{Q}}_\ell$$

obtained by composing endomorphisms and then taking the trace induces the perfect duality

(8.1)

$$H^i(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd^0(V)) \times H^{2-i}(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd^0(V)) \to H^2(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, \bar{\mathbb{Q}}_\ell).$$

For $i = 1$, the bilinear form (8.1) is symplectic. We conclude that $H^2(\bar{C}\otimes_{\mathbb{F}_q}\mathbb{F}, j_*\mathcal{E}nd^0(V)) = 0$ and that $H^1(\bar{C}\otimes_{\mathbb{F}_q}\mathbb{F}, j_*\mathcal{E}nd^0(V))$ is even dimensional. But dim $H^1(\bar{C}\otimes_{\mathbb{F}_q}\mathbb{F}, \bar{\mathbb{Q}}_\ell) = 2g$ thus $H^1(\bar{C}\otimes_{\mathbb{F}_q}\mathbb{F}, j_*\mathcal{E}nd(V))$ is even dimensional as well. We define

$$2d = \dim\, H^1(\bar{C}\otimes_{\mathbb{F}_q}\mathbb{F}, j_*\mathcal{E}nd(V)).$$

Conjecture 8.1. (Deligne's conjecture)

i) There are finitely many Weil numbers a_i, b_j of weight between 0 and $2d$ such that
$$N(n) = \sum_i a_i^n - \sum_j b_j^n$$

ii) If $M \neq \emptyset$, there is precisely one of the numbers a_i, b_j of weight $2d$ and moreover, it is one of the a_i and is equal to q^d.

An example where $M = \emptyset$ is given by $\bar{C} = \mathbb{P}^1$, C is the complement of 3 rational points $\{0, 1, \infty\}$, the rank r is 2 and the V_s are unipotent, so in particular, the Swan conductor at the 3 points is 0. Indeed, fixing ℓ', the inertia groups $I(s)$ at the 3 points, which depend on the choice of a base point, can be chosen so the product over the 3 points of the $\xi_{\ell'}$ is equal to 1. Thus the set $\{V_s, s = 0, 1, \infty\}$ is defined by 3 unipotent matrices A_0, A_1, A_∞ in $\mathrm{GL}(2, \bar{\mathbb{Q}}_\ell)$ such that $A_0 \cdot A_1 \cdot A_\infty = 1$. Since $A_0 \cdot A_1$ is then unipotent, A_0 and A_1, and thus A_∞, lie in the same Borel subgroup of $GL(2, \bar{\mathbb{Q}}_\ell)$. Thus the 3 matrices have one common eigenvector. Since the tame fundamental group is spanned by the images of $I(0), I(1), I(\infty)$, a $\bar{\mathbb{Q}}_\ell$-sheaf of rank 2 with $V \otimes K_s$ isomorphic to V_s is not irreducible. Thus $M = \emptyset$.

Two further examples are computed in [14]. For the first case [14, section 7], $C = \mathbb{P}^1 \setminus D$ where D is a reduced degree 4 divisor, with unipotent $V_{\bar{s}}$. The answer is $N(n) = q^n$. For the second case, $C = \mathbb{P}^1 \setminus D$ where D is a reduced non-irreducible degree 3 divisor with unipotent $V_{\bar{s}}$ with only one Jordan block (a condition which could be forced by the irreducibility condition for V). Then $N(n) = q^n$ as well.

Appendix A

In this appendix we gather a few facts on how to recognize through their closed points affine schemes of finite type as subschemes of affine schemes not necessarily of finite type.

Proposition A.1. *Let k be an algebraically closed field, let Y be an affine k-scheme. Then the map*

$$Z \mapsto Z(k)$$

embeds the set of reduced closed subschemes $Z \hookrightarrow Y$ of finite type into the power set $\mathcal{P}(Y(k))$.

Proof. Choose a filtered direct system $B_\alpha \subset B = k(Y)$ of affine k-algbras (of finite type), such that $B = \varinjlim_\alpha B_\alpha$. Set $Y_\alpha = \operatorname{Spec} B_\alpha$. Consider two closed subschemes

$$Z_1 = \operatorname{Spec} B/I_1 \hookrightarrow Y, \quad Z_2 = \operatorname{Spec} B/I_2 \hookrightarrow Y \tag{A.1}$$

of finite type over k such that $Z_1(k) = Z_2(k) \subset Y(k)$. After replacing the direct system α by a cofinal subsystem we can assume that $B_\alpha \to B/I_1$ and $B_\alpha \to B/I_2$ are surjective. Hilbert's Nullestellensatz for the closed subschemes $Z_1 \hookrightarrow Y_\alpha$ and $Z_2 \to Y_\alpha$ implies $I_1 \cap B_\alpha = I_2 \cap B_\alpha$. So $I_1 = I_2$ and the closed subschemes (A.1) agree. □

Proposition A.2. *Let k be a characteristic 0 field, let $K \supset k$ be an algebraically closed field extension. Let Y be an affine scheme over k, and $Z \hookrightarrow Y \otimes_k K$ be a closed embedding of an affine scheme of a finite type. If the subset $Z(K)$ of $Y(K)$ is invariant under the automorphism group of K over k, then there is a reduced closed subscheme $Z_0 \hookrightarrow Y$ of finite type over k such that*

$$(Z \hookrightarrow Y \otimes_k K) = (Z_0 \hookrightarrow Y) \otimes_k K.$$

Proof. Let $G = \operatorname{Aut}(K/k)$, $B = k(Y)$, $Z = \operatorname{Spec}((B \otimes_k K)/I)$. The G-stability of $Z(K) \subset Y(K)$ and Proposition A.1 imply that $I \subset B \otimes_k K$ is stable under G. Then [6, Sec. V.10.4] implies that $I_0 = I^G \subset B$ satisfies $I_0 \otimes_k K = I$. Set $Z_0 = \operatorname{Spec} B/I_0$.

□

Proposition A.3. *Let k be an algebraically closed field, let $\varphi : Z \to Y$ be an integral k-morphism of affine schemes, with Z of finite type over k. Then there is a uniquely defined reduced closed subscheme $X \hookrightarrow Y$ of finite type over k such that*

$$\varphi(Z(k)) = X(k).$$

Proof. Write $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$, for commutative k-algebras B, C with C of finite type over k. Without loss of generality assume that B and C are reduced. There are finitely many elements of C which span C as a k-algebra. They are integral over B. This defines finitely many minimal polynomials, thus finitely many coefficients of those polynomials in B. Thus there is an affine k-algebra of finite type $B_0 \subset B$ containing them all. It follows that C is finite over B_0. Choose a filtered inverse system $Y_\alpha = \operatorname{Spec} B_\alpha$ of affine k-schemes of finite type, such that $B_\alpha \subset B$ and

$$Y = \operatorname{Spec} B = \varprojlim_\alpha Y_\alpha.$$

The morphisms $\varphi_\alpha : Z \xrightarrow{\varphi} Y \to Y_\alpha$ are all finite. Let $X_\alpha = \operatorname{Spec} C_\alpha \hookrightarrow Y_\alpha$ be the (reduced) image of φ_α. We obtain finite ring extensions $C_\alpha \subset C$. By Noether's basis theorem the filtered direct system C_α stabilizes at some α_0. Then

$$X = \operatorname{Spec} C_{\alpha_0} = \varprojlim_\alpha \operatorname{Spec} C_\alpha \hookrightarrow Y$$

is of finite type over k and satisfies $\varphi(Z(k)) = X(k)$. □

Proposition A.4. *Let k be an algebraically closed field of characteristic 0, let Y be an affine k-scheme, such that $Y = \operatorname{Spec} B = \varprojlim_n Y_n$, $n \in \mathbb{N}$ is the projective limit of reduced affine schemes Y_n of finite type. If the transition morphisms induce bijections $Y_{n+1}(k) \xrightarrow{\cong} Y_n(k)$ on closed points, then there is a $n_0 \in \mathbb{N}$ such that $Y_n \to Y_{n_0}$ is an isomorphism for all $n \geq n_0$. In particular, $Y \to Y_{n_0}$ is an isomorphism as well.*

Proof. Applying Zariski's Main Theorem [2, Thm.4.4.3], one constructs inductively affine schemes of finite type $\bar{Y}_n$, $\bar{Y}_0 = Y_0$, together with an open embedding $Y_n \hookrightarrow \bar{Y}_n$, such that the transition morphisms $Y_{n+1} \to Y_n$ extend to finite transition morphisms $\bar{Y}_{n+1} \to \bar{Y}_n$. On the other hand, the assumption implies that the morphisms $Y_{n+1} \to Y_n$ are birational on every irreducible component. So the same property holds true for $\bar{Y}_{n+1} \to \bar{Y}_n$. One thus has a factorization $\widetilde{Y_0} \to \bar{Y}_n \to Y_0$ for all n, where $\widetilde{Y_0} \to Y_0$ the normalization morphism. Since $\widetilde{Y_0}$ is of finite type, there is a n_0 such that $\bar{Y}_n \to \bar{Y}_{n_0}$ is an isomorphism for all $n \geq n_0$. Thus the composite morphism $Y_n \to Y_{n_0} \to \bar{Y}_{n_0}$ is an open embedding for all $n \geq n_0$, and thus $Y_{n+1} \to Y_n$ is an open embedding as well. Since it induces a bijection on points, and the Y_n are reduced, the transition morphisms $Y_{n+1} \to Y_n$ are isomorphisms for $n \geq n_0$. □

Remark A.5. If in Proposition A.4, one assumes in addition that the transition morphisms $Y_{n+1} \to Y_n$ are finite, then one does not need Zariski's Main Theorem to conclude.

Appendix B

In the proof of Corollary 4.5 we claim the existence of a curve with certain properties. The Bertini argument given in [27, p. 201] for the construction of such a curve is, as such, not correct. We give a complete proof here relying on Hilbert irreducibility instead of Bertini.

