Approximation by linear combinations of translates of a single function

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Abstract

We study approximation of periodic functions by arbitrary linear combinations of n translates of a single function. We construct some linear methods of this approximation for univariate functions in the class induced by the convolution with a single function, and prove upper bounds of the L^p-approximation convergence rate by these methods, when $n \to \infty$, for $1 \le p \le \infty$. We also generalize these results to classes of multivariate functions defined as the convolution with the tensor product of a single function. In the case $p = 2$, for this class, we also prove a lower bound of the quantity characterizing best approximation of by arbitrary linear combinations of n translates of arbitrary function.

Keywords: Function spaces induced by the convolution with a given function ; Approximation by arbitrary linear combinations of n translates of a single function.

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1 Introduction

The present paper continues investigating the problem of function approximation by arbitrary linear combinations of n translates of a single function which has been studied in [1, 3]. In the last papers, some linear methods were constructed for approximation of periodic functions in a class induced by the convolution with a given function, and prove upper bounds of the L^p-approximation convergence rate by these methods, when $n \to \infty$, for the case $1 < p < \infty$. The main technique of the proofs of the results is based on Fourier analysis, in particular, the multiplier theory. However, this technique cannot be extended to the two important cases $p = 1$ and $p = \infty$. In the present paper, we aim at this approximation problem for the cases $p = 1$ and $p = \infty$ by using a different technique. For convenience of presentation we will do this for $1 \le p \le \infty$.

We shall begin our discussion here by introducing notation used throughout the paper. In this regard, we merely follow closely the presentation in [1, 3]. The d-dimensional torus denoted by $\mathbb{T}^d$ is the cross product of d copies of the interval $[0, 2\pi]$ with the identification of the end points. When $d = 1$, we merely denote the d-torus by $\mathbb{T}$. Functions on $\mathbb{T}^d$ are identified with functions on $\mathbb{R}^d$ which are 2π periodic in each variable. Denote by $L^p(\mathbb{T}^d),\ 1 \leq p \leq \infty$, the space of integrable functions on $\mathbb{T}^d$ equipped with the norm

$$\|f\|_p \ := \begin{cases} (2\pi)^{-d/p} \left(\int_{\mathbb{T}^d} |f(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{\boldsymbol{x}\in\mathbb{T}^d} |f(\boldsymbol{x})|, & p = \infty. \end{cases}$$

We will consider only real valued functions on $\mathbb{T}^d$. However, all the results in this paper are true for the complex setting. Also, we will use Fourier series of a real valued function in complex form.

Here, we use the notation $\mathbb{N}_m$ for the set $\{1, 2, \ldots, m\}$. For vectors $\boldsymbol{x} := (x_l : l \in \mathbb{N}_d)$ and $\boldsymbol{y} := (y_l : l \in \mathbb{N}_d)$ in $\mathbb{T}^d$ we use $(\boldsymbol{x}, \boldsymbol{y}) := \sum_{l\in\mathbb{N}_d} x_l y_l$ for the inner product of $\boldsymbol{x}$ with $\boldsymbol{y}$. Also, for notational convenience we allow $\mathbb{N}_0$ and $\mathbb{Z}_0$ to stand for the empty set. Given any integrable function f on $\mathbb{T}^d$ and any lattice vector $\boldsymbol{j} = (j_l : l \in \mathbb{N}_d) \in \mathbb{Z}^d$, we let $\widehat{f}(\boldsymbol{j})$ denote the $\boldsymbol{j}$-th Fourier coefficient of f defined by the equation

$$\widehat{f}(\boldsymbol{j}) \ := \ (2\pi)^{-d} \int_{\mathbb{T}^d} f(\boldsymbol{x})\, e^{-i(\boldsymbol{j},\boldsymbol{x})}\, d\boldsymbol{x}.$$

Frequently, we use the superscript notation $\mathbb{B}^d$ to denote the cross product of d copies of a given set $\mathbb{B}$ in $\mathbb{R}^d$.

Let $S^{'}(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d$. Every $f \in S^{'}(\mathbb{T}^d)$ can be identified with the formal Fourier series

$$f = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \widehat{f}(\boldsymbol{j}) e^{i(\boldsymbol{j},.)},$$

where the sequence $(\widehat{f}(\boldsymbol{j}) : \quad \boldsymbol{j} \in \mathbb{Z}^d)$ forms a tempered sequence.

Let $\lambda : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be a bounded function. With the univariate λ we associate the multivariate tensor product function λ_d given by

$$\lambda_d(\boldsymbol{x}) := \prod_{l=1}^{d} \lambda(x_l), \quad \boldsymbol{x} = (x_l : l \in \mathbb{N}_d),$$

and introduce the function $\varphi_{\lambda,d}$, defined on $\mathbb{T}^d$ by the equation

$$\varphi_{\lambda,d}(\boldsymbol{x}) \ := \ \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \lambda_d(\boldsymbol{j})\, e^{i(\boldsymbol{j},\boldsymbol{x})}. \tag{1.1}$$

Moreover, in the case that $d = 1$ we merely write φ_λ for the univariate function $\varphi_{\lambda,1}$. We introduce a subspace of $L^p(\mathbb{T}^d)$ defined as

$$\mathcal{H}_{\lambda,p}(\mathbb{T}^d) := \left\{ f : f = \varphi_{\lambda,d} * g,\ g \in L^p(\mathbb{T}^d) \right\},$$

with norm

$$\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)} \ := \|g\|_p,$$

where $f_1 * f_2$ is the convolution of two functions f_1 and f_2 on $\mathbb{T}^d$.

As in [1, 3], we are concerned with the following concept. Let $\mathbb{W}$ be a prescribed subset of $L^p(\mathbb{T}^d)$ and $\psi \in L^p(\mathbb{T}^d)$ be a given function. We are interested in the approximation in $L^p(\mathbb{T}^d)$-norm of all functions $f \in \mathbb{W}$ by arbitrary linear combinations of n translates of the function ψ, that is, by the functions in the set $\big\{\psi(\cdot - \boldsymbol{y}_l): \ \boldsymbol{y}_l \in \mathbb{T}^d, l \in \mathbb{N}_n\big\}$ and measure the error in terms of the quantity

$$M_n(\mathbb{W}, \psi)_p := \sup_{f \in \mathbb{W}} \inf \left\{ \left\| f - \sum_{l \in \mathbb{N}_n} c_l \psi(\cdot - \boldsymbol{y}_l) \right\|_p : c_l \in \mathbb{R}, \boldsymbol{y}_l \in \mathbb{T}^d \right\}.$$

The aim of the present paper is to investigate the convergence rate, when $n \to \infty$, of $M_n(U_{\lambda,p}(\mathbb{T}^d), \psi)_p$ for $1 \le p \le \infty$, where

$$U_{\lambda,p}(\mathbb{T}^d) := \Big\{ f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d) : \quad \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)} \le 1 \Big\}$$

is the unit ball in $\mathcal{H}_{\lambda,p}(\mathbb{T}^d)$. We shall also obtain a lower bound for the convergence rate as $n \to \infty$ of the quantity

$$M_n(U_{\lambda,2}(\mathbb{T}^d))_2 := \inf \Big\{ M_n(U_{\lambda,2}(\mathbb{T}^d), \psi)_2 : \psi \in L^2(\mathbb{T}^d) \Big\},$$

which gives information about the best choice of ψ.

