Event-triggered state estimation for nonlinear systems aided by machine learning

D.C. Huong, T.N. Nguyen, H.T. Le

Abstract—**This paper considers the event-triggered state estimation problem with the aid of machine learning for nonlinear systems subject to external disturbances in the state and output vectors. First, we develop a recurrent neural network (RNN) learning algorithm to predict the nonlinear systems. Second, we design a discrete-time event-triggered mechanism and a state observer based on this mechanism for the RNN model. This discrete-time event-triggered state observer significantly reduces the utilization of communication resources. Third, we establish a sufficient condition to ensure that the state observer can robustly estimate the state vector of the recurrent neural network. Finally, we provide an illustrative example to verify the merit of the proposed method.**

Index Terms—**Nonlinear systems, discrete-time event-triggered mechanism, disturbances, linear matrix inequality (LMI).**

In many engineering fields, the information of state vectors of dynamical systems is usually required for control design or fault detection. However, due to technical or economic reasons, the true state vectors of the systems are not available. Therefore, dynamical state estimation becomes an important application in different areas ranging from control engineering, robotics, tracking and navigation (see, for example, [1], [2], [3], [4]).

There are many approaches in the literature dealing with the design of state observers for the purpose of state estimation (see, for example, [5]-[14]). It is noted that all the state observers in the references [5]-[14] are designed based on time-triggered schemes, i.e., observer designs require system data for each sampling instant, which may lead to the wastage of communication resources in practical applications. To overcome this drawback, some event-triggered state observers were proposed [15]-[26] in order to maintain the desired performance while reducing the utilization of communication resources. In particular, an event-triggered extended state observer was proposed in [15] for nonlinear systems with disturbances, where nonlinear functions are continuously differentiable, while a robust state observer on the basis of the continuous dynamic event-triggered mechanism was designed in [16] for a class of nonlinear systems where the nonlinear function satisfies the one-sided Lipschitz and quadratically inner-bounded conditions. Noting that the nonlinear functions in [15] and [16] have some restriction, therefore the event-triggered state observers in these references cannot applied to general nonlinear systems. Moreover, since the dynamic event-triggered mechanism in [16] depends on continuous supervision of the event-driven condition, it may cause strict requirement for driven-condition.

On the other hand, in recent years, machine learning has attracted a lot of research attentions due to the increase in its applications in many fileds as fault-tolerant control, cybersecurity, real-time control, and optimization problems. With the rapid development of computational resources, machine learning techniques have been widely utilized to solve important problem such as classification and regression. RNNs are belong to machine learning techniques which have received considerable attention in recent years due to their advantages in learning ability, parallel computation, industrial automation, and function approximation [27], [28], [29], [30]. Recently, in [31], the authors considered the design of model predictive control systems for nonlinear processes that utilize an ensemble of models to predict nonlinear dynamics. They first trained a RNN model by using extensive open-loop simulation data to capture process dynamics in a certain operating region such that the modeling error between the RNN model and the actual nonlinear process model was sufficiently small. Then, they utilized the RNN model as the prediction model in order to achieve closed-loop state boundedness and convergence to the origin. However, [31] did not consider the issue of external disturbances in the RNN as well as the design of discrete-time event-triggered state observers for the RNN prediction model.

Motivated by the works [15], [16] and [31], in this paper, we focus on the design of discrete-time event-triggered state observers for the nonlinear systems subject to external disturbances with the aid of machine learning. The contributions of this paper are: (1) The nonlinear system is predicted by a RNN model subject to external disturbances; (2) a discrete-time event-triggered mechanism is designed and utilized for designing event-triggered state observers of the RNN model; (3) the event-triggered the observer is new and can reduce the utilization of communication resources while maintaining the desired robust estimation performance; and (4) a sufficient condition for the existence of the event-triggered state observer is established and unknown observer matrices are obtained by solving a convex optimization problem.

Notation: For a matrix A, A^T denotes its transpose. $\mathbb{R}^n$ is the $n-$ dimensional linear vector space over the reals with the Euclidean norm. $||x||$ denotes Euclidean norm of vector $x \in \mathbb{R}^n$. If $x^T P x > 0, \forall x \neq 0$, then matrix P is called positive definite and denoted by $P > 0$. sym$\Big\{M\Big\}$ denotes the notation $M + M^T$. $*$ denotes the entries of a matrix implied by symmetry. $||g||_{\mathcal{L}_2^n} = \sqrt{\int_0^\infty g^T(t)g(t)dt}$ is the $\mathcal{L}_2^n$ norm of

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the vector function $g : [0, \infty) \to \mathbb{R}^n$ and $\mathcal{L}_2^n = \{g : [0, \infty) \to \mathbb{R}^n : ||g||_{\mathcal{L}_2^n} < \infty\}$.

I. Preliminaries and problem statement

Let us consider the following nonlinear systems subject to disturbances

$$\begin{aligned} \dot{x}(t) &= F(x,u,\omega) = f(x(t)) + g(x(t))u(t) \\ &\quad + h(x(t))\omega(t),\ t \geq 0, \quad (1) \\ x(0) &= x_0, \quad (2) \\ y(t) &= Cx(t) + Ed(t), \quad (3) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ is the output vector, $\omega(t) \in \mathbb{R}^{n_\omega}$ is the disturbance in the state vector, $d(t) \in \mathbb{R}^{n_d}$ is the disturbance in the output vector. $F(x, u, \omega)$ is a nonlinear function with respect to x, u and ω. $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ are sufficiently smooth vector and matrix functions of dimensions $n \times 1$, $n \times m$ and $n \times n_\omega$, respectively. $C \in \mathbb{R}^{p \times n}$, $E \in \mathbb{R}^{p \times n_d}$ are constant matrices.