Let X be in $\mathrm{Sm}_{\mathbb{F}_q}$.

Proposition B.1. *For $V \in \mathcal{R}_r(X)$ irreducible and a closed point $x \in X$, there is an irreducible smooth curve $C/\mathbb{F}_q$ and a morphism $\psi : C \to X$ such that*

- *$\psi^*(V)$ is irreducible,*
- *x is in the image of ψ.*

Lemma B.2. *For an irreducible $\bar{\mathbb{Q}}_\ell$-étale sheaf V on X there is a connected étale covering $X' \to X$ with the following property:*
For a smooth irreducible curve $C/\mathbb{F}_q$ and a morphism $\psi : C \to X$ the implication

$$C \times_X X' \text{ irreducible} \quad \Longrightarrow \quad \psi^*(V) \text{ irreducible}$$

holds.

Proof. Choose a finite normal extension R of $\mathbb{Z}_\ell$ with maximal ideal $\mathfrak{m} \subset R$ such that V is induced by a continuous representation

$$\rho : \pi_1(X) \to GL(R, r).$$

Let H_1 be the kernel of $\pi_1(X) \to GL(R/\mathfrak{m}, r)$ and let G be the image of ρ. The subgroup

$$H_2 = \bigcap_{\nu \in \mathrm{Hom}(H_1, \mathbb{Z}/\ell)} \ker(\nu)$$

is open normal in $\pi_1(X)$ according to [3, Th. Finitude]. Indeed observe that $H_1/H_2 = H_1^{\mathrm{ab}}/\ell$ is Pontryagin dual to $H^1_{\text{ét}}(X_{H_1}, \mathbb{Z}/\ell)$, where X_{H_1} is the étale covering of X associated to H_1. Since the image of H_1 in G is pro-ℓ, and therefore pro-nilpotent, any morphism of pro-finite groups $K \to \pi_1(X)$ satisfies:

$$(K \to \pi_1(X)/H_2 \quad \text{surjective }) \Longrightarrow \quad (K \to G \text{ surjective }).$$

(Use [6, Cor. I.6.3.4].)

Finally, let $X' \to X$ be the Galois covering corresponding to H_2. $\square$

Proof of Proposition B.1. We can assume that X is affine. By Proposition 4.3 we can, after some twist, assume that V is étale. Let X' be as in the lemma. By Noether normalization, e.g. [16, Corollary 16.18], there is a finite generically étale morphism

$$f : X \to \mathbb{A}^d.$$

Let $U \subset \mathbb{A}^d$ be an open dense subscheme such that $f^{-1}(U) \to U$ is finite étale. Let $y \in \mathbb{A}^d$ be the image of x. Choose a linear projection $\pi : \mathbb{A}^d \to \mathbb{A}^1$ and set $z = \pi(y)$ and consider the map $h : U \to \mathbb{A}^1$. By definition, $U_{k(\mathbb{A}^1)} \subset \mathbb{A}^{d-1}_{k(\mathbb{A}^1)}$.

Let $F = k(\Gamma) \supset k(\mathbb{A}^1)$ be a finite extension such that $X' \otimes_{k(\mathbb{A}^1)} F$ is irreducible and the smooth curve $\Gamma \to \mathbb{A}^1$ contains a closed point z' with $k(z') = k(y)$.

It is easy to see that there is an $\hat{F}$-point in $U_{k(\mathbb{A}^1)}$ which specializes to y. By Hilbert irreducibility, see [15, Cor. A.2], we find an F-point $u \in U_{k(\mathbb{A}^1)}$ which specializes to y and such that u does not split in $X' \times_{\mathbb{A}^1} \Gamma$.

Let $v \in X$ be the unique point over u. By the going-down theorem [7, Thm. V.2.4.3] the closure $\overline{\{v\}}$ contains x. Finally, we let C be the normalization of $\overline{\{v\}}$. □

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Hanoi lectures on the arithmetic of hyperelliptic curves

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1 Introduction

Manjul Bhargava and I have recently proved a result on the average order of the 2-Selmer groups of the Jacobians of hyperelliptic curves of a fixed genus $n \geq 1$ over $\mathbb{Q}$, with a rational Weierstrass point [2, Thm 1]. A surprising fact which emerges is that the average order of this finite group is equal to 3, independent of the genus n. This gives us a uniform upper bound of $\frac{3}{2}$ on the average rank of the Mordell-Weil groups of their Jacobians over $\mathbb{Q}$. As a consequence, we can use Chabauty's method to obtain a uniform bound on the number of points on a majority of these curves, when the genus is at least 2.

We will state these results more precisely below, after some general material on hyperelliptic curves with a rational Weierstrass point. We end with a short discussion of hyperelliptic curves with two rational points at infinity. I want to thank Manjul Bhargava, Ngô Báo Châu, Brian Conrad, and Jerry Wang for their comments.

2 Hyperelliptic curves with a marked Weierstrass point

For another treatment of this basic material, see [5]. Chevalley considers the more general case of a double cover of a curve of genus 0 in [3, Ch IV,§9].

Let k be a field and let C be a complete, smooth, connected curve over k of genus $n \geq 1$. Let O be a k-rational point of C, and let $U = C - \{O\}$ be the corresponding affine curve. The k-algebra $H^0(U, \mathscr{O}_U)$ of functions on C which are regular outside of O is a Dedekind domain with unit group k^*. The subset $L(mO)$ of functions with a pole of order $\leq m$ at O and regular elsewhere is a finite-dimensional k-vector space.

We henceforth assume that the vector space $L(2O)$ has dimension equal to 2. There cannot be a function having a simple pole at O and regular elsewhere, as that would give an isomorphism of C with $\mathbb{P}^1$ (and we have assumed that the genus of C is greater than 0). Hence $L(2O)$ is spanned by the constant function 1 and a function x with a double pole at O. We normalize the function x by fixing a non-zero tangent vector v to C at the point O and choosing a uniformizing parameter π in the completion of the function field at O with the property that $\frac{d}{dv}(\pi) = 1$. We then scale x so that $x = \pi^{-2} + \cdots$ in the completion. This depends only on the choice of tangent vector v, not on the choice of uniformizing parameter π adapted to v. The other functions in $L(2O)$ with this property all have the form $x + c$, where c is a constant in k. If we replace the tangent vector v by $v^* = uv$ with $u \in k^*$, then $x^* = u^2x + c$.

It follows that the space $L((2n-1)O)$ contains the vectors $\{1, x, x^2, \ldots x^{n-1}\}$. Since these functions have different orders of poles at O, they are linearly independent. But the dimension of $L((2n-1)O)$ is equal $(2n-1) + (1-n) = n$ by the theorem of Riemann-Roch. Hence these powers of x give a basis for $L((2n-1)O)$. Since they all lie in the subspace $L((2n-2)O)$, they give a basis for that space too. Hence the dimension of $L((2n-2)O)$ is equal to the genus n. It follows from the Riemann-Roch theorem that the divisor $(2n-2)O$ is canonical.

The Riemann-Roch theorem also shows that the dimension of $L((2n)O)$ is equal to $n+1$, so a basis is given by the vectors $\{1, x, x^2, \ldots, x^n\}$. Similarly, the dimension of $L((2n+1)O)$ is equal to $n+2$. Hence there is a function y with a pole of exact order $(2n+1)$ at O, which cannot be equal to a polynomial in x. We use the uniformizing parameter π to normalize the function y by insisting that $y = \pi^{-(2n+1)} + \cdots$ in the completion. Again, this depends only on the tangent vector v. The other functions in $L((2n+1)O)$ with this property all have the form $y + q_n(x)$, where $q_n(x)$ is a polynomial of degree $\leq n$ with coefficients in k. If we replace v by $v^* = uv$ with $u \in k^*$, then $y^* = u^{2n+1}y + q_n(u^2x)$.

It is then easy to show that the algebra $H^0(U, \mathscr{O}_U)$ is generated over k by the two functions x and y, and that they satisfy a single polynomial relation $G(x, y) = 0$ of the form

$$y^2 + p_n(x)y = x^{2n+1} + p_{2n}(x) = F(x),$$

where p_n and p_{2n} are polynomials in x of degree $\leq n$ and $\leq 2n$ respectively. Indeed, the $(3n+4)$ vectors $\{y^2, x^ny, x^{n-1}y, \ldots, xy, y, x^{2n+1}, x^{2n}, \ldots, x, 1\}$ all lie in the vector space $L((4n+2)O)$, which has dimension $3n+3$. Hence they are linearly dependent. Since there are no linear relations in the spaces with poles of lesser order, this relation must involve a non-zero multiple of y^2 and a non-zero multiple of x^{2n+1}. By our normalization, we can scale the relation so that the multiple is 1. Hence the k-algebra $H^0(U, \mathscr{O}_U)$ is a quotient of the ring $k[x, y]/(G(x, y) = 0)$. Since the k-algebra $k[x] + yk[x]$ gives the correct dimensions of $L(mO)$ for all $m \geq 0$, there are no further relations, and the affine curve $U = C - \{O\}$ is defined by an equation of this form. The affine curve U is non-singular if and only if a certain universal polynomial Δ in the coefficients of $p_n(x)$ and $p_{2n}(x)$ takes a non-zero value in k [5, Thm 1.7]. Of course, changing the choice of the functions x and y in $L(2O)$ and $L((2n+1)O)$ changes the equation of the affine curve.

In the case when the genus of C is equal to 1, the pair (C, O) defines an elliptic curve over the field k. The polynomial relation above is Tate's affine equation for U (see [8, §2])

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

and the condition for smoothness is the non-vanishing of the discriminant $\Delta(a_1, a_2, a_3, a_4, a_6)$. The closure of this affine curve defines a smooth cubic in $\mathbb{P}^2$. For $n \geq 2$, the closure of the affine equation of degree $2n+1$ in $\mathbb{P}^2$ is not smooth, but one has a smooth model for C defined by gluing [6, Ch II, Ex 2.14].