This paper is organized in the following manner. In Section 2, we give the necessary background from Fourier analysis and construct a method for approximation of functions in the univariate case. In Section 3, we extend the method of approximation developed in Section 2 to the multivariate case, in particular, prove upper bounds for the approximation error and convergence rate, we also prove a lower bound of $M_n(U_{\lambda,2}(\mathbb{T}^d))_2$.

2 Univariate approximation

In this section, we construct a linear method in the form of a linear combination of translates of a function φ_β defined as in (1.1) for approximation of univariate functions in $\mathcal{H}_{\lambda,p}(\mathbb{T})$. We give upper bounds of the approximation error for various λ and β.

Let $\lambda, \beta, \vartheta: \ \mathbb{R} \to \mathbb{R}$ be given 2-times continuously differentiable functions and ϑ be such that

$$\vartheta(x) := \begin{cases} 1, & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}], \\ 0, & \text{if } x \notin (-1, 1). \end{cases}$$

Corresponding to these functions we define the functions $\mathcal{G}$ and H_m as

$$\mathcal{G}(x) := \frac{\lambda(x)}{\beta(x)}, \quad H_m(x) := \sum_{k \in \mathbb{Z}} \vartheta(k/m) \mathcal{G}(k)\, e^{ikx}. \tag{2.2}$$

For a function $f \in \mathcal{H}_{\lambda,p}(\mathbb{T})$ represented as $f = \varphi_\lambda * g$, $g \in L^p(\mathbb{T})$, we define the operator

$$Q_{m,\beta}(f) := \frac{1}{2m+1} \sum_{k=0}^{2m} V_m(g) \left(\frac{k}{2m+1} \right) \varphi_\beta \left(\cdot - \frac{k}{2m+1} \right), \tag{2.3}$$

where $V_m(g) := H_m * g$. Finally, we define for a function $h: \mathbb{R} \to \mathbb{R}$,

$$\sigma_m(h; f)(x) := \sum_{k \in \mathbb{Z}} h(k/m) \widehat{f}_k e^{ikx}.$$

Let us obtain upper estimates for the error of approximating a function $f \in \mathcal{H}_{\lambda,p}(\mathbb{T})$ by the trigonometric polynomial $Q_{m,\beta}(f)$ a linear combination of $2m+1$ translates of the function φ_β.

Definition 2.1 *A 2-times continuously differentiable function $\psi : \mathbb{R} \to \mathbb{R}$ is called a function of monotone type if there exists a positive constant c_0 such that*

$$|\psi(x)| \geq c_0|\psi(y)|, \quad |\psi^{''}(x)| \geq c_0|\psi^{''}(y)| \quad \text{for all} \;\; 2|y| \geq |x| \geq |y|/2.$$

We put

$$\varepsilon_m := J_m(\lambda) + \sup_{|x|\in[-m,m]} \left(|\mathcal{G}(x)| + m^2 \sup_{|x|\in[-m,m]} |\mathcal{G}^{''}(x)| \right) J_m(\beta),$$

where for a 2-times continuously differentiable function ψ,

$$J_m(\psi) := \int_{|x|\geq m} \left(\left|\frac{\psi(x)}{m}\right| + \left|x\psi^{''}(x)\right| \right) dx.$$

Theorem 2.2 *Let $1 \leq p \leq \infty$. Assume that the functions λ, β are of monotone type. Then there exists a positive constant c such that for all $f \in \mathcal{H}_{\lambda,p}(\mathbb{T})$ and $m \in \mathbb{N}$,*

$$\|f - Q_{m,\beta}(f)\|_p \leq c\varepsilon_m \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T})}.$$

Before we give the proof of the above theorem, we recall a lemma proved in [6], [7].

Lemma 2.3 *Let $1 \leq p \leq \infty$, $f \in L^p(\mathbb{T})$ and $h : \mathbb{R} \to \mathbb{R}$ be 2-times continuously differentiable function, supported on $[-1,1]$. Then there exists a constant c_1 independent of f, h, m such that*

$$\|\sigma_m(h; f)||_p \leq c_1 \|h^{''}\|_\infty \|f\|_p.$$

We also need a Landau's inequality for derivatives [4].

Lemma 2.4 *Let $f \in L^\infty(\mathbb{R})$ be 2-times continuously differentiable function. Then*

$$\|f^{'}\|_\infty^2 \leq 4\|f\|_\infty \|f^{''}\|_\infty.$$

In particular,

$$\|f^{'}\|_\infty \leq \|f\|_\infty + \|f^{''}\|_\infty.$$

Proof. (Proof of Theorem 2.2) Let $f \in \mathcal{H}_{\lambda,p}(\mathbb{T})$ be represented as $\varphi_{\lambda,d} * g$ for some $g \in L^p(\mathbb{T})$. We define the kernel $P_m(x,t)$ for $x, t \in \mathbb{T}$ as

$$P_m(x,t) := \frac{1}{2m+1} \sum_{k=0}^{2m} \varphi_\beta \left(x - \frac{k}{2m+1} \right) H_m \left(\frac{k}{2m+1} - t \right).$$

It is easy to obtain from the definition (2.3) that

$$Q_{m,\beta}(f)(x) = \frac{1}{2\pi} \int_{\mathbb{T}} P_m(x,t) g(t)\, dt.$$

We now use equation (1.1), the definition of the trigonometric polynomial H_m given in equation (2.2) and the easily verified fact, for $k, s \in \mathbb{Z}, s \in [-m, m]$, that

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m} e^{ik(t-(\ell/2m+1))}e^{is((\ell/2m+1)-t)} = \begin{cases} 0, & \text{if } \frac{k-s}{2m+1} \notin \mathbb{Z}, \\ e^{i(k-k_m)t}, & \text{if } \frac{k-s}{2m+1} \in \mathbb{Z}, \end{cases}$$

to conclude that

$$P_m(x,t) = \sum_{k\in\mathbb{Z}} \gamma(k)e^{ikx}e^{-ik_m t},$$

where $\gamma(k) = \vartheta(k_m/m)\mathcal{G}(k_m)\beta(k)$ and $k_m \in [-m, m]$ satisfy $(k - k_m)/(2m + 1) \in \mathbb{Z}$. Hence,

$$\begin{aligned} Q_{m,\beta}(f)(x) &= \sum_{k>m} \gamma(k)e^{ikx}\widehat{g}(k_m) + \sum_{k<-m} \gamma(k)e^{ikx}\widehat{g}(k_m) + \sum_{k=-m}^{m} \gamma(k)e^{ikx}\widehat{g}(k_m) \\ &=: \mathcal{A}_m(x) + \mathcal{B}_m(x) + \mathcal{C}_m(x). \end{aligned}$$

Consequently,

$$\|f - Q_{m,\beta}(f)\|_p \;\leq\; \|\mathcal{A}_m\|_p + \|\mathcal{B}_m\|_p + \|f - \mathcal{C}_m\|_p. \tag{2.4}$$

