EXHAUSTION OF HYPERBOLIC COMPLEX MANIFOLDS AND RELATIONS TO THE SQUEEZING FUNCTION

NINH VAN THU, TRINH HUY VU AND NGUYEN QUANG DIEU[1,2]

ABSTRACT. The purpose of this article is twofold. The first aim is to characterize an n-dimensional hyperbolic complex manifold M exhausted by a sequence $\{\Omega_j\}$ of domains in $\mathbb{C}^n$ via an exhausting sequence $\{f_j\colon \Omega_j \to M\}$ such that $f_j^{-1}(a)$ converges to a boundary point $\xi_0 \in \partial\Omega$ for some point $a \in M$. Then, our second aim is to show that any spherically extreme boundary point must be strongly pseudoconvex.

1. INTRODUCTION

Let M be an n-dimensional hyperbolic complex manifold. Let $\{M_j\}$ and $\{\Omega_j\}$ be two sequences of open subsets in M and $\mathbb{C}^n$ respectively. Suppose that M *can be exhausted by* $\{\Omega_j\}$ via an exhausting sequence $\{f_j\colon \Omega_j \to M_j \subset M\}$ in the sense that f_j is a biholomorphism from M_j onto Ω_j for all $j \geqslant 1$ and $\cup_{j=1}^{\infty} M_j = M$. Then it is a natural problem is to describe M in terms of Ω. In the case that $\Omega_j = \Omega$ for all $j \geqslant 1$, this problem is called the *union problem* (cf. [BBMV21, FSi81]). In 1977, J. E. Fornæss and L. Stout [FS77] proved that if $\Omega = \mathbb{B}^n$, then M is biholomorphically equivalent to $\mathbb{B}^n$. More generality, M is biholomorphically equivalent to Ω if it is a homogeneous bounded domain in $\mathbb{C}^n$. For further results about the union problem the reader may also consult the references [FSi81, Liu18, NT21, BBMV21].

Now let us fix a point $a \in M$ and assume that $\lim \Omega_j = \Omega$ (see Definition 2.1 for the notion of this limit). Then, we consider the behavior of the sequence $\{f_j^{-1}(a)\} \subset \Omega$. In the case when $\{f_j^{-1}(a)\}$ converges to a point $p \in \Omega$, M is biholomorphically equivalent to Ω (cf. Corollary 2.4 in Section 2). Therefore, we especially pay attention to the case that $\{f_j^{-1}(a)\}$ converges to a boundary point $\xi_0 \in \partial\Omega$.

In the first part of this paper, we give a characterization of our manifold M in term of the behavior of the orbit $\{f_j^{-1}(a)\}$. To do this, let us fix positive integers $m_1, \ldots, m_{n-1}$ and let $P(z')$ be a $(1/m_1, \ldots, 1/m_{n-1})$-homogeneous polynomial given by

$$P(z) = \sum_{wt(K)=wt(L)=1/2} a_{KL} {z'}^K \bar{z}'^L,$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \bar{a}_{LK}$, satisfying that $P(z') > 0$ whenever $z' \neq 0$. Here and in what follows, $z := (z_1, \ldots, z_{n-1})$ and $wt(K) := \sum_{j=1}^{n-1} \frac{k_j}{2m_j}$ denotes the weight of any multi-index $K = (k_1, \ldots, k_{n-1}) \in \mathbb{N}^{n-1}$ with respect to $\Lambda := (1/m_1, \ldots, 1/m_{n-1})$. Then the general ellipsoid D_P in $\mathbb{C}^n$ ($n \geqslant 1$), defined in [NNTK19] by

$$D_P := \{(z', z_n) \in \mathbb{C}^n\colon |z_n|^2 + P(z') < 1\}.$$

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Throughout this paper, we assume that the domain D_P is a WB-domain, i.e., D_P is strongly pseudoconvex at every boundary point outside the set $\{(0', e^{i\theta}) : \theta \in \mathbb{R}\}$ (cf. [AGK16]).

For any $s, r \in (0, 1]$, inspired by [Liu18, Lemma 2.5] let us define D_P^s and $D_P^{s,r}$ respectively by

$$D_P^s := \{z \in \mathbb{C}^n : |z_n - b|^2 + sP(z') < s^2\};$$
$$D_P^{s,r} := \{z \in \mathbb{C}^n : |z_n - b|^2 + \frac{s}{r}P(z') < s^2\},$$

where $s = 1 - b$. We note that $D_P^{s,1} = D_P^s$ and the property that $\lim \psi_j^{-1}(D_P^s) = D_P$ for a certain family $\{\psi_j\} \subset \mathrm{Aut}(D_P)$ (cf. Lemma 4.1 in Section 4) plays a key role in the proofs of our main theorems below.

Indeed, we first prove the following theorem.

Theorem 1.1. *Let M be an n-dimensional hyperbolic complex manifold and let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Let $\{\Omega_j\}$ be a sequence of subdomains of D_P such that $D_P^s \subset \Omega_j \subset D_P, j \geqslant 1$. Suppose also that M can be exhausted by $\{\Omega_j\}$ via an exhausting sequence $\{f_j : D_P \supset \Omega_j \to M_j \subset M\}$. If there exists a point $a \in M$ such that the sequence $D_P^{s,r} \ni \eta_j := f_j^{-1}(a)$ converges to $(0', 1)$ in D_P for some fixed $r \in (0, 1)$, then M is biholomorphically equivalent to D_P.*

Remark 1.1. Notice that the point $p = (0', 1)$ is (P, s)-extreme for each domain Ω_j (cf. [NNC21, Definition 1.1] for the notion of (P, s)-extreme points) and the convergence of a sequence of points in $D_P^{s,r}$ to p is exactly the Λ-nontangential convergence introduced in [NN19, Definition 3.4]. Therefore, [NT21, Theorem 1.1] yields Theorem 1.1 for the case that $\Omega_j = D_P$ for all $j \geqslant 1$. However, our proof here is quite different and more simple. In particular, we do not need the condition that Ω_j converges to D_P and the proof shows that D_P is not necessary to be a WB-domain.