We need the following assumptions:
(H_1) $||F(x, u, \omega)|| \leq M$;
(H_2) $||F(z, u, \omega) - F(\bar{z}, u, \omega)|| \leq L_z||z - \bar{z}||$, $\forall z, \bar{z} \in D$, $\forall u \in U$ and $\forall \omega \in W$, where D is an open neighborhood around the origin, $U = \{u \in \mathbb{R}^m : u_i \in [u_i^{\min}, u_i^{\max}],\ i = 1, 2, \ldots, m\}$, $W = \{\omega \in \mathbb{R}^{n_\omega} : ||\omega|| \leq \bar{\omega}\}$ and M, L_z, $\bar{\omega}$ are positive scalars.

II. Main Results

A. Recurrent neural network (RNN)

Let us now consider the following RNN model which is utilized to approximate the nonlinear system (1):

$$\begin{aligned} \dot{\tilde{x}}(t) &= F_{rnn}(\tilde{x}, u, \omega) \\ &= A\tilde{x}(t) + \Theta^T \overline{y}(t),\ t \geq 0, \quad (4) \\ \tilde{x}(0) &= x_0, \quad (5) \\ \tilde{y}(t) &= C\tilde{x}(t) + Ed(t), \quad (6) \end{aligned}$$

where $\tilde{x}(t) \in \mathbb{R}^n$ is the RNN state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $\overline{y}(t) = \begin{bmatrix} \overline{y}^1(t) & \overline{y}^2(t) & \overline{y}^3(t) \end{bmatrix}^T \in \mathbb{R}^{n+m+n_\omega}$, $\overline{y}^1(t) = \begin{bmatrix} \overline{y}_1 & \ldots & \overline{y}_n \end{bmatrix}^T = \begin{bmatrix} \sigma(\tilde{x}_1) & \ldots & \sigma(\tilde{x}_n) \end{bmatrix}^T$, $\overline{y}^2(t) = \begin{bmatrix} \overline{y}_{n+1} & \ldots & \overline{y}_{n+m} \end{bmatrix}^T = \begin{bmatrix} u_1 & \ldots & u_m \end{bmatrix}^T$, $\overline{y}^3(t) = \begin{bmatrix} \overline{y}_{n+m+1} & \ldots & \overline{y}_{n+m+n_\omega} \end{bmatrix}^T = \begin{bmatrix} \omega_1 & \ldots & \omega_{n_\omega} \end{bmatrix}^T$, where $\sigma(\cdot)$ is the nonlinear activation function. $A = \mathrm{diag}\{-a_1, -a_2, \ldots, -a_n\}$ and $\Theta = \begin{bmatrix} \theta_1 & \ldots & \theta_n \end{bmatrix} \in \mathbb{R}^{(n+m+n_\omega)\times n}$, $\theta_i = b_i \begin{bmatrix} w_{i1} & \ldots & w_{i(n+m+n_\omega)} \end{bmatrix}^T$, $a_i > 0$, b_i are constants, w_{ij} is the weight connecting the jth input to the ith neuron, $i = 1, \ldots, n$, $j = 1, \ldots, n + m + n_\omega$ are matrices to be optimized during training.

In this paper we assume that activation function $\sigma(.)$ and external disturbance $d(t)$ in the output vector satisfy the following assumptions:
(H_3) There are positive scalars $\sigma_i > 0$ for $i = 1, 2, \ldots, n$ such that

$$|\sigma_i(s_1) - \sigma_i(s_2)| \leq \sigma_i|s_1 - s_2|, \forall s_1, s_2 \in \mathbb{R}. \quad (7)$$

(H_4) The external disturbance $d(t)$ is norm bounded, i.e. there exist a positive constant $\bar{d}$ such that

$$||d(t)|| \leq \bar{d},\ \forall t \geq 0. \quad (8)$$

We now develop the RNN learning algorithm in [30], [31] to obtain an optimal weight matrix θ_i^* such that state vector $\tilde{x}(t)$ of the RNN can estimate state vector $x(t)$ of nonlinear system (1). For this, we first express the dynamic behavior of each state of the system (1) by the following equation:

$$\dot{x}_i(t) = -a_i x_i + \theta_i^* \overline{y}(t) + \nu_i(t), \quad (9)$$

where

$$\nu_i(t) = F_i(x, u, \omega) - (F_{rnn})_i(\tilde{x}, u, \omega), \quad (10)$$

and optimal weight vector θ_i^* is defined as below:

$$\theta_i^* = \arg\min_{|\theta_i| \leq \theta_m} \Big\{ \sum_{k=1}^{N} |F_i(x_k, u_k, \omega_k) + a_i x_k - \theta_i^T \overline{y}_k| \Big\}, \quad (11)$$

where $\theta_m > 0$ and N is the number of data samples utilized for training.

It is assumed that RNN model (4) and nonlinear system (1) satisfy the following sufficiently small modeling error

$$||\nu(t)|| = ||F(x, u, \omega) - F_{rnn}(\tilde{x}, u, \omega)|| \leq \bar{\nu}, \bar{\nu} > 0. \quad (12)$$

We now denote the state error as $e(t) = \tilde{x}(t) - x(t) \in \mathbb{R}^n$. Then, it follows from (4) and (9) that

$$\dot{e}_i(t) = -a_i e_i(t) + (\theta_i(t) - \theta_i^*)^T \overline{y}_i(t) - \nu_i(t). \quad (13)$$

The weight $\theta_i(t)$ is updated during the training process as below:

$$\dot{\theta}_i(t) = -\eta_i \overline{y}(t) e_i(t),\ i = 1, 2, \ldots, n, \quad (14)$$

where the learning rate η_i is a positive define $(n + m + n_\omega) \times (n + m + n_\omega)$ matrix.