For each $j \in \mathbb{N}$, we define the functions $\Lambda_{j,m}(x)$, $\mathcal{J}_m(x)$, $\mathcal{K}_{j,m}(x)$, $\mathcal{D}_{j,m}(x)$ and the set $I_{j,m}$ as follows

$$\Lambda_{j,m}(x) := \beta(mx + j(2m+1)), \qquad \mathcal{J}_m(x) := \mathcal{G}(mx),$$

$$\mathcal{K}_{j,m}(x) := \Lambda_{j,m}(x)\vartheta(x)\mathcal{J}_m(x), \qquad \mathcal{D}_{j,m}(x) := \sum_{k\in I_{j,m}} \gamma(k)e^{ikx}\widehat{g}(k_m),$$

$$I_{j,m} := \{k \in \mathbb{Z} : \quad (2m+1)j - m \leq k \leq (2m+1)j + m\}.$$

Then we have

$$\mathcal{A}_m(x) = \sum_{j\in\mathbb{N}}\sum_{k\in I_{j,m}} \gamma(k)e^{ikx}\widehat{g}(k_m) = \sum_{j\in\mathbb{N}} \mathcal{D}_{j,m}(x), \tag{2.5}$$

and for all $k \in I_{j,m}$,

$$\begin{aligned} \gamma(k) &= \beta(k)\vartheta(k_m/m)\mathcal{G}(k_m) = \beta(j(2m+1) + k_m)\vartheta(k_m/m)\mathcal{G}(k_m) \\ &= \Lambda_{j,m}(k_m/m)\vartheta(k_m/m)\mathcal{G}(k_m) = \Lambda_{j,m}(k_m/m)\vartheta(k_m/m)\mathcal{J}_m(k_m/m) = \mathcal{K}_{j,m}(k_m/m). \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{D}_{j,m}(x) &= \sum_{k\in I_{j,m}} \gamma(k)e^{ikx}\widehat{g}(k_m) = \sum_{k_m\in[-m,m]} \mathcal{K}_{j,m}(k_m/m)e^{i(j(2m+1)+k_m)x}\widehat{g}(k_m) \\ &= e^{ij(2m+1)x}\sum_{k_m\in[-m,m]} \mathcal{K}_{j,m}(k_m/m)e^{ik_m x}\widehat{g}(k_m) = e^{ij(2m+1)x}\sigma_m(\mathcal{K}_{j,m}; g). \end{aligned}$$

Therefore, by Lemma 2.3, there exists a constant c_1 such that

$$\|\mathcal{D}_{j,m}\|_p \leq c_1\|(\mathcal{K}_{j,m})''\|_\infty\|g\|_p.$$

Then it follows from (2.5) that

$$\|\mathcal{A}_m\|_p \leq \sum_{j\in\mathbb{N}} \|\mathcal{D}_{j,m}\|_p \leq c_1 \sum_{j\in\mathbb{N}} \|(\mathcal{K}_{j,m})^{''}\|_\infty \|g\|_p. \tag{2.6}$$

From the definition of $\mathcal{K}_{j,m}$, $\operatorname{supp}\vartheta \subset [-1,1]$, and $\|\vartheta\|_\infty \leq 2\|\vartheta^{'}\|_\infty \leq 4\|\vartheta^{''}\|_\infty$, we deduce that

$$\begin{aligned}\|(\mathcal{K}_{j,m})^{''}\|_\infty &\leq 4\|\vartheta^{''}\|_\infty \sup_{x\in[-1,1]} \Big(|\Lambda_{j,m}(x)\mathcal{J}_m(x)| + |(\Lambda_{j,m}\mathcal{J}_m)^{'}(x)| + |(\Lambda_{j,m}\mathcal{J}_m)^{''}(x)|\Big)\\ &\leq 4\|\vartheta^{''}\|_\infty \Bigg[\sup_{x\in I_{j,m}} \Big(|\beta(x)| + m|\beta^{'}(x)| + m^2|\beta^{''}(x)|\Big) \sup_{x\in[-m,m]} |\mathcal{G}(x)|\\ &+ m \sup_{x\in I_{j,m}} \Big(|\beta(x)| + m|\beta^{'}(x)|\Big) \sup_{x\in[-m,m]} |\mathcal{G}^{'}(x)| + m^2 \sup_{x\in I_{j,m}} |\beta(x)| \sup_{x\in[-m,m]} |\mathcal{G}^{''}(x)|\Bigg].\end{aligned}$$

Hence,

$$\|(\mathcal{K}_{j,m})^{''}\|_\infty \leq 4\|\vartheta^{''}\|_\infty \sup_{x\in I_{j,m}} \Big(|\beta(x)| \ + \ m|\beta^{'}(x)| + m^2|\beta^{''}(x)|\Big) \sup_{x\in[-m,m]} \Big(|\mathcal{G}(x)| + \ m|\mathcal{G}^{'}(x)| + m^2|\mathcal{G}^{''}(x)|\Big)$$

for all $j \in \mathbb{N}$. Therefore, it follows from (2.6) that

$$\begin{aligned}\|\mathcal{A}_m\|_p \leq 4c_1\|\vartheta^{''}\|_\infty \sum_{j\in\mathbb{N}} \sup_{x\in I_{j,m}} \Big(|\beta(x)| + m|\beta^{'}(x)| + m^2|\beta^{''}(x)|\Big) \times\\ \times \sup_{x\in[-m,m]} \Big(|\mathcal{G}(x)| + m|\mathcal{G}^{'}(x)| + m^2|\mathcal{G}^{''}(x)|\Big) \|g\|_p.\end{aligned}$$

So, by Lemma 2.4, we have

$$\|\mathcal{A}_m\|_p \quad \leq \quad 16c_1\|\vartheta^{''}\|_\infty \sum_{j\in\mathbb{N}} \sup_{x\in I_{j,m}} \Big(|\beta(x)| + m^2|\beta^{''}(x)|\Big) \sup_{x\in[-m,m]} \Big(|\mathcal{G}(x)| + m^2|\mathcal{G}^{''}(x)|\Big) \|g\|_p. \tag{2.7}$$

Since the function α, β is of monotone type, there exists a constant c_0 such that

$$|\alpha(x)| \geq c_0|\alpha(y)|, |\alpha^{''}(x)| \geq c_0|\alpha^{''}(y)|, |\beta(x)| \geq c_0|\beta(y)|, |\beta^{''}(x)| \geq c_0|\beta^{''}(y)| \tag{2.8}$$

for all $4|y| \geq |x| \geq |y|/4$. Hence,

$$\sup_{|x|\in I_{j,m}} |\beta(x)| \leq \frac{c_0}{m} \int_{|x|\in I_{j,m}} |\beta(x)|dx,$$

$$\sup_{|x|\in I_{j,m}} |m^2\beta^{''}(x)| \leq c_0 m \int_{|x|\in I_{j,m}} |\beta^{''}(x)|dx.$$

So,

$$\sum_{j\in\mathbb{N}} \sup_{|x|\in I_{j,m}} \Big(|\beta(x)| + |m^2\beta^{''}(x)|\Big) \leq c_0 \int_{|x|\geq m} \left(\frac{|\beta(x)|}{m} + \big|m\beta^{''}(x)\big|\right) dx \leq c_0 J_m(\beta).$$

Combining this with (2.7), we obtain that

$$\|\mathcal{A}_m\|_p \le 16c_0c_1\|\vartheta''\|_\infty\varepsilon_m\|g\|_p. \tag{2.9}$$

Similarly,

$$\|\mathcal{B}_m\|_p \le 16c_0c_1\|\vartheta''\|_\infty\varepsilon_m\|g\|_p. \tag{2.10}$$