All of this works over a general field k, but there are some important simplifications when the characteristic of k does not divide $2(2n+1)$. First, if the characteristic of k is not equal to 2, we can uniquely choose $y^* = y - p_n(x)/2$ to complete the square of the above equation and obtain one of the simpler form

$$y^2 = x^{2n+1} + c_1 x^{2n} + c_2 x^{2n-1} + \cdots + c_{2n} x + c_{2n+1} = F(x).$$

The automorphism ι of C defined by $\iota(x, y) = (x, -y)$ is the unique involution which fixes the rational point O, and y is the unique normalized element in $L((2n+1)O)$ which is taken to its negative. The automorphism ι acts as -1 on the space of holomorphic differentials, which is spanned by $\{dx/2y, xdx/2y, \ldots, x^{n-1}dx/2y\}$. The differential $dx/2y$ has divisor $(2n-2)O$ and the differential $-x^{n-1}dx/2y$ is dual to the tangent vector v at O. In this case, the fact that U is smooth is equivalent to the non-vanishing of the discriminant of the polynomial $F(x)$, and the polynomial Δ is given by the formula $\Delta = 4^{2n}\,\mathrm{disc}(F)$ (see [5, 1.6]).

Next, when the characteristic of k does not divide $2n+1$, we can replace x by $x - c_1/(2n+1)$ to obtain an equation of the form

$$y^2 = x^{2n+1} + c_2 x^{2n-1} + \cdots + c_{2n} x + c_{2n+1} = F(x).$$

This equation is uniquely determined by the triple (C, O, v), where v is a non-zero tangent vector at the point O. In particular, the moduli problem of triples (C, O, v) is rigid, and represented by the complement of the discriminant hypersurface $(\Delta = 0)$ in affine space of dimension $2n$. The automorphism ι of (C, O) defines an isomorphism from (C, O, v) to $(C, O, -v)$. If we replace $v^* = uv$ with $u \in k^*$, then $x^* = u^2 x$ and $y^* = u^{2n+1} y$. The coefficients c_m in the polynomial $F(x)$ are scaled by the factor u^{2m}, and the discriminant Δ of the model is scaled by the factor $u^{2(2n+1)(2n)}$ in k^*.

3 The height of the pair (C, O)

We first assume that $k = \mathbb{Q}$ is the field of rational numbers. To each pair (C, O) we will associate a positive real number $H(C, O)$, its height. Choose a non-zero tangent vector v at the point O so that the coefficients c_m of the corresponding equation of (C, O, v) are all integers with the property that no prime p has the property that p^{2m} divides c_m for all m. We call such an equation minimal. Then v is unique up to sign, and the integers c_m which appear in this minimal equation are uniquely determined by the pair (C, O). We then define

$$H(C, O) = \mathrm{Max}\{|c_m|^{(2n+1)(2n)/m}\}.$$

The factor $(2n+1)(2n)$ is added in the exponent so that the height $H(C, O)$ and the discriminant Δ have the same homogeneous degree. Clearly there are only finitely many pairs (C, O) with $H(C, O) < X$ for any positive real number X, so the height gives a convenient way to enumerate hyperelliptic curves over $\mathbb{Q}$ of a fixed genus n with a rational Weierstrass point. The number of pairs with $H(C, O) < X$ grows like a constant times $X^{(2n+3)/(4n+2)}$.

In the case when the genus of C is equal to 1, the minimal equation has the form

$$y^2 = x^3 + c_2 x + c_3$$

with c_2 and c_3 both integers, not respectively divisible by p^4 and p^6 for any prime p. We note that this is not necessarily a global minimal model at the primes $p = 2$ and $p = 3$ (cf. [6, Ch VII]). The discriminant is given by the formula $\Delta = 2^4(-4c_2^3 - 27c_3^2)$ and the height is given by the formula $H(C, O) = \mathrm{Max}\{|c_2|^3, |c_3|^2\}$. The number of elliptic curves with height less than X grows like a constant times $X^{5/6}$.

More generally, suppose that k is a number field, and that (C, O) is a pair over k. Choose a non-zero tangent vector v so that the equation determined by the triple (C, O, v)

$$y^2 = x^{2n+1} + c_2x^{2n-1} + \cdots + c_{2n+1}$$

has coefficients in the ring A of integers of k. We define the height $H(C, O)$ by modifying the naive height of the point $(c_2, c_3, \ldots, c_{2n+1})$ in weighted projective space, using the notion of "size" defined in [4]. Namely, define the fractional ideal

$$I = \{\alpha \in k : \alpha^4c_2, \alpha^6c_3, \ldots, \alpha^{4n+2}c_{2n+1} \in A\}.$$

Then I contains A and $I = A$ if and only if the coefficients c_m are not all divisible by P^{2m}, for every non-zero prime ideal P of A. We define the height of the pair by

$$H(C, O) = (N(I))^{(2n+1)(2n)} \prod_{v|\infty} \mathrm{Max}\{|c_m|_v^{(2n+1)(2n)/m}\},$$

where the product is taken over all infinite places v of k. The product formula shows that this definition is independent of the choice of non-zero tangent vector v. When $k = \mathbb{Q}$, the choice of a minimal integral equation gives $N(I) = 1$ and we are reduced to the previous definition. In general, the number of pairs with $H(C, O) < X$ is finite, and again grows like a constant (depending on the arithmetic of k) times $X^{(2n+3)/(4n+2)}$ (cf. [4, Thm A]).

Let S be a real-valued function on pairs (C, O) over k. We say that the average value of S is equal to L if the ratios

$$(\sum_{H(C,O)<X} S(C, O))/(\sum_{H(C,O)<X} 1)$$

tend to the limiting value L as $X \to \infty$. If R is a property of pairs (C, O) over $\mathbb{Q}$, we define the function S_R on pairs by $S_R(C, O) = 1$ if the pair satisfies property R and $S_R(C, O) = 0$ otherwise. We say that the proportion of pairs satisfying property R is equal to r if the ratios

$$(\sum_{H(C,O)<X} S_R(C, O))/(\sum_{H(C,O)<X} 1)$$

tend to the limiting value r as $X \to \infty$. If this limit exists, then clearly $0 \le r \le 1$. If the *liminf* is greater than r, we say the proportion is greater than r.

For example, let R be the property that O is the *only* k-rational point of the curve C. When the genus of C satisfies $n \geq 2$ we suspect that the proportion of pairs (C, O) with this property is equal to 1. When the genus of C is equal to 1, we suspect that this proportion is equal to $\frac{1}{2}$.

4 The 2-torsion subgroup and the 2-descent

Let (C, O) be a pair as above, defined over a field k whose characteristic is not equal to 2. Let

$$y^2 = F(x) = x^{2n+1} + c_1 x^{2n} + \cdots .$$

be an affine equation for $U = C - \{O\}$. In this section we will use the separable polynomial $F(x)$ to describe the 2-torsion subgroup $J[2]$ of the Jacobian J of C as a finite group scheme over k. We will then explicitly calculate the map in Galois cohomology involved in the 2-descent. For more details, see [7].

Since $\mathrm{disc}(F) \neq 0$, the k-algebra $L = k[x]/(F(x))$ is étale. Let λ be the image of x in L, so $L = k + k\lambda + \cdots + k\lambda^{2n}$. Let k^s denote a separable closure of k and let $G = \mathrm{Gal}(k^s/k)$. The set $\mathrm{Hom}(L, k^s)$ of homomorphisms of k-algebras has cardinality $2n+1$ and has a left action of G, so defines a homomorphism $G \to S_{2n+1}$ up to conjugacy. We will see that the kernel of this homomorphism fixes the subfield of k^s generated by the 2-torsion points in the Jacobian.

Since $C(k)$ is non-empty, the points of the Jacobian $J(K)$ over any extension field K of k are isomorphic to the quotient of the abelian group of divisors of degree zero on C which are rational over K by the subgroup of principal divisors $\mathrm{div(f)}$ with f in $K(C)^*$. For each root β of the polynomial $F(x)$ in k^s, we define the point $P_\beta = (\beta, 0)$ on C and the divisor $d_\beta = (P_\beta) - (O)$ of degree zero. The class of d_β has order 2 in the Jacobian, as $2d_\beta = \mathrm{div}(\mathrm{x} - \beta)$. It follows from the Riemann-Roch theorem that the $2n+1$ classes d_β in $J[2](k^s)$ satisfy a single linear relation over $\mathbb{Z}/2\mathbb{Z}$:

$$\sum_\beta d_\beta = \mathrm{div(y)}.$$

They therefore span a finite subgroup of order 2^{2n}. Since this is the order of the full group $J[2](k^s)$, we have found a presentation of the 2-torsion over the separable closure. The Galois group acts on the $2n+1$ classes d_β through the homomorphism $G \to S_{2n+1}$, so we have an isomorphism of group schemes over k

$$J[2] \cong \mathrm{Res}_{L/k}\, \mu_2/\mu_2 \cong (\mathrm{Res}_{L/k}\, \mathbb{G}_m/\mathbb{G}_m)[2],$$

where Res denotes the restriction of scalars. Since $2n+1$ is odd, we have a splitting

$$\mathrm{Res}_{L/k}\, \mu_2 = \mu_2 \oplus (\mathrm{Res}_{L/k}\, \mu_2)_{N=1},$$

where the latter subgroup is the kernel of the norm map $N : \mathrm{Res}_{L/k}\, \mu_2 \to \mu_2$. Hence $J[2] \cong (\mathrm{Res}_{L/k}\, \mu_2)_{N=1}$. This splitting also allows us to compute the Galois cohomology groups

$$H^0(k, J[2]) = J[2](k) = \{\alpha \in L^* : \alpha^2 = N(\alpha) = 1\}$$

$$H^1(k, J[2]) = (L^*/L^{*2})_{N\equiv 1},$$

where the subscript $N \equiv 1$ means that the norm of a class in (L^*/L^{*2}) is a square in k^*.