The following lemma provides an upper bound for the error $e(t)$:

Lemma 3. Let assumption (H_2) and condition (12) be satisfied. The following inequality holds:

$$||e(t)|| \leq \frac{\bar{\nu}}{L_x}(e^{L_x t} - 1),\ t > 0. \quad (15)$$

Remark 1: In [31], the authors considered a RNN learning algorithm without external disturbances ($\omega(t) \equiv 0$) for nonlinear systems with external disturbances of the form (1). Therefore, the error $e(t)$ between state vector of the RNN and state vector of the nonlinear system satisfying the following inequality:

$$||e(t)|| \leq \frac{L_\omega \bar{\omega} + \bar{\nu}}{L_x}(e^{L_x t} - 1),\ t > 0,\ \omega(t) \in W. \quad (16)$$

Unlike the method in [31], in this paper, we consider a RNN learning algorithm with external disturbances ($\omega(t) \not\equiv 0$) and thus, term $L_\omega \bar{\omega}$ in the upper bound of the error $e(t)$ in (16) is removed. The effect of the external disturbances will be minimized when we solve the robust state estimation for the RNN subject to external disturbances in the next section.

B. Event-triggered state observers

Let us first express system (4)-(6) into the following form

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + \Theta_{\tilde{x}}\sigma(\tilde{x}(t)) + \Theta_u u(t) + \Theta_\omega \omega(t) \ t \geq 0, \\ \tilde{x}(0) &= x_0, && (17) \\ \tilde{y}(t) &= C\tilde{x}(t) + Ed(t), && (18) \end{aligned}$$

where $\begin{bmatrix} \Theta_{\tilde{x}} & \Theta_u & \Theta_\omega \end{bmatrix} = \Theta^T$.

Next, we propose the following event-triggered state observer for system (17)-(18):

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + \Theta_{\tilde{x}}\sigma(\hat{x}(t)) + \Theta_u u(t) \\ &\quad + L(\tilde{y}(t_k) - C\hat{x}(t_k + k(t)h), \ t \in [t_k, t_{k+1}), \end{aligned} \tag{19}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $\tilde{x}(t)$, L is the gain matrix to be designed, $k(t) = \lfloor \frac{t-t_k}{h} \rfloor = \max\Big\{ s \in \mathbb{N} \mid s \leq \frac{t-t_k}{h} \Big\}$, $\{t_k h\}_{k\in\mathbb{N}}$ ($k \in \mathbb{N}$, $t_0 h = 0 < t_1 h < \cdots < t_k h < \ldots$) is the measurement transmission instant sequence and consider the following event-triggered mechanism (ETM):

$$\begin{aligned} &t_{k+1}h = \\ &\min_{\ell > t_k, \ \ell \in \mathbb{N}^+} \Big\{ \ell h \mid \mathcal{H}(\tilde{y}(\ell h), \tilde{y}(t_k), \delta(t_k)) > 0 \Big\}, \ k \in \mathbb{N}, \end{aligned} \tag{20}$$

where $\delta(t_k) = \delta_1 + \delta_2 e^{-\delta_3 t}$ and $\mathcal{H}(\tilde{y}(\ell h), \tilde{y}(t_k), \delta(t_k)) = \delta_4 ||\tilde{y}(\ell h) - \tilde{y}(t_k)||^2 - \delta_5 ||\tilde{y}(t_k)||^2 - \delta(t_k)$, $\delta_i, \ i = 1,2,3,4,5$ are positive scalars.

The state observer (19) has a constraint that the current sampled measurement $\tilde{y}(\ell h)$ is only transmitted when the event-triggered condition (20) is satisfied.

From the above definition of $k(t)$, we have $k(t) = \max_{\ell}\Big\{ \ell \in \mathbb{N} \mid t_k \leq \ell h + t_k \leq t \Big\}$. This implies that

$$t - h < t_k + k(t)h \leq t. \tag{21}$$

Let us define

$$\begin{aligned} e_{\tilde{y}}(t) &= \tilde{y}(t_k) - \tilde{y}(t_k + k(t)h), && (22) \\ \tau(t) &= t - (t_k + k(t)h). && (23) \end{aligned}$$

Then we obtain $0 \leq \tau(t) < h$.

By denoting $\overline{x}(t) = \tilde{x}(t) - \hat{x}(t)$ and $\sigma^{\Delta}_{\tilde{x}z}(t) = \sigma(\tilde{x}(t)) - \sigma(\hat{x}(t))$, from (19), (22) and (23), we obtain

$$\begin{aligned} \dot{\overline{x}}(t) &= A\overline{x}(t) + \Theta_{\tilde{x}}\sigma^{\Delta}_{\tilde{x}z}(t) + \Theta_\omega \omega(t) \\ &\quad - LC\overline{x}(t - \tau(t)) - Le_{\tilde{y}}(t) \\ &\quad - LEd_\tau(t), \ t \in [t_k, t_{k+1}), && (24) \\ \overline{x}(\theta) &= \phi(0) - \psi(0), \ \theta \in [-h, 0], && (25) \end{aligned}$$

where $d_\tau(t) = d(t - \tau(t))$.

Remark 2: The discrete-time event-triggered state observer (19) is different from the one in [15], [16], [17], [18], since the term $C\hat{x}(t)$ is replaced by $C\hat{x}(t_k + k(t)h) = C\hat{x}(t - \tau(t))$, which is more reality because the measurement received by the observers after a network delay $\tau(t)$.

In the following, we will design the gain matrix L such that, under zero initial condition, estimation error $\overline{x}(t)$ with a time-varying delay $\tau(t)$ satisfies the following inequality

$$||\overline{x}||_{\mathcal{L}_2^n} \leq \lambda_\omega ||\omega||_{\mathcal{L}_2^{n_\omega}} + \lambda_d ||d||_{\mathcal{L}_2^{n_d}}, \tag{26}$$

where $\lambda_\omega > 0$ and $\lambda_d > 0$ will be optimized.