Next, we will estimate $\|f-\mathcal{C}_m\|_p$. Notice that $\gamma(k) = \vartheta(k/m)\mathcal{G}(k)\beta(k) = \vartheta(k/m)\lambda(k)$ for $k \in [-m, m]$, and then

$$\sigma_m(\vartheta; f)(x) = \sum_{k\in\mathbb{Z}} \vartheta(k/m)\widehat{f}(k)e^{ikx} = \sum_{k=-m}^{m} \vartheta(k/m)\lambda(k)\widehat{g}(k)e^{ikx} = \sum_{k=-m}^{m} \gamma(k)\widehat{g}(k)e^{ikx} = \mathcal{C}_m(x),$$

and therefore,

$$\|f - \mathcal{C}_m\|_p = \|f - \sigma_m(\vartheta; f)\|_p. \tag{2.11}$$

We define the functions $S(x), \Phi_{j,m}(x)$ and $\Psi_{j,m}(x)$ as

$$S(x) := \vartheta(x) - \vartheta(x/2), \quad \Phi_{j,m}(x) := \lambda(2^j mx), \quad \Psi_{j,m}(x) := S(x)\Phi_{j,m}(x).$$

Clearly, we have that

$$(\vartheta(k/(2^{j+1}m)) - \vartheta(k/(2^j m)))\lambda(k) = S(k/(2^j m))\Phi_{j,m}(k/(2^j m)) = \Psi_{j,m}(k/(2^j m)),$$

which together with

$$\begin{aligned}\sigma_{2^{j+1}m}(\vartheta; f) - \sigma_{2^j m}(\vartheta; f) &= \sum_{k\in\mathbb{Z}}(\vartheta(k/(2^{j+1}m)) - \vartheta(k/(2^j m))\widehat{f}(k)e^{ikx} \\ &= \sum_{k\in\mathbb{Z}}(\vartheta(k/(2^{j+1}m)) - \vartheta(k/(2^j m)))\lambda(k)\widehat{g}(k)e^{ikx}\end{aligned}$$

implies that

$$\sigma_{2^{j+1}m}(\vartheta; f) - \sigma_{2^j m}(\vartheta; f) = \sum_{k\in\mathbb{Z}} \Psi_{j,m}(k/(2^j m))\widehat{g}(k)e^{ikx} = \sigma_{2^j m}(\Psi_{j,m}; g).$$

Then by Lemma 2.3, we obtain

$$\|\sigma_{2^{j+1}m}(\vartheta; f) - \sigma_{2^j m}(\vartheta; f)\|_p \le c_1\|\Psi''_{j,m}\|_\infty\|g\|_p. \tag{2.12}$$

Moreover, from the definition of $\Psi_{j,m}$, supp$S \subset [-2, -1/2]\cup[1/2, 2]$, and $\|S\|_\infty \le 2\|S'\|_\infty \le 4\|S''\|_\infty \le 8\|\vartheta''\|_\infty$, we have that

$$\begin{aligned}|\Psi''_{j,m}(x)| &= |S''(x)\Phi_{j,m}(x) + 2S'(x)\Phi'_{j,m}(x) + S(x)\Phi''_{j,m}(x)| \\ &\le 8\|\vartheta''\|_\infty \sup_{|x|\in[1/2,2]} \left(|\Phi_{j,m}(x)| + \Phi'_{j,m}(x)| + |\Phi''_{j,m}(x)|\right) \\ &\le 16\|\vartheta''\|_\infty \sup_{|x|\in[1/2,2]} \left(|\Phi_{j,m}(x)| + |\Phi''_{j,m}(x)|\right) \\ &= 16\|\vartheta''\|_\infty \sup_{|x|\in[2^{j-1}m,2^{j+1}m]} \left(|\lambda(x)| + (2^j m)^2|\lambda''(x)|\right) \\ &\le 64\|\vartheta''\|_\infty \sup_{|x|\in[2^{j-1}m,2^{j+1}m]} \left(|\lambda(x)| + |x^2\lambda''(x)|\right).\end{aligned}$$

Combining this and (2.12), we deduce

$$\|\sigma_{2^{j+1}m}(\vartheta;f)-\sigma_{2^jm}(\vartheta;f)\|_p \le 64c_1\|\vartheta''\|_\infty \sup_{|x|\in[2^{j-1}m,2^{j+1}m]}\Big(|\lambda(x)|+|x^2\lambda''(x)|\Big)\,\|g\|_p.$$

Therefore, by (2.11) and $\lim_{m\to\infty}\|f-\sigma_{2^jm}(\vartheta;f)\|_p=0$, we have that

$$\begin{aligned}\|f-\mathcal{C}_m\|_p &\le \sum_{j=0}^{\infty}\|\sigma_{2^{j+1}m}(\vartheta;f)-\sigma_{2^jm}(\vartheta;f)\|_p\\ &\le 64c_1\|\vartheta''\|_\infty\sum_{j=0}^{\infty}\sup_{|x|\in[2^{j-1}m,2^{j+1}m]}\Big(|\lambda(x)|+|x^2\lambda''(x)|\Big)\,\|g\|_p. \end{aligned}\tag{2.13}$$

Since (2.8),

$$\sup_{|x|\in[2^{j-1}m,2^{j+1}m]}|\lambda(x)|\le\frac{c_0}{2^jm}\int_{|x|\in[2^jm,2^{j+1}m]}|\lambda(x)|dx\le\frac{c_0}{m}\int_{|x|\in[2^jm,2^{j+1}m]}|\lambda(x)|dx,$$

and

$$\sup_{|x|\in[2^{j-1}m,2^{j+1}m]}|x^2\lambda''(x)|\le 2c_0\int_{|x|\in[2^jm,2^{j+1}m]}|x\lambda''(x)|dx.$$

So,

$$\sum_{j=0}^{\infty}\sup_{|x|\in[2^{j-1}m,2^{j+1}m]}\Big(|\lambda(x)|+|x^2\lambda''(x)|\Big)\le 2c_0\int_{|x|\ge m}\left(\frac{|\lambda(x)|}{m}+|x\lambda''(x)|\right)dx=2c_0J_m(\lambda).$$

Hence, by (2.13), we deduce

$$\|f-\mathcal{C}_m\|_p\le 128c_0c_1\|\vartheta''\|_\infty\varepsilon_m\|g\|_p.\tag{2.14}$$

Combining (2.9), (2.10) and (2.14) we have

$$\|f-Q_{m,\beta}(f)\|_p\ \le c\varepsilon_m\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T})}.$$

∎

From the above theorem, by letting $\lambda=\beta$, we obtain the following corollary.

Corollary 2.5 *Let $1\le p\le\infty$ and λ be of monotone type. Then there exists a positive constant c such that for all $f\in\mathcal{H}_{\lambda,p}(\mathbb{T})$ and $m\in\mathbb{N}$,*

$$\|f-Q_{m,\lambda}(f)\|_p\le cJ_m(\lambda)\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T})}.$$

Definition 2.6 *Let $r,\kappa\in\mathbb{R}$. A function $f:\mathbb{R}\to\mathbb{R}$ will be called a mask of type (r,κ) if f is an even, 2 times continuously differentiable such that for $t\ge1$, $f(t)=|t|^{-r}(\log(|t|+1))^{-\kappa}F(\log|t|)$ for some $F:\mathbb{R}\to\mathbb{R}$ such that $|F^{(k)}(t)|\le a_1$ for all $t\ge1, k=0,1,2$.*

Theorem 2.7 *Let $1\le p\le\infty$, $1<r<\infty,\kappa\in\mathbb{R}$ and the function λ be a mask of type (r,κ). Then there exists a positive constant c such that for all $f\in\mathcal{H}_{\lambda,p}(\mathbb{T})$ and $m\in\mathbb{N}$,*

$$\|f-Q_{m,\lambda}(f)\|_p\le cm^{-r}(\log m)^{-\kappa}\ \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T})}.$$