The homomorphism $2 : J \to J$ is a separable isogeny, so is surjective on points over k^s. The kernel is the group scheme $J[2]$, so taking the long exact sequence in Galois cohomology, we obtain a short exact sequence

$$0 \to J(k)/2J(k) \xrightarrow{\delta} H^1(k, J[2]) \to H^1(k, J)[2] \to 0.$$

If $P = (a, b)$ is a k-rational point on the curve C with $b \neq 0$, and $d = (P) - (O)$ is the class of the corresponding divisor of degree zero in $J(k)$, then the image $\delta(d)$ is the class of $(a - \lambda)$ in $H^1(k, J[2]) = (L^*/L^{*2})_{N\equiv 1}$ [7, Thm 1.2]. Note that $(a - \lambda)$ is an element of L^* with $N(a - \lambda) = b^2$ in k^*.

We remark that the elementary nature of the 2-torsion is almost a defining property of hyperelliptic curves with a marked Weierstrass point. For a general curve of genus $n \geq 1$ over the field k (of characteristic $\neq 2$), the 2-torsion on the Jacobian is rational over k^s and generates a finite Galois extension $M = k(J[2](k^s))$ of k. The Galois group of M/k acts $\mathbb{Z}/2\mathbb{Z}$-linearly on $J[2](k^s) \cong (\mathbb{Z}/2\mathbb{Z})^{2n}$ and preserves the Weil pairing $\langle , \rangle : J[2] \times J[2] \to \mu_2$, which is strictly alternating and non-degenerate. Hence the group $\mathrm{Gal}(M/k)$ is isomorphic to a subgroup of the finite symplectic group $\mathrm{Sp}_{2n}(2)$. When the curve is hyperelliptic with a k-rational Weierstrass point, the Weil pairing is given on the generators of $J[2]$ by

$$\langle d_\beta, d_\beta \rangle = +1$$
$$\langle d_\beta, d_{\beta'} \rangle = -1,$$

and the Galois group of M/k is isomorphic to the subgroup of $S_{2n+1} \subset \mathrm{Sp}_{2n}(2)$ which is determined by the étale algebra L.

We will see in the final section that the situation is similar (but a bit more complicated) for a hyperelliptic curve of genus $n \geq 2$ with a pair of k-rational points $\{O, O'\}$ which are switched by the hyperelliptic involution ι. In that case, the Galois group of M/k is isomorphic to a subgroup of $S_{2n+2} \subset \mathrm{Sp}_{2n}(2)$.

5 The 2-Selmer group

We henceforth assume that $k = \mathbb{Q}$, although we expect that the results in this section will extend to the case when k is a number field [9]. Let (C, O) be a hyperelliptic curve of genus $n \geq 1$ with a $\mathbb{Q}$-rational Weierstrass point O. The group $H^1(\mathbb{Q}, J[2])$ is infinite, but contains an important finite subgroup, the 2-Selmer group $\mathrm{Sel}(J, 2)$. This is the subgroup of cohomology classes whose restriction to $H^1(\mathbb{Q}_v, J[2])$ lies in the image $\delta(J(\mathbb{Q}_v)/2J(\mathbb{Q}_v))$ of the local descent map, for all places v [8, §7]. The assertion that the subgroup $\mathrm{Sel}(J, 2)$ defined in this manner is *finite* is the first half of the Mordell-Weil theorem; the proof uses the finiteness of the class group and the finite generation of the unit group for number fields. Since the 2-Selmer group contains the image of $J(\mathbb{Q})/2J(\mathbb{Q})$ under the inclusion δ, an upper bound on its order gives an upper bound on the rank of the finitely generated group $J(\mathbb{Q})$.

Here is a simple example, which illustrates the partial computation of a 2-Selmer group. Suppose that C is given by an integral equation $y^2 = F(x) = x^{2n+1} + \cdots$. Assume further that the polynomial $F(x)$ is irreducible and that the discriminant of $F(x)$ is square-free. Then the the algebra $L = k[x]/(F(x))$ is a number field with ring of integers $A_L = \mathbb{Z}[x]/F(x)$. In this case, one can show that the local image $\delta(J(\mathbb{Q}_p)/2J(\mathbb{Q}_p))$ is contained in the unit subgroup of elements with even valuation in $(L_p^*/L_p^{*2})_{N\equiv 1}$ for all finite primes p. It is equal to the unit subgroup when p is odd, and has index 2^n in the unit subgroup when $p = 2$. Hence the 2-Selmer group is a subgroup of the finite group $(L^{*(2)}/L^{*2})_{N\equiv 1}$ consisting of those elements in $(L^*/L^{*2})_{N\equiv 1}$ which have even valuation at all finite primes. To see that this group is finite, note that we have an exact sequence

$$1 \to (A_L^*/A_L^{*2})_{N=1} \to (L^{*(2)}/L^{*2})_{N\equiv 1} \to \mathrm{Pic}(A_L)[2] \to 1,$$

where the map to $\mathrm{Pic}(A_L)[2]$ takes the class of α with $(\alpha) = \mathfrak{a}^2$ to the class of the ideal $\mathfrak{a}$. The 2-Selmer group is the subgroup of this finite group which is defined by the local descent conditions at the places $v = 2$ and $v = \infty$. If we assume further that $F(x)$ has only one real root, so that $(A_L^*/A_L^{*2})_{N=1}$ has order 2^n by the unit theorem, then the only local conditions remaining are at the place $v = 2$.

In general, the local conditions at a finite set of bad places for C, which always include $v = 2$ and $v = \infty$, can be difficult to compute. It is therefore much easier to obtain an upper bound on the order of the Selmer group $\mathrm{Sel}(J, 2)$ than it is to determine its exact order. For some explicit computations with elliptic curves, see [6, Ch X]. The main result in [2, Th 1] gives the average order of this group, when we consider *all* hyperelliptic curves with a marked Weierstrass point over $\mathbb{Q}$.

Proposition 1 *When the pairs (C, O) of a fixed genus $n \geq 1$ are ordered by height, the average order of the group $\mathrm{Sel}(J, 2)$ is equal to 3.*

Let m be the rank of the Mordell-Weil group $J(\mathbb{Q})$. Since we have the inequalities $2m \leq 2^m \leq \#\,\mathrm{Sel}(J, 2)$ we obtain the following corollary.

Corollary 2 *When the pairs (C, O) of a fixed genus $n \geq 1$ are ordered by height, the average rank of the Mordell-Weil group $J(\mathbb{Q})$ is less than or equal to $\frac{3}{2}$.*

More precisely, the *limsup* of the average rank is less than or equal to $\frac{3}{2}$, as we do not know that the limit defining the average rank exists. We suspect that the limit does exist, and is equal to $\frac{1}{2}$.

The proof of Proposition 1 has an algebraic and an analytic part. The algebraic part of the proof identifies the elements in the 2-Selmer group of J, for any pair (C, O) of genus n over $\mathbb{Q}$, with certain orbits in a fixed linear representation of the split special odd orthogonal group $\mathrm{SO}(W) = \mathrm{SO}_{2n+1}$ over $\mathbb{Q}$. Specifically, we study the stable orbits of $\mathrm{SO}(W)$ on the highest weight submodule $V = \mathrm{Sym}^2(W)_0$ in the symmetric square of the standard representation. The vectors in this representation can be identified with self-adjoint operators $T : W \to W$ of trace 0, and a vector is stable if its characteristic polynomial $F_T(x)$ has a non-zero discriminant. Associated to a stable orbit, we obtain a pencil of quadrics in projective space of dimension $2n + 1$ with smooth base locus. The Fano variety of maximal linear subspaces of the base locus is a homogeneous space of order 2 for the Jacobian of the hyperelliptic curve defined by the equation $y^2 = F_T(x)$. The orbits corresponding to classes in the Selmer group are those operators T where the Fano variety has points over $\mathbb{Q}_v$ for all places v; we call these orbits locally solvable. When $n = 1$, the representation $\mathrm{Sym}^2(W)_0$ of $\mathrm{SO}_3 = \mathrm{PGL}_2$ is given by the action on the space of binary quartic forms $q(x, y)$, a vector is stable if the quartic form has a non-zero discriminant, and the Fano variety is the curve $z^2 = q(x, y)$.

Having identified classes in the Selmer group with locally solvable orbits, the analytic part of the proof estimates the number of locally soluble integral orbits of height less than X as $X \to \infty$. The average value of the order of the Selmer group actually appears as a sum $3 = 2 + 1$, where 2 is equal to the

Tamagawa number of SO_{2n+1}. This adèlic volume computation, together with some delicate arguments from the geometry of numbers, gives the average number of non-distinguished orbits (corresponding to the non-trivial classes in the Selmer group). The distinguished orbits (which all appear near a cusp of the fundamental domain) cannot be estimated by volume arguments. However, since they correspond to the trivial class in each Selmer group, the average number of these orbits is 1.

Since the average rank of $J(\mathbb{Q})$ is less than or equal to $\frac{3}{2}$, and this upper bound is less than the genus n of the curve C once $n \geq 2$, one can use the method of Chabauty (as refined by Coleman) to provide explicit bounds for the number of rational points on a majority (= a proportion greater than $\frac{1}{2}$) of the pairs (C, O). Here is a sample result, which is due to B. Poonen and M. Stoll. A slightly weaker result is obtained in [2, Cor 4].

Corollary 3 *If $n \geq 3$, a majority of the pairs (C, O) have at most 7 rational points, and a positive proportion of the pairs have only one rational point – the Weierstrass point O.*

To be more precise, we do not yet know that the limits defining these proportions exist. What they show is that the *liminf* of the ratios is $> \frac{1}{2}$ in the first case, and is > 0 in the second.