The following theorem allows us to obtain the gain matrix L and the minimized levels λ_ω and λ_d.

Theorem 1. Let assumptions (H_3) and (H_4) be satisfied. For given positive numbers β and $0 < \xi < 1$, the event-triggered state observer design problem (26) with minimized levels $\lambda_\omega > 0$ and $\lambda_d > 0$ is solvable if there exist matrices $P > 0$, $Q > 0$, $R > 0$, Z, X, non-singular S and positive scalars $\bar{\lambda}_\omega$, $\bar{\lambda}_d$ and ϵ such that the following convex optimization problem is feasible:

$$\min(\xi\bar{\lambda}_\omega + (1-\xi)\bar{\lambda}_d) \tag{27}$$

subject to

$$\hat{\Xi}(\nu) = \begin{bmatrix} \Xi(\nu) & \nabla \\ * & -\epsilon I_n, \end{bmatrix} < 0, \ \forall \nu \in \{0,1\}, \tag{28}$$

$$\Delta = \begin{bmatrix} \text{diag}(R,R) & Z \\ * & \text{diag}(R,R) \end{bmatrix} > 0, \tag{29}$$

where

$$\begin{aligned} \Xi(\nu) &= \text{sym}\{\Psi_\nu^T P \Gamma\} + e_1^T Q e_1 - e_3^T Q e_3 \\ &\quad + h^2 e_4^T R_2 e_4 - \Lambda^T \Delta \Lambda + e_1^T e_1 - \lambda_\omega e_8^T e_8 \\ &\quad - \lambda_d e_9^T e_9 - \text{sym}\Big\{ \begin{bmatrix} e_1^T & e_4^T \end{bmatrix} \overline{S} e_4 \Big\}, \\ &\quad + \text{sym}\Big\{ \begin{bmatrix} e_1^T & e_4^T \end{bmatrix} \overline{S} A e_1 \\ &\quad - \begin{bmatrix} e_1^T & e_4^T \end{bmatrix} \overline{X} C e_2 \\ &\quad - \begin{bmatrix} e_1^T & e_4^T \end{bmatrix} \overline{X} e_7 \\ &\quad + \begin{bmatrix} e_1^T & e_4^T \end{bmatrix} \overline{S} \Theta_\omega e_8 \\ &\quad - \begin{bmatrix} e_1^T & e_4^T \end{bmatrix} \overline{X} E e_9 \Big\}, \\ &\quad + \epsilon \sigma_{\max}^2 e_1^T e_1, \sigma_{\max} = \max\{\sigma_1, \ldots, \sigma_n\}, \\ &\quad \overline{S} = \begin{bmatrix} \beta S \\ S \end{bmatrix}, \ \overline{X} = \begin{bmatrix} \beta X \\ X \end{bmatrix}, \\ &\quad \Psi_\nu = \begin{bmatrix} e_1^T & \nu h e_5^T + (1-\nu) h e_6^T \end{bmatrix}^T, \\ &\quad \Gamma = \begin{bmatrix} e_4^T & (e_1^T - e_3^T) \end{bmatrix}^T, \ \Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \end{bmatrix}^T, \\ &\quad \Lambda_1 = \begin{bmatrix} (e_1 - e_2)^T & \sqrt{3}(e_1 + e_2 - 2e_5)^T \end{bmatrix}, \\ &\quad \Lambda_2 = \begin{bmatrix} (e_2 - e_3)^T & \sqrt{3}(e_2 + e_3 - 2e_6)^T \end{bmatrix}, \\ \nabla &= \begin{bmatrix} e_1^T & e_4^T \end{bmatrix} \overline{S} \Theta_{\tilde{x}} \begin{bmatrix} I_n & 0_{n\times n} \end{bmatrix}, \\ &\quad e_i = \begin{bmatrix} e_i^1 & e_i^2 \end{bmatrix} \in \mathbb{R}^{n\times(6n+p+n_\omega+n_d)}, \\ &\quad i = 1, \ldots, 6, \ e_i^1 = \begin{bmatrix} 0_{n\times(i-1)n} & I_n \end{bmatrix}, \\ &\quad e_i^2 = \begin{bmatrix} 0_{n\times(6-i)n} & 0_{n\times(p+n_\omega+n_d)} \end{bmatrix}, \\ &\quad e_7 = \begin{bmatrix} 0_{p\times 6n} & I_p & 0_{p\times(n_\omega+n_d)} \end{bmatrix} \\ &\quad \in \mathbb{R}^{p\times(6n+p+n_\omega+n_d)}, \\ &\quad e_8 = \begin{bmatrix} 0_{n_\omega\times 6n} & 0_{n_\omega\times p} & I_{n_\omega} & 0_{n_\omega\times(n_\omega+n_d)} \end{bmatrix} \\ &\quad \in \mathbb{R}^{n_\omega\times(6n+p+n_\omega+n_d)}, \\ &\quad e_9 = \begin{bmatrix} 0_{n_d\times 6n} & 0_{n_\omega\times p} & 0_{n_d\times n_d} & I_{n_d} \end{bmatrix} \\ &\quad \in \mathbb{R}^{n_d\times(6n+p+n_\omega+n_d)}. \end{aligned}$$

The observer gain matrix L is obtained as

$$L = S^{-1}X. \quad (30)$$

Proof. Let us first denote $\nu = \nu(t) = \frac{\tau(t)}{h}$, $\tilde{e}(t) = \begin{bmatrix} \overline{x}^T(t) & \int_{t-h}^{t} \overline{x}^T(s)ds \end{bmatrix}^T$ and consider the following Lyapunov functional candidate:

$$\begin{aligned} V(t) &= \tilde{e}^T(t)P\tilde{e}(t) + \int_{t-h}^{t} \overline{x}^T(s)Q\overline{x}(s)ds \\ &+ h\int_{-h}^{0}\int_{t+\eta}^{t} \dot{\overline{x}}^T(s)R\dot{\overline{x}}(s)ds. \end{aligned} \quad (31)$$

