A NOTE ON EXHAUSTION OF HYPERBOLIC COMPLEX MANIFOLDS

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Abstract. The purpose of this article is to investigate a hyperbolic complex manifold M exhausted by a pseudoconvex domain Ω in $\mathbb{C}^n$ via an exhausting sequence $\{f_j\colon \Omega \to M\}$ such that $f_j^{-1}(a)$ converges to a boundary point $\xi_0 \in \partial\Omega$ for some point $a \in M$.

1. introduction

Let M and Ω be two complex manifolds. One says that Ω *can exhaust* M or M *can be exhausted by* Ω if for any compact subset K of M there is a holomorphic embedding $f_K\colon \Omega \to M$ such that $f_K(\Omega) \supset K$. In this case, one may consider a sequence $f_j\colon \Omega \to M$ so that for any compact subset K of M there is a number $j_0(K) > 0$ so that $f_j(\Omega) \supset K$ for $j > j_0(K)$. Throughout this paper, we call such a sequence $\{f_j\}$ an *exhausting sequence* (see [FS77, Fr83, FM95]).

In [Fr86, Theorem 1], there exists a bounded domain D in $\mathbb{C}^n$ such that D can exhaust any domain in $\mathbb{C}^n$. In addition, the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ can exhaust many complex manifods, which are not biholomorphically equivalent to each other (see [For04, FS77]). However, if M in addition is hyperbolic then M must be biholomorphically equivalent to $\mathbb{B}^n$ (cf. [FS77]). Furthermore, any n-dimensional hyperbolic complex manifold, exhausted by a homogeneous bounded domain D in $\mathbb{C}^n$, is biholomorphically equivalent to D. As a consequence, although the polydisc $\mathbb{U}^n$ and the unit ball $\mathbb{B}^n$ are both homogeneous and there is a domain U in $\mathbb{B}^n$ that contains almost all of $\mathbb{B}^n$, i.e., $\mathbb{B}^n\backslash U$ has measure zero (cf. [FS77, Theorem 1]), and is biholomorphically equivalent to $\mathbb{U}^n$, yet $\mathbb{U}^n$ cannot exhaust the unit ball $\mathbb{B}^n$ since it is well-known that $\mathbb{U}^n$ is not biholomorphically equivalent to $\mathbb{B}^n$.

Let M be a hyperbolic complex manifold exhausted by a bounded domain $\Omega \subset \mathbb{C}^n$ via an exhausting sequence $\{f_j\colon \Omega \to M\}$. Let us fix a point $a \in M$. Then, thanks to the boundedness of Ω, without loss of generality we may assume that $f_j^{-1}(a) \to \xi_0 \in \overline{\Omega}$ as $j \to \infty$. If $\xi_0 \in \Omega$, then one always has that M is biholomorphically equivalent to Ω (cf. Lemma 2.3 in Section 2).

The purpose of this paper is to investigate such a complex manifold M with $\xi_0 \in \partial\Omega$. More precisely, our first main result is the following theorem.

Theorem 1.1. *Let M be an $(n+1)$-dimensional hyperbolic complex manifold and let Ω be a pseudoconvex domain in $\mathbb{C}^{n+1}$ with C^∞-smooth boundary. Suppose that M can be exhausted by Ω via an exhausting sequence $\{f_j : \Omega \to M\}$. If there exists a point $a \in M$ such that the sequence $f_j^{-1}(a)$ converges Λ-nontangentially to an h-extendible boundary point $\xi_0 \in \partial\Omega$ (see Definition 2.2 in Section 2 for definitions of the Λ-nontangential*

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convergence and of the h-extendibility), then M is biholomorphically equivalent to the associated model M_P for Ω at ξ_0.

When ξ_0 is a strongly pseudoconvex boundary point, we do not need the condition that the sequence $f_j^{-1}(a)$ converges Λ-nontangentially to ξ_0 as $j \to \infty$. Moreover, in this circumstance, the model M_P is in fact biholomorphically equivalent to $M_{|z|^2}$, which is biholomorphically equivalent to the unit ball $\mathbb{B}^{n+1}$. More precisely, our second main result is the following theorem.

Theorem 1.2. *Let M be an $(n+1)$-dimensional hyperbolic complex manifold and let Ω be a pseudoconvex domain in $\mathbb{C}^{n+1}$. Suppose that $\partial\Omega$ is $\mathcal{C}^2$-smooth boundary near a strongly pseudoconvex boundary point $\xi_0 \in \partial\Omega$. Suppose also that M can be exhausted by Ω via an exhausting sequence $\{f_j : \Omega \to M\}$. If there exists a point $a \in M$ such that the sequence $\eta_j := f_j^{-1}(a)$ converges to ξ_0, then M is biholomorphically equivalent to the unit ball $\mathbb{B}^{n+1}$.*

Notice that Theorem 1.2 is a local version of [DZ19, Theorem 1.1] and [Fr83, Theorem I] (see Corollary 3.4 in Section 3). We note that their proofs are based on the boundary estimate of the Fridman invariant and of the squeezing function for strongly pseudoconvex domains. However, in order to prove Theorem 1.1 and Theorem 1.2, we shall use the scaling technique, achieved recently in [Ber06, DN09, NN19].