6 Even hyperelliptic curves

The curves C with a marked Weierstrass point O are often referred to as odd hyperelliptic curves, as (when the characteristic of k is not equal to 2) they have an equation of the form

$$y^2 = F(x) = x^{2n+1} + c_1 x^{2n} + \cdots$$

where the separable polynomial $F(x)$ has odd degree. We now make some general remarks on the "even" case, which is not treated in our paper but for which similar results are expected to hold. For more details, we refer the reader to the PhD thesis of X. Wang [9].

Let k be a field (not of characteristic 2) and let C be a complete, smooth, connected curve over k of genus $n \geq 1$. Let (O, O') be a pair of distinct k-rational points on C with $L((O) + (O'))$ of dimension equal to 2, and let $U = C - \{O, O'\}$ be the corresponding smooth affine curve. A similar analysis to the one we did above shows that the k-algebra $H^0(U, \mathscr{O}_U)$ is generated by functions x (with poles at O and O' of order 1) and y (with poles at O and O' of order $n+1$). These functions can be normalized satisfy a single equation of the form

$$y^2 = F(x) = x^{2n+2} + c_1 x^{2n+1} + \cdots,$$

where $F(x)$ has $2n+2$ distinct roots in k^s. The automorphism ι of C defined by $\iota(x, y) = (x, -y)$ is the unique involution which switches the two rational points O and O'.

The function y is the unique normalized vector in $L((n+1)(O) + (O'))$ which lies in the minus eigenspace for ι. When the characteristic of k does not divide $2n+2$, we can modify the function x in $L((O) + (O'))$ by a constant so that the the above equation has $c_1 = 0$. Then the equation depends only on the data $(C, (O, O'))$ and the choice of a non-zero tangent vector v to C at O. If we replace v by $v^* = uv$ with $u \in k^*$, the coefficients of the equation are scaled: $c_m^* = u^m c_m$. When k is a global field we can define the height of a triple $(C, (O, O'))$ by considering the coefficients $(c_2, c_3, \ldots, c_{2n+2})$ of this equation as a point in weighted projective space and taking its size as above [4]. Since there are only finitely many triples of a fixed genus $n \geq 1$ having height less than any real number X, we can define the average of a real-valued function S on triples $(C, (O, O'))$ as before.

The 2-torsion subgroup $J[2]$ of the Jacobian is a bit more complicated to describe. It is generated by the differences of the $2n+2$ Weierstrass points on C (none of which may be rational over k). Let $L = k[x]/(F(x))$, which is an étale k algebra of rank $2n+2$. Then we have an isomorphism of finite group schemes over k

$$J[2] \cong ((\mathrm{Res}_{L/k}\, \mu_2)_{N=1})/\mu_2.$$

The cohomology groups of $J[2]$ are also a bit more difficult to calculate. For example, the abelian group $\{\alpha \in L^* : \alpha^2 = N(\alpha) = 1\}/\{\pm 1\}$ maps into $H^0(k, J[2])$, but this map may not be surjective. This complicates matters somewhat in the 2-descent.

The class of the divisor $d = (O) - (O')$ of degree zero is well-defined in $J(k)/2J(k)$. It is usually a non-trivial element in this quotient of the Mordell-Weil group, although there are some triples where d is divisible by 2. When the class of d is non-trivial in $J(k)/2J(k)$, it gives rise to a non-trivial class in the 2-Selmer group. We should mention that Abel [1] found a beautiful criterion, in terms of the continued fraction of the square root of $F(x)$ in the completion $k((1/x))$, for the class of d to be of finite order in the Jacobian $J(k)$.

In the even case, we expect the average order of the 2-Selmer group of the Jacobian to be equal to $6 = 4 + 2$. The proof is similar in structure to the odd case. First the classes in the Selmer group are identified with the locally solvable orbits of the adjoint quotient $\mathrm{PSO}_{2n+2} = \mathrm{PSO}(W)$ of the split special even orthogonal group SO_{2n+2} over $\mathbb{Q}$ on the representation $V = \mathrm{Sym}^2(W)_0$ [9]. Then the average number of these orbits will be determined using arguments from the geometry of numbers. The contribution of 4 should come from the Tamagawa number of PSO_{2n+2} over $\mathbb{Q}$ and the contribution of 2 from the distinguished classes in the Selmer group whose orbits lie near the cusp. From the average order of the 2-Selmer group, one can deduce that the average rank of the Mordell-Weil group of the Jacobian is bounded above by $\frac{5}{2} = \frac{3}{2} + 1$.

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Realising unstable modules as the cohomology of spaces and mapping spaces

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Abstract

This report discuss the question wether or not a given unstable module is the mod-p cohomology of a space. One first discuss some classical results and give their relations to homotopy theory and geometric topology. Then one describes more recent results, emphasizing the use of the space $map(B\mathbb{Z}/p, X)$.

1 Introduction

Let p be a prime number, in all the sequel H^*X will denote the the mod p singular cohomology of the topological space X. All spaces X will be supposed p-complete and connected.
The singular mod-p cohomology is endowed with various structures :

- it is a graded $\mathbb{F}_p$-algebra, commutative in the graded sense,
- it is naturally a module over the algebra of stable cohomology operations which is known as the mod p Steenrod algebra and denoted by $\mathcal{A}_p$.

The Steenrod algebra is generated by element Sq^i of degree $i > 0$ if $p = 2$, β and P^i of degree 1 and $2i(p-1) > 0$ if $p > 2$. These elements satisfy certain relations named the Adem relations. As an example in the mod-2 case the relations write:

$$Sq^a Sq^b = \sum_0^{[a/2]} \binom{a-2t}{b-t-1} Sq^{a+b-t} Sq^t$$

There are two types of relations for $p > 2$:

$$P^a P^b = \sum_0^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t} P^t$$

for $a, b > 0$, et :

$$P^a \beta P^b = \sum_0^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta P^{a+b-i} P^t + \sum_0^{[(a-1)/p]} (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} P^{a+b-i} \beta P^t$$

for $a, b > 0$.
An easy consequence of these relations is :

Theorem 1.1 *The elements Sq^{2^i} if $p=2$; and β and P^{p^i} if $p>2$ form a minimal set of multiplicative generators.*

There is more structure. The cohomology of a space is an unstable module, which means that for a cohomology class x

- $Sq^i(x)=0$ if $i>|x|$ if $p=2$,
- $\beta^\epsilon P^i(x)=0$ if $\epsilon+2i>|x|$, if $p>2$,

One denotes by $\mathcal{U}$ the abelian category of unstable modules.

As an example consider $H^*B\mathbb{Z}/2 \otimes \mathbb{F}_2[u]$, $|u|=1$; $H^*B\mathbb{Z}/p \cong E(t)\otimes\mathbb{F}_p[x]$, $|t|=1$ and $|x|=2$.
One has $Sq^1(u)=u^2$; resp. $\beta(t)=u$ and $P^1(x)=x^p$.
These, the Cartan formula that gives the action on products, the restriction axiom that tells that

- $Sq^d=x^2$ if $|x|=d$ $(p=2)$;
- $P^ix=x^p$ if $|x|=2i$ $(p>2$.

and the instability completely determine the action.

The definition of the suspension of an unstable module is central in the theory. This is motivated by the suspension theorem for the cohomology of ΣX:

$$\tilde{H}^*\Sigma X\cong\Sigma\tilde{H}^*X$$

with ΣM defined by:

- $(\Sigma M)^n\cong M^{n-1}$,
- $\theta(\Sigma x)=\Sigma\theta(x)$.

or

$$\Sigma M\cong M\otimes\Sigma\mathbb{F}_p$$

The category of algebras that are unstable modules, and such that the above properties relating the two structures hold is called the category of unstable algebras and denoted by $\mathcal{K}$.

It is a very classical question in homotopy theory to ask wether a certain unstable $\mathcal{A}_p$-module is the mod p cohomology of a space. The Hopf invariant one problem is a very famous example, it is the following one.
Given a map $f\colon S^{2n-1}\to S^n$, consider the cone C_f of the map. The reduced cohomology of C_f is of dimension 1 in dimension n and $2n$, trivial elsewhere. Denote by g_n (resp. g_{2n}) a generator in degree n (resp. $2n$). The Hopf invariant of f is defined (up to a sign) by the equation

$$g_n^2=H(f)g_{2n}$$

If n is odd one works with mod-2 cohomology. The question is to decide whether $H(f)$ can take the value 1.
Here are two examples: the self-map of S^1, $z \mapsto z^2$ whose cone is $\mathbb{R}P^2$, the Hopf map $S^3 \to S^3/S^1 \cong S^2$ whose cone is $\mathbb{C}P^2$, have both Hopf invariant 1.
Because of the restriction axiom for unstable algebras the equation above rewrites as:

$$Sq^n g_n = H(f) g_{2n}$$

So one can reformulate the Hopf invariant one question as follows. Let k be a given integer, does there exists a 2-cells space, with one cell in dimension h a second one in dimension $n+h$ related by the operation Sq^n

$$\begin{array}{ccc} h & & n+h \\ \mathbb{F}_2 & \ldots 0 \ldots & \mathbb{F}_2 \end{array}$$
$$\mathbb{F}_2 \xrightarrow{Sq^n} \mathbb{F}_2$$

In fact doing that one modifies the question by going to the stable homotopy world. Because of 1.1 for such a complex to exist n must be a power of 2. So the problem reduces to complexes as:

$$\begin{array}{ccc} h & & h+2^k \\ \mathbb{F}_2 & \ldots 0 \ldots & \mathbb{F}_2 \end{array}$$
$$\mathbb{F}_2 \xrightarrow{Sq^{2^k}} \mathbb{F}_2$$

It corresponds in terms of the Adams spectral sequence for spheres to decide wether or not the elements h_i of the first line of the E_2-term persists to infinity.
The problem was solved by John Frank Adams using secondary operations in mod 2 cohomology in a celebrated paper [Ad60], the only values of k for which this holds are $0, 1, 2, 3$. Later Adams et Michael Atiyah gave a proof based on Adams operations in K-theory [AA66].
This is strongly linked to a geometrical problem : for which values of k does there exists a Lie group structure (or a somewhat weaker structure, *e.g.* H-space structure) on the sphere S^h? Outside of S^0 the sphere needs to be of odd dimension by elementary differential geometry. The answer is that the only possible values are $1, 3, 7$.