Taking derivative of $V(t)$ in t, we obtain

$$\begin{aligned} \dot{V}(t) &= 2\tilde{e}^T(t)P\dot{\tilde{e}}(t) + \overline{x}^T(t)Q\overline{x}(t) \\ &\quad -\overline{x}^T(t-h)Q\overline{x}(t-h) + h^2\dot{\overline{x}}^T(t)R\dot{\overline{x}}(t) \\ &\quad -h\int_{t-h}^{t} \dot{\overline{x}}^T(s)R\dot{\overline{x}}(s)ds \quad (32) \\ &= 2\zeta^T(t)\Psi_\nu^T P\Gamma\zeta(t) + \zeta^T(t)[e_1^TQe_1 - e_3^TQe_3]\zeta(t) \\ &\quad +h^2\zeta^T(t)(e_4^TRe_4)\zeta(t) - h\int_{t-\tau(t)}^{t} \dot{\overline{x}}^T(s)R\dot{\overline{x}}(s)ds \\ &\quad -h\int_{t-h}^{t-\tau(t)} \dot{\overline{x}}^T(s)R\dot{\overline{x}}(s)ds, \quad (33) \end{aligned}$$

where

$$\begin{aligned} \zeta(t) &= \begin{bmatrix} \zeta_1(t) & \zeta_2(t) & \zeta_3(t) & \zeta_4(t) \end{bmatrix}^T, \\ &\zeta_1(t) = \begin{bmatrix} \overline{x}^T(t) & \overline{x}^T(t-\tau(t)) & \overline{x}^T(t-h) \end{bmatrix}, \\ &\zeta_2(t) = \begin{bmatrix} \dot{\overline{x}}^T(t) & \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t} \overline{x}^T(s)ds \end{bmatrix}, \\ &\zeta_3(t) = \begin{bmatrix} \frac{1}{h-\tau(t)}\int_{t-h}^{t-\tau(t)} \overline{x}^T(s)ds \end{bmatrix}, \\ &\zeta_4(t) = \begin{bmatrix} e_{\tilde{y}}^T(t) & \omega^T(t) & d_\tau^T(t) \end{bmatrix}. \end{aligned} \quad (34)$$

From the Wirtinger-based integral inequality (see, [32]), we have

$$\begin{aligned} &-h\int_{t-\tau(t)}^{t} \dot{\overline{x}}^T(s)R\dot{\overline{x}}(s)ds \\ &\le -h\Big\{\frac{1}{\tau(t)}[\overline{x}(t)-\overline{x}(t-\tau(t))]^TR[\overline{x}(t)-\overline{x}(t-\tau(t))] \\ &+\frac{3}{\tau(t)}[\overline{x}(t)+\overline{x}(t-\tau(t)) - \frac{2}{\tau(t)}\int_{t-\tau(t)}^{t} \overline{x}^T(s)ds]^TR \\ &\times\Big[\overline{x}(t)+\overline{x}(t-\tau(t)) - \frac{2}{\tau(t)}\int_{t-\tau(t)}^{t} \overline{x}^T(s)ds\Big]\Big\}, \end{aligned} \quad (35)$$

and

$$\begin{aligned} &-h\int_{t-h}^{t-\tau(t)} \dot{\overline{x}}^T(s)R\dot{\overline{x}}(s)ds \\ &\le -h\Big\{\frac{1}{h-\tau(t)}[\overline{x}(t-\tau(t))-\overline{x}(t-h)]^TR \\ &\times[\overline{x}(t-\tau(t))-\overline{x}(t-h)] + \frac{3}{h-\tau(t)}\Big[\overline{x}(t-\tau(t)) \\ &+\overline{x}(t-h) - \frac{2}{h-\tau(t)}\int_{t-h}^{t-\tau(t)} \overline{x}^T(s)ds\Big]^TR \\ &\times\Big[\overline{x}(t-\tau(t))+\overline{x}(t-h) - \frac{2}{h-\tau(t)}\int_{t-h}^{t-\tau(t)} \overline{x}^T(s)ds\Big]\Big\}, \end{aligned} \quad (36)$$

Form (35) and (36), we obtain

$$\begin{aligned} &-h\int_{t-\tau(t)}^{t} \dot{\overline{x}}^T(s)R\dot{\overline{x}}(s)ds \\ &\le -\frac{1}{\nu}\zeta^T(t)(e_1-e_2)^TR(e_1-e_2)\zeta(t) \\ &-\frac{3}{\nu}\zeta^T(t)(e_1+e_2-2e_5)^TR(e_1+e_2-2e_5)\zeta(t), \end{aligned} \quad (37)$$

and

$$\begin{aligned} &-h\int_{t-h}^{t-\tau(t)} \dot{\overline{x}}^T(s)R\dot{\overline{x}}(s)ds \\ &\le -\frac{1}{1-\nu}\zeta^T(t)(e_2-e_3)^TR(e_2-e_3)\zeta(t) \\ &-\frac{3}{1-\nu}\zeta^T(t)(e_2+e_3-2e_6)^TR(e_2+e_3-2e_6)\zeta(t). \end{aligned} \quad (38)$$

By using assumption (H_3) and Cauchy inequality [33], we obtain:

$$\begin{aligned} &2e^T(t)\overline{S}\Theta_{\tilde{x}}\sigma_{\tilde{x}z}^{\Delta}(t) \le \epsilon^{-1}e^T(t)\overline{S}\Theta_{\tilde{x}}\Theta_{\tilde{x}}^T\overline{S}^Te(t) \\ &+\epsilon(\sigma_{\tilde{x}z}^{\Delta}(t))^T\sigma_{\tilde{x}z}^{\Delta}(t) \\ &\le \epsilon^{-1}e^T(t)\overline{S}\Theta_{\tilde{x}}\Theta_{\tilde{x}}^T\overline{S}^Te(t) + \epsilon\sigma_{\max}^2\overline{x}^T(t)\overline{x}(t). \end{aligned} \quad (39)$$