Proof. Since the function λ be a mask of type (r,κ) and $r>1$,

$$\int_{|x|\geq m}\left|\frac{\lambda(x)}{m}\right|dx \leq a_1\int_{|x|\geq m}\frac{|x|^{-r}(\log(|x|+1))^{-\kappa}}{m}dx \leq a_2 m^{-r}(\log(m+1))^{-\kappa} \quad \forall m\in\mathbb{N}. \tag{2.15}$$

On the other hand,

$$\int_{|x|\geq m}|x\lambda''(x)|dx \leq \int_{|x|\geq m}|x|\,\Big((|x|^{-r}(\log(|x|+1))^{-\kappa})''|F(\log|x|)|$$

$$+\,2(|x|^{-r}(\log(|x|+1))^{-\kappa})'|F'(\log|x|)|/|x| + (|x|^{-r}(\log(|x|+1))^{-\kappa})|F''(\log|x|)-F'(\log|x|)|/x^2\Big)dx$$

$$\leq a_1\int_{|x|\geq m}|x|\,\Big((|x|^{-r}(\log(|x|+1))^{-\kappa})'' + 2(|x|^{-r}(\log(|x|+1))^{-\kappa})'/|x| + 2(|x|^{-r}(\log(|x|+1))^{-\kappa})/x^2\Big)\,dx$$

$$\leq a_3 m^{-r}(\log(m+1))^{-\kappa}.$$

Hence, by (2.15), we deduce

$$J_m(\lambda)\leq a_4 m^{-r}(\log(m+1))^{-\kappa}.$$

From this and Corollary 2.5, we complete the proof. □

Corollary 2.8 *For $1\leq p\leq\infty$, $1<r<\infty$ and $\lambda(x)=\beta(x)=x^{-r}$ for $x\neq 0$, $\mathcal{H}_{\lambda,p}(\mathbb{T})$ becomes the Korobov space $K_p^r(\mathbb{T})$. Then we have the estimate as in [1]:*

$$M_n(U_{\lambda,p}(\mathbb{T}),\kappa_r)_p\leq cm^{-r}$$

where κ_r is the Korobov function.

Definition 2.9 *A function $f:\mathbb{R}\to\mathbb{R}$ is called a function of exponent type if f is 2 times continuously differentiable and there exists a positive constant s such that $f(t)=e^{-s|t|}F(|t|)$ for some decreasing function $F:[0,+\infty)\to(0,+\infty)$.*

Theorem 2.10 *Let $1\leq p\leq\infty$, $1<r<\infty$, $\kappa\in\mathbb{Z}$, the function λ be a mash of type (r,κ), the function β of exponent type. Then there exists a positive constant c such that for all $f\in\mathcal{H}_{\lambda,p}(\mathbb{T})$ and $m\in\mathbb{N}$, we have*

$$\|f-Q_{m,\beta}(f)\|_p\leq cm^{-r}(\log(m+1))^{-\kappa}\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T})}.$$

Proof. We will use the notation in the proof of Theorem 2.2. For $k\in I_{j,m}$ we have $k_m=k-j(2m+1)$ and then

$$|\gamma(k)| = \left|\beta(k_m+j(2m+1))\vartheta(k_m/m)\frac{\lambda(k_m)}{\beta(k_m)}\right|$$

$$= e^{-sj(2m+1))}\frac{|\lambda(k_m)F(k_m+j(2m+1))|}{|F(k_m)|}\leq b_1 e^{-sj(2m+1))}.$$

Hence,

$$\left\|\sum_{k\in I_{j,m}}\gamma(k)e^{ikx}\widehat{g}(k_m)\right\|_p \;\leq\; 3b_1 m e^{-sj(2m+1))}\|g\|_p.$$

This implies that

$$\|\mathcal{A}_m\|_p = \left\| \sum_{j\in\mathbb{N}} \sum_{k\in I_{j,m}} \gamma(k)e^{ikx}\widehat{g}(k_m) \right\|_p \\ \leq 3b_1 \sum_{j\in\mathbb{N}} me^{-sj(2m+1))}\|g\|_p \leq b_2 m^{-r}(\log(m+1))^{-\kappa}\|g\|_p. \tag{2.16}$$

Similarly,

$$\|\mathcal{B}_m\|_p \leq b_2 m^{-r}(\log(m+1))^{-\kappa}\|g\|_p. \tag{2.17}$$

We also known that in the proof of Theorem 2.2 that

$$\|f - \mathcal{C}_m\|_p \leq b_3 \sum_{j=0}^{\infty} \sup_{|x|\in[2^{j-1}m, 2^{j+1}m]} \left(|\lambda(x)| + |x^2\lambda''(x)|\right) \|g\|_p. \tag{2.18}$$

We see that

$$\sup_{|x|\in[2^{j-1}m,2^{j+1}m]} |\lambda(x)| \leq b_4 \int_{|x|\in[2^j m, 2^{j+1}m]} \frac{|\lambda(x)|}{|x|} dx$$

$$\sup_{|x|\in[2^{j-1}m,2^{j+1}m]} |x^2\lambda''(x)| \leq b_4 \int_{|x|\in[2^j m, 2^{j+1}m]} |x\lambda''(x)| dx.$$

So,

$$\sum_{j=0}^{\infty} \sup_{|x|\in[2^{j-1}m,2^{j+1}m]} \left(|\lambda(x)| + |x^2\lambda''(x)|\right) \leq b_4 \int_{|x|\geq m} \left(\frac{|\lambda(x)|}{|x|} + \left|x\lambda''(x)\right| \right) dx.$$

Hence, by (2.18), we deduce that

$$\|f - \mathcal{C}_m\|_p \leq b_3 b_4 \|g\|_p \int_{|x|\geq m} \left(\frac{|\lambda(x)|}{|x|} + \left|x\lambda''(x)\right| \right) dx \leq b_5 m^{-r}(\log(m+1))^{-\kappa}\|g\|_p.$$

Combining this, (2.16), (2.17) and (2.4), we complete the proof. ∎

3 Multivariate approximation

In this section, we make use of the univariate operators $Q_{m,\lambda}$ to construct multivariate operators on sparse Smolyak grids for approximation of functions from $\mathcal{H}_{\lambda,p}(\mathbb{T}^d)$. Based on this approxiation with certain restriction on the function λ we prove an upper bound of $M_n(U_{\lambda,p}(\mathbb{T}^d), \varphi_{\lambda,d})_p$ for $1 \leq p \leq \infty$ as well as a lower bound of $M_n(U_{\lambda,2}(\mathbb{T}^d))_2$. The results obtained in this section generalize some results in [1, 2].

3.1 Error estimates for functions in the space $\mathcal{H}_{\lambda,p}(\mathbb{T}^d)$

For $\boldsymbol{m} \in \mathbb{N}^d$, let the multivariate operator $Q_{\boldsymbol{m}}$ in $\mathcal{H}_{\lambda,p}(\mathbb{T}^d)$ be defined by

$$Q_{\boldsymbol{m}} := \prod_{j=1}^{d} Q_{m_j,\lambda}, \tag{3.19}$$

where the univariate operator $Q_{m_j,\lambda}$ is applied to the univariate function f by considering f as a function of variable x_j with the other variables held fixed, $\mathbb{Z}^d_+ := \{\boldsymbol{k} \in \mathbb{Z}^d : \ k_j \ge 0, \ j \in \mathbb{N}_d\}$ and k_j denotes the jth coordinate of $\boldsymbol{k}$.