By applying Theorem 1.2 and Lemma 2.3, we also prove that if a hyperbolic complex manifold M is exhausted by a general ellipsoid D_P provided that D_P is a WB-domain (see Section 4 for the definitions of D_P and WB-domains), then M is either biholomorphically equivalent to D_P or to the unit ball $\mathbb{B}^n$ (cf. Proposition 4.1 in Section 4). In particular, when D_P is an ellipsoid E_m ($m \in \mathbb{Z}_{\geqslant 1}$), given by

$$E_m = \left\{(z, w) \in \mathbb{C}^2 \colon |w|^2 + |z|^{2m} < 1\right\},$$

in fact Proposition 4.1 is a generalization of [Liu18, Theorem 1].

The organization of this paper is as follows: In Section 2 we provide some results concerning the normality of a sequence of biholomorphisms and the h-extendibility. In Section 3, we give our proofs of Theorem 1.1 and Theorem 1.2. Finally, the proof of Proposition 4.1 will be introduced in Section 4.

2. The normality and the h-extendibility

2.1. **The normality of a sequence of biholomorphisms.** First of all, we recall the following definition (see [GK87] or [DN09]).

Definition 2.1. Let $\{\Omega_i\}_{i=1}^\infty$ be a sequence of open sets in a complex manifold M and Ω_0 be an open set of M. The sequence $\{\Omega_i\}_{i=1}^\infty$ is said to converge to Ω_0 (written $\lim \Omega_i = \Omega_0$) if and only if

(i) For any compact set $K \subset \Omega_0$, there is an $i_0 = i_0(K)$ such that $i \geqslant i_0$ implies that $K \subset \Omega_i$; and
(ii) If K is a compact set which is contained in Ω_i for all sufficiently large i, then $K \subset \Omega_0$.

Next, we recall the following proposition, which is a generalization of the theorem of H. Cartan (see [DN09, GK87, DT04]).

Proposition 2.1. *Let $\{A_i\}_{i=1}^{\infty}$ and $\{\Omega_i\}_{i=1}^{\infty}$ be sequences of domains in a complex manifold M with $\lim A_i = A_0$ and $\lim \Omega_i = \Omega_0$ for some (uniquely determined) domains A_0, Ω_0 in M. Suppose that $\{f_i : A_i \to \Omega_i\}$ is a sequence of biholomorphic maps. Suppose also that the sequence $\{f_i : A_i \to M\}$ converges uniformly on compact subsets of A_0 to a holomorphic map $F : A_0 \to M$ and the sequence $\{g_i := f_i^{-1} : \Omega_i \to M\}$ converges uniformly on compact subsets of Ω_0 to a holomorphic map $G : \Omega_0 \to M$. Then one of the following assertions holds.*

(i) *The sequence $\{f_i\}$ is compactly divergent, i.e., for each compact set $K \subset A_0$ and each compact set $L \subset \Omega_0$, there exists an integer i_0 such that $f_i(K) \cap L = \varnothing$ for $i \geqslant i_0$; or*

(ii) *There exists a subsequence $\{f_{i_j}\} \subset \{f_i\}$ such that the sequence $\{f_{i_j}\}$ converges uniformly on compact subsets of A_0 to a biholomorphic map $F : A_0 \to \Omega_0$.*

Remark 2.2. By [Ber94, Proposition 2.1] or [DN09, Proposition 2.2] and by the hypotheses of Theorem 1.1 and Theorem 1.2, it follows that for each compact subset $K \Subset M$ and each neighborhood U of ξ_0 in $\mathbb{C}^{n+1}$, there exists an integer $j_0 = j_0(K)$ such that $K \subset f_j(\Omega \cap U)$ for all $j \geqslant j_0$. Consequently, the sequence of domains $\{f_j(\Omega \cap U)\}$ converges to M.

We will finish this subsection by recalling the following lemma (cf. [Fr83, Lemma 1.1]).

Lemma 2.3 (see [Fr83])**.** *Let M be a hyperbolic complex manifold of complex dimension n. Assume that M can be exhausted by Ω via an exhausting sequence $\{f_j : \Omega \to M\}$, where Ω is a bounded domain in $\mathbb{C}^n$. Suppose that there is an interior point $a \in M$ such that $f_j^{-1}(a) \to p \in \Omega$. Then, M is biholomorphically equivalent to Ω.*

2.2. **The h-extendibility.** In this subsection, we recall some definitions and notations given in [Cat84, Yu95].