Now we consider the case that $\{a_j\} \subset \Omega \cap U$ converges Λ-tangentially to $p = 0$ in the sense that for any $0 < r < 1$ there exists $j_r \in \mathbb{N}$ such that $a_j \notin D_{s,r}$ for all $j \geqslant j_r$, we do not know whether M is biholomorphically equivalent to D_P. However, the following theorem shows that M is biholomorphically equivalent to the unit ball $\mathbb{B}^n$ provided that all $\partial\Omega_j$ share a small neighborhood of the point $(0', 1)$ with ∂D_P to which the sequence of points converges Λ-tangentially. More precisely, we prove the following theorem.

Theorem 1.2. *Let M be an n-dimensional hyperbolic complex manifold. Let $\{\Omega_j\}$ be a sequence of subdomains of D_P such that $\Omega_j \cap U = D_P \cap U$, $j \geqslant 1$, for a fixed neighborhood U of the origin in $\mathbb{C}^n$. Suppose also that M can be exhausted by $\{\Omega_j\}$ via an exhausting sequence $\{f_j : D_P \supset \Omega_j \to M_j \subset M\}$. If there exists a point $a \in M$ such that the sequence $\eta_j := f_j^{-1}(a)$ converges Λ-tangentially to $(0', 1)$ in D_P, then M is biholomorphically equivalent to the unit ball $\mathbb{B}^n$.*

Remark 1.2. In order to prove Theorem 1.2, we shall show without loss of generality that the sequence $\{\psi_j^{-1}(\eta_j)\}$ converges nontangentially to some boundary $p \in \partial D_P \cap \{z_n = 0\}$. Thanks to the fact that D_P is a WB-domain, that point is strongly pseudoconvex and hence the proof easily follows from Theorem 3.1. However, if D_P is not a WB-domain, i.e. the point p may not be strongly pseudoconvex, but it is h-extendible in the sense of J. Yu (cf. [Yu95]), then M would be biholomorphically equivalent to some model by Theorem 1.1 in [NT21].

Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Suppose that $\xi_0 \in \partial\Omega$ is a boundary orbit accumulation point, i.e. there exists a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\eta_j := \varphi_j(a)$ converges to $\xi_0 \in \partial\Omega$ for some point $a \in \Omega$. Here and in what follows, $\mathrm{Aut}(\Omega)$ denotes the automorphism group of Ω. Then, one notices that our domain Ω is exhausted by itself via the sequence $\varphi_j \colon \Omega \to \Omega$. In addition, as an application of Theorem 1.1 and Theorem 1.2, we show that if Ω is a subdomain of D_P and $\Omega \cap U = D_P \cap U$ for a fixed neighborhood U of $(0', 1)$ in $\mathbb{C}^n$, then Ω must be biholomorphically equivalent to either D_P or $\mathbb{B}^n$ (cf. Corollary 4.2 and Corollary 4.3 in Section 4).

Now we move to the second part of this paper. Let Ω be a domain in $\mathbb{C}^n$ and $q \in \Omega$. For a holomorphic embedding $f \colon \Omega \to \mathbb{B}^n := \mathbb{B}(0; 1)$ with $f(q) = 0$, one sets

$$\sigma_{\Omega,f}(q) := \sup \{r > 0 \colon B(0; r) \subset f(\Omega)\},$$

where $\mathbb{B}^n(z; r) \subset \mathbb{C}^n$ denotes the ball of radius r with center at z. Then the *squeezing function* $\sigma_\Omega : \Omega \to \mathbb{R}$ is defined in [DGZ12] as

$$\sigma_\Omega(q) := \sup_f \{\sigma_{\Omega,f}(q)\}.$$

Note that $0 < \sigma_\Omega(z) \leqslant 1$ for any $z \in \Omega$ and it is obvious that the squeezing function is invariant under biholomorphisms.

Now we recall the definition of spherically extreme boundary points (cf. [KZ16]). Indeed, a boundary point $p \in \partial\Omega$ is said to be locally spherically extreme if there exist a neighborhood U of p and a ball $\mathbb{B}(c(p); R)$ in $\mathbb{C}^n$ of some radius R, center at some point $c(p)$ such that $\partial\Omega \cap U$ is $\mathcal{C}^2$-smooth, $\Omega \cap U \subset \mathbb{B}(c(p); R)$, and $p \in \partial\Omega \cap \partial\mathbb{B}(c(p); R)$.

By using the scaling method, K.-T. Kim and L. Zhang [KZ16, Theorem 3.1] proved that if a domain in $\mathbb{C}^n$ admits a locally spherically extreme boundary point p, then

$$\lim_{\Omega \cap U \ni q \to p} \sigma_{\Omega \cap U}(q) = 1,$$

where U is a small neighborhood of p. Of course, we may not have that $\lim\limits_{\Omega \ni q \to p} \sigma_\Omega(q) = 1$ (see [FN21, Theorem 1]). It is known that every strongly convex boundary point is locally spherically extreme. However, the following theorem points out that every locally spherically extreme point is also strongly pseudoconvex.

Theorem 1.3. *Let Ω be a domain with $\mathcal{C}^2$ smooth boundary near the point $p \in \partial\Omega$. Suppose that Ω admits p as a locally spherically extreme point. Then, Ω is strongly pseudoconvex at p.*

As an application of Theorem 1.3, in the union problem, the hyperbolic complex manifold M must be biholomorphically equivalent to $\mathbb{B}^n$ provided that $\{f_j^{-1}(a)\}$ converges to a spherically extreme boundary point $\xi_0 \in \partial\Omega$ (see Theorem 3.1 and Corollary 5.1 in Sections 3 and 5 respectively for more details). More generally, if $\lim\limits_{z \to \xi_0} \sigma_\Omega(z) = 1$, then M is also biholomorphically equivalent to $\mathbb{B}^n$ (cf. Corollary 5.3).

The organization of this paper is as follows: In Section 2 we provide some results concerning the normality of a sequence of biholomorphisms. Next, we introduce a proof of Theorem 3.1. Then, in Section 4 we give our proofs of Theorem 1.1 and Theorem 1.2. Finally, the proof of Theorem 1.3 will be introduced in Section 5.

2. The normality

First of all, we recall the following definition (see [GK87] or [DN09]).