Set $\mathbb{Z}^d_{-1} := \{\boldsymbol{k} \in \mathbb{Z}^d : \ k_j \ge -1, \ j \in \mathbb{N}_d\}$. For $k \in \mathbb{Z}_{-1}$, we define the univariate operator T_k in $\mathcal{H}_{\lambda,p}(\mathbb{T})$ by

$$T_k := \mathrm{I} - Q_{2^k,\lambda}, \ k \ge 0, \quad T_{-1} := \mathrm{I},$$

where I is the identity operator. If $\boldsymbol{k} \in \mathbb{Z}^d_{-1}$, we define the mixed operator $T_{\boldsymbol{k}}$ in $\mathcal{H}_{\lambda,p}(\mathbb{T}^d)$ in the manner of the definition of (3.19) as

$$T_{\boldsymbol{k}} := \prod_{i=1}^d T_{k_i}.$$

Set $|\boldsymbol{k}| := \sum_{j \in \mathbb{N}_d} |k_j|$ for $\boldsymbol{k} \in \mathbb{Z}^d_{-1}$ and $\boldsymbol{k}^{-\kappa}_{(2)} = \prod_{j=1}^d (k_j+2)^{-\kappa}$.

Lemma 3.1 *Let $1 \le p \le \infty$, $1 < r < \infty, 0 \le \kappa < \infty$ and the function λ be a mask of type (r,κ). Then we have for any $f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)$ and $\boldsymbol{k} \in \mathbb{Z}^d_{-1}$,*

$$\|T_{\boldsymbol{k}}(f)\|_p \ \le \ C\boldsymbol{k}^{-\kappa}_{(2)} 2^{-r|\boldsymbol{k}|}\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)}$$

with some constant C independent of f and $\boldsymbol{k}$.

Proof. We prove the lemma by induction on d. For $d=1$ it follows from Theorems 2.7. Assume the lemma is true for $d-1$. Set $\boldsymbol{x}' := \{x_j : \ j \in \mathbb{N}_{d-1}\}$ and $\boldsymbol{x} = (\boldsymbol{x}', x_d)$ for $\boldsymbol{x} \in \mathbb{R}^d$. We temporarily denote by $\|f\|_{p,\boldsymbol{x}'}$ and $\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^{d-1}),\boldsymbol{x}'}$ or $\|f\|_{p,x_d}$ and $\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}),x_d}$ the norms applied to the function f by considering f as a function of variable $\boldsymbol{x}'$ or x_d with the other variable held fixed, respectively. For $\boldsymbol{k} = (\boldsymbol{k}', k_d) \in \mathbb{Z}^d_{-1}$, we get by Theorems 2.7 and the induction assumption

$$\begin{aligned}
\|T_{\boldsymbol{k}}(f)\|_p \ &= \ \|\|T_{\boldsymbol{k}'}T_{k_d}(f)\|_{p,\boldsymbol{x}'}\|_{p,x_d} \ \ll \ \|2^{-r|\boldsymbol{k}'|}\boldsymbol{k}'^{-\kappa}_{(2)}\|T_{k_d}(f)\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^{d-1}),\boldsymbol{x}'}\|_{p,x_d} \\
&= \ 2^{-r|\boldsymbol{k}'|}\boldsymbol{k}'^{-\kappa}_{(2)}\|\|T_{k_d}(f)\|_{p,x_d}\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^{d-1}),\boldsymbol{x}'} \\
&\ll \ 2^{-r|\boldsymbol{k}'|}\boldsymbol{k}'^{-\kappa}_{(2)}\|2^{-rk_d}(k_d+2)^{-\kappa}\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}),x_d}\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^{d-1}),\boldsymbol{x}'} \\
&= \ 2^{-r|\boldsymbol{k}|}\prod_{j=1}^d (k_j+2)^{-\kappa}\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)}.
\end{aligned}$$

∎

Let the univariate operator q_k be defined for $k \in \mathbb{Z}_+$, by

$$q_k := \ Q_{2^k,\lambda} - Q_{2^{k-1},\lambda}, \ k > 0, \quad q_0 := \ Q_{1,\lambda},$$

and in the manner of the definition of (3.19), the multivariate operator $q_{\boldsymbol{k}}$ for $\boldsymbol{k} \in \mathbb{Z}^d_+$, by

$$q_{\boldsymbol{k}} := \prod_{j=1}^d q_{k_j}.$$

For $\boldsymbol{k} \in \mathbb{Z}^d_+$, we write $\boldsymbol{k} \to \infty$ if $k_j \to \infty$ for each $j \in \mathbb{N}_d$.

Theorem 3.2 *Let $1 \le p \le \infty$, $1 < r < \infty, 0 \le \kappa < \infty$ and the function λ be a mask of type (r, κ). Then every $f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)$ can be represented as the series*

$$f \ = \ \sum_{\boldsymbol{k} \in \mathbb{Z}^d_+} q_{\boldsymbol{k}}(f) \tag{3.20}$$

converging in L^p-norm, and we have for $\boldsymbol{k} \in \mathbb{Z}^d_+$,

$$\|q_{\boldsymbol{k}}(f)\|_p \ \le \ C 2^{-r|\boldsymbol{k}|} \boldsymbol{k}_{(2)}^{-\kappa} \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)} \tag{3.21}$$

with some constant C independent of f and $\boldsymbol{k}$.

Proof. Let $f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)$. In a way similar to the proof of Lemma 3.1, we can show that

$$\|f - Q_{2^{\boldsymbol{k}}}(f)\|_p \ \ll \ \max_{j \in \mathbb{N}_d} 2^{-rk_j} k_j^{\kappa} \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)},$$

and therefore,

$$\|f - Q_{2^{\boldsymbol{k}}}(f)\|_p \to 0, \ \boldsymbol{k} \to \infty,$$

where $2^{\boldsymbol{k}} = (2^{k_j} : \ j \in \mathbb{N}_d)$. On the other hand,

$$Q_{2^{\boldsymbol{k}}} \ = \ \sum_{s_j \le k_j, \ j \in \mathbb{N}_d} q_{\boldsymbol{s}}(f).$$

This proves (3.20). To prove (3.21) we notice that from the definition it follows that

$$q_{\boldsymbol{k}} \ = \ \sum_{e \subset \mathbb{N}_d} (-1)^{|e|} T_{\boldsymbol{k}^e},$$

where $\boldsymbol{k}^e$ is defined by $k_j^e = k_j$ if $j \in e$, and $k_j^e = k_j - 1$ if $j \notin e$. Hence, by Lemma 3.1

$$\|q_{\boldsymbol{k}}(f)\|_p \ \le \ \sum_{e \subset \mathbb{N}_d} \|T_{\boldsymbol{k}^e}(f)\|_p \ \ll \ \sum_{e \subset \mathbb{N}_d} 2^{-r|\boldsymbol{k}^e|} (\boldsymbol{k}^e_{(2)})^{-\kappa} \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)} \ \ll \ 2^{-r|\boldsymbol{k}|} \boldsymbol{k}_{(2)}^{-\kappa} \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)}.$$

□

For approximation of $f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)$, we introduce the linear operator $P_m, m \in \mathbb{N}$, by

$$P_m(f) := \sum_{|\boldsymbol{k}| \le m} q_{\boldsymbol{k}}(f). \tag{3.22}$$

We give an upper bound for the error of the approximation of functions $f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)$ by the operator P_m in the following theorem.