Let Ω be a smooth pseudoconvex domain in $\mathbb{C}^{n+1}$ and $p \in \partial\Omega$. Let ρ be a local defining function for Ω near p. Suppose that the multitype $\mathcal{M}(p) = (1, m_1, \ldots, m_n)$ is finite. (See [Cat84] for the notion of multitype.) Let us denote by $\Lambda = (1/m_1, \ldots, 1/m_n)$. Then, there are distinguished coordinates $(z, w) = (z_1, \ldots, z_n, w)$ such that $p = 0$ and $\rho(z, w)$ can be expanded near 0 as follows:

$$\rho(z, w) = \mathrm{Re}(w) + P(z) + R(z, w),$$

where P is a Λ-homogeneous plurisubharmonic polynomial that contains no pluriharmonic terms, R is smooth and satisfies

$$|R(z, w)| \leqslant C \left(|w| + \sum_{j=1}^{n} |z_j|^{m_j} \right)^{\gamma},$$

for some constant $\gamma > 1$ and $C > 0$. Here and in what follows, a polynomial P is called Λ-homogeneous if

$$P(t^{1/m_1} z_1, t^{1/m_2} z_2, \ldots, t^{1/m_n} z_n) = tP(z),\ \forall t > 0, \forall z \in \mathbb{C}^n.$$

Definition 2.2 (see [NN19])**.** The domain $M_P = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} \colon \mathrm{Re}(w) + P(z) < 0\}$ is called an *associated model* of Ω at p. A boundary point $p \in \partial\Omega$ is called *h-extendible* if its associated model M_P is *h-extendible*, i.e., M_P is of finite type (see [Yu94, Corollary

2.3]). In this circumstance, we say that a sequence $\{\eta_j = (\alpha_j, \beta_j)\} \subset \Omega$ *converges Λ-nontangentially to p* if $|\mathrm{Im}(\beta_j)| \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$ and $\sigma(\alpha_j) \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$, where

$$\sigma(z) = \sum_{k=1}^{n} |z_k|^{m_k}.$$

Throughout this paper, we use $\lesssim$ and $\gtrsim$ to denote inequalities up to a positive multiplicative constant. Moreover, we use $\approx$ for the combination of $\lesssim$ and $\gtrsim$. In addition, $\mathrm{dist}(z, \partial\Omega)$ denotes the Euclidean distance from z to $\partial\Omega$. Furthermore, for $\mu > 0$ we denote by $\mathcal{O}(\mu, \Lambda)$ the set of all smooth functions f defined near the origin of $\mathbb{C}^n$ such that

$$D^u \overline{D}^v f(0) = 0 \text{ whenever } \sum_{j=1}^{n} (u_j + v_j) \frac{1}{m_j} \leqslant \mu.$$

If $n = 1$ and $\Lambda = (1)$ then we use $\mathcal{O}(\mu)$ to denote the functions vanishing to order at least μ at the origin (cf. [Cat84, Yu95]).

3. Proofs of Theorem 1.1 and Theorem 1.2

This section is devoted to our proofs of Theorem 1.1 and Theorem 1.2. First of all, let us recall the definition of the Kobayashi infinitesimal pseudometric and the Kobayashi pseudodistance as follows:

Definition 3.1. Let M be a complex manifold. The Kobayashi infinitesimal pseudometric $F_M \colon M \times T^{1,0}M \to \mathbb{R}$ is defined by

$$F_M(p, X) = \inf \{c > 0 \mid \exists\, f \colon \Delta \to M \text{ holomorphic with } f(0) = p, f'(0) = X/c\},$$

for any $p \in M$ and $X \in T^{1,0}M$, where Δ is the unit open disk of $\mathbb{C}$. Moreover, the Kobayashi pseudodistance $d_M^K \colon M \times M \to \mathbb{R}$ is defined by

$$d_M^K(p, q) = \inf_{\gamma} \int_0^1 F_M(\gamma(t), \gamma'(t)) dt,$$

for any $p, q \in M$ where the infimum is taken over all differentiable curves $\gamma : [0, 1] \to M$ joining p and q. A complex manifold M is called hyperbolic if $d_M^K(p, q)$ is actually a distance, i.e., $d_M^K(p, q) > 0$ whenever $p \neq q$.

Next, we need the following lemma, whose proof will be given in the Appendix for the convenience of the reader, and the following proposition.

Lemma 3.1. *Assume that $\{D_j\}$ is a sequence of domains in $\mathbb{C}^{n+1}$ converging to a model M_P of finite type. Then, we have*

$$\lim_{j\to\infty} F_{D_j}(z, X) = F_{M_P}(z, X), \ \forall (z, X) \in M_P \times \mathbb{C}^{n+1}.$$

Moreover, the convergence takes place uniformly over compact subsets of $M_P \times \mathbb{C}^{n+1}$.