Definition 2.1. Let $\{\Omega_i\}_{i=1}^{\infty}$ be a sequence of open sets in a complex manifold M and Ω_0 be an open set of M. The sequence $\{\Omega_i\}_{i=1}^{\infty}$ is said to converge to Ω_0 (written $\lim \Omega_i = \Omega_0$) if and only if

(i) For any compact set $K \subset \Omega_0$, there is an $i_0 = i_0(K)$ such that $i \geqslant i_0$ implies that $K \subset \Omega_i$; and

(ii) If K is a compact set which is contained in Ω_i for all sufficiently large i, then $K \subset \Omega_0$.

Next, we recall the following proposition, which is a generalization of the theorem of H. Cartan (see [DN09, GK87, DT04]).

Proposition 2.1. *Let $\{A_i\}_{i=1}^{\infty}$ and $\{\Omega_i\}_{i=1}^{\infty}$ be sequences of domains in complex manifolds M and N respectively with $\dim M = \dim N$, $\lim A_i = A_0$, and $\lim \Omega_i = \Omega_0$ for some (uniquely determined) domains A_0 in M and Ω_0 in N. Suppose that $\{f_i : A_i \to \Omega_i\}$ is a sequence of biholomorphic maps. Suppose also that the sequence $\{f_i : A_i \to N\}$ converges uniformly on compact subsets of A_0 to a holomorphic map $F : A_0 \to N$ and the sequence $\{g_i := f_i^{-1} : \Omega_i \to M\}$ converges uniformly on compact subsets of Ω_0 to a holomorphic map $G : \Omega_0 \to M$. Then, one of the following assertions holds:*

(i) *The sequence $\{f_i\}$ is compactly divergent, i.e., for each compact set $K \subset A_0$ and each compact set $L \subset \Omega_0$, there exists an integer i_0 such that $f_i(K) \cap L = \varnothing$ for $i \geqslant i_0$; or*

(ii) *There exists a subsequence $\{f_{i_j}\} \subset \{f_i\}$ such that the sequence $\{f_{i_j}\}$ converges uniformly on compact subsets of A_0 to a biholomorphic map $F : A_0 \to \Omega_0$.*

Remark 2.1. In [DN09], Do Duc Thai and the first author proved the above proposition for the case that $M = N$, but the same proof can give the result in the above proposition.

The following proposition is inspired by Theorem 3.2 in [FSi81].

Proposition 2.2. *Let X, Y be complex manifolds of dimension n. Let $\{M_j\}_{j=1}^{\infty}$ be a sequence of domains in X that converges to a domain M and $\{\Omega_j\}$ be a sequence of domains in Y that converges to a taut domain Ω. Let $\{f_j : M_j \to \Omega_j\}$ be a normal sequence of biholomorphisms. Suppose that there exists a point $a \in M$ satisfying the following conditions:*

(i) $\lim\limits_{j\to\infty} f_j(a) = b \in \Omega$;

(ii) $\overline{\lim\limits_{j\to\infty}} F_{M_j}(a, \xi) > 0, \ \forall \xi \neq 0.$

Then, $\{f_j\}$ contains a subsequence that converges uniformly on compacta to a biholomorphic map $f : M \to \Omega$.

We claim no originality for the following fact which is a slight extension of Hurwitz's theorem.

Lemma 2.3. *Let X, Y be complex manifolds of complex dimension n and let $\{f_j\}$ be a sequence of injective holomorphic maps from X to Y. Suppose that $\{f_j\}$ converges locally uniformly to a non-constant holomorphic map $f : X \to Y$. Then, f is also injective on X.*

Proof. Suppose that there exist two distinct points $z, w \in X$ such that $f(z) = f(w) = \lambda \in Y$. Then we choose disjoint neighbourhoods U, V of z and w, respectively, such that each of them is biholomorphic to the unit ball in $\mathbb{C}^n$. By Hurwitz's theorem there exists n_0 such that $\lambda \in f_{n_0}(U) \cap f_{n_0}(V)$. This is a contradiction to injectivity of f_{n_0}. □

Proof of Proposition 2.2. Since $\{f_j\}$ is normal and $f_j(a) \to b \in \Omega$ as $j \to \infty$, without loss of generality we may assume that f_j converges uniformly on compacta to a holomorphic map $f\colon M \to \overline{\Omega}$ such that $f(a) = b$. By the invariance property of the infinitesimal Kobayshi metric we obtain

$$F_{M_j}(a, \xi) = F_{\Omega_j}(f_j(a), f_j'(a)\xi),\ \forall j \geqslant 1,\ \forall \xi \in T_a^{\mathbb{C}}(M).$$

Since Ω is taut, by letting $j \to \infty$ while applying Lemma 3.1 in [NT21] we obtain for each $\xi \neq 0$ that

$$0 < \overline{\lim_{j\to\infty}} F_{M_j}(a, \xi) = F_\Omega(b, f'(a)\xi).$$

It implies that f' is invertible at a, and hence at every point of M by Hurwitz's theorem. Therefore, by the inverse function theorem, f is an open map on M, and so $f(M) \subset \Omega$. By Lemma 2.3 f is indeed one to one entirely on M.

Finally, because of the biholomorphism from M to $f(M) \subset \Omega$ and the tautness of Ω, it follows that the sequence $f_j^{-1} : \Omega_j \to M_j \subset M$ is normal. Moreover, since $f_j(a) \to b \in \Omega$, it yields the sequences f_j and f_j^{-1} are not compactly divergent. Therefore, by Proposition 2.1, after passing to a subsequence, f_j converges uniformly on compacta to a biholomorphic map $f : M \to \Omega$.

□

By Proposition 2.2, we obtain the following corollary, which is a generalization of [Fr83, Lemma 1.1].