Theorem 3.3 *Let $1 \le p \le \infty$, $1 < r < \infty, 0 \le \kappa < \infty$ and the function λ be a mask of type (r, κ). Then, we have for every $m \in \mathbb{N}$ and $f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)$,*

$$\|f - P_m(f)\|_p \ \le \ C\, 2^{-rm} m^{d-1-\kappa} \, \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)}$$

with some constant C independent of f and m.

Proof. From Theorem 3.2 we deduce that

$$\begin{aligned}\|f - P_m(f)\|_p &= \left\| \sum_{|\boldsymbol{k}|>m} q_{\boldsymbol{k}}(f) \right\|_p \leq \sum_{|\boldsymbol{k}|>m} \|q_{\boldsymbol{k}}(f)\|_p \\ &\ll \sum_{|\boldsymbol{k}|>m} 2^{-r|\boldsymbol{k}|}\boldsymbol{k}_{(2)}^{-\kappa}\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)} \ll \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)} \sum_{|\boldsymbol{k}|>m} 2^{-r|\boldsymbol{k}|}\boldsymbol{k}_{(2)}^{-\kappa} \\ &\ll 2^{-rm}m^{d-1-\kappa}\,\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)}.\end{aligned}$$

□

3.2 Convergence rate

We choose a positive integer $m \in \mathbb{N}$, a lattice vector $\boldsymbol{k} \in \mathbb{Z}_+^d$ with $|\boldsymbol{k}| \leq m$ and another lattice vector $\boldsymbol{s} = (s_j : j \in \mathbb{N}_d) \in \prod_{j\in\mathbb{N}_d} Z[2^{k_j+1}+1]$ to define the vector $\boldsymbol{y}_{\boldsymbol{k},\boldsymbol{s}} = \left(\frac{2\pi s_j}{2^{k_j+1}+1} : j \in \mathbb{N}_d\right)$. The Smolyak grid on $\mathbb{T}^d$ consists of all such vectors and is given as

$$G^d(m) := \left\{\boldsymbol{y}_{\boldsymbol{k},\boldsymbol{s}} : |\boldsymbol{k}| \leq m, \boldsymbol{s} \in \otimes_{j\in\mathbb{N}_d} Z[2^{k_j+1}+1]\right\}.$$

A simple computation confirms, for $m \to \infty$ that

$$|G^d(m)| = \sum_{|\boldsymbol{k}|\leq m} \prod_{j\in\mathbb{N}_d} (2^{k_j+1}+1) \asymp 2^d m^{d-1},$$

so, $G^d(m)$ is a sparse subset of a full grid of cardinality 2^{dm}. Moreover, by the definition of the linear operator P_m given in equation (3.22) we see that the range of P_m is contained in the subspace

$$\operatorname{span}\{\varphi_{\lambda,d}(\cdot - \boldsymbol{y}) : \boldsymbol{y} \in G^d(m)\}.$$

Other words, P_m defines a multivariate method of approximation by translates of the function $\varphi_{\lambda,d}$ on the sparse Smolyak grid $G^d(m)$. An upper bound for the error of this approximation of functions from $\mathcal{H}_{\lambda,p}(\mathbb{T}^d)$ is given in Theorem 3.3.

Now, we are ready to prove the next theorem, thereby establishing an upper bound of $M_n(U_{\lambda,p}, \varphi_{\lambda,d})_p$.

Theorem 3.4 *If $1 \leq p \leq \infty$, $1 < r < \infty, 0 \leq \kappa < \infty$ and the function λ be a mask of type (r,κ), then*

$$M_n(U_{\lambda,p}(\mathbb{T}^d), \varphi_{\lambda,d})_p \ll n^{-r}(\log n)^{r(d-1)-\kappa}.$$

Proof. If $n \in \mathbb{N}$ and m is the largest positive integer such that $|G^d(m)| \leq n$, then $n \asymp 2^m m^{d-1}$ and by Theorem 3.3 we have that

$$M_n(U_{\lambda,p}(\mathbb{T}^d), \varphi_{\lambda,d})_p \leq \sup_{f\in U_{\lambda,p}(\mathbb{T}^d)} \|f - P_m(f)\|_p \ll 2^{-rm}m^{d-1-d\kappa} \asymp n^{-r}(\log n)^{r(d-1)-\kappa}.$$

□

For $p = 2$, we are able to establish a lower bound for $M_n(U_{\lambda,2}(\mathbb{T}^d), \varphi_{\lambda,d})_2$. We prepare some auxiliary results. Let $\mathbb{P}_q(\mathbb{R}^l)$ be the set of algebraic polynomials on $\mathbb{R}^l$ of total degree at most q, and

$$\mathbb{E}^m := \{\boldsymbol{t} = (t_j : j \in \mathbb{N}_m) : |t_j| = 1, j \in \mathbb{N}_m\}.$$

We define the polynomial maifold

$$\mathbb{M}_{m,l,q} := \left\{(p_j(\boldsymbol{u}) : j \in \mathbb{N}_m) : p_j \in \mathbb{P}_q(\mathbb{R}^l), j \in \mathbb{N}_m, \boldsymbol{u} \in \mathbb{R}^l\right\}.$$

Denote by $\|\boldsymbol{x}\|_2$ the Euclidean norm of a vector $\boldsymbol{x}$ in $\mathbb{R}^m$. The following lemma was proven in [5].

Lemma 3.5 *Let $m, l, q \in \mathbb{N}$ satisfy the inequality $l \log(\frac{4emq}{l}) \leq \frac{m}{4}$. Then there is a vector $\boldsymbol{t} \in \mathbb{E}^m$ and a positive constant c such that*

$$\inf\{\|\boldsymbol{t} - \boldsymbol{x}\|_2 : \boldsymbol{x} \in \mathbb{M}_{m,l,q}\} \geq c\, m^{1/2}.$$

Theorem 3.6 *If $1 < r < \infty, 0 \leq \kappa < \infty$ and the function λ be a mask of type (r, κ), then we have that*

$$n^{-r}(\log n)^{r(d-2)-d\kappa} \ll M_n(U_{\lambda,2})_2 \ll n^{-r}(\log n)^{r(d-1)-\kappa}. \quad (3.23)$$

Proof. The upper bound of (3.23) is in Theorem 3.4. Let us prove the lower bound by developing a technique used in the proofs of [5, Theorem 1.1] and [1, Theorem 4.4] . For a positive number a we define a subset $\mathbb{H}(a)$ of lattice vectors by

$$\mathbb{H}(a) := \left\{\boldsymbol{k} = (k_j : j \in \mathbb{N}_d) \in \mathbb{Z}^d : \prod_{j \in \mathbb{N}_d} |k_j| \leq a\right\}.$$

Notice that $|\mathbb{H}(a)| \asymp a(\log a)^{d-1}$ when $a \to \infty$. To apply Lemma 3.5, for any $n \in \mathbb{N}$, we take $q = \lfloor n(\log n)^{-d+2}\rfloor + 1$, $m = 5(2d+1)\lfloor n \log n\rfloor$ and $l = (2d+1)n$. With these choices we obtain

$$|\mathbb{H}(q)| \asymp m \quad (3.24)$$

and

$$q \asymp m(\log m)^{-d+1} \quad (3.25)$$

as $n \to \infty$. Moreover, we have that

$$\lim_{n\to\infty} \frac{l}{m} \log\left(\frac{4emq}{l}\right) = \frac{1}{5},$$

and therefore, the assumption of Lemma 3.5 is satisfied for $n \to \infty$.