Proposition 3.2 (see [NN19]). *Assume that $\{D_j\}$ is a sequence of domains in $\mathbb{C}^{n+1}$ converging to a model M_P of finite type. Assume also that ω is a domain in $\mathbb{C}^k$ and $\sigma_j : \omega \to D_j$ is a sequence of holomorphic mappings such that $\{\sigma_j(a)\} \Subset M_P$ for some $a \in \omega$. Then $\{\sigma_j\}$ contains a subsequence that converges locally uniformly to a holomorphic map $\sigma : \omega \to M_P$.*

Now we are ready to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let ρ be a local defining function for Ω near ξ_0 and the multitype $\mathcal{M}(\xi_0) = (1, m_1, \dots, m_n)$ is finite. In what follows, denote by $\Lambda = (1/m_1, \dots, 1/m_n)$. Since ξ_0 is an h-extendible point, there exist local holomorphic coordinates (z, w) in which $\xi_0 = 0$ and Ω can be described in a neighborhood U_0 of 0 as follows:

$$\Omega \cap U_0 = \{\rho(z,w) = \mathrm{Re}(w) + P(z) + R_1(z) + R_2(\mathrm{Im} w) + (\mathrm{Im} w)R(z) < 0\},$$

where P is a Λ-homogeneous plurisubharmonic real-valued polynomial containing no pluriharmonic terms, $R_1 \in \mathcal{O}(1, \Lambda), R \in \mathcal{O}(1/2, \Lambda)$, and $R_2 \in \mathcal{O}(2)$. (See the proof of Theorem 1.1 in [NN19] or the proof of Lemma 4.11 in [Yu95].)

By assumption, there exists a point $a \in M$ such that the sequence $\eta_j := f_j^{-1}(a)$ converges Λ-nontangentially to ξ_0. Without loss of generality, we may assume that the sequence $\{\eta_j\} \subset \Omega \cap U_0$ and we write $\eta_j = (\alpha_j, \beta_j) = (\alpha_{j1}, \dots, \alpha_{jn}, \beta_j)$ for all j. Then, the sequence $\{\eta_j := f^{-1}(a)\}$ has the following properties:

(a) $|\mathrm{Im}(\beta_j)| \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$;
(b) $|\alpha_{jk}|^{m_k} \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$ for $1 \leqslant k \leqslant n$.

For the sequence $\{\eta_j = (\alpha_j, \beta_j)\}$, we define a sequence of points $\eta_j' = (\alpha_{j1}, \dots, \alpha_{jn}, \beta_j + \epsilon_j)$, where $\epsilon_j > 0$, such that η_j' is in the hypersurface $\{\rho = 0\}$ for all j. We note that $\epsilon_j \approx \mathrm{dist}(\eta_j, \partial\Omega)$. Now let us consider the sequences of dilations Δ^{ϵ_j} and translations $L_{\eta_j'}$, defined respectively by

$$\Delta^{\epsilon_j}(z_1, \dots, z_n, w) = \left(\frac{z_1}{\epsilon_j^{1/m_1}}, \dots, \frac{z_n}{\epsilon_j^{1/m_n}}, \frac{w}{\epsilon_j}\right)$$

and

$$L_{\eta_j'}(z, w) = (z, w) - \eta_j' = (z - \alpha_j', w - \beta_j').$$

Under the change of variables $(\tilde{z}, \tilde{w}) := \Delta^{\epsilon_j} \circ L_{\eta_j'}(z, w)$, i.e.,

$$\begin{cases} w - \beta_j' = \epsilon_j \tilde{w} \\ z_k - \alpha_{jk}' = \epsilon_j^{1/m_k} \tilde{z}_k, \; k = 1, \dots, n, \end{cases}$$

one can see that $\Delta^{\epsilon_j} \circ L_{\eta_j'}(\alpha_j, \beta_j) = (0, \cdots, 0, -1)$ for all j. Moreover, as in [NN19], after taking a subsequence if necessary, we may assume that the sequence of domains $\Omega_j := \Delta^{\epsilon_j} \circ L_{\eta_j'}(\Omega \cap U_0)$ converges to the following model

$$M_{P,\alpha} := \{(\tilde{z}, \tilde{w}) \in \mathbb{C}^n \times \mathbb{C} \colon \mathrm{Re}(\tilde{w}) + P(\tilde{z} + \alpha) - P(\alpha) < 0\},$$

which is obviously biholomorphically equivalent to the model M_P. Without loss of generality, in what follows we always assume that $\{\Omega_j\}$ converges to M_P.

Now we first consider the sequence of biholomorphisms $F_j := T_j \circ f_j^{-1} \colon M \supset f_j(\Omega \cap U_0) \to \Omega_j$, where $T_j := \Delta^{\epsilon_j} \circ L_{\eta_j'}$. Since $F_j(a) = (0', -1)$ and $f_j(\Omega \cap U_0)$ converges to M as $j \to \infty$ (see Remark 2.2), by Proposition 3.2, without loss of generality, we may assume that the sequence F_j converges uniformly on every compact subset of M to a holomorphic map F from M to $\mathbb{C}^{n+1}$. Note that $F(M)$ contains a neighborhood of $(0', -1)$ and $F(M) \subset \overline{M_P}$.

Since F_j is normal, by the Cauchy theorem it follows that $\{J(F_j)\}$ converges uniformly on every compact subsets of M to $J(F)$, where $J(F)$ denotes the Jacobian determinant of F. However, by the Cartan theorem, $J(F_j)(z)$ is nowhere zero for any j because F_j is a biholomorphism. Then, the Hurwitz theorem implies that $J(F)$ is either a zero function or nowhere zero. In the case that $JF \equiv 0$, F is regular at no point of M. As $F(M)$ contains a neighborhood of $(0', -1)$, the Sard theorem shows that F is regular outside a proper subvariety of M, which is a contradiction. This yields JF is nowhere zero and hence F is regular everywhere on M. By [FS77, Lemma 0], it follows that $F(M)$ is open and $F(M) \subset M_P$.