Corollary 2.4. *Let $\{M_j\}_{j=1}^\infty$ be a sequence of domains in an n-dimensional hyperbolic complex manifold M such that $\lim M_j = M$ and $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$ converging to a taut domain $\Omega \subset \mathbb{C}^n$. Let $\{f_j : M_j \to \Omega_j\}$ be a normal sequence of biholomorphisms. If there exists a point $a \in M$ such that $f_j(a)$ converges to a point $b \in \Omega$, then $\{f_j\}$ contains a subsequence that converges uniformly on compacta to a biholomorphic map $f : M \to \Omega$.*

Proof. We first prove that $F_{M_j}(a, \xi) \downarrow F_M(a, \xi)$ for all $a \in M$ and for all and $\xi \in T_a^{\mathbb{C}}(M)$. Indeed, by the decreasing property of the infinitesimal Kobayashi distance we see that the sequence $\{F_{M_j}(a, \xi)\}_{j\geqslant 1}$ is decreasing and bounded from below by $F_M(a, \xi)$.

On the other hand, for each $j \geqslant 1$, let $f_j : \Delta \to M$ be a holomorphic map with

$$f_j(0) = a, f_j'(0) = \lambda\xi, \frac{1}{\lambda} \leqslant F_M(a, \xi) + \frac{1}{j}.$$

Notice that $f_j((1-\frac{1}{j})\Delta)$ is relatively compact in M for any $j \geqslant 1$, and thus it is included in some $M_{k(j)}$. By considering the map $\tilde{f}_j : \Delta \to M_{k(j)}$ defined by

$$\tilde{f}_j(z) := f_j\Big((1 - \frac{1}{j})z\Big),\ z \in \Delta,$$

we obtain

$$F_{M_{k(j)}}(a, \xi) \leqslant \frac{1}{\lambda\Big(1 - \frac{1}{j}\Big)},\ j \geqslant 1.$$

Therefore, one has that

$$\Big(1-\frac{1}{j}\Big)F_{M_{k(j)}}(a,\xi)\leqslant F_M(a,\xi)+\frac{1}{j},,\ j\geqslant 1.$$

By letting $j\to\infty$, we get

$$\lim_{j\to\infty}F_{M_j}(a,\xi)\leqslant F_M(a,\xi).$$

Hence, $\lim\limits_{j\to\infty}F_{M_j}(a,\xi)=F_M(a,\xi)$ for all $a\in M$ and $\xi\in T_a^{\mathbb{C}}(M)$, as desired. The proof now follows from Proposition 2.2. □

Next, we need the following lemma which is essentially well-known (cf. [NNTK19]).

Lemma 2.5 (see [NNTK19]). *Let P be a weighted homogeneous polynomial with weight $(m_1,\ldots,m_{n-1})$ given by (1) such that $P(z')>0$ for all $z'\in\mathbb{C}^{n-1}\backslash\{0'\}$. Then,* $\mathrm{Aut}(D_P)$ *contains the following automorphisms $\phi_{a,\theta}$, defined by*

$$(z',z_n)\mapsto\left(\frac{(1-|a|^2)^{1/2m_1}}{(1-\bar{a}z_n)^{1/m_1}}z_1,\ldots,\frac{(1-|a|^2)^{1/2m_{n-1}}}{(1-\bar{a}z_n)^{1/m_{n-1}}}z_{n-1},e^{i\theta}\frac{z_n-a}{1-\bar{a}z_n}\right), \tag{1}$$

where $a\in\Delta:=\{z\in\mathbb{C}\colon |z|<1\}$ and $\theta\in\mathbb{R}$.

3. The strongly pseudoconvexity

The following theorem is a slight generalization of [NT21, Theorem 1.2].

Theorem 3.1. *Let M be an n-dimensional hyperbolic complex manifold and let Ω be a pseudoconvex domains in $\mathbb{C}^n$. Suppose that $\partial\Omega$ is $\mathcal{C}^2$-smooth boundary near a strongly pseudoconvex boundary point $\xi_0\in\partial\Omega$. In addition, let $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$ such that $\Omega_j\cap U=\Omega\cap U$, $j\geqslant 1$, for some neighborhood U of ξ_0. Suppose also that M can be exhausted by $\{\Omega_j\}$ via an exhausting sequence $\{f_j:\Omega_j\to M_j\subset M\}$. If there exists a point $a\in M$ such that the sequence $\eta_j:=f_j^{-1}(a)$ converges to ξ_0, then M is biholomorphically equivalent to the unit ball $\mathbb{B}^n$.*

Proof. We shall follow the proof of [NT21, Theorem 1.2] with minor modifications. Indeed, let U be a neighborhood of ξ_0 given in the statement of the theorem and let ρ be a local defining function for Ω near ξ_0. We may assume that $\xi_0=0$. After a linear change of coordinates, one can find local holomorphic coordinates $w=(w',w_n)$, defined on a neighborhood $U_0\subset U$ of ξ_0, such that

$$\rho(w)=\mathrm{Re}(w_n)+\sum_{j=1}^{n-1}|w_j|^2+O(|w_n|\|w'\|+\|w'\|^3)$$

By [DN09, Proposition 3.1] (or Subsection 3.1 in [Ber06] for the case $n=1$), for each point η in a small neighborhood of the origin, there exists an automorphism Φ_η of $\mathbb{C}^n$ such that

$$\rho(\Phi_\eta^{-1}(z))-\rho(\eta)=\mathrm{Re}(z_n)+\sum_{j=1}^{n-1}|z_j|^2+O(|z_n|\|z'\|+\|z'\|^3).$$

We now define an anisotropic dilation Δ^ϵ by

$$\Delta^\epsilon(z)=\left(\frac{z_1}{\sqrt{\epsilon}},\ldots,\frac{z_{n-1}}{\sqrt{\epsilon}},\frac{z_n}{\epsilon}\right).$$

For each $\eta \in \partial\Omega$, if we set $\rho^\epsilon_\eta(z) = \epsilon^{-1}\rho \circ \Phi_\eta^{-1} \circ (\Delta^\epsilon)^{-1}(z)$, then

$$\rho^\epsilon_\eta(z) = \mathrm{Re}(z_n) + \sum_{j=1}^{n-1} |z_j|^2 + O(\sqrt{\epsilon}).$$