The Kervaire invariant one problem is another example, it is equivalent to the existence of complexes as is shown below:

$$\begin{tikzcd}[row sep=small] n & n+1 & & n+2^k \\ \mathbb{F}_2 \arrow[r, bend left, "Sq^1"] \arrow[rrr, bend right, "Sq^{2^k}"'] & \mathbb{F}_2 & \ldots 0 \ldots & \mathbb{F}_2 \end{tikzcd}$$

or

$$\begin{tikzcd}
n & n+1 & & n+2^k & n+1+2^k \\
\mathbb{F}_2 \arrow[r, bend left, "Sq^1"] \arrow[rrr, bend right, "Sq^{2^k}"'] & \mathbb{F}_2 \arrow[rrr, bend left, "Sq^{2^k}"] & \dots 0 \dots & \mathbb{F}_2 \arrow[r, bend right, "Sq^1"'] & \mathbb{F}_2
\end{tikzcd}$$

Such complexes are known to exist if $k = 0, 1, 2, 3, 4, 5, 6$ do not exist if $n > 7$ after the recent work of M. Hill, M. Hopkins and D. Ravenel [HHR]. Their proof depends on an equivariant cohomology theory linked to an height 4 formal group law. The case $n = 7$ remains unsolved.
Again this question has a geometric counterpart. The question being to know wether or not a stably framed manifolds is cobordant or not to an exotic differentiable sphere. Here are examples that are not $S^1 \times S^1$, $S^3 \times S^3$, $S^7 \times S^7$ with the framing induced by the Lie group, or octonion structure. Homotopy theory tells that such examples can only occur in dimension $2^k - 2$, corresponding to the elements h_i^2 of the second line of the E_2-term of the Adams spectral sequence for spheres.

The preceding examples (as well as others) give evidences for the following "Local Realisation Conjecture" (LCR) done in a slightly more restricted form by Nick Kuhn [K95].
Let M_1 and M_2 be two unstable modules. Assume M_i is the reduced cohomology of a space X_i, and that one is given a map $f \colon \Sigma^k X_2 \to X_1$ that induces the trivial map in cohomology. Then the long exact sequence splits and the cohomology of the cone of f is an element in $\mathrm{E}xt^1_{\mathcal{U}}(M_1, \Sigma^{k+1} M_2)$. The most famous examples have been described above as $\tilde{H}^* \mathbb{R}P^2$ and $\tilde{H}^* \mathbb{C}P^2$.

Conjecture 1.1 *Let M_1 and M_2 be two finite unstable modules. Let k be an integer that is large enough. Then any non-trivial extension*

$$E \in \mathrm{E}xt^1_{\mathcal{U}}(M_1, \Sigma^k M_2)$$

is not the cohomology of a space.

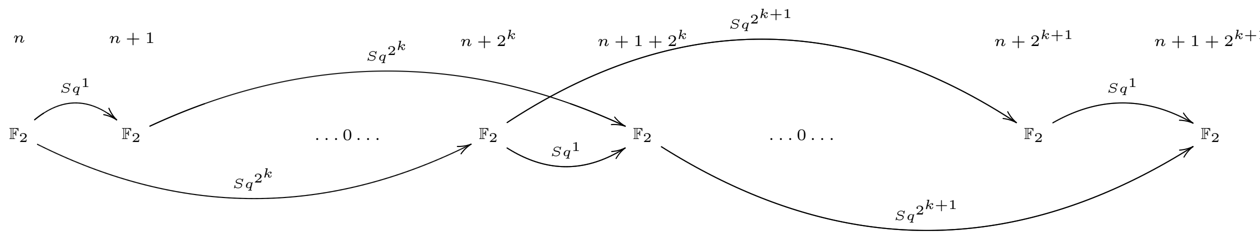
The construction above can be generalised as follows. Suppose given a map $f \colon X_2 \to X_1$ and assume it can be factored as a composition of n-maps g_i, $1 \leq i \leq n$, inducing the trivial map in reduced cohomology. One says that f has Adams filtration at least n. Splicing together the extensions obtained from the maps g_i one gets an element in $Ext^n_{\mathcal{U}}(\tilde{H}^* X_1, \tilde{\Sigma}^n H^* X_2)$. This a way to construct the Adams spectral sequence.

In this talk one will now describe another way to get results about the realisation problem. One will consider a certain unstable module M, assume it is the reduced cohomology of a space X, and then consider mapping spaces $map(S, X)$, may be pointed, and get contradictions by looking at the the cohomology of the mapping space. One option for the space S is to choose S^n. In this case one will consider the space of pointed maps. If $n = 1$ one has at hand the Eilenberg-Moore spectral sequence to evaluate the cohomology of the space of pointed loops. More generally (for any n there is a generalisation of the former, induced by the Goodwillie-Arone tower. The first case is studied in [**?**], the second one in [K08]. This is not what one will describe here. However it is worth to mention that what makes possible to use this tools is the nice behaviour of these spectral sequence with respect to the action of the Steenrod algebra. In particular in the case of the Eilennberg-Moore spectral sequence the properties of the E_2-term

$$\mathrm{T}or_{H^*X}^{-i,*}(\mathbb{F}_p, \mathbb{F}_p)$$

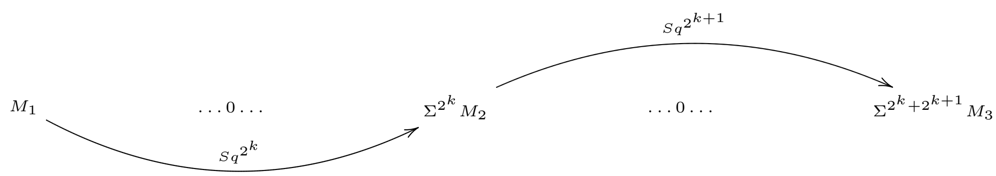
as $\mathcal{A}_p$-module are well understood ([S98].
Below is the type of results one can get :

Theorem 1.2 *Let k be large enough. There does not exist a complex which has as reduced cohomology the following module :*



More generally, one would like to have a result as described informally below (at $p = 2$) :

Theorem 1.3 *Let M_1, M_2, M_3 be given finite modules. Let k be large enough. As soon as there exists $x \in M_1$ such that $Sq^{2^{k+1}}Sq^{2^k}x \neq 0$ there does not exist a complex which has as reduced cohomology "looking like":the following :*



One can also consider the space of all maps with $S = B\mathbb{Z}/p$. in this case the Bousfield-Kan spectral sequence for the cohomology of the mapping space degenerates because of the properties of $H^*B\mathbb{Z}/p$ as an object of the categories $\mathcal{U}$ and $\mathcal{K}$. This is what one is going to do, and show that information about the algebraic structure of the category $\mathcal{U}$ allows to get substantial results.
The results described below are (most of the time, but not all) of a more qualitative nature. Here is an example [GS12] , conjectured by Kuhn and Stanley Kochman :

Theorem 1.4 *[Gérald Gaudens, L. Schwartz] Let X be a space such that H^*X is finitely generated as an $\mathcal{A}_p$-module. Then H^*X is finite.*

The LRC-conjecture implies 1.4, this follows from what will be described later as "Kuhn's trick". The proofs depends on the algebraic structure of the category $\mathcal{U}$, and as said above, on the cohomology of mapping spaces.
This result will be a consequence of :

Theorem 1.5 *[G. Gaudens, Nguyen The Cuong, L. Schwartz] Let X be an m-cone, for some m. If $QH^*X \in \mathcal{U}_n$, then $QH^*X \in \mathcal{U}_0$*

In the next section one describes m-cones and discuss some qualitative results that motivates interest for spaces with nilpotent cohomology. In section 3 one describes a first filtration of the category $\mathcal{U}$, a second one is described in section 5.

2 m-cones and finite Postnikov systems

There are two, dual to some extent, ways to construct spaces in homotopy theory. The first one is by attaching cells. One says that a space is 0-cone if it is contractible, an m-cone is the homotopy cofiber (the cone) of any map from a space A to an $m-1$-cone. It is clear that an m-cone has cup-length less than $m+1$. This means that any $(m+1)$-fold product is trivial. In particular any element of positive degree is nilpotent. The following theorem [FHLT] gives restrictions on the cohomology of an n-cone.

Theorem 2.1 *[Yves Félix, Stephen Halperin, Jean Michel Lemaire, Jean Claude Thomas] If X is 1-connected and the homology is finite dimensional in each degree then*

$$depth(H_*(\Omega X;\mathbb{F}_p)) \leq cat(X)$$

The depth of a graded connected k-algebra R (possibly infinity) is the largest n such that $Ext^i_R(k,R)=\{0\}$, $i<n$, $cat(X)$ denotes the Lusternick-Schnirelman category of X. This is the minimum number of elements of covering of X by contractible subspaces.

Here is the second way to construct spaces: a 1-Postnikov system or GEM (generalized Eilenberg-Mac Lane space) is a product (may be infinite) of usual Eilenberg-Mac Lane $K(\pi,n)$-spaces. An m-Postnikov system is the homotopy fiber of an $(m-1)$-Postnikov system into a GEM. The m-th Postnikov tower $P_m(X)$ of a space X is a particular case of an m-system.