Provided that $\Delta > 0$, from the reciprocally convex combination inequality [34], the following inequality is satisfied:

$$\begin{aligned} &-h\int_{t-\tau(t)}^{t} \dot{\overline{x}}^T(s)R\dot{\overline{x}}(s)ds - h\int_{t-h}^{t-\tau(t)} \dot{\overline{x}}^T(s)R\dot{\overline{x}}(s)ds \\ &\le -\zeta^T(t)\Lambda^T\Delta\Lambda\zeta(t). \end{aligned} \quad (40)$$

Denoting $\overline{e}(t) = \begin{bmatrix} \overline{x}^T(t) & \dot{\overline{x}}^T(t) \end{bmatrix}^T$ and applying the free weighting matrix technique [35], from (24) we obtain

$$\begin{aligned} 0 &= 2\overline{e}^T(t)\overline{S}\Big[-\dot{\overline{x}}(t) + A\overline{x}(t) + \Theta_\omega\omega(t) \\ &\quad -LC\overline{x}(t-\tau(t)) - Le_{\tilde{y}}(t) - LEd_\tau(t)\Big] \quad (41) \\ &= -2\overline{e}^T(t)\overline{S}\dot{\overline{x}}(t) + 2\overline{e}^T(t)\overline{S}A\overline{x}(t) \\ &\quad -2\overline{e}^T(t)\overline{X}C\overline{x}(t-\tau(t)) - 2\overline{e}^T(t)\overline{X}e_{\tilde{y}}(t) \\ &\quad +2\overline{e}^T(t)\overline{S}\Theta_\omega\omega(t) - 2\overline{e}^T(t)\overline{X}Ed_\tau(t), \end{aligned} \quad (42)$$

where S is a non-singular matrix with appropriate dimensions, β is a scalar and $X = SL$.

Let us denote $\bar{\lambda}_\omega = \lambda_\omega^2$ and $\bar{\lambda}_d = \lambda_d^2$. From equations (35) to (42), we obtain

$$\begin{aligned} &\dot{V}(t) + \overline{x}^T(t)\overline{x}(t) - \lambda_\omega^2\omega^T(t)\omega(t) \\ &-\lambda_d^2d_\tau^T(t)d_\tau(t) \le \zeta^T(t)\hat{\Xi}(\nu)\zeta(t), \end{aligned} \quad (43)$$

where $\nu \in (0,1)$.

Therefore, inequality

$$\dot{V}(t) + \overline{x}^T(t)\overline{x}(t) - \lambda_\omega^2\omega^T(t)\omega(t) - \lambda_d^2d_\tau^T(t)d_\tau(t) < 0 \quad (44)$$

is satisfied if $\Delta > 0$ and $\hat{\Xi}(\nu) < 0,\ \forall\nu \in (0,1)$. Since $\hat{\Xi}(\nu)$ is convex with respective to ν, $\hat{\Xi}(\nu) < 0\ \forall\nu \in \{0,1\}$ implies $\hat{\Xi}(\nu) < 0\ \forall\nu \in (0,1)$.

By intergrating both sides of (44) from $t = 0$ to ∞, we obtain

$$V(\infty) - V(0) + ||\overline{x}||^2_{\mathcal{L}_2^n} - \lambda_\omega^2 ||\omega||^2_{\mathcal{L}_2^{n_\omega}} - \lambda_d^2 ||d||^2_{\mathcal{L}_2^{n_d}} \le 0. \quad (45)$$

Therefore, under zero initial condition, we obtain the following inequality

$$||\overline{x}||_{\mathcal{L}_2^n} \le \lambda_\omega ||\omega||_{\mathcal{L}_2^{n_\omega}} + \lambda_d ||d||_{\mathcal{L}_2^{n_d}}. \quad (46)$$

Note that minimisation of λ_ω and λ_d can be performed by minimisation of $\bar{\lambda}_\omega$ and $\bar{\lambda}_d$, and the optimal value of λ_ω and λ_d are obtained as $\lambda_\omega = \sqrt{\bar{\lambda}_\omega}$ and $\lambda_d = \sqrt{\bar{\lambda}_d}$. Moreover, since S is invertible, from relation $X = SL$, the gain matrix L can be obtained as $L = S^{-1}X$. The proof is completed. The following algorithm allows us to design the event-triggered state observer as defined in (19).

Algorithm 1

Stage 1 (Estimating nonlinear systems by using RNN): Check assumptions (H_1) and (H_2). Using the newly RNN to train the nonlinear systems, obtain a RNN of the form (22)-(23).
Stage 2 (Designing an ETM): Given scalars $h > 0$, δ_i, $i = 1, 2, 3, 4, 5$, obtain an ETM of the form (25).
Stage 3 (Designing a dynamic state observer):
Step 1: Check conditions (7) and (8).
Step 2: Given a scalar $0 < \xi < 1$ and a positive scalar β, solve the convex optimization problem (32)-(34) to obtain matrices $P > 0$, $Q > 0$, $R > 0$, $S > 0$, X, non-singular S and positive scalars $\bar{\lambda}_\omega$, $\bar{\lambda}_d$, δ, minimized levels λ_ω, λ_d and observer gain L.
Step 3: Obtain an observer of the form (24).