By assumption, the sequence $\eta_j := f_j^{-1}(a)$ converges to ξ_0. Then, we associate with a sequence of points $\tilde{\eta}_j = (\eta_{j1}, \ldots, \eta_{j(n-1)}, \eta_{jn} + \epsilon_j)$, $\epsilon_j > 0$, such that $\tilde{\eta}_j$ is in the hypersurface $\{\rho = 0\}$. Then $\Delta^{\epsilon_j} \circ \Phi_{\tilde{\eta}_j}(\eta_j) = (0, \ldots, 0, -1)$ and one can see that $\Delta^{\epsilon_j} \circ \Phi_{\tilde{\eta}_j}(\{\rho = 0\})$ is defined by an equation of the form

$$\mathrm{Re}(z_n) + \sum_{j=1}^{n-1} |z_j|^2 + O(\sqrt{\epsilon_j}) = 0.$$

Therefore, it follows that, after taking a subsequence if necessary, $\widetilde{\Omega}_j := \Delta^{\epsilon_j} \circ \Phi_{\tilde{\eta}_j}(U_0 \cap \Omega)$ converges to the following domain

$$\mathcal{E} := \{\hat{\rho} := \mathrm{Re}(z_n) + \sum_{j=1}^{n-1} |z_j|^2 < 0\}, \tag{2}$$

which is biholomorphically equivalent to the unit ball $\mathbb{B}^n$.

Now, let us consider the sequence of biholomorphisms $F_j := T_j \circ f_j^{-1} \colon M \supset f_j(\Omega \cap U_0) \to \widetilde{\Omega}_j$, where $T_j := \Delta^{\epsilon_j} \circ \Phi_{\tilde{\eta}_j}$ for all $j \geqslant 1$. By [Ber94, Proposition 2.1] or [DN09, Proposition 2.2], since $\lim_{j\to\infty} f_j^{-1}(a) = \xi_0$ and ξ_0 is strongly pseudoconvex, it follows that for every compact subset $K \Subset M$ there exists $j_0 = j_0(K) > 0$ that for $j > j_0$ we have $f_j^{-1}(K) \subset \Omega \cap U_0$ and then $K \subset f_j(\Omega \cap U_0)$. Consequently, the sequence of domains $\{f_j(\Omega \cap U_0) = f_j(\Omega_j \cap U_0)\}$ converges to M as $j \to \infty$. In addition, since $\widetilde{\Omega}_j$ converges to the taut domain $\mathcal{E}$ and $F_j(a) = (0', -1)$ for all $j \geqslant 1$, by [DN09, Theorem 3.11] the sequence $\{F_j\}$ is normal. Therefore, Corollary 2.4 shows that, after taking some subsequence, we may assume that F_j converges uniformly on compacta to a biholomorphism from M onto $\mathcal{E}$, and hence the proof is complete. □

4. Proofs of Theorem 1.1 and Theorem 1.2

This section is devoted to proofs of Theorem 1.1 and Theorem 1.2. To do this, we first need the following lemma which is a generalization of [Liu18, Lemma 2.5].

Lemma 4.1. *Let $\{\psi_j\} \subset \mathrm{Aut}(D_P)$ be a sequence of automorphisms*

$$\psi_j(z, w) = \left(\frac{\sqrt[2m_1]{1 - |a_j|^2}}{\sqrt[m_1]{1 + \bar{a}_j z_n}} z_1, \ldots, \frac{\sqrt[2m_{n-1}]{1 - |a_j|^2}}{\sqrt[m_{n-1}]{1 + \bar{a}_j z_n}} z_{n-1}, \frac{z_n + a_j}{1 + \bar{a}_j z_n} \right),$$

where $a_j \in (0, 1)$ with $\lim \alpha_j = 1$. Then, for any $s \in (0, 1)$ we have $\psi_j^{-1}(D_P^s) \to D_P$ as $j \to \infty$.

Remark 4.1. In [Liu18, Lemma 2.5], B. Liu consider the case that $P(z') = |z'|^2$, i.e. D_P is the unit ball $\mathbb{B}^n$, and D_P^s is a ball center at $(0', b)$ with radius $1 - b$. However, the limit of $\psi_j^{-1}(D_P^s)$ must be the ellipsoid $\left\{ |z_n|^2 + \dfrac{1}{1-b}|z'|^2 < 1 \right\}$, which is strictly smaller than the unit ball $\mathbb{B}^n$. Therefore, the proof of his main theorem should be adjusted.

Proof of Lemma 4.1. A computation shows that

$$\left|\frac{z_n+a_j}{1+a_j z_n}-b\right|^2+sP\left(\frac{\sqrt[2m_1]{1-|a_j|^2}}{\sqrt[m_1]{1+\bar a_j w}}z_1,\dots,\frac{\sqrt[2m_{n-1}]{1-|a_j|^2}}{\sqrt[m_{n-1}]{1+\bar a_j z_n}}z_{n-1}\right)<s^2$$

$$\Leftrightarrow \left|\frac{z_n+a_j}{1+a_j z_n}-b\right|^2+s\frac{1-|a_j|^2}{|1+a_j z_n|^2}P(z)<s^2$$

$$\Leftrightarrow \left|w-\frac{b(1-a_j)}{1+a_j-2a_jb}\right|^2+\frac{(1-b)(1+a_j)}{1+a_j-2a_jb}P(z)<\frac{1+a_j-2b}{1+a_j-2a_jb}+\left|\frac{b(1-a_j)}{1+a_j-2a_jb}\right|^2.$$

Moreover, by a straightforward calculation, one has that

$$\lim_{j\to\infty}\frac{b(1-a_j)}{1+a_j-2a_jb}=0,\ \lim_{j\to\infty}\frac{(1-b)(1+a_j)}{1+a_j-2a_jb}=1,\ \lim_{j\to\infty}\frac{1+a_j-2b}{1+a_j-2a_jb}=1.$$

This yields $\psi_j^{-1}(D_P^s)\to D_P$ as $j\to\infty$. □

Proof of Theorem 1.1. By the invariance of $D_P^{s,r}$, D_P under the rotation $(z',z_n)\mapsto(z',e^{i\theta}z_n)$ for $\theta\in\mathbb{R}$ satisfying that $\mathrm{Im}(e^{i\theta}\eta_{jn})=0$, without loss of generality we may assume that $\mathrm{Im}(\eta_{jn})=0$ for every $j\geqslant 1$.