Corollary 2.2 *Let X be a 1-connected m-cone, assume that the cohomology is finite dimensional in any degree. Then the p-localisation of X is never a finite p-local Postnikov system.*

This is to be compared with [LS89]:

Theorem 2.3 *[Jean Lannes, L. Schwartz] Let $P_n(X)$ be a 1-connected n-Postnikov tower such that $H^*P_n(X)$ is non-trivial. Then the reduced cohomology $\tilde{H}^*P_nX$ contains a non nilpotent element.*

A finite 1-connected Postnikov tower is never an n-cone, because the cup length of an n-cone is bounded by $n+1$. Nevertheless, Jiang Dong Hua [JDG] has shown there exists a 3-stage Postnikov system with nilpotent cohomology.

3 The Krull filtration on $\mathcal{U}$

The category of unstable modules. $\mathcal{U}$ has a natural filtration: the Krull filtration, by thick subcategories stable under colimits

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \ldots \subset \mathcal{U}$$

Brcause of the degree filtration the simple objects are the $\Sigma^n\mathbb{F}_p$.
The subcategory $\mathcal{U}_0$ is the largest thick sub-category generated by simple objects and stable under colimits. It is the subcategory of locally finite modules. An unstable module is locally finite if the span over $\mathcal{A}_p$ of any $x \in M$ is finite.

Having defined by induction $\mathcal{U}_n$ one defines $\mathcal{U}_{n+1}$ as follows. One first introduces the quotient category $\mathcal{U}/\mathcal{U}_n$ whose objects are the same of those of $\mathcal{U}$ but where morphisms in $\mathcal{U}$ that have kernel and cokernel in $\mathcal{U}_0$ are formally inverted. Then $(\mathcal{U}/\mathcal{U}_n)_0$ is defined as above and $\mathcal{U}_{n+1}$ is the pre-image of this subcategory in $\mathcal{U}$ *via* the canonical projection. This construction works for any abelian category. One refers to [Gab] for details.
This induces a filtration on any unstable module M, one has [S94]:

Theorem 3.1 *Let $M \in \mathcal{U}$ and $K_n(M)$ be the largest sub-object of M that is in $\mathcal{U}_n$, then*

$$M = \cup_n K_n(M)$$

As examples one has

- $\Sigma^k F(n) \in \mathcal{U}_n \setminus \mathcal{U}_{n-1}$, the unstable modules $F(n)$ are the canonical generators of $\mathcal{U}$, generated in degree n by ι_n and $\mathbb{F}_2$-basis $Sq^I\iota_n$, I an admissible multi-index of excess less than n;
- $H^*B\mathbb{Z}/2 \cong \mathbb{F}_2[u]$, does not belong to $\mathcal{U}_n$ any n but,
- $H^*B\mathbb{Z}/2$ is a Hopf algebra and the n-th step of the primitive filtration $P_nH^*B\mathbb{Z}/2$ is in $\mathcal{U}_n$.

There is a characterisation of the Krull filtration in terms of a functor introduced by Lannes and denoted T.

Definition 3.2 *The functor $T\colon \mathcal{U} \to \mathcal{U}$ is left adjoint to the functor $M \mapsto H^*B\mathbb{Z}/p \otimes M$. As the unstable module splits up as the direct sum $\mathbb{F}_p \oplus \tilde{H}^*B\mathbb{Z}/p$.*
The functor T is isomorphic to the direct sum of the identity functor and of the functor $\bar{T}$ left adjoint of $M \mapsto \tilde{H}^*B\mathbb{Z}/p \otimes M$.

It is easy to compute $T(\Sigma^n\mathbb{F}_p)$ and show that it is isomorphic to $\Sigma^n\mathbb{F}_p$.
The functor T has wonderful properties that will be shortly described at the end of this section. As a consequence one gets [S94]

Theorem 3.3 *The following two conditions are equivalent:*

- $M \in \mathcal{U}_n$,
- $\bar{T}^{n+1}(M) = \{0\}$.

There is also a characterisation of objects in $\mathcal{U}_n$ of combinatorial nature (as soon as they are of finite dimension in any degree) [S06]:

Theorem 3.4 *A finitely generated unstable $\mathcal{A}_2$-module M is in $\mathcal{U}_n$ if only if its Poincaré series $\Sigma_n a_n t^n$ has the following property. There exits an integer k so that the coefficient a_d can be non trivial only for the values of d such that if $\alpha(d-i) \leq n$, for some $0 \leq i \leq k$.*

In this statement $\alpha(k)$ is the number of 1 in the 2-adic expansion of k. A similar statement holds for $p > 2$.

Let $\mathcal{F}$ be the category of functors from finite dimensional $\mathbb{F}_p$-vector spaces to all vector spaces. Define a functor $f : \mathcal{U} \to \mathcal{F}$ by (here V is a finite dimensional $\mathbb{F}_p$-vector space)

$$f(M)(V) = \mathrm{H}om_{\mathcal{U}}(M, H^*(BV))^* = T_V(M)^0$$

It makes the following diagram commutes :

$$\begin{tikzcd}
\mathcal{U}_0 \ldots \arrow[d, "f"] & \mathcal{U}_{n-1} \arrow[r, hook] \arrow[d, "f"] & \mathcal{U}_n \arrow[r, hook] \arrow[d, "f"] & \mathcal{U} \arrow[d, "f"] \\
\mathcal{F}_0 \ldots & \mathcal{F}_{n-1} \arrow[r, hook] & \mathcal{F}_n \arrow[r, hook] & \mathcal{F}
\end{tikzcd}$$

$\mathcal{F}_n$ is the sub-category of polynomial functors of degree less than n, which is defined as follows. Let $F \in \mathcal{F}$, let $\Delta(F) \in \mathcal{F}$ defined by

$$\Delta(F)(V) = ker(F(V \oplus \mathbb{F}_p) \to F(V))$$

Then by definition $F \in \mathcal{F}_n$ if and only if $\Delta^{n+1}(F) = 0$.
As an an example $V \mapsto V^{\otimes n}$ is in $\mathcal{F}_n$.

As announced above below are the main properties of the functor T_V, [La92], [S94]

Theorem 3.5 *[Lannes] The functor T_V commutes with colimits (as a left adjoint). It is exact. Moreover there is a canonical isomorphism*

$$T_V(M_1 \otimes M_2) \cong T_V(M_1) \otimes T_V(M_2)$$

A special case of the last property if $M_1 = \Sigma\mathbb{F}_p$ it writes as

$$T_V(\Sigma M) \cong \Sigma T_V(M)$$

It follows from 3.3 and the preceding theorem that

Corollary 3.6 *If $M \in \mathcal{U}_m$ and $N \in \mathcal{U}_n$ then $M \otimes N \in \mathcal{U}_{m+n}$*

4 Special cases of Kuhn's conjectures

The following has been conjectured by N. Kuhn. It is implied by the LRC, however not equivalent.

Theorem 4.1 *[Gaudens, Schwartz] Let X be a space such that $H^*X \in \mathcal{U}_n$ then $H^*X \in \mathcal{U}_0$.*

As said above he following corollary was also conjectured by Kuhn and sometimes before by Stanley Kochman.

Corollary 4.2 *Let X be a space. If H^*X has finitely many generators as unstable module it is finite.*

Indeed, if an unstable module has finitely many generators it is in $\mathcal{U}_n$ for some n, because it is a quotient of a finite direct sum of $F(k)'s$. Then by 1.4 it is in $\mathcal{U}_0$. But a finitely generated locally finite unstable module is finite.

Denote, as usual, by QH^*X the quotient of indecomposable elements ofH^*X:

$$QH^*X \cong \tilde{H}^*X/(\tilde{H}^*X)^2$$

Theorem 4.1 is a consequence of :

Theorem 4.3 *Let X be an m-cone, for some m or more generally a space so that any element in $\tilde{H}^*X$ is nilpotent. If $QH^*X \in \mathcal{U}_n$, then $QH^*X \in \mathcal{U}_0$*

As observed above the reduced cohomology of an n-cone is nilpotent, by that one means that any element is nilpotent.

There are two cases to distinguish.
In he first one there is a non-nilpotent element in the cohomology of X. Then, as $\tilde{H}^*X$ contains non nilpotent element in degrees d with $\alpha(d)$ arbitrary large, section 5 and 3.4 imply that $H^*X \notin \mathcal{U}_n$ for all n.
In the second case, $\tilde{H}^*X$ is nilpotent in the sense defined above. In particular there are non trivial algebra maps from H^*X into a polynomial algebra. So

$$\mathrm{H}om_{\mathcal{K}}(H^*X, H^*BV) = *$$

This at least true for $p = 2$. The case $p > 2$ is also true but needs the results of [LZ86] and the action of $\mathcal{A}_p$.
If $H^*X \in \mathcal{U}_n$, then $QH^*X \in \mathcal{U}_n$ and by the theorem $QH^*X \in \mathcal{U}_0$.
Under this hypothesis proposition 3.9.7 of [S94] (se also [DW]) implies that $H^*X \in \mathcal{U}_0$, in fact there is even an equivalence proposition 6.4.5 of the same reference).

Note that all of these results above are unstable. The cohomology of the Eilenberg-Mac Lane spectrum $H\mathbb{Z}/p$ is free monogenic but infinite.

In [K95] Kuhn proved the corollary under additional hypothesis, using the Hopf invariant one theorem. One key step is a reduction depending on Lannes' mapping space theorem which is going to be described in section 6. In [S98] the corollary is proved for $p = 2$ using the Eilenberg-Moore spectral sequence, the argument is claimed to extend to all primes. However it is observed that one has to take care of a differential d_{p-1} in the Eilenberg-Moore spectral sequence. As Gaudens observed the method of [S98] does not work without some more hypothesis, alike the triviality of the Bockstein homomorphism.

For $p = 2$ in [K08] Kuhn gives a proof depending on the Goodwillie-Arone spectral sequence. Manfred Stelzer and his student s get results for $p > 2$, however observed that the proof do not extend directly for $p > 2$.
The theorem is proved now using only the Bott-Samelson theorem and Lannes' mapping space theorem.