III. A NUMRICAL EXAMPLE

Consider nonlinear system of the form (1)-(3), where

$$\begin{aligned}
F(x,u,\omega) &= \begin{bmatrix} F_1(x,u,\omega) \\ F_2(x,u,\omega) \\ F_3(x,u,\omega) \\ F_4(x,u,\omega) \end{bmatrix}, \\
F_1(x,u,\omega) &= -x_1(t)x_2(t) - 0.25\cos t, \\
F_2(x,u,\omega) &= -0.0081x_2(t) + 0.5\cos t, \\
F_3(x,u,\omega) &= 24.10^{-4}x_1(t) - 0.13x_3(t) \\
&\quad +10^{-4}x_4(t) + 0.5\cos t, \\
F_4(x,u,\omega) &= -0.5x_4(t) + 0.5\sin t + 0.5\cos t.
\end{aligned}$$

For this example, we can check that assumptions (H_1), (H_2) satisfy on $\mathcal{D} = \{x \in \mathbb{R}^4 | |x_i| \le 35\}$ with $M = 1.2252.10^3$, $L_x = 8.3666$, respectively.

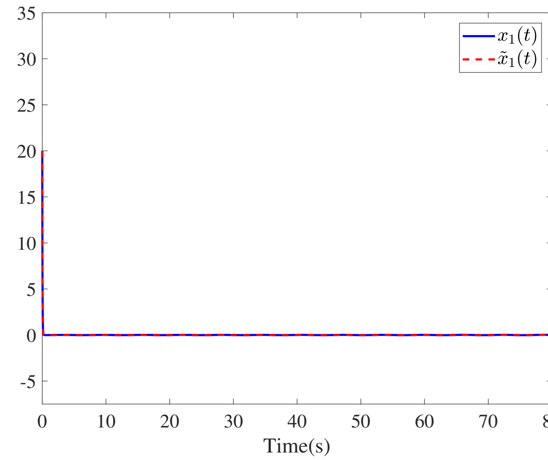

Fig. 1. $x_1(t)$ and $\tilde{x}_1(t)$

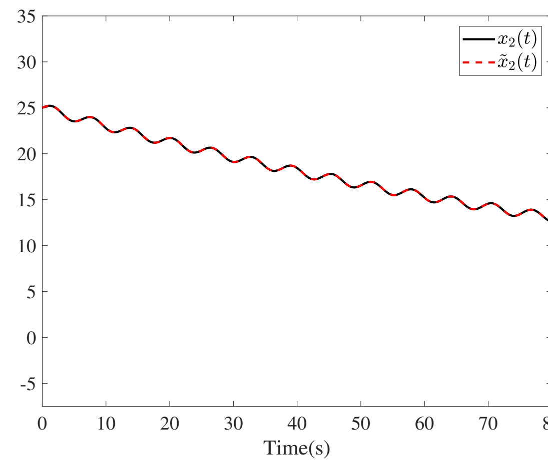

Fig. 2. $x_2(t)$ and $\tilde{x}_2(t)$

By using (9), (11), (13) and (14), a RNN model of the form (17)-(18) which approximates the nonlinear system (1)-(3) is obtained, where

$$\begin{aligned}
A &= \mathrm{diag}\{-25.0022, -0.0081, -0.13, -0.5\}, \\
\Theta_{\tilde{x}} &= \begin{bmatrix} 0.6516 & -0.3158 & -0.0236 & 0.0106 \\ 0 & -0.0001 & 0 & 0 \\ 0.044 & -0.0211 & -0.0013 & -0.0006 \\ 0.0001 & -0.0001 & 0 & 0 \end{bmatrix}, \\
\Theta_u &= \begin{bmatrix} -0.007 \\ 0 \\ 0.0002 \\ 0.5 \end{bmatrix}, \; \Theta_\omega = \begin{bmatrix} -0.7642 \\ 1 \\ 1.0002 \\ 1 \end{bmatrix}.
\end{aligned}$$

Figures 1-4 show the responses of $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and $\tilde{x}_1(t)$, $\tilde{x}_2(t)$, $\tilde{x}_3(t)$, $\tilde{x}_4(t)$, respectively. It is clear from figures 1-4 that the RNN can estimate the four states of the nonlinear system subject to unknown disturbance $\omega(t)$, as expected.

Given scalars $h = 0.02$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.2$, $\delta_4 = 0.35$, $\delta_5 = 0.4$, we obtain an ETM of the form (20).

Let us consider the activation $\sigma(.)$ as $\sigma_i(x_i) = \tanh x_i(t)$, $i = 1, 2, 3, 4$ and the external disturbance in the output vector as $Ed(t) = 0.01\sin t$, $t \ge 0$. It is hot hard to