We now consider the sequence of automorphisms $\{\psi_j\}\subset \mathrm{Aut}(D_P)$, given by

$$\psi_j(z)=\left(\frac{\sqrt[2m_1]{1-|a_j|^2}}{\sqrt[m_1]{1+\bar a_j z_n}}z_1,\dots,\frac{\sqrt[2m_{n-1}]{1-|a_j|^2}}{\sqrt[m_{n-1}]{1+\bar a_j z_n}}z_{n-1},\frac{z_n+a_j}{1+\bar a_j z_n}\right),$$

where $a_j=\mathrm{Re}(\eta_{jn})=\eta_{jn}\in\mathbb{R}$ for all $j\geqslant 1$. Since $a_j\to 1$ as $j\to\infty$, Lemma 4.1 yields

$$\lim_{j\to\infty}\psi_j^{-1}(D_P^{s,r})=D_{P,r};\ \lim_{j\to\infty}\psi_j^{-1}(\Omega_j)=D_P,$$

where $D_{P,r}:=D_{P/r}=\left\{z\in\mathbb{C}^n\colon |z_n|^2+\frac{1}{r}P(z')<1\right\}$. Moreover, since $\psi_j^{-1}(\eta_j)=\left(\frac{a_{j1}}{\lambda_j^{1/2m_1}},\dots,\frac{a_{j(n-1)}}{\lambda_j^{1/2m_{n-1}}},0\right)\in D_{P,r}\cap\{z_n=0\}$, where $\lambda_j=1-|a_j|^2$ for all $j\geqslant 1$ and $D_{P,r}\cap\{z_n=0\}\Subset D_P\cap\{z_n=0\}$, by passing to a subsequence if necessary, we may assume that $\psi_j^{-1}(\eta_j)$ converges to some point $p\in D_P$ (see Figure 1 below). Therefore, by Corollary 2.4 we conclude that $\psi_j^{-1}\circ f_j^{-1}$ converges uniformly on compacta to a biholomorphic map $F\colon M\to D_P$, and thus the proof is complete. □

Proof of Theorem 1.2. For each $j\geqslant 1$, choose $\theta_j\in\mathbb{R}$ such that that $\mathrm{Im}(e^{i\theta_j}\eta_{jn})=0$. Since $\mathrm{Im}(\eta_{jn})\to 0$ as $j\to\infty$, one has that $\theta_j\to 0$ as $j\to\infty$. Moreover, by shrinking U if necessary we may also assume that $R_{\theta_j}(\Omega_j)\cap U=D_P\cap U$ for all $j\geqslant 1$. Therefore, by the invariance of D_P under the rotation $R_{\theta_j}\colon(z',z_n)\mapsto(z',e^{i\theta_j}z_n)$ for $j\geqslant 1$, without loss of generality we assume that $\mathrm{Im}(\eta_{jn})=0$ for every $j\geqslant 1$ and $\{\Omega_j\}$ is a sequence of subdomains of D_P such that $\Omega_j\cap U=D_P\cap U$ for all $j\geqslant 1$.

We now consider the sequence of automorphisms $\{\psi_j\}\subset \mathrm{Aut}(D_P)$, given by

$$\psi_j(z)=\left(\frac{\sqrt[2m_1]{1-|a_j|^2}}{\sqrt[m_1]{1+\bar a_j z_n}}z_1,\dots,\frac{\sqrt[2m_{n-1}]{1-|a_j|^2}}{\sqrt[m_{n-1}]{1+\bar a_j z_n}}z_{n-1},\frac{z_n+a_j}{1+\bar a_j z_n}\right),$$

where $a_j=\mathrm{Re}(\eta_{jn})=\eta_{jn}\in\mathbb{R}$ for all $j\geqslant 1$. Since $a_j\to 1$ as $j\to\infty$, Lemma 4.1 yields

$$\lim_{j\to\infty}\psi_j^{-1}(\Omega_j)=\lim_{j\to\infty}\psi_j^{-1}(\Omega_j\cap U)=\lim_{j\to\infty}\psi_j^{-1}(D_P\cap U)=D_P.$$

Let us set $b_j = \psi_j^{-1}(\eta_j)$ for all $j \geqslant 1$. Then, a straightforward computation shows that

$$b_j = \psi_j^{-1}(\eta_j) = \Big(\frac{\eta_{j1}}{\lambda_j^{1/2m_1}}, \ldots, \frac{\eta_{j(n-1)}}{\lambda_j^{1/2m_{n-1}}}, 0\Big) \in D_P \cap \{z_n = 0\},$$

where $\lambda_j = 1 - |a_j|^2$ for all $j \geqslant 1$.

Since $\{\eta_j\}$ converges Λ-tangentially to $(0', 1)$, it follows that there exists a sequence $\{r_j\} \subset (0, 1)$ with $r_j \to 1$ as $j \to \infty$ such that

$$|\eta_{jn} - 1 - s|^2 + \frac{s}{r_j} P(\eta_j') > s^2, \ \forall j \geqslant 1.$$

This implies that

$$\begin{aligned} P(b_j') = \frac{1}{\lambda_j} P(\eta_j') &\geqslant \frac{2r_j(1-a_j)}{1-a_j^2} - \frac{r_j}{s} \frac{|1-a_j|^2}{1-a_j^2} \\ &\geqslant \frac{2r_j}{1+a_j} - \frac{r_j}{s} \frac{(1-a_j)}{1+a_j} \end{aligned}$$

for all $j \geqslant 1$. Therefore, we obtain that $P(b_j') \to 1$ as $j \to \infty$, and hence by passing to a subsequence if necessary, we may assume that $\psi_j^{-1}(\eta_j)$ converges to some strongly pseudoconvex boundary point $p \in \partial D_P \cap \{z_n = 0\}$ (see Figure 2 below). Thus, by Theorem 3.1 we conclude that $\psi_j^{-1} \circ f_j^{-1}$ converges uniformly on compacta to a biholomorphic map $F \colon M \to \mathbb{B}^n$, and thus the proof is complete. □