5 The nilpotent filtration

Above one has considered spaces so that any element in $\tilde{H}^*X$ is nilpotent and on introduced the terminology "nilpotent" for the cohomology. The restriction axiom allows to express this in term of the action of the Steenrod algebra. More precisely (for $p = 2$) it is equivalent to ask that the operation $Sq_0 \colon x \mapsto Sq^{|x|}x$ is "nilpotent" on any element. It makes it possible to extend this definition to any unstable module.

Definition 5.1 *One says that an unstable module M is nilpotent if for any $x \in M$ there exists k such that $Sq^k x = 0$.*

In particular an unstable module is 0-connected. A suspension is nilpotent. In fact one has the following:

Proposition 5.2 *An unstable module M is nilpotent if and only if it is the colimit of unstable modules which have a finite filtration whose quotients are suspensions.*

This allows to extend easily the definition for $p > 2$.
More generally one can define a filtration on $\mathcal{U}$. It is filtered by subcategories $\mathcal{N}il_s$, $s \geq 0$, $\mathcal{N}il_s$ is the smallest thick subcategory stable under colimits and containing k-suspensions.

$$\mathcal{U} = \mathcal{N}il_0 \supset \mathcal{N}il_1 \supset \mathcal{N}il_2 \supset \ldots \supset \mathcal{N}il_s \supset \ldots$$

By very definition *any $M \in \mathcal{N}il_s$ is $(s-1)$-connected..*

Proposition 5.3 *Any M has a convergent decreasing filtration $\{M_s\}_{s \geq 0}$ with $M_s/M_{s+1} \cong \Sigma^s R_s(M)$ where $R_s(M)$ is a reduced unstable module, i.e. does not contain a non trivial suspension.*

Only the second part of the proposition needs a small argument see [S94] or [K95]. The following results are easy consequences of the commutation of T with suspension, the definition, and of 3.5. Just the last needs a small amount of additional care because T does not commutes with limits.

Proposition 5.4 *One has the following properties*

- *if $M \in \mathcal{N}il_m$, $N \in \mathcal{N}il_n$ then $M \otimes N \in \mathcal{N}il_{m+n}$;*
- *if $M \in \mathcal{N}il_m$ then $T(M) \in \mathcal{N}il_m$,*
- *$M \in \mathcal{U}_n$ if and only if for any s $f(R_s(M)) \in \mathcal{F}_n$.*

The following is easy:

Proposition 5.5 *The indecomposable elements of an augmented unstable algebra are in $\mathcal{N}il_1$.*

Let us introduce (following N. Kuhn) for M an unstable module a function $w_M \colon \mathbb{N} \to \mathbb{Z} \cup \infty$.

$$w_M(i) = \deg f(R_i(M))$$

The following lemma is a consequence of 3.6 and 5.4:

Lemma 5.6 $M \in \mathcal{N}il_s \Rightarrow T(M) \in \mathcal{N}il_s$
Let M be such that $w_M(i) \leq i$, $w_{\mathbb{T}(M)}$ the tensor algebra on M. Then the function $w_{\mathbb{T}(M)}$ has the same property.

Below are two statements that imply 4.3

Let X be a space, define $w_X = w_{H^*X}$ and $q_X = w_{QH^*X}$.

Theorem 5.7 (Gaudens, Nguyen T. Cuong, Schwartz) *Let X be such that $\tilde{H}^*X \in \mathcal{N}il_1$. The function q_X either is equal to 0 or $q_X - Id$ takes at least one positive (non zero) value.*

Theorem 5.8 (Gaudens, Schwartz) *Let X be such that $\tilde{H}^*X \in \mathcal{N}il_1$. The function w_X either is equal to 0 or $w_X - Id$ takes arbitrary large values.*

6 Lannes' theorem and Kuhn's reduction, beginning of the proof

X p-complete, 1-connected, assume that TH^*X is finite dimensional in each degree. Following François Xavier Dehon and Gaudens hese conditions could be relaxed using Morel's machinery of profinite spaces. The following theorem of Lannes is the major geometrical application of 3.5. It has lot of applications, in particular in the theory of p-compact groups (Dwyer and Wilkerson) and of p-local groups (Robert Oliver). The evaluation map :

$$B\mathbb{Z}/P \times map(B\mathbb{Z}/p, X) \to X$$

induces a map in cohomology :

$$H^*X \to H^*B\mathbb{Z}/P \otimes H^*map(B\mathbb{Z}/P, X)$$

and by adjunction

$$TH^*X \to H^*map(B\mathbb{Z}/p, X)$$

Theorem 6.1 (Lannes) *Under the hypothesis mentionned above the natural map $TH^*X \to H^*\mathrm{map}(B\mathbb{Z}/p, X)$ is an isomorphism of unstable algebras.*

Kuhn considers the homotopy cofiber $\Delta(X)$, of the natural map $x \to map(B\mathbb{Z}/p, X)$. reduction is to consider the cofiber $\Delta(X)$ of $X \to \mathrm{map}(B\mathbb{Z}/p, X)$. Then 6.1 immediatly yields :

$$H^*(\Delta(X)) \cong \bar{T}H^*X$$

As a consequence if $H^*X \in \mathcal{U}_n \setminus \mathcal{U}_{n-1}$, then $H^*\Delta(X) \in \mathcal{U}_{n-1} \setminus \mathcal{U}_{n-2}$.

Given an augmented unstable algebra K he indecomposable functor Q does commute with T :

$$T(Q(K)) \cong Q(TK)$$

but this is not true with $\bar{T}$. However if K is a Hopf algebra it is true, in particular let Z be an H-space, then (CCS)

Proposition 6.2 *$QH^*\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z) = \bar{T}^n QH^*Z$.*

On the way one notes that these authors proved the following beautiful result :

Theorem 6.3 (Castellana, Crespo, Scherer) *Let X be an H-space such that $QH^*X \in \mathcal{U}_n$, then $QH^*\Omega X \in \mathcal{U}_{n-1}$*

In order to prove 5.7 or 5.8 one shows a space cannot be such that $\tilde{H}^*X \in \mathcal{N}il_1$ and such that q_X is not 0 and les or equal to Id (one adapts in the second case). Kuhn's reduction allows us to suppose that the reduced mod-p cohomology is exactly s-nilpotent, $s > 0$ and that $R_s(H^*X) \in \mathcal{U}_1 \setminus \mathcal{U}_0$.

Let Z be $\Omega\Sigma X$, then $H^*Z \cong \mathbb{T}(\tilde{H}^*X)$

The first part of the proof consists of i the following chain of implications:

- $q_X \leq Id \Rightarrow w_X \leq Id$, in fact this hols for any unstable algebra K;
- $w_X \leq Id \Rightarrow w_Z \leq Id$, this follows from 5.6;
- $w_Z \leq Id \Rightarrow \bar{T}^n H^*Z$ is $(ns-1)$-connected;
- $\bar{T}^n H^*Z$ $(ns-1)$-connected $\Rightarrow map_*(B^{\wedge n}, Z)$ $(ns-1)$-connected, this follows from 6.2.

It follows that :

Proposition 6.4 *$\tilde{H}^*Z \in \mathcal{N}il_s$, $\bar{T}^n(H^*Z)$ is $(ns-1)$-connected, thus $map_*(B^{\wedge n}, Z)$ $(ns-1)$-connected.*

Then, one gets a non trivial algebraic map (of unstable algebras)

$$\varphi_s^* : H^*Z \to \Sigma^s R_s H^*Z \to \Sigma^s F(1) \subset \Sigma^s \tilde{H}^* B\mathbb{Z}/p\ .$$

It cannot factor through $H^*\Sigma^{s-1}K(\mathbb{Z}/p, 2)$, because there are no non trivial map from an s-suspension (and thus from an unstable module in $\mathcal{N}il_s$) to an $(s-1)$-suspension of a reduced module as, $H^*K(\mathbb{Z}/p, 2)$, indded

Proposition 6.5 *$H^*K(\mathbb{Z}/P, 2)$ is reduced.*

7 End of the proof, obstruction theory

The contradiction comes from the fact that using obstruction theory one can construct a factorisation.
Construction of φ_s, $K(\mathbb{Z}/p, 2)$ and obstruction theory
The existence of a map realising φ_s^* is a consequence (using Lannes' theorem) of the Hurewicz theorem because $\mathrm{map}_*(B\mathbb{Z}/p, Z)$ is $(s-1)$-connected.

$K(\mathbb{Z}/p,2)$ is built up, starting with $\Sigma B\mathbb{Z}/p$, as follows (Milnor's construction). There is a filtration $* = C_0 \subset C_1 = \Sigma B\mathbb{Z}/p \subset C_2 \subset \ldots \subset \cup_n C_n = K(\mathbb{Z}/p,2)$, a diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & B^{*n+1} & \longrightarrow & B^{*n+2} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & C_n & \longrightarrow & C_{n+1} & \longrightarrow & \cdots \end{array}$$

and cofibrations, up to homotopy

$$\Sigma^{n-1} B^{\wedge n} \to C_{n-1} \to C_n$$

$$\Sigma^{n-2+s} B^{\wedge n} \to \Sigma^{s-1} C_{n-1} \to \Sigma^{s-1} C_n$$

The obstructions to extend $\varphi_s \colon \Sigma^s B\mathbb{Z}/p \to Z$ to $\Sigma^{s-1} K(\mathbb{Z}/p,2)$ are in the groups

$$[\Sigma^{n+s-2}(B\mathbb{Z}/p)^{\wedge n}, Z] = \pi_{n+s-2}\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$$

but $\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$ is $(ns-1)$-connected. As $ns - 1 \geq n + s - 2$ they are trivial. It follows one can do the extension, this is a contradiction.
To prove the last theorem it is now enough to observe that $w_{\Delta(X)} = w_X - 1$...

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