Next, we shall prove that F is one-to-one. Indeed, let $z_1, z_2 \in M$ be arbitrary. Fix a compact subset $L \Subset M$ such that $z_1, z_2 \in L$. Then, by Remark 2.2 there is a $j_0(L) > 0$ such that $L \subset f_j(\Omega \cap U_0)$ and $F_j(L) \subset K \Subset M_P$ for all $j > j_0(L)$, where K is a compact subset of M_P. By Lemma 3.1 and the decreasing property of Kobayashi distance, one has

$$\begin{aligned} d^K_M(z_1, z_2) &\leqslant d^K_{f_j(\Omega\cap U_0)}(z_1, z_2) = d^K_{\Omega_j}(F_j(z_1), F_j(z_2))) \leqslant C \cdot d^K_{M_P}(F_j(z_1), F_j(z_2)) \\ &\leqslant C\left(d^K_{M_P}(F(z_1), F(z_2)) + d^K_{M_P}(F_j(z_1), F(z_1)) + d^K_{M_P}(F_j(z_2), F(z_2))\right), \end{aligned}$$

where $C > 1$ is a positive constant. Letting $j \to \infty$, we obtain

$$d^K_M(z_1, z_2) \leqslant C \cdot d^K_{M_P}(F(z_1), F(z_2)).$$

Since M is hyperbolic, it follows that if $F(z_1) = F(z_2)$, then $z_1 = z_2$. Consequently, F is one-to-one, as desired.

Finally, because of the biholomorphism from M to $F(M) \subset M_P$ and the tautness of M_P (cf. [Yu95]), it follows that the sequence $F_j^{-1} = f_j \circ T_j^{-1} : T_j(\Omega \cap U_0) \to f_j(\Omega\cap U_0) \subset M$ is also normal. Moreover, since $T_j \circ f_j^{-1}(a) = (0', -1) \in M_P$, it follows that the sequence $T_j \circ f_j^{-1}$ is not compactly divergent. Therefore, by Proposition 2.1, after taking some subsequence we may assume that $T_j \circ f_j^{-1}$ converges uniformly on every compact subset of M to a biholomorphism from M onto M_P. Hence, the proof is complete. ☐

Remark 3.3. If M is a bounded domain in $\mathbb{C}^{n+1}$, the normality of the sequence F_j^{-1} can be shown by using the Montel theorem. Thus, the proof of Theorem 1.1 simply follows from Proposition 2.1.

Proof of Theorem 1.2. Let ρ be a local defining function for Ω near ξ_0. We may assume that $\xi_0 = 0$. After a linear change of coordinates, one can find local holomorphic coordinates $(\tilde z, \tilde w) = (\tilde z_1, \cdots, \tilde z_n, \tilde w)$, defined on a neighborhood U_0 of ξ_0, such that

$$\rho(\tilde z, \tilde w) = \mathrm{Re}(\tilde w) + \sum_{j=1}^n |\tilde z_j|^2 + O(|\tilde w|\|\tilde z\| + \|\tilde z\|^3).$$

By [DN09, Proposition 3.1] (or Subsection 3.1 in [Ber06] for the case $n = 1$), for each point η in a small neighborhood of the origin, there exists an automorphism Φ_η of $\mathbb{C}^n$ such that

$$\rho(\Phi_\eta^{-1}(z, w)) - \rho(\eta) = \mathrm{Re}(w) + \sum_{j=1}^n |z_j|^2 + O(|w|\|z\| + \|z\|^3).$$

Let us define an anisotropic dilation Δ^ϵ by

$$\Delta^\epsilon(z_1, \cdots, z_n, w) = \left(\frac{z_1}{\sqrt{\epsilon}}, \cdots, \frac{z_n}{\sqrt{\epsilon}}, \frac{w}{\epsilon}\right).$$

For each $\eta \in \partial\Omega$, if we set $\rho^\epsilon_\eta(z,w) = \epsilon^{-1}\rho \circ \Phi_\eta^{-1} \circ (\Delta^\epsilon)^{-1}(z,w)$, then

$$\rho^\epsilon_\eta(z,w) = \mathrm{Re}(w) + \sum_{j=1}^n |z_j|^2 + O(\sqrt{\epsilon}).$$

By assumption, the sequence $\eta_j := f_j^{-1}(a)$ converges to ξ_0. Then, we define a sequence of points $\eta'_j = (\eta_{j1}, \cdots, \eta_{jn}, \eta_{j(n+1)} + \epsilon_j)$, $\epsilon_j > 0$, such that η'_j is in the hypersurface $\{\rho = 0\}$. Then $\Delta^{\epsilon_j} \circ \Phi_{\eta'_j}(\eta_j) = (0, \cdots, 0, -1)$ and one can see that $\Delta^{\epsilon_j} \circ \Phi_{\eta'_j}(\{\rho = 0\})$ is defined by an equation of the form

$$\mathrm{Re}(w) + \sum_{j=1}^n |z_j|^2 + O(\sqrt{\epsilon_j}) = 0.$$

Therefore, it follows that, after taking a subsequence if necessary, $\Omega_j := \Delta^{\epsilon_j} \circ \Phi_{\eta'_j}(U_0^-)$ converges to the following domain

$$\mathcal{E} := \{\hat{\rho} := \mathrm{Re}(w) + \sum_{j=1}^n |z_j|^2 < 0\}, \tag{1}$$

which is biholomorphically equivalent to the unit ball $\mathbb{B}^{n+1}$.