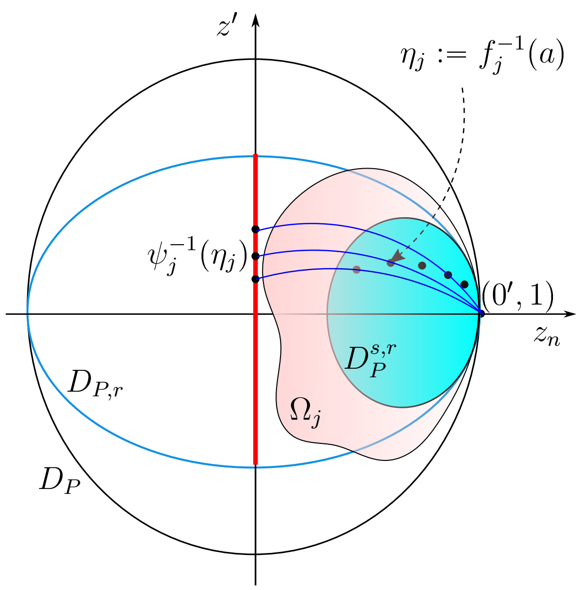

FIGURE 1. Λ-nontangential convergence

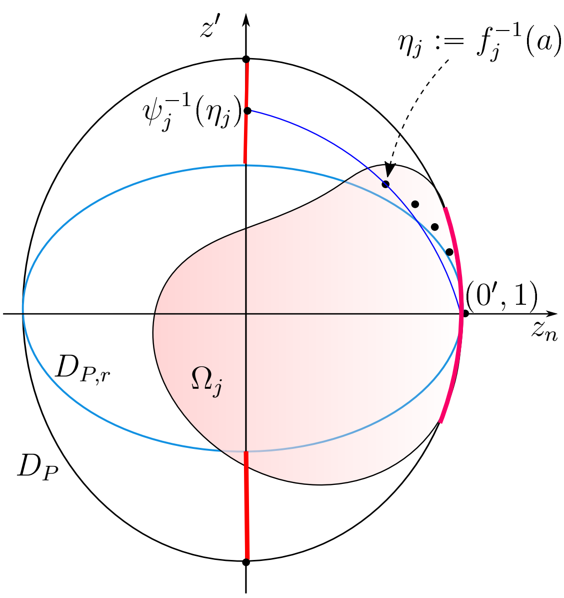

FIGURE 2. Λ-tangential convergence

We now consider a pseudoconvex domain Ω in $\mathbb{C}^n$ with noncompact automorphism group. Roughly speaking, there exists a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\eta_j := \varphi_j(a)$ converges to a boundary point $\xi_0 \in \partial\Omega$ for some $a \in \Omega$. In [NN19], the first and last authors showed that if $\xi_0 \in \partial\Omega$ is an h-extendible boundary point and $\eta_j := \varphi_j(a)$ converges Λ-nontangentially to ξ_0 (cf. [NN19, Definition 3.4]), then Ω is biholomorphically equivalent to the model

$$M_P := \{z \in \mathbb{C}^n \colon \mathrm{Re}(z_n) + P(z') < 0\}.$$

However, we notice that our domain Ω is exhausted by Ω via the sequence $\varphi_j \colon \Omega \to \Omega$. Moreover, in the case that $P(z') > 0$ whenever $z' \in \mathbb{C}^{n-1}\backslash\{0\}$ and D_P is a WB-domain, by Theorem 1.1 and Theorem 1.2 we have the following corollaries.

Corollary 4.2. *Let Ω be a subdomain of D_P and $\Omega \cap U = D_P \cap U$ for a fixed neighborhood U of $(0', 1)$ in $\mathbb{C}^n$. Suppose that there exists a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\eta_j := \varphi_j(a)$ converges to ξ_0 for some $a \in \Omega$. Then, one of the following assertions holds:*

(i) *Ω is biholomorphically equivalent to M_P;*
(ii) *Ω and D_P are biholomorphically equivalent to $\mathbb{B}^n$.*

For the case when $\max\{m_1, \ldots, m_{n-1}\} > 1$, [CP01, Main Theorem] shows that D_P is not biholomorphically equivalent to $\mathbb{B}^n$. Therefore, Corollary 4.2 yields the following corollary.

Corollary 4.3. *Let Ω be a subdomain of D_P and $\Omega \cap U = D_P \cap U$ for a fixed neighborhood U of $(0', 1)$ in $\mathbb{C}^n$. Suppose that there exists a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\{\varphi_j(a)\}$ converges to ξ_0 for some $a \in \Omega$. If $\max\{m_1, \ldots, m_{n-1}\} > 1$, then $\{\varphi_j(a)\}$ must converge Λ-nontangentially to ξ_0.*

5. Spherically extreme boundary points

In this section, we are going to give a proof of Theorem 1.3. Then, several corollaries are also given.

Proof of Theorem 1.3. Let Ω be a bounded domain with $\mathcal{C}^2$ smooth boundary in a neigborhood U of the point $p \in \partial\Omega$. Suppose that Ω admits p as a locally spherically extreme point in the sense that the unit ball tangent to $\partial\Omega$ at p. We will show that Ω is strongly pseudoconvex at p. Notice that we do not assume apriori that Ω is pseudoconvex near p.

For the simplicity of exposition we may assume $p = (0, \ldots, 0, 1) \in \partial\Omega$. Moreover, by a rotation of coordinates and the implicit function theorem we may assume that near p, Ω admits a local defining function ρ taking the form

$$\rho(z) := y_n - \varphi(z', x_n),$$

where φ is $\mathcal{C}^2$ smooth near the origin $(0, \ldots, 0) \in \mathbb{R}^{2n-1}$ and satisfies $\varphi(0, \ldots, 0) = 1$. Since the function

$$\psi(z) := \|z'\|^2 + x_n^2 + \varphi(z', x_n)^2$$

attains its local maximum at the origin, for $1 \leqslant i \leqslant n-1$ we obtain

$$0 = \frac{\partial\psi}{\partial z_i}(0) = \frac{\partial\psi}{\partial x_n}(0).$$

An easy application of the chain rule yields

$$\frac{\partial\varphi}{\partial z_i}(0) = \frac{\partial\varphi}{\partial x_n}(0) = 0.$$