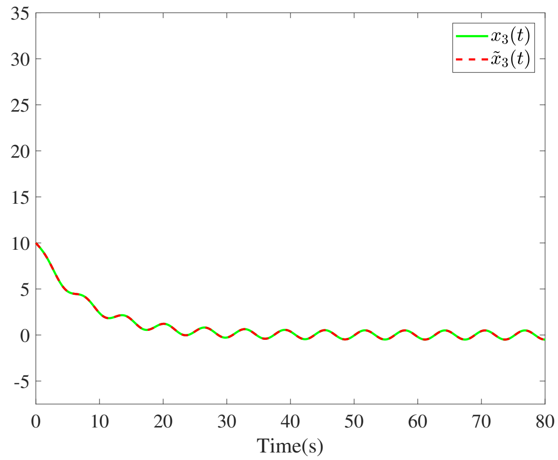

Fig. 3. $x_3(t)$ and $\tilde{x}_3(t)$

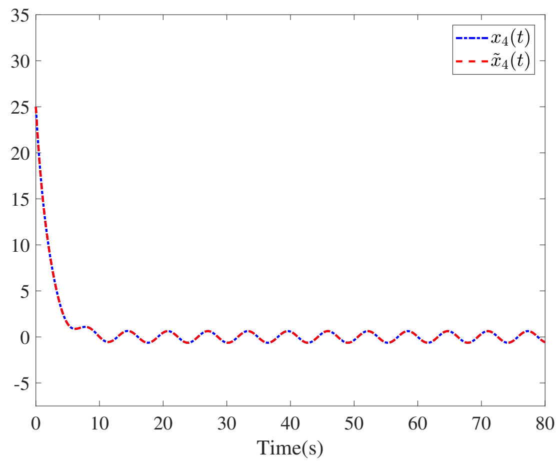

Fig. 4. $x_4(t)$ and $\tilde{x}_4(t)$

see conditions (7) and (8) are satisfied. Given $0 < \xi = 0.6 < 1$ and $\beta = 10.5$, by following Stage 3 of Algorithm 1, we obtain minimized levels $\lambda_\omega = 0.4545$, $\lambda_d = 1.3617 \times 10^{-4}$ and gain matrix $L = \begin{bmatrix} 16.9461 \\ -0.0240 \\ 1.1136 \\ -0.0234 \end{bmatrix}$.

Figure 5 depicts the triggering instants and intervals of ETM (10) while Figures 6-9 show the responses of $\tilde{x}(t)$ and its estimation.

IV. Conclusion

In this paper, we have considered the problem of estimating state vectors of nonlinear systems with the aid of machine learning. A RNN model which predicts the nonlinear system has been first obtained. Then a discrete-time event-triggered mechanism and a state observer based on this mechanism for the RNN model have been designed. A sufficient condition in terms of a convex optimization problem has been established to ensure that the event-triggered state observer can robustly estimate the state vector of the RNN model. Finally, a numerical example and simulation results are given to illustrate the effectiveness of our proposed design method.

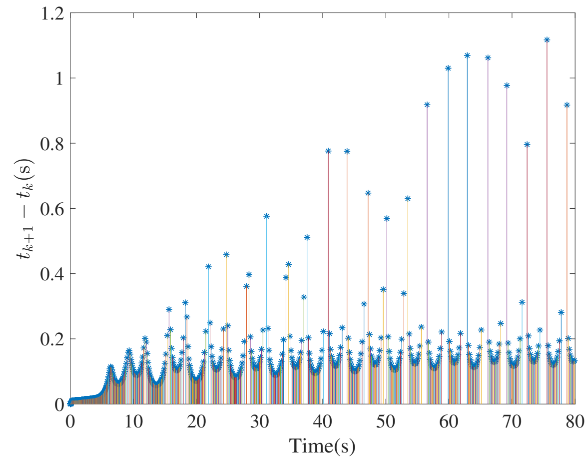

Fig. 5. Triggering instants and intervals of (20)

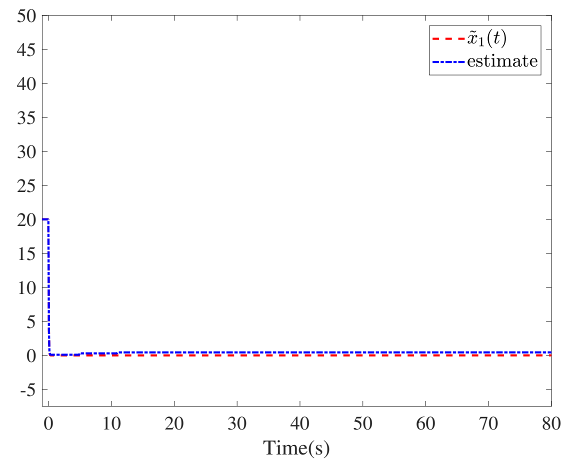

Fig. 6. $\tilde{x}_1(t)$ and its estimation

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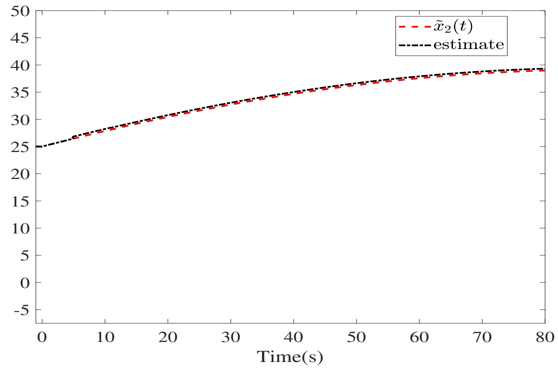

Fig. 7. $\tilde{x}_2(t)$ and its estimation

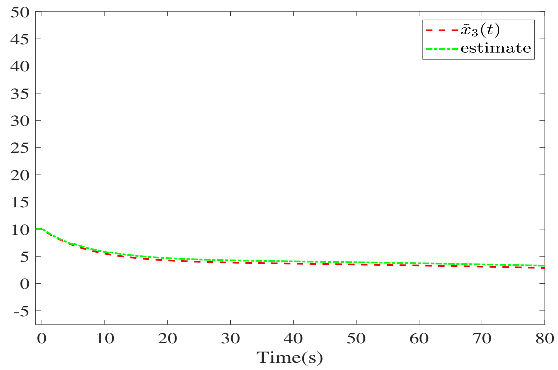

Fig. 8. $\tilde{x}_3(t)$ and its estimation

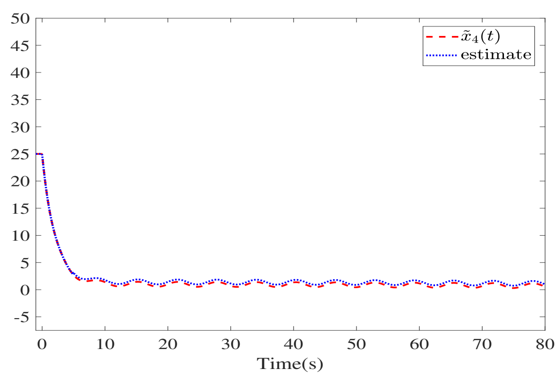

Fig. 9. $\tilde{x}_4(t)$ and its estimation

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