Now let us consider the sequence of biholomorphisms $F_j := T_j \circ f_j^{-1} \colon M \supset f_j(\Omega \cap U_0) \to T_j(\Omega \cap U_0)$, where $T_j := \Delta^{\epsilon_j} \circ \Phi_{\eta'_j}$. Since $F_j(a) = (0', -1)$, by [DN09, Theorem 3.11], without loss of generality, we may assume that the sequence F_j converges uniformly on every compact subset of M to a holomorphic map F from M to $\mathbb{C}^{n+1}$. Note that $F(M)$ contains a neighborhood of $(0', -1)$ and $F(M) \subset \overline{M_P}$. Following the argument as in the proof of Theorem 1.1, we conclude that F is a biholomorphism from M onto $\mathcal{E}$, and thus M is biholomorphically equivalent to $\mathbb{B}^{n+1}$, as desired. □

By Lemma 2.3 and Theorem 1.2, we obtain the following corollary, proved by F. S. Deng and X. J. Zhang [DZ19, Theorem 2.4] and by B. L. Fridman [Fr83, Theorem I].

Corollary 3.4. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^2$-smooth boundary. If a bounded domain Ω can be exhausted by D, then Ω is biholomorphically equivalent to D or the unit ball $\mathbb{B}^n$.*

4. Exhausting a complex manifold by a general ellipsoid

In this section, we are going to prove that if a hyperbolic complex manifold M can be exhausted by a general ellipsoid D_P provided that it is a WB-domain (see the definitions of D_P and WB-domains below), then M is biholomorphically equivalent to either D_P or the unit ball B^n.

First of all, let us fix n positive integers $m_1, \ldots, m_{n-1}$ and denote by $\Lambda := \left(\frac{1}{m_1}, \ldots, \frac{1}{m_{n-1}}\right)$. We assign weights $\frac{1}{m_1}, \ldots, \frac{1}{m_{n-1}}, 1$ to $z_1, \ldots, z_n$. For an $(n-1)$-tuple $K = (k_1, \ldots, k_{n-1}) \in \mathbb{Z}^{n-1}_{\geqslant 0}$, denote the weight of K by

$$wt(K) := \sum_{j=1}^{k-1} \frac{k_j}{m_j}.$$

Next, we consider the general ellipsoid D_P in $\mathbb{C}^n$ ($n \geqslant 2$), defined by

$$D_P := \{(z', z_n) \in \mathbb{C}^n \colon |z_n|^2 + P(z') < 1\},$$

where

$$P(z') = \sum_{wt(K)=wt(L)=1/2} a_{KL} {z'}^K \bar{z'}^L, \tag{2}$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \bar{a}_{LK}$, satisfying that $P(z') > 0$ whenever $z' \in \mathbb{C}^{n-1} \setminus \{0'\}$. We would like to emphasize here that the polynomial P given in (2) is Λ-homogeneous and the assumption that $P(z') > 0$ whenever $z' \neq 0$ ensures that D_P is bounded in $\mathbb{C}^n$ (cf. [NNTK19, Lemma 6]). Moreover, since $P(z') > 0$ for $z' \neq 0$ and by the Λ-homogeneity, there are two constants $c_1, c_2 > 0$ such that

$$c_1 \sigma_\Lambda(z') \leqslant P(z') \leqslant c_2 \sigma_\Lambda(z'),\ \forall z' \in \mathbb{C}^{n-1},$$

where $\sigma_\Lambda(z') = |z_1|^{m_1} + \cdots + |z_{n-1}|^{m_{n-1}}$. In addition, D_P is called a WB-domain if it is strongly pseudoconvex at every boundary point outside the set $\{(0', e^{i\theta}) \colon \theta \in \mathbb{R}\}$ (cf. [AGK16]).

Now we prove the following proposition.

Proposition 4.1. *Let M be a n-dimensional hyperbolic complex manifold. Suppose that M can be exhausted by the general ellipsoid D_P via an exhausting sequence $\{f_j : D_P \to M\}$. If D_P is a WB-domain, then M is biholomorphically equivalent to either D_P or the unit ball $\mathbb{B}^n$.*

Remark 4.2. The possibility that M is biholomorphic to the unit ball $\mathbb{B}^n$ is not excluded because D_P can exhaust the unit ball $\mathbb{B}^n$ by [FM95, Corollary 1.4].