Now we look at the complex tangent plane at p. For this, we note that

$$\frac{\partial\rho}{\partial z_i}(p)=\frac{1}{2}[\frac{\partial\rho}{\partial x_i}(p)-i\frac{\partial\rho}{\partial y_i}(p)]=0\ \forall 1\leqslant i\leqslant n-1,$$
$$\frac{\partial\rho}{\partial z_n}(p)=\frac{1}{2}[\frac{\partial\rho}{\partial x_n}(p)-i\frac{\partial\rho}{\partial y_n}(p)]=-\frac{i}{2}.$$

Hence the complex tangent at p reduces to $\mathbb{C}^{n-1}\times\{0\}$. Now we suppose for the sake of obtaining a contradiction that p is not a strongly pseudoconvex of $\partial\Omega$. Then, by the structure of the complex tangent at p, we may find $(t_1,\ldots,t_{n-1})\in\mathbb{C}^{n-1}\backslash\{0\}$ such that

$$\sum_{1\leqslant j,k\leqslant n-1}\frac{\partial^2\rho}{\partial z_j\partial\bar z_k}(0)t_j\bar{t_k}<0,$$

or equivalently

$$\sum_{1\leqslant j,k\leqslant n-1}\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}(0)t_j\bar{t_k}>0. \tag{3}$$

Since $\psi(z',0)\leqslant 1$ on a neighborhood of $0\in\mathbb{C}^{n-1}$, by Taylor expansion theorem the following estimate holds true for all $\varepsilon>0$ small enough

$$\begin{aligned}1&\geqslant\varepsilon^2(|t_1|^2+\cdots+|t_{n-1}|^2)+\varphi(\varepsilon t_1,\ldots,\varepsilon t_{n-1})^2\\&\geqslant\varepsilon^2(|t_1|^2+\cdots+|t_{n-1}|^2)+\left(1+\frac{\varepsilon^2}{2}\sum_{1\leqslant j,k\leqslant n-1}\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}(0)t_j\bar{t_k}+o(\varepsilon^2)\right)^2\\&>1+\varepsilon^2(|t_1|^2+\cdots+|t_{n-1}|^2)+\varepsilon^2\sum_{1\leqslant j,k\leqslant n-1}\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}(0)t_j\bar{t_k}+o(\varepsilon^2).\end{aligned}$$

After rearranging the above estimate and letting $\epsilon\to 0$ we arrive at a contradiction to (3). □

By Theorem 3.1 and Theorem 1.3, we obtain the following corollary.

Corollary 5.1. *Let M be an n-dimensional hyperbolic complex manifold and let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Suppose that $\partial\Omega$ admits a spherically extreme boundary point ξ_0 in a neighborhood of which the boundary $\partial\Omega$ is $\mathcal{C}^2$-smooth. In addition, let $\{\Omega_j\}$ be a subdomains of Ω such that $\Omega_j\cap U=\Omega\cap U$, $j\geqslant 1$, for some neighborhood U of ξ_0 in $\mathbb{C}^n$. Suppose also that M can be exhausted by $\{\Omega_j\}$ via an exhausting sequence $\{f_j:\Omega\supset\Omega_j\to M_j\subset M\}$. If there exists a point $a\in M$ such that the sequence $\eta_j:=f_j^{-1}(a)$ converges to ξ_0, then M is biholomorphically equivalent to the unit ball $\mathbb{B}^n$.*

We note that if $p\in\partial\Omega$ is a spherically extreme boundary point, then $\lim\limits_{\Omega\ni z\to p}\sigma_\Omega(z)=1$ (see [KZ16, Theorem 3.1]). Hence, the above corollary easily follows from the following corollaries.

Corollary 5.2. *Let M be an n-dimensional hyperbolic complex manifold and let $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$. Suppose that M can be exhausted by $\{\Omega_j\}$ via an exhausting sequence $\{f_j:\Omega_j\to M_j\subset M\}$. Suppose also that there exists a point $a\in M$ such that*

$$\lim_{n\to\infty}\sigma_{\Omega_j}(\eta_j)=1,$$

where $\eta_j := f_j^{-1}(a)$ *for all* $j \geqslant 1$*. Then,* M *is biholomorphically equivalent to the unit ball* $\mathbb{B}^n$*.*

Proof. By the assumption on the sequence $\{\eta_j\}$, there exists a sequence of injective holomorphic maps $G_j : \Omega_j \to \mathbb{B}^n$ such that $G_j(\eta_j) = 0$ and $G_j(\Omega_j)$ exhausts $\mathbb{B}^n$. Thus the sequence

$$\tilde{G}_j := G_j \circ f_j^{-1} : M_j \to G_j(\Omega_j)$$

satisfies $\tilde{G}_j(a) = 0$ for all $j \geqslant 1$. By Montel theorem, the sequence is also normal. Thus, we may apply Corollary 2.4 to complete the proof. □

By Corollary 5.2, one obtains the following corollary.

Corollary 5.3. *Let* M *be an* n*-dimensional hyperbolic complex manifold and let* Ω *be a domain in* $\mathbb{C}^n$*. Suppose that* M *can be exhausted by* Ω *via an exhausting sequence* $\{f_j : \Omega \to M_j \subset M\}$*. Assume that there exists a point* $a \in M$ *such that the sequence* $\eta_j := f_j^{-1}(a)$ *converges to* $\xi_0 \in \partial\Omega$ *and*

$$\lim_{q \to \xi_0} \sigma_\Omega(q) = 1.$$

Then, M *is biholomorphically equivalent to the unit ball* $\mathbb{B}^n$*.*

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Ninh Van Thu

School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, No. 1 Dai Co Viet, Hai Ba Trung, Hanoi, Vietnam
Email address: thu.ninhvan@hust.edu.vn

Trinh Huy Vu

Department of Mathematics, Vietnam National University at Hanoi, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam
Email address: trinhhuyvu1508@gmail.com

Nguyen Quang Dieu

[1] Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Hanoi, Vietnam

[2] Thang Long Institute of Mathematics and Applied Sciences, Nghiem Xuan Yem, Hoang Mai, HaNoi, Vietnam
Email address: ngquang.dieu@hnue.edu.vn