Now, let us specify the polynomial manifold $\mathbb{M}_{m,l,q}$. To this end, we put $\zeta := q^{-r}m^{-1/2}(\log q)^{-d\kappa}$ and let $\mathbb{Y}$ be the set of trigonometric polynomials on $\mathbb{T}^d$, defined by

$$\mathbb{Y} := \left\{f = \zeta \sum_{\boldsymbol{k} \in \mathbb{H}(q)} a_{\boldsymbol{k}} t_{\boldsymbol{k}} : \mathbf{t} = (t_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{H}(q)) \in \mathbb{E}^{|\mathbb{H}(q)|}\right\}.$$

If $f \in \mathbb{Y}$ and

$$f = \zeta \sum_{\boldsymbol{k} \in \mathbb{H}(q)} a_{\boldsymbol{k}} t_{\boldsymbol{k}},$$

then $f = \varphi_{\lambda,d} * g$ for some trigonometric polynomial g such that

$$\|g\|^2_{L^2(\mathbb{T}^d)} \leq \zeta^2 \sum_{\boldsymbol{k}\in\mathbb{H}(q)} |\lambda(\boldsymbol{k})|^{-2}.$$

Since

$$\begin{aligned}\zeta^2 \sum_{\boldsymbol{k}\in\mathbb{H}(q)} |\lambda(\boldsymbol{k})|^{-2} &\leq \zeta^2 q^{2r} \sum_{\boldsymbol{k}\in\mathbb{H}(q)} \left|\log \prod_{j=1}^{d} k_j\right|^{2\kappa} \\ &\leq \zeta^2 q^{2r} \sum_{\boldsymbol{k}\in\mathbb{H}(q)} \left|\sum_{j=1}^{n} \log k_j\right|^{2d\kappa} \leq \zeta^2 q^{2r} (\log q)^{2d\kappa} |\mathbb{H}(q)| = m^{-1}|\mathbb{H}(q)|,\end{aligned}$$

by (3.24) that there is a positive constant c such that $\|g\|_{L^2(\mathbb{T}^d)} \leq c$ for all $n \in \mathbb{N}$. Therefore, we can either adjust functions in $\mathbb{Y}$ by dividing them by c, or we can assume without loss of generality that $c = 1$, and obtain $\mathbb{Y} \subseteq U_{\lambda,2}(\mathbb{T}^d)$.

We are now ready to prove the lower bound for $M_n(U_{\lambda,2}(\mathbb{T}^d))_2$. We choose any $\varphi \in L^2(\mathbb{T}^d)$ and let v be any function formed as a linear combination of n translates of the function φ:

$$v = \sum_{j\in\mathbb{N}_n} c_j \varphi(\cdot - \boldsymbol{y}_j).$$

By the well-known Bessel inequality we have for a function

$$f = \zeta \sum_{\boldsymbol{k}\in\mathbb{H}(q)} a_{\boldsymbol{k}} t_{\boldsymbol{k}} \in \mathbb{Y},$$

that

$$\|f - v\|^2_{L^2(\mathbb{T}^d)} \geq \zeta^2 \sum_{\boldsymbol{k}\in\mathbb{H}(q)} \left| t_{\boldsymbol{k}} - \frac{\widehat{\varphi}(\boldsymbol{k})}{\zeta} \sum_{j\in\mathbb{N}_n} c_j e^{i(\boldsymbol{y}_j,\boldsymbol{k})} \right|^2. \tag{3.26}$$

We introduce a polynomial manifold so that we can use Lemma 3.5 to get a lower bound for the expressions on the left hand side of inequality (3.26). To this end, we define the vector $\mathbf{c} = (c_j : j \in \mathbb{N}_n) \in \mathbb{R}^n$ and for each $j \in \mathbb{N}_n$, let $\boldsymbol{z}_j = (z_{j,l} : l \in \mathbb{N}_d)$ be a vector in $\mathbb{C}^d$ and then concatenate these vectors to form the vector $\boldsymbol{z} = (\boldsymbol{z}_j : j \in \mathbb{N}_n) \in \mathbb{C}^{nd}$. We employ the standard multivariate notation

$$\boldsymbol{z}_j^{\boldsymbol{k}} = \prod_{l\in\mathbb{N}_d} z_{j,l}^{k_l}$$

and require vectors $\boldsymbol{w} = (\boldsymbol{c}, \boldsymbol{z}) \in \mathbb{R}^n \times \mathbb{C}^{nd}$ and $\boldsymbol{u} = (\boldsymbol{c}, \operatorname{Re}\boldsymbol{z}, \operatorname{Im}\boldsymbol{z}) \in \mathbb{R}^l$ to be written in concatenate form. Now, we introduce for each $\boldsymbol{k} \in \mathbb{H}(q)$ the polynomial $\boldsymbol{q}_{\boldsymbol{k}}$ defined at $\boldsymbol{w}$ as

$$\boldsymbol{q}_{\boldsymbol{k}}(\boldsymbol{w}) := \frac{\widehat{\varphi}(\boldsymbol{k})}{\zeta} \sum_{\boldsymbol{j}\in\mathbb{H}(q)} c_{\boldsymbol{j}} \boldsymbol{z}^{\boldsymbol{j}}.$$

We only need to consider the real part of $\boldsymbol{q}_{\boldsymbol{k}}$, namely, $\boldsymbol{p}_{\boldsymbol{k}} = \operatorname{Re}\boldsymbol{q}_{\boldsymbol{k}}$ since we have that

$$\inf\left\{ \sum_{\boldsymbol{k}\in\mathbb{H}(q)} \left| t_{\boldsymbol{k}} - \frac{\widehat{\varphi}(\boldsymbol{k})}{\zeta} \sum_{j\in\mathbb{N}_n} c_j e^{i(\boldsymbol{y}_j,\boldsymbol{k})} \right|^2 : c_j \in \mathbb{R}, \boldsymbol{y}_j \in \mathbb{T}^d \right\} \geq \inf\left\{ \sum_{\boldsymbol{k}\in\mathbb{H}(q)} |t_{\boldsymbol{k}} - p_{\boldsymbol{k}}(\boldsymbol{u})|^2 : \boldsymbol{u} \in \mathbb{R}^l \right\}.$$

Therefore, by Lemma 3.5 and (3.25) we conclude there is a vector $\boldsymbol{t}^0 = (t^0_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{H}(q)) \in \mathbb{E}^{h_q}$ and the corresponding function

$$f^0 = \zeta \sum_{\boldsymbol{k} \in \mathbb{H}(q)} t^0_{\boldsymbol{k}} \chi_{\boldsymbol{k}} \in \mathbb{Y}$$

for which there is a positive constant c such that for every v of the form

$$v = \sum_{j \in \mathbb{N}_n} c_j \varphi(\cdot - \boldsymbol{y}_j),$$

we have that

$$\|f^0 - v\|_{L^2(\mathbb{T}^d)} \geq c\zeta m^{\frac{1}{2}} = q^{-r}(\log q)^{-d\kappa} \asymp n^{-r}(\log n)^{r(d-2)-d\kappa}$$

which proves the lower bound of (3.23). ∎

Similar to the proof of the above theorem, we can prove the following theorem for the case $-\infty < \kappa < 0$.

Theorem 3.7 *If* $1 < r < \infty, -\infty < \kappa < 0$ *and the function* λ *be a mask of type* (r, κ), *then we have that*

$$n^{-r}(\log n)^{r(d-2)-\kappa} \ \ll \ M_n(U_{\lambda,2}(\mathbb{T}^d))_2 \ \ll \ n^{-r}(\log n)^{r(d-1)-d\kappa}.$$

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