Proof of Proposition 4.1. Let q be an arbitrary point in M. Then, thanks to the boundedness of D_P, after passing to a subsequence if necessary we may assume that the sequence $\{f_j^{-1}(q)\}_{j=1}^{\infty}$ converges to a point $p \in \overline{D_P}$ as $j \to \infty$.

We now divide the argument into two cases as follows:

Case 1. $f_j^{-1}(q) \to p \in D_P$. Then, it follows from Lemma 2.3 that M is biholomorphically equivalent to D_P.

Case 2. $f_j^{-1}(q) \to p \in \partial D_P$. Let us write $f_j^{-1}(q) = (a_j', a_{jn}) \in D_P$ and $p = (a', a_n) \in \partial D_P$. As in [NNTK19], for each $j \in \mathbb{N}^*$ we consider $\psi_j \in \operatorname{Aut}(D_P)$, defined by

$$\psi_j(z) = \left(\frac{(1-|a_{jn}|^2)^{1/m_1}}{(1-\bar{a}_{jn} z_n)^{2/m_1}} z_1, \ldots, \frac{(1-|a_{jn}|^2)^{1/m_{n-1}}}{(1-\bar{a}_{jn} z_n)^{2/m_{n-1}}} z_{n-1}, \frac{z_n - a_{jn}}{1-\bar{a}_{jn} z_n} \right).$$

Then $\psi_j \circ f_j(q) = (b_j, 0)$, where

$$b_j = \left(\frac{a_{j1}}{(1-|a_{jn}|^2)^{1/m_1}}, \ldots, \frac{a_{j(n-1)}}{(1-|a_{jn}|^2)^{1/m_{n-1}}} \right),\ \forall j \in \mathbb{N}^*.$$

Without loss of generality, one may assume that $b_j \to b \in \mathbb{C}^{n-1}$ as $j \to \infty$.

Since D_P is a WB-domain, two possibilities may occur:

Subcase 1: $p = (a', a_n)$ is a strongly pseudoconvex boundary point. In this subcase, it follows directly from Theorem 1.2 that M is biholomorphically equivalent to $\mathbb{B}^n$.

Subcase 2: $p = (0', e^{i\theta})$ is a weakly pseudoconvex boundary point. In this subcase, one must have $a_j' \to 0'$ and $a_{jn} \to e^{i\theta}$ as $j \to \infty$. Denote by $\rho(z) := |z_n|^2 - 1 + P(z')$ a defining function for D_P. Then $\operatorname{dist}(a_j, \partial D_P) \approx -\rho(a_j) = 1 - |a_{jn}|^2 - P(a_j')$. Suppose that $\{a_j\}$ converges Λ-nontangentially to p, i.e., $P(a_j') \approx \sigma_\Lambda(a_j') \lesssim \mathrm{dist}(a_j, \partial D_P)$, or equivalently $P(a_j') \leqslant C(1-|a_{jn}|^2 - P(a_j'))$, $\forall j \in \mathbb{N}^*$, for some $C > 0$. This implies that

$$P(a_j') \leqslant \frac{C}{1+C}(1-|a_{jn}|^2),\ \forall j \in \mathbb{N}^*,$$

and thus $P(b_j) = \dfrac{1}{1-|a_{jn}|^2}P(a_j') \leqslant \dfrac{C}{1+C} < 1$, $\forall j \in \mathbb{N}^*$. This yields $\psi_j \circ f_j^{-1}(q) = (b_j, 0) \to (b,0) \in D_P$ as $j \to \infty$. So, again by Lemma 2.3 one concludes that M is biholomorphically equivalent to D_P.

Now let us consider the case that the sequence $\{a_j\}$ does not converge Λ-nontangentially to p, i.e., $P(a_j') \geqslant c_j \mathrm{dist}(a_j, \partial D_P)$, $\forall j \in \mathbb{N}^*$, where $0 < c_j \to +\infty$. This implies that $P(a_j') \geqslant c_j'(1-|a_{jn}|^2 - P(a_j'))$, $\forall j \in \mathbb{N}^*$, for some $0 < c_j' \to +\infty$, and hence

$$P(a_j') \geqslant \frac{c_j'}{1+c_j'}(1-|a_{jn}|^2),\ \forall j \in \mathbb{N}^*.$$

Thus, one obtains that $P(b_j) = \dfrac{1}{1-|a_{jn}|^2}P(a_j') \geqslant \dfrac{c_j'}{1+c_j'}$, which implies that $P(b) = 1$. Consequently, $\psi_j \circ f_j^{-1}(q)$ converges to the strongly pseudoconvex boundary point $p' = (b, 0)$ of ∂D_P. Hence, as in Subcase 1, it follows from Theorem 1.2 that M is biholomorphically equivalent to $\mathbb{B}^n$.

Therefore, altogether, the proof of Proposition 4.1 finally follows. □

Appendix

Proof of Lemma 3.1. We shall follow the proof of [Yu95, Theorem 2.1] with minor modifications. To do this, let us fix compact subsets $K \Subset M_P$ and $L \Subset \mathbb{C}^{n+1}$. Then it suffices to prove that $F_{D_j}(z, X)$ converges to $F_{M_P}(z, X)$ uniformly on $K \times L$. Indeed, suppose otherwise. Then, there exist $\epsilon_0 > 0$, a sequence of points $\{z_{j_\ell}\} \subset K$, and a sequence $X_{j_\ell} \subset L$ such that

$$|F_{D_{j_\ell}}(z_{j_\ell}, X_{j_\ell}) - F_{M_P}(z_{j_\ell}, X_{j_\ell})| > \epsilon_0,\ \forall\ \ell \geqslant 1.$$

By the homogeneity of the Kobayashi metrics $F(z, X)$ in X, we may assume that $\|X_{j_\ell}\| = 1$ for all $\ell \geqslant 1$. Moreover, passing to subsequences, we may also assume that $z_{j_\ell} \to z_0 \in K$ and $X_{j_\ell} \to X_0 \in L$ as $\ell \to \infty$. Since M_P is taut (see [Yu95, Theorem 3.13]), for each $(z, X) \in M_P \times \mathbb{C}^{n+1}$ with $X \neq 0$, there exists an analytic disc $\varphi \in \mathrm{Hol}(\Delta, M_P)$ such that $\varphi(0) = z$ and $\varphi'(0) = X/F_{M_P}(z, X)$. This implies that $F_{M_P}(z, X)$ is continuous on $M_P \times \mathbb{C}^{n+1}$. Hence, we obtain

$$F_{M_P}(z_{j_\ell}, X_{j_\ell}) \to F_{M_P}(z_0, X_0),$$

and thus we have

$$|F_{D_{j_\ell}}(z_{j_\ell}, X_{j_\ell}) - F_{M_P}(z_0, X_0)| > \epsilon_0/2 \tag{3}$$

for ℓ big enough.

By definition, for any $\delta \in (0,1)$ there exists a sequence of analytic discs $\varphi_{j_\ell} \in \mathrm{Hol}(\Delta, D_{j_\ell})$ such that $\varphi_{j_\ell}(0) = z_0, \varphi_{j_\ell}'(0) = \lambda_{j_\ell} X_{j_\ell}$, where $\lambda_{j_\ell} > 0$, and

$$F_{D_{j_\ell}}(z_{j_\ell}, X_{j_\ell}) \geqslant \frac{1}{\lambda_{j_\ell}} - \delta.$$

It follows from Proposition 3.2 that every subsequence of the sequence $\{\varphi_{j_\ell}\}$ has a subsequence converging to some analytic disc $\psi \in \mathrm{Hol}(\Delta, M_P)$ such that $\psi(0) = z_0, \psi'(0) = \lambda X_0$, for some $\lambda > 0$. Thus, one obtains that

$$F_{M_P}(z_0, X_0) \leqslant \frac{1}{|\psi'(0)|}$$

for any such ψ. Therefore, one has

$$\liminf_{\ell\to\infty} F_{D_{j_\ell}}(z_{j_\ell}, X_{j_\ell}) \geqslant F_{M_P}(z_0, X_0) - \delta. \tag{4}$$

On the other hand, as in [Yu95], by the tautness of M_P, there exists an analytic disc $\varphi \in \mathrm{Hol}(\Delta, M_P)$ such that $\varphi(0) = z_0, \varphi'(0) = \lambda X_0$, where $\lambda = 1/F_{M_P}(z_0, X_0)$.

Now for $\delta \in (0,1)$, let us define an analytic disc $\psi^\delta_{j_\ell} : \Delta \to \mathbb{C}^{n+1}$ by settings:

$$\psi^\delta_{j_\ell}(\zeta) := \varphi((1-\delta)\zeta) + \lambda(1-\delta)(X_{j_\ell} - X_0) + (z_{j_\ell} - z_0) \text{ for all } \zeta \in \Delta.$$

Since $\varphi((1-\delta)\overline{\Delta})$ is a compact subset of M_P and $X_{j_\ell} \to X_0$, $z_{j_\ell} \to z_0$ as $\ell \to \infty$, it follows that $\psi^\delta_{j_\ell}(\Delta) \subset D_{j_\ell}$ for all sufficiently large ℓ, that is, $\psi^\delta_{j_\ell} \in \mathrm{Hol}(\Delta, D_{j_\ell})$. Moreover, by construction, $\psi^\delta_{j_\ell}(0) = z_{j_\ell}$ and $\left(\psi^\delta_{j_\ell}\right)'(0) = (1-\delta)\lambda X_{j_\ell}$. Therefore, again by definition, one has

$$F_{D_{j_\ell}}(z_{j_\ell}, X_{j_\ell}) \leqslant \frac{1}{(1-\delta)\lambda} = \frac{1}{(1-\delta)} F_{M_P}(z_0, X_0)$$

for all large ℓ. Thus, letting $\delta \to 0^+$, one concludes that

$$\limsup_{\ell\to\infty} F_{D_{j_\ell}}(z_{j_\ell}, X_{j_\ell}) \leqslant F_{M_P}(z_0, X_0). \tag{5}$$

By (4), (5), and (3), we obtain a contradiction. Hence, the proof is complete